Symmetries and solutions of geometric flows

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Abstract

In this paper, we investigate the group-invariant solutions of the hyperbolic geometric flow on Riemann surfaces by the application of Lie groups in differential equations, including solutions of separation variables, traveling wave type and radial solutions. In the proceeding of reduction, there come elliptic, hyperbolic, and mixed-types of equations. For the elliptic equations, some exact solutions are found, while for the hyperbolic, and mixed-types equations, some implicit solutions are found. We furthermore investigate weather there will be global solution or blow up. Actually, for any given initial metric on $\mathbb{R}^2$ in certain class of metrics, one can always choose suitable initial velocity such that the solution exists for all time; Moreover, if the initial velocity does not satisfy the condition, then the solution blows up at finite time.

1. Introduction

The hyperbolic geometric flow, i.e.,the hyperbolic version of Ricci flow has been introduced by Kong and Liu [3]

$$\frac{\partial^2}{\partial t^2} g_{ij} = -2R_{ij} = -R g_{ij}, \tag{1.1}$$

where R is the scalar curvature.

On Riemann surface $(\mathcal{M}^2, g)$, the above equation (1.1) can be rewritten as scalar equation (see [4])

$$u_{tt} = \Delta \ln u, \tag{1.2}$$

while the metric is written locally in the following form

$$g_{ij} = u(x, y, t)\delta_{ij}.$$

For the hyperbolic geometric flow, we refer to [4].

In this paper, we find exact solutions of equation (1.2), by looking for group invariance and doing some reduction [6]. After reduction, there appear many types of equations with two independent variables, including elliptic, hyperbolic, and mixed-type ones.

Given a further view, in the Ricci case, exact solutions come out easily for the reduced elliptic equations, including solutions of separation variables, traveling wave type, and radial solutions, We will see them in Section 2 and in the Appendix. The difference between the original Ricci flow and the hyperbolic version lies in the reduced hyperbolic, and mixed-types, such as

$$(e^w - 1)w_{zz} - w_{yy} = -e^w w_z^2, \tag{1.3}$$

$$w_{tt} - (1 + a^2)e^{-w} w_{zz} = -w_t^2, \tag{1.4}$$

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where $w = \ln u(x, y, t)$, and $z = t - x$ in the hyperbolic equation (1.3), $z = x + ay$ in the mixed equation (1.4), with a any given constant. Regard of this, we focus on the Cauchy problem of the hyperbolic equations: when will there be global solutions, and when will they blow up? That is discussed in Section 3. And both equation (1.3), and (1.4) share great similarity with the one studied by Kong, Liu and Xu [4]. With a reference to that, we get the following similar theorems. First of all, we are interested in the following "initial" metric on a surface of topological type $\mathbb{R}^2$. For equation (1.3), the initial metric

$$z = 0 : \quad u(x, y, t) = u_0(y). \tag{1.5}$$

For equation (1.4), the initial metric

$$t = 0 : \quad u(x, y, t) = u_0(z), \tag{1.6}$$

where u_0 is a smooth function with bound C^2 norm and satisfies

$$1 < m \leq u_0 \leq M < \infty. \tag{1.7}$$

Theorem 1.1 If $u_1(x)$ is a smooth function with bounded C^1 norm and satisfies

$$u_1(x) \geq \sqrt{\frac{1}{u_0(x) - 1}} |u_0'(x)|, \quad \forall x \in \mathbb{R}\,, \tag{1.8}$$

then the cauchy problem

$$\begin{cases} (e^w - 1)w_{tt} - w_{xx} = -e^w w_t^2, \\ t = 0 : \quad e^w = u_0(x), \quad (e^w)_t = u_1(x) \end{cases} \tag{1.9}$$

has a unique global smooth solution for all $t \in \mathbb{R}$.

Theorem 1.2 Suppose $u_0'(x) \not\equiv 0$, and suppose further that there exist a point $x_0 \in \mathbb{R}$, such that

$$u_0'(x) < 0.$$

For the following initial velocity

$$u_1(x) = \sqrt{\frac{1}{u_0(x) - 1}} u_0'(x), \quad \forall x \in \mathbb{R},$$

the solution of the above Cauchy problem blows up.

The paper is organized as follows. In Section 2, we observe the symmetries of the hyperbolic geometric flow and find some exact solutions. In Section 3, we investigate the global existence and blow up of solutions on the Riemann surface of topological type $\mathbb{R}^2$. Finally, in the Appendix we discuss the symmetries and exact solutions of the Ricci flow, compared with the hyperbolic geometric flow.

2. Hyperbolic geometric flow

Consider the equation (1.1), for any given surface, its metric is locally conformal to Euclidean metric

$$g_{ij} = u(x, y, t)\delta_{ij},$$

where $u(x, y, t) > 0$. Thus

$$\frac{\partial^2}{\partial t^2} u = \frac{\Delta \ln u}{u} u.$$

That is

$$u_{tt} = \Delta \ln u.$$

Let $w = \ln u$, then

$$e^w w_{tt} + e^w w_t^2 - w_{xx} - w_{yy} = 0. \tag{2.10}$$

Note that (x, y) is local coordinate in this section.

2.1 Symmetry Groups

We consider the one-parameter group of infinitesimal transformations (x, y, t, w) (see [10]), given by

$$\begin{aligned} x^* &= x + \varepsilon\xi(x, y, t, w) + o(\varepsilon^2), \\ y^* &= y + \varepsilon\eta(x, y, t, w) + o(\varepsilon^2), \\ t^* &= t + \varepsilon\tau(x, y, t, w) + o(\varepsilon^2), \\ w^* &= w + \varepsilon\phi(x, y, t, w) + o(\varepsilon^2), \end{aligned} \tag{2.11}$$

where ε is group parameter.

It is required that equation (2.10) be invariant under transformation (2.11), and this yields a system of over determined, linear equations for the infinitesimals (x, y, t, w)

$$\begin{cases} \xi = \xi(x, y), \\ \eta = \eta(x, y), \\ \tau = c_1 + c_2 t, \\ \phi = 2c_2 - 2\xi_x, \\ \xi_x - \eta_y = 0, \\ \eta_x + \xi_y = 0. \end{cases}$$

In the view of last two equations, we know that that is equivalent to require that $\xi + i\eta = F(z), z = x + iy$, is holomorphic. Similar to the Ricci case in the Appendix, let $F(z) = k_1 z + k_2$ be linear, where k_1, k_2 are complex constants. That is

$$\begin{cases} \tau = c_1 + c_2 t, \\ \xi = c_3 + c_4 x + c_5 y, \\ \eta = c_6 - c_5 x + c_4 y, \\ \phi = 2c_2 - 2c_4, \end{cases}$$

where c_i $(i = 1, 2, \cdots, 6)$ are real constants.

And the associated vector fields for the one-parameter Lie group of infinitesimal transformations are

$$\begin{cases} V_1 = \partial_t, \\ V_2 = \partial_x, \\ V_3 = \partial_y, \\ V_4 = t\partial_t + 2\partial_w, \\ V_5 = y\partial_x - x\partial_y, \\ V_6 = x\partial_x + y\partial_y - 2\partial_w. \end{cases} \tag{2.12}$$

So the following transformations leave the solutions of equation (2.10) invariant,

$$\begin{aligned} &G_1 : (x, y, t, w) \to (x, y, t + \varepsilon, w), \\ &G_2 : (x, y, t, w) \to (x + \varepsilon, y, t, w), \\ &G_3 : (x, y, t, w) \to (x, y + \varepsilon, t, w), \\ &G_4 : (x, y, t, w) \to (x, y, e^{\varepsilon} t, w + 2\varepsilon), \\ &G_5 : (x, y, t, w) \to (y \sin\varepsilon - x \cos\varepsilon, -x \sin\varepsilon - y \cos\varepsilon, t, w), \\ &G_6 : (x, y, t, w) \to (e^{\varepsilon} x, e^{\varepsilon} y, t, w - 2\varepsilon). \end{aligned}$$

Theorem 2.1 If $w = f(x, y, t)$ is a solution to the equation (2.10), then so is

$$\begin{aligned}
w_1 &= f(x, y, t-\varepsilon),\\
w_2 &= f(x-\varepsilon, y, t),\\
w_3 &= f(x, y-\varepsilon, t),\\
w_4 &= f(x, y, e^{-\varepsilon}t) + 2\varepsilon,\\
w_5 &= f(-y\sin\varepsilon - x\cos\varepsilon, x\sin\varepsilon - y\cos\varepsilon, t),\\
w_6 &= f(e^{-\varepsilon}x, e^{-\varepsilon}y, t) - 2\varepsilon.
\end{aligned}$$

2.2 Reductions and Solutions

With a reference to Xu [10], we have the following theorem,

Theorem 2.2 The operators in (2.12) generate an optimal system S

$$\begin{array}{lll}
(a) & V_6 + a_4V_4 + a_5V_5, \quad a_6 \neq 0.\\
(b) & V_5 + a_4V_4, \quad a_6 = 0, \quad a_5 \neq 0.\\
(c_1) & V_4, \quad a_6 = a_5 = 0, \quad a_4 \neq 0.\\
(c_2) & V_4 + V_2, \quad a_6 = a_5 = 0, \quad a_4 \neq 0.\\
(c_3) & V_4 + V_3, \quad a_6 = a_5 = 0, \quad a_4 \neq 0.\\
(d_1) & V_1, \quad a_6 = a_5 = a_4 = 0, \quad a_1 \neq 0.\\
(d_2) & V_1 + V_2, \quad a_6 = a_5 = a_4 = 0, \quad a_1 \neq 0.\\
(d_3) & V_1 - V_3, \quad a_6 = a_5 = a_4 = 0, \quad a_1 \neq 0.\\
(e) & V_2 + a_3V_3, \quad a_6 = a_5 = a_4 = a_1 = 0, \quad a_2 \neq 0.\\
(f) & V_3, \quad a_6 = a_5 = a_4 = a_1 = a_2 = 0.
\end{array}$$

By this, now we will discuss the reduction and solutions of equation (2.10).

(a)

$$\begin{aligned}
V &= V_6 + a_4V_4 + a_5V_5\\
&= (x + a_5y)\partial_x + (y - a_5x)\partial_y + a_4t\partial_t + (2a_4 - 2)\partial_w.
\end{aligned}$$

Let $a_4 = 1, a_5 = a$, where a is an arbitrary constant, then

$$V = (x + ay)\partial_x + (y - ax)\partial_y + t\partial_t.$$

The corresponding characteristic equations are

$$\frac{dx}{x + ay} = \frac{dy}{y - ax} = \frac{dt}{t}.$$

It leads to the invariance

$$u = \frac{x + ay}{t}, \quad v = \frac{ax - y}{t}, \quad h = e^w.$$

That is, the invariant solution takes the form

$$w = \ln h(u, v).$$

Therefore, the equation (2.10) can be reduced to

$$h^2(u^2h_{uu} + 2uvh_{uv} + v^2h_{vv} + 2uh_u + 2vh_v) - 2(1 + a^2)(h_{uu}h + h_{vv}h - h_u^2 - h_v^2) = 0. \tag{2.13}$$

Now look for the "traveling wave solution " of the above equation (2.13). Let $z = u + \delta v, h = h(z)$, thus

$$h^2(z^2h'' + 2zh') - 2(1 + a^2)(1 + \delta^2)(h''h - (h')^2) = 0.$$

Denote $\lambda = (1+a^2)(1+\delta^2) > 0$, and we have

$$z^2 h' - 2\lambda \frac{h'}{h} = c_1,$$

with c_1 an arbitrary constant. It has the solutions

$$h(z) = \frac{2\lambda}{z^2},$$

if $c_1 = 0$. And the original metric

$$u(x,y,t) = h(z) = \frac{2\lambda t^2}{(x+ay+\delta(ax-y))^2}$$

for $z = u + \delta v = \frac{x+ay+\delta(ax-y)}{t}$. And it always has singularity at the line $x + ay + \delta(ax - y) = 0$. Moreover, when $t \to +\infty, u(x,y,t) \to +\infty$. It means that the metric dilates infinitely.

Acting by $G_1, G_2, \cdots, G_6$, one obtain more solutions of equation (2.10) by Theorem 2.1,

$$u(x,y,t) = e^{-2\varepsilon_6+2\varepsilon_4} \frac{2\lambda \bar{t}^2}{(\bar{x}+a\bar{y}+\delta(a\bar{x}-\bar{y}))^2},$$

where

$$\begin{aligned}
\bar{x} &= -xe^{-\varepsilon_6}\cos\varepsilon_5 - ye^{-\varepsilon_6}\sin\varepsilon_5 - \varepsilon_2,\\
\bar{y} &= xe^{-\varepsilon_6}\sin\varepsilon_5 - ye^{-\varepsilon_6}\cos\varepsilon_5 - \varepsilon_3,\\
\bar{t} &= e^{-\varepsilon_4}t - \varepsilon_1,
\end{aligned}$$

and ε_i $(i = 1, 2, \cdots, 6)$ are parameters.

(b)

$$\begin{aligned}
V &= V_5 + a_4 V_4\\
&= y\partial_x - x\partial_y + a_4 t\partial_t + 2a_4\partial_w,
\end{aligned}$$

with invariance

$$r = \sqrt{x^2+y^2}, \quad h = t^{-2}e^w.$$

That is, the invariant solution takes the form

$$w = \ln h(r) + 2\ln t.$$

And the equation (2.10) reduces to

$$(h''h - (h')^2)r + h'h - 2h^3 r = 0.$$

Since $u(x,y,t) > 0$, we find the following solution

$$h(r) = \frac{1 + (\tan(\frac{\ln r - c_2}{2c_1}))^2}{4r^2c_1^2}.$$

Therefore, the metric

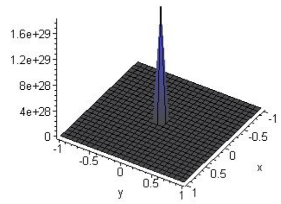
$$u(x,y,t) = h(r)t^2 = \frac{1 + (\tan(\frac{\ln r - c_2}{2c_1}))^2}{4r^2c_1^2} t^2,$$

where $r = \sqrt{x^2+y^2}$. It has singularity at $(0,0,t)$, as in Figure 1.

Acting by $G_1, G_2, \cdots, G_6$, one obtains more solutions of equation (2.10),

$$u(x,y,t) = e^{-2\varepsilon_6+2\varepsilon_4}\bar{t}^2 \frac{1 + (\tan(\frac{\ln \bar{r} - c_2}{2c_1}))^2}{4\bar{r}^2c_1^2},$$

Figure 1: Radial case



where

$$\begin{aligned}\bar{t} &= e^{-\varepsilon_4}t - \varepsilon_1,\\ \bar{r}^2 &= (-xe^{-\varepsilon_6}\cos\varepsilon_5 - ye^{-\varepsilon_6}\sin\varepsilon_5 - \varepsilon_2)^2\\ &\quad + (xe^{-\varepsilon_6}\sin\varepsilon_5 - ye^{-\varepsilon_6}\cos\varepsilon_5 - \varepsilon_3)^2.\end{aligned}$$

(c1)

$$V = V_4 = t\partial_t + 2\partial_w,$$

with invariance

$$x, \quad y, \quad h = t^{-2}e^w.$$

That is, the invariant solution takes the form

$$w = \ln h(x, y) + 2\ln t.$$

And the equation (2.10) reduces to

$$h_{xx}h - h_x^2 + h_{yy}h - h_y^2 - 2h^3 = 0.$$

Notice that, it is elliptic. Now look for the "traveling wave solution" of the above equation. Let $z = x + \delta y, h = h(z)$, therefore, we have

$$(1 + \delta^2)(\frac{h'}{h})' = 2h.$$

Denote $\lambda = 1 + \delta^2$, and since $u(x, y, t) > 0$, we have solutions

(c11)

$$h(z) = \frac{-1}{4c_1}[1 + (\tan(\frac{\sqrt{-\lambda c_1}(z + c_2)^2}{2\lambda c_1}))^2], \quad \text{if } c_1 < 0;$$

(c12)

$$h(z) = \frac{(1 + \delta^2)}{(z + c)^2}, \quad \text{if } c_1 = 0.$$

Hence the metric $u(x, y, t) = h(z)t^2$, taking the form of separation variables, with $z = x + \delta y$, and $h(z)$ takes forms in cases (c11), (c12), respectively.

For case (c11), $u(x, y, t)$ has no singularity. While for case (c12), weather $u(x, y, t)$ has singularity or not depends on the value of $h(0)$. Actually, if $h(0) = \frac{(1+\delta^2)}{c^2}$ is small enough, i.e. c^2 is large enough, then $z + c$ will nowhere be zero, since $z = x + \delta y$ is always small for the snake of local coordinate (x, y) around $(0, 0)$. So $u(x, y, t)$ has no singularity if $h(0)$ is small enough. Conversely, if $h(0)$ is large enough, i.e. c^2 is small, then $u(x, y, t)$ has singularity at the surface $x + \delta y + c = 0$, in this case.

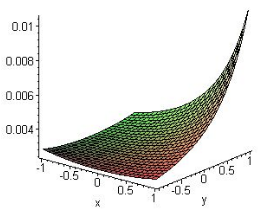


Figure 2: case (c11)

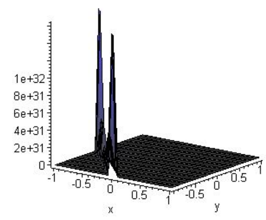


Figure 3: case (c12) $c = 1$

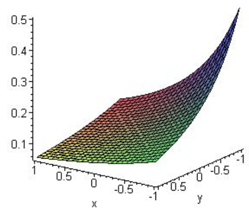


Figure 4: case (c12) $c = 4$

In case (c11), Figure 2 is always smooth, and in case (c12), viewing Figure3 and Figure4, they are different depending on the constant c.

Also, acting by $G_1, G_2, \cdots, G_6$, one obtain more solutions of equation (2.10) by Theorem 2.1,

$$u(x,y,t) = \frac{-1}{4c_1}\{1 + (\tan(\frac{\sqrt{-\lambda c_1}(\bar{z}+c_2)^2}{2\lambda c_1}))^2\}e^{-2\varepsilon_6+2\varepsilon_4}\bar{t}^2, \quad \text{if } c_1 < 0;$$

$$u(x,y,t) = \frac{2(1+\delta^2)}{(\bar{z}+c)^2}e^{-2\varepsilon_6+\varepsilon_4}\bar{t}, \quad \text{if } c_1 = 0;$$

where

$$\begin{aligned} \bar{t} &= e^{-\varepsilon_4}t - \varepsilon_1, \\ \bar{z} &= (-xe^{-\varepsilon_6}\cos\varepsilon_5 - ye^{-\varepsilon_6}\sin\varepsilon_5 - \varepsilon_2) \\ &\quad + \delta(xe^{-\varepsilon_6}\sin\varepsilon_5 - ye^{-\varepsilon_6}\cos\varepsilon_5 - \varepsilon_3). \end{aligned}$$

(d2)

$$V = V_1 + V_2 = \partial_t + \partial_x,$$

with invariance

$$z = t - x, \quad y.$$

And the equation (2.10) reduces to

$$w_{zz} + w_{yy} = e^w(w_{zz} + w_z^2), \tag{2.14}$$

which is actually a mixed equation, and we will have more discussion about it in section 3. Now look for the "traveling wave solution" of the above equation (2.14). Let $\xi = z + \delta y$, denote $\lambda = 1 + \delta^2$, more exactly, we have

$$e^w - \lambda w = c_1\xi + c_2.$$

That is

$$e^w - \lambda w = c_1(x - t + \delta y) + c_2.$$

And we have the following pictures for (y, x, w) at different time. They seem different to each other, by Figure 5, and Figure 6. Here we set $c_1 = c_2 = 1, \delta = 1$. If we set $y = 0$, then we have another kind of picture for (t, w, x), viewing Figure 7.

(d3)

$$V = V_1 - V_3 = \partial_t - \partial_y$$

with invariance

$$z = t + y, \quad x.$$

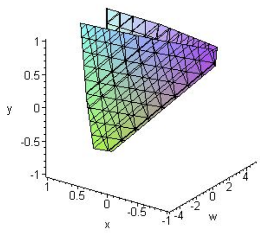


Figure 5: case (d2) $t = 0.1$

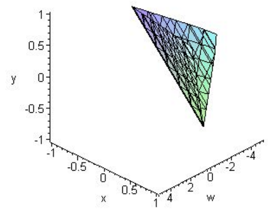


Figure 6: case (d2) $t = 1$

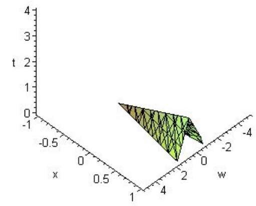


Figure 7: case (d2) $y = 0$

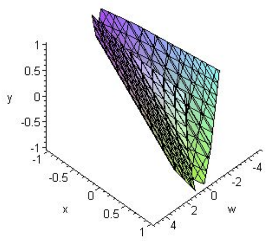


Figure 8: case (d3) $t = 0.1$

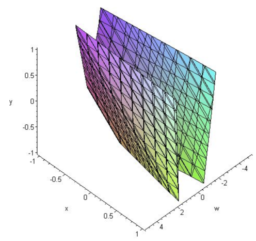


Figure 9: case (d3) $t = 1$

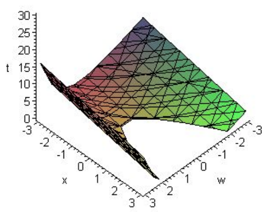


Figure 10: case (d3) $y = 0$

That is to say, the invariant solution takes the form

$$w = w(z, x).$$

And the equation (2.10) reduces to

$$w_{zz} + w_{xx} = e^w(w_{zz} + w_z^2). \tag{2.15}$$

Also, it is of mixed-type. Now look for the "traveling wave solution " of the above equation (2.15). Let $\eta = z + \delta x = t + y + \delta x$, denote $\lambda = 1 + \delta^2$, it follows that

$$e^w - \lambda w = c_1\eta + c_2.$$

That is

$$e^w - \lambda w = c_1(t + y + \delta x) + c_2.$$

It is similar to case (d2). Set $c_1 = c_2 = 1, \delta = 1$, and we have pictures for different time, referring to Figure 8, and Figure 9. If we set $y = 0$, then we have another kind of picture for (t, w, x), viewing Figure 10.

(e)

$$V = V_2 + aV_3 = \partial_x + a\partial_y,$$

again, with invariance

$$z = x - ay, \quad t.$$

That is to say, the invariant solution takes the form

$$w = w(z, t).$$

Thus the equation (2.10) reduces to

$$e^w(w_{tt} + w_t^2) = w_{zz} + a^2 w_{zz}. \tag{2.16}$$

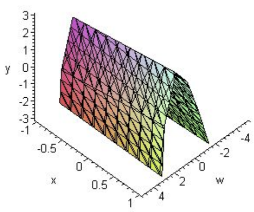


Figure 11: case (e) $t = 0.1$

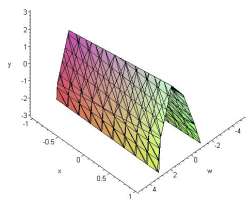


Figure 12: case (e)$t = 2$

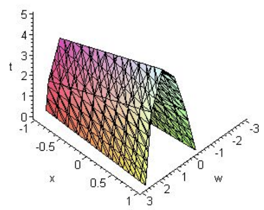


Figure 13: case (e)$y = 0$

Now look for the "traveling wave solution " of the above equation (2.16). Let $\tau = z - \delta t = x - ay - \delta t$, we have

$$e^w - (1 + a^2)w = c_1\tau + c_2.$$

That is

$$e^w - (1 + a^2)w = c_1(x - ay - \delta t) + c_2.$$

It is similar to case (d2). Set $c_1 = c_2 = 1, \delta = 1, a = 2$, we have pictures for different time, viewing Figure 11, and Figure 12. If we set $y = 0$, then we have another kind of picture for (t, w, x), viewing Figure 13.

3. Global existence and blow up of solutions

In this section, suppose the Riemann surface — of topological type $\mathbb{R}^2$, has the following initial metric (1.5), (1.6), (1.7). So here, (x, y) is global coordinates on the Riemann surface. The discussion in this section is similar to Kong, Liu and Xu [4].

For equation (2.14), which is equivalent to

$$(e^w - 1)w_{zz} - w_{yy} = -e^w w_z^2, \tag{3.17}$$

it is equation of mixed-type. That is to say, when $e^w - 1 > 0$, it is hyperbolic, while if $e^w - 1 < 0$, it is elliptic. Now, let us investigate the hyperbolic case explicitly. Let

$$w_2 = w_z, \quad w_3 = w_y,$$

then (3.17) can be rewritten as the following quasilinear system of first order

$$\begin{cases} w_z = w_2, \\ (w_3)_z - (w_2)_y = 0, \\ (w_2)_z - \dfrac{1}{e^w - 1}(w_3)_y = -\dfrac{e^w}{e^w - 1}w_2^2. \end{cases} \tag{3.18}$$

Introduce

$$p = w_2 + \sqrt{\frac{1}{e^w - 1}}w_3, \quad q = w_2 - \sqrt{\frac{1}{e^w - 1}}w_3.$$

Denote

$$\lambda = \sqrt{\frac{1}{e^w - 1}}, \quad \kappa = -\frac{e^w}{4(e^w - 1)},$$

we have

$$\begin{cases} w_z = \dfrac{p+q}{2}, \\ p_z - \lambda p_y = \kappa(p^2 + 3pq), \\ q_z + \lambda q_y = \kappa(q^2 + 3pq). \end{cases} \tag{3.19}$$

Observe that $z = t - x$. By an abuse of notation, replace z by t, y by x. And the Cauchy problem

$$\begin{cases} (e^w - 1)w_{tt} - w_{xx} = -e^w w_t^2, \\ t = 0: \quad e^w = u_0(x), \quad (e^w)_t = u_1(x), \end{cases} \tag{3.20}$$

is equivalent to

$$\begin{cases} w_t = \dfrac{p+q}{2}, \\ p_t - \lambda p_x = \kappa(p^2 + 3pq), \\ q_t + \lambda q_x = \kappa(q^2 + 3pq), \\ t = 0: \quad w = \ln u_0(x), \quad p = p_0(x), \quad q = q_0(x), \end{cases} \tag{3.21}$$

where

$$\begin{cases} p_0(x) = \dfrac{u_1(x)}{u_0(x)} + \sqrt{\dfrac{1}{u_0(x) - 1}} \dfrac{u_0'(x)}{u_0(x)}, \\ q_0(x) = \dfrac{u_1(x)}{u_0(x)} - \sqrt{\dfrac{1}{u_0(x) - 1}} \dfrac{u_0'(x)}{u_0(x)}, \\ \lambda = \sqrt{\dfrac{1}{e^w - 1}}, \quad \kappa = -\dfrac{e^w}{4(e^w - 1)}. \end{cases} \tag{3.22}$$

Theorem 3.1 If $u_1(x)$ is a smooth function with bounded C^1 norm and satisfies

$$u_1(x) \geq \sqrt{\frac{1}{u_0(x) - 1}} |u_0'(x)|, \quad \forall x \in \mathbb{R}, \tag{3.23}$$

then the Cauchy problem (3.21), hence (3.20) has a unique global smooth solution for all $t \in \mathbb{R}$.

Lemma 3.1 In the existence domain of the smooth solution of the Cauchy problem(3.21), it holds that

$$\begin{aligned} 0 \leq p(t,x) \leq \sup_{y \in \mathbb{R}} p_0(y), \\ 0 \leq q(t,x) \leq \sup_{y \in \mathbb{R}} q_0(y). \end{aligned} \tag{3.24}$$

Proof Given any point (t, x), there're two characteristics passing through it, defined by $\xi = \xi_\pm(\tau; t, x)$, satisfying

$$\begin{cases} \dfrac{d\xi_\pm}{d\tau} = \pm\lambda(\tau; \xi_\pm(\tau; t, x)), \\ \xi_\pm(t; t, x) = x, \end{cases} \tag{3.25}$$

respectively. So along the characteristic $\xi = \xi_-(\tau; t, x)$, it holds that

$$p(t,x) = p_0(\xi_-(\tau; t, x)) \exp\left(\int_0^t \kappa(p + 3q)(\tau; \xi_-(\tau; t, x)) d\tau\right). \tag{3.26}$$

Noting(3.22) and (3.23), we have

$$p_0(x) \geq 0, \quad \forall x \in \mathbb{R},$$

and

$$\kappa \leq -\frac{1}{4} < 0. \tag{3.27}$$

Thus, we obtain

$$p(t, x) \geq 0.$$

Similarly, we have

$$q(t, x) \geq 0.$$

With respect to the fact (3.27), and note that (3.26), we know

$$0 \leq p(t, x) \leq p_0(\xi_-(0; t, x)) \leq \sup_{y \in \mathbb{R}} p_0(y),$$

and similar for $q(t, x)$. □

Since we have assume $u_1(x)$ is a smooth function with bounded C^1 norm, thus

$$\begin{cases} 0 \leq p_0(x) \leq \sup\limits_{y \in \mathbb{R}} p_0(y) \triangleq P_0 < \infty, \\ 0 \leq q_0(x) \leq \sup\limits_{y \in \mathbb{R}} q_0(y) \triangleq Q_0 < \infty. \end{cases} \tag{3.28}$$

Now we can estimate $w(t, x)$ first of all. Note the first equation in (3.21),

$$w(t, x) = \ln u_0(x) + \int_0^t \frac{p+q}{2}(\tau, x) d\tau.$$

By lemma 3.1, we have

$$\ln u_0(x) \leq w(t, x) \leq \ln u_0(x) + \frac{P_0 + Q_0}{2} t. \tag{3.29}$$

Next we have to estimate p_x and q_x. Let

$$r = p_x, \quad s = q_x,$$

then we have

Lemma 3.2

$$\begin{cases} r_t - \lambda r_x = \kappa((3p + 2q)r + 3ps) + \dfrac{\kappa\lambda}{2}(p - q)(p^2 + 3pq), \\ s_t + \lambda s_x = \kappa((3q + 2q)s + 3qr) + \dfrac{\kappa\lambda}{2}(p - q)(q^2 + 3pq). \end{cases} \tag{3.30}$$

Proof By a direct calculation, we can easily prove (3.30). □

Denote

$$r_0(x) \triangleq p_0'(x), \quad s_0(x) \triangleq q_0'(x).$$

Lemma 3.3 In the existence domain of the smooth solution, it holds that

$$|r(t, x)|, |s(t, x)| \leq \max\{\sup_{y \in \mathbb{R}} |r_0(y)|, \sup_{y \in \mathbb{R}} |s_0(y)|\} + C_1(\exp(G_0 t) - 1), \tag{3.31}$$

where C_1, G_0 are some constants.

Proof Let

$$\begin{cases} A = \kappa(3p + 2q), \quad B = 3\kappa p, \\ \bar{A} = \kappa(3q + 2p), \quad \bar{B} = 3\kappa q, \\ h_1 = \dfrac{\kappa\lambda}{2}(p - q)(p^2 + 3pq), \\ h_2 = \dfrac{\kappa\lambda}{2}(p - q)(q^2 + 3pq), \end{cases} \tag{3.32}$$

then the system (3.30) can be rewritten as

$$\begin{cases} r_t - \lambda r_x = Ar + Bs + h_1, \\ s_t + \lambda s_x = \bar{A}s + \bar{B}r + h_2. \end{cases} \tag{3.33}$$

By Lemma 3.1, $A, \bar{A}, B, \bar{B} \geq 0, A \leq B, \bar{A} \leq \bar{B}$. According to the terminology in Kong [2], system (3.33) is weakly disspative. Therefore , it follows from Theorem 2.3 in Kong [2] that

$$|r(t,x)|, |s(t,x)| \leq \max\{\sup_{y\in I(0)} |r_0(y)|, \sup_{y\in I(0)} |s_0(y)|\} + \int_0^t \max(\sup_{I(s)} |h_1|, \sup_{I(s)} |h_2|)ds, \tag{3.34}$$

where

$$I(0) = [\xi_-(0;t,x), \xi_+(0;t,x)],$$

$$I(s) = \{(s,\xi), \xi \in [\xi_-(s;t,x), \xi_+(s;t,x)]\}.$$

Note that we have assumed that $1 < m \leq u_0(x) \leq M < \infty$, and with the estimation of (3.29), hence

$$\frac{1}{4}\frac{m}{(Me^{G_0 t} - 1)^{\frac{3}{2}}} \leq |\kappa\lambda| \leq \frac{1}{4}\frac{Me^{G_0 t}}{(m-1)^{\frac{3}{2}}}.$$

Notice that

$$\int_0^t \exp(G_0\tau)d\tau = \frac{1}{G_0}(e^{G_0 t} - 1).$$

Therefore, it follows that

$$\int_0^t \max(\sup_{I(s)} |h_1|, \sup_{I(s)} |h_2|)ds \leq C_1(e^{G_0 t} - 1).$$

Combine with (3.34), we get the result. □

We have estimated $w(t,x)$ by (3.29), furthermore,

$$(w_x)_t = \frac{1}{2}(p_x + q_x) = \frac{1}{2}(r+s).$$

That is

$$w_x = w_x(0,x) + \frac{1}{2}\int_0^t (r+s)(\tau,x)d\tau.$$

With respect to (3.31), we know that

$$|w_x(t,x)| \leq \frac{|u_0'(x)|}{u_0(x)} + (\sup_{y\in\mathbb{R}} |r_0(y)| + \sup_{y\in\mathbb{R}} |s_0(y)| + C_1)t + \frac{C_1}{G_0}(e^{G_0 t} - 1).$$

Proof of Theorem 3.1:

Proof Following the estimation of $w(t,x)$ in (3.29), we have

$$\sqrt{\frac{1}{Me^{G_0 t} - 1}} \leq \lambda \leq \sqrt{\frac{1}{m-1}}.$$

Referring to [4], for any interval $[a,b]$, introduce the following triangle domain

$$\triangle_{[a,b]} = \{(t,x) \mid a + \sqrt{\frac{1}{m-1}}t \leq x \leq b + \sqrt{\frac{1}{M-1}}t\}.$$

In order to prove Theorem 3.1, it suffices to prove that, for any interval $[a,b]$, the Cauchy problem (3.20) has a unique smooth solution in $\triangle_{[a,b]}$. Recall all the estimation we have made above, here comes the priori estimation for any point (t,x) in $\triangle_{[a,b]}$,

$$0 \leq p(t,x) \leq \sup_{y\in[a,b]} p_0(y) \leq P_0, \tag{3.35}$$

$$0 \le q(t,x) \le \sup_{y\in[a,b]} q_0(y) \le Q_0, \tag{3.36}$$

$$|r(t,x)|, |s(t,x)| \le \max\{\sup_{y\in[a,b]} |r_0(y)|, \sup_{y\in[a,b]} |s_0(y)|\} + C_1(\exp(G_0 t_{[a,b]}) - 1) < \infty, \tag{3.37}$$

$$\ln m \le w(t,x) \le \ln M + \frac{P_0+Q_0}{2} t_{[a,b]}, \tag{3.38}$$

$$|w_x(t,x)| \le \frac{\sup_{x\in[a,b]} |u_0'(x)|}{u_0(x)} + (\sup_{y\in[a,b]} |r_0(y)| + \sup_{y\in[a,b]} |s_0(y)| + C_1)t_{[a,b]} + \frac{C_1}{G_0}(e^{G_0 t_{[a,b]}} - 1), \tag{3.39}$$

where

$$t_{[a,b]} = \frac{b-a}{\sqrt{\frac{1}{m-1}} - \sqrt{\frac{1}{M-1}}}. \tag{3.40}$$

The above priori estimation implies that the Cauchy problem (3.21), has a unique smooth solution on the whole triangle $\triangle_{[a,b]}$, which proves Theorem 3.1. □

Theorem 3.2 Suppose $u_0'(x) \not\equiv 0$, and suppose further that there exist a point $x_0 \in \mathbb{R}$, such that

$$u_0'(x) < 0.$$

For the following initial velocity

$$u_1(x) = \sqrt{\frac{1}{\exp(u_0(x)-1)}} u_0'(x), \quad \forall x \in \mathbb{R},$$

the solution of the Cauchy problem (3.21) blows up.

Proof As above, it suffices to study the equivalent Cauchy problem,

$$\begin{cases} w_t = \dfrac{p+q}{2}, \\ p_t - \lambda p_x = \kappa(p^2 + 3pq), \\ q_t + \lambda q_x = \kappa(q^2 + 3pq), \\ t = 0: \quad w = \ln u_0(x), \quad p = p_0(x), \quad q = 0. \end{cases} \tag{3.41}$$

And it reduces to

$$\begin{cases} w_t = \dfrac{p}{2}, \\ p_t - \lambda p_x = \kappa p^2, \\ t = 0: \quad w = \ln u_0(x), \quad p = p_0(x). \end{cases} \tag{3.42}$$

Note (3.27), so

$$p_t - \lambda p_x \le -\frac{1}{4}p^2.$$

Along the characteristic $\xi = \xi_-(\tau; t, x)$, it follows that

$$p(t,x) \le \frac{p_0(\xi_-(0;t,x))}{1 + \frac{1}{4}p_0(\xi_-(0;t,x))t}. \tag{3.43}$$

In particular,

$$p(t, \xi_-(t; 0, x_0)) \le \frac{p_0(x_0)}{1 + \frac{1}{4}p_0(x_0)t}. \tag{3.44}$$

Note that we have done some replacements at the beginning, so here the t, is actually $t - x$, may take negative value. Knowing that $p_0(x_0) < 0$, we have

$$\frac{p_0(x_0)}{1 + \frac{1}{4}p_0(x_0)t} \searrow -\infty, \quad \text{as} \quad t \nearrow -\frac{4}{p_0(x_0)},$$

which implies

$$p(t, \xi_-(t; 0, x_0)) \searrow -\infty,$$

with respect to the fact

$$\frac{dp}{d\tau} = \kappa p^2 < 0,$$

along the characteristic $\xi = \xi_-(\tau; 0, x_0)$, and (3.44). Thus the solution of the Cauchy problem (3.21) blows up. □

Remark 1 Actually, there is another chance for system (3.42) to blow up. Now assume $u_0'(x) \leq 0$, hence $p_0(x) \leq 0$, and

$$p_0(x_0) = \inf_{x \in \mathbb{R}} p_0(x) < 0.$$

Under these hypothesis,

$$\begin{aligned} &|p_0(\xi_-(0; t, x))| < |p_0(x_0)|, \\ &p_0(\xi_-(0; t, x)) > p_0(x_0). \end{aligned} \tag{3.45}$$

Noting (3.43),

$$|p(t, x)| \leq \frac{|p_0(x_0)|}{1 - \frac{1}{4}|p_0(x_0)|t}, \tag{3.46}$$

for $t < T_1$, where $T_1 = \frac{4}{|p_0(x_0)|}$. Following (3.42), we have

$$w(t, x) = \ln u_0(x) + \int_0^t \frac{p}{2}(\tau, x) d\tau.$$

Note that (3.46),

$$|w(t, x)| \leq \ln M - 2\ln(1 - \frac{1}{4}|p_0(x_0)|t).$$

Set

$$t_* = \frac{4(1 - \sqrt{M})}{|p_0(x_0)|} < T_1,$$

thus we have $w(t_*, x) = 0$. That's to say, $w(t, x)$ has been 0, hence κ, λ have both been tending to $+\infty$, before t reaches T_1, which implies the system (3.42) has been blowed up in this way.

For equation (2.15), it is similar to the above case of equation (3.17), and we will have no more discussion.

For equation (2.16), that is

$$w_{tt} - (1 + a^2)e^{-w} w_{zz} = -w_t^2. \tag{3.47}$$

Setting $\tilde{z} = \sqrt{1 + a^2} z$, it is just the equation appearing in Kong, Liu and Xu [4], in which weather there is global solution or solution blowing up has been studied.

Notice that, here $\tilde{z} = \sqrt{1 + a^2}(x - ay)$, and when $a = 0$, it reduce to the case — surface of topological type $\mathbb{R}^2$, with initial metric at $t = 0 : ds^2 = u_0(x)(dx^2 + dy^2)$, which has been studied in Kong, Liu and Xu [4].

4. Appendix : Ricci flow on Riemann surface

The Ricci flow

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij} = -R g_{ij},$$

where R is the scalar curvature. For any surface, its metric is locally conformal to Euclidean metric,

$$g_{ij} = u(x, y, t)\delta_{ij},$$

where $u(x, y, t) > 0$. Thus

$$\frac{\partial}{\partial t}u = \frac{\Delta \ln u}{u}u.$$

That is

$$u_t = \Delta \ln u.$$

Let $w = \ln u$, it becomes

$$w_{xx} + w_{yy} - e^w w_t = 0. \tag{4.48}$$

4.1 Symmetry Groups

We consider the one-parameter group of infinitesimal transformations (x, y, t, w) (see [10]), given by

$$\begin{aligned} x^* &= x + \varepsilon\xi(x, y, t, w) + o(\varepsilon^2), \\ y^* &= y + \varepsilon\eta(x, y, t, w) + o(\varepsilon^2), \\ t^* &= t + \varepsilon\tau(x, y, t, w) + o(\varepsilon^2), \\ w^* &= w + \varepsilon\phi(x, y, t, w) + o(\varepsilon^2), \end{aligned} \tag{4.49}$$

where ε is group parameter.

It is required that equation (4.48) be invariant under transformation (4.49), and this yields a system of over determined, linear equations for the infinitesimals (x, y, t, w),

$$\begin{cases} \xi = \xi(x, y), \\ \eta = \eta(x, y), \\ \tau = c_1 + c_2 t, \\ \phi = c_2 - 2\xi_x, \\ \xi_x - \eta_y = 0, \\ \eta_x + \xi_y = 0. \end{cases}$$

In the view of last two equations, we know that that is equivalent to require that $\xi + i\eta = F(z), z = x + iy$, is holomorphic.

Now consider the simplest case, let $F(z) = k_1 z + k_2$ be linear, where k_1, k_2 are complex constants. Hence

$$\begin{cases} \tau = c_1 + c_2 t, \\ \xi = c_3 + c_4 x + c_5 y, \\ \eta = c_6 - c_5 x + c_4 y, \\ \phi = c_2 - 2c_4, \end{cases}$$

where c_i $(i = 1, 2, \cdots, 6)$ are real constants.

And the associated vector fields for the one-parameter Lie group of infinitesimal transformations are

$$\begin{cases} V_1 = \partial_t, \\ V_2 = \partial_x, \\ V_3 = \partial_y, \\ V_4 = t\partial_t + \partial_w, \\ V_5 = y\partial_x - x\partial_y, \\ V_6 = x\partial_x + y\partial_y - 2\partial_w. \end{cases} \tag{4.50}$$

So the following transformations leave the solutions of equation (4.48) invariant,

$$\begin{aligned}
&G_1 : (x, y, t, w) \rightarrow (x, y, t + \varepsilon, w),\\
&G_2 : (x, y, t, w) \rightarrow (x + \varepsilon, y, t, w),\\
&G_3 : (x, y, t, w) \rightarrow (x, y + \varepsilon, t, w),\\
&G_4 : (x, y, t, w) \rightarrow (x, y, e^{\varepsilon} t, w + \varepsilon),\\
&G_5 : (x, y, t, w) \rightarrow (y \sin\varepsilon - x\cos\varepsilon, -x\sin\varepsilon - y\cos\varepsilon, t, w),\\
&G_6 : (x, y, t, w) \rightarrow (e^{\varepsilon} x, e^{\varepsilon} y, t, w - 2\varepsilon).
\end{aligned}$$

Theorem 4.1 If $w = f(x, y, t)$ is a solution to the equation (4.48), then so is

$$\begin{aligned}
w_1 &= f(x, y, t - \varepsilon),\\
w_2 &= f(x - \varepsilon, y, t),\\
w_3 &= f(x, y - \varepsilon, t),\\
w_4 &= f(x, y, e^{-\varepsilon} t) + \varepsilon,\\
w_5 &= f(-y\sin\varepsilon - x\cos\varepsilon, x\sin\varepsilon - y\cos\varepsilon, t),\\
w_6 &= f(e^{-\varepsilon} x, e^{-\varepsilon} y, t,) - 2\varepsilon.
\end{aligned}$$

4.2 Reductions and Solutions

With reference to Xu [10], we have the following theorem,

Theorem 4.2 The operators in (4.50) generate an optimal system S:

$$\begin{aligned}
&(a) && V_6 + a_4V_4 + a_5V_5, \quad a_6 \neq 0.\\
&(b) && V_5 + a_4V_4, \quad a_6 = 0, \quad a_5 \neq 0.\\
&(c_1) && V_4, \quad a_6 = a_5 = 0, \quad a_4 \neq 0.\\
&(c_2) && V_4 + V_2, \quad a_6 = a_5 = 0, \quad a_4 \neq 0.\\
&(c_3) && V_4 + V_3, \quad a_6 = a_5 = 0, \quad a_4 \neq 0.\\
&(d_1) && V_1, \quad a_6 = a_5 = a_4 = 0, \quad a_1 \neq 0.\\
&(d_2) && V_1 + V_2, \quad a_6 = a_5 = a_4 = 0, \quad a_1 \neq 0.\\
&(d_3) && V_1 - V_3, \quad a_6 = a_5 = a_4 = 0, \quad a_1 \neq 0.\\
&(e) && V_2 + a_3V_3, \quad a_6 = a_5 = a_4 = a_1 = 0, \quad a_2 \neq 0.\\
&(f) && V_3, \quad a_6 = a_5 = a_4 = a_1 = a_2 = 0.
\end{aligned}$$

Following Theorem 4.1, we will discuss the reduction and exact solutions of equation (4.48).

(a)

$$\begin{aligned}
V &= V_6 + a_4V_4 + a_5V_5\\
&= (x + a_5y)\partial_x + (y - a_5x)\partial_y + a_4t\partial_t + (a_4 - 2)\partial_w.
\end{aligned}$$

Set $a_4 = 1, a_5 = a, a$ is an arbitrary constant, then

$$V = (x + ay)\partial_x + (y - ax)\partial_y + t\partial_t - \partial_w.$$

The corresponding characteristic equations are

$$\frac{dx}{x + ay} = \frac{dy}{y - ax} = \frac{dt}{t} = \frac{dw}{-1},$$

leading to the invariance

$$u = \frac{x + ay}{t}, \qquad v = \frac{ax - y}{t}, \qquad h = te^{w}.$$

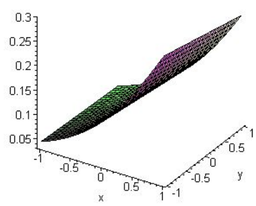


Figure 14: case (a1) $t = 1$

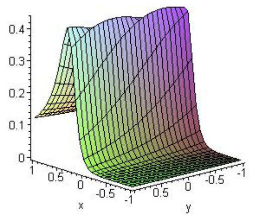


Figure 15: case (a1) $t = 0.1$

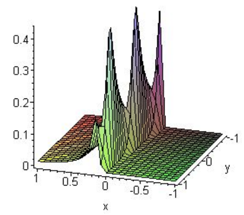


Figure 16: case (a1) $t = 0.001$

That is the invariant solution takes the form

$$w = \ln h(u, v) - \ln t.$$

Thus equation (4.48) can be reduced to

$$(1 + a^2)(h_{uu}h + h_{vv}h) - (1 + a^2)(h_u^2 + h_v^2) + uh_u h^2 + vh_v h^2 + h^3 = 0. \tag{4.51}$$

Now look for the "traveling wave solution " of the above equation. Let $z = u + \delta v$, $h = h(z)$, then we get

$$(1 + a^2)(1 + \delta^2)(h''h - (h')^2) + zh'h^2 + h^3 = 0.$$

Denote $\lambda = (1 + a^2)(1 + \delta^2) > 0$, one obtains the solutions,

(a1)

$$h(z) = \frac{c_1^2 e^{\frac{c_1}{\lambda}z}}{-\lambda e^{\frac{c_1}{\lambda}z} + zc_1 e^{\frac{c_1}{\lambda}z} + c_2 c_1^2},$$

if $h'(0) \neq 0$, that is, $c_1 \neq 0$.

And the original metric,

$$u(x, y, t) = \frac{h}{t} = \frac{c_1^2 e^{\frac{c_1}{\lambda}z}}{-\lambda e^{\frac{c_1}{\lambda}z} + zc_1 e^{\frac{c_1}{\lambda}z} + c_2 c_1^2} \cdot \frac{1}{t}, \qquad u(x, y, t) \geq 0,$$

where $z = u + \delta v = \frac{x+ay+\delta(ax-y)}{t}$.

(i) When t is small, z may tend to $+\infty$ or $-\infty$ arbitrarily. so, $u(x, y, t)$ may have singularity, which depends on the weather c_1, c_2 are positive or negative, i.e. the boundary condition of $h(z)$.

(ii) When t is large enough, $z \to 0$, the $u(x, y, t)$ has no singularity, no matter whatever the boundary condition of $h(z)$ is. Actually, given $h(0) = \frac{c_1^2}{-\lambda + c_2 c_1^2}$, $-\lambda + c_2 c_1^2 \neq 0$. So $-\lambda e^{\frac{c_1}{\lambda}z} + zc_1 e^{\frac{c_1}{\lambda}z} + c_2 c_1^2 \neq 0$, when $z \to 0$, which also implies the metric $u(x, y, t)$, has no singularity when t is large enough.

What is more, it can be verified by the shape of $u(x, y, t)$, viewing Figure 14, Figure 15 and Figure 16. Here we set $a = 0.1, \delta = 0.2, c_1 = 1, c_2 = 10$.

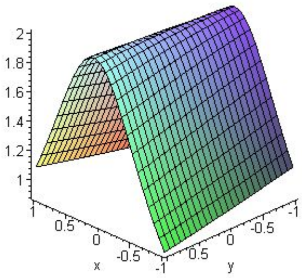
(a2)

$$h(z) = \frac{2\lambda}{z^2 + c},$$

if $h'(0) = 0$, that is, $c_1 = 0$. So

$$u(x, y, t) = \frac{2\lambda}{z^2 + c}\frac{1}{t} = \frac{2\lambda t}{(x + ay + \delta(ax - y))^2 + ct^2}.$$

Since $u(x,y,t) > 0$, thus $h(0) = \frac{2\lambda}{c} > 0$, i.e., $c > 0$. Therefore, $u(x,y,t)$ has no singularity in this case, viewing the following picture, and we know that when $t \to +\infty, u(x,y,t) \to 0$.



$t = 1$. Here we set $c = 1$.

And if we set $a_4 = 2, a_5 = 0$, then

$$V = V_6 + 2a_4V_4 = x\partial_x + y\partial_y + 2t\partial_t.$$

The characteristic equations are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dt}{2t},$$

leading to the invariant form

$$\xi = \frac{x}{\sqrt{t}}, \eta = \frac{y}{\sqrt{t}}, w(\xi,\eta) = e^w.$$

Hence the equation (4.48) reduces to

$$w_{\xi\xi} + w_{\eta\eta} = \frac{w_\xi^2 + w_\eta^2}{w} - \frac{1}{2}w(\xi w_\xi + \eta w_\eta).$$

Look for its "traveling wave solution", let $z = \xi + \delta\eta, w = w(z)$, we obtain

$$(1+\delta^2)(\frac{w'}{w})' + \frac{1}{2}zw' = 0.$$

And we have solution,

$$w = \frac{2(1+\delta^2)}{z^2} = \frac{2(1+\delta^2)}{(\xi+\delta\eta)^2}.$$

Then

$$u(x,y,t) = w(\xi,\eta) = \frac{2(1+\delta^2)t}{(x+\delta y)^2},$$

and it always has singularity at the surface $x + \delta y = 0$. But this time, when $t \to +\infty, u(x,y,t) \to +\infty$.

Acting by $G_1, G_2, \cdots, G_6$, one obtains more solutions of equation (4.48) by Theorem 4.1,

$$u(x,y,t) = e^{-2\varepsilon_6+\varepsilon_4}\frac{c_1^2 e^{\frac{c_1}{\lambda}\bar{z}}}{-\lambda e^{\frac{c_1}{\lambda}\bar{z}} + \bar{z}c_1 e^{\frac{c_1}{\lambda}\bar{z}} + c_2c_1^2}\cdot\frac{1}{\bar{t}},$$

$$u(x,y,t) = e^{-2\varepsilon_6+\varepsilon_4}\frac{2\lambda}{(\bar{z})^2 + c}\cdot\frac{1}{\bar{t}},$$

where

$$\bar{z} = \frac{\bar{x} + a\bar{y} + \delta(a\bar{x} - \bar{y})}{\bar{t}},$$

$$\bar{x} = -xe^{-\varepsilon_6}\cos\varepsilon_5 - ye^{-\varepsilon_6}\sin\varepsilon_5 - \varepsilon_2,$$

$$\bar{y} = xe^{-\varepsilon_6}\sin\varepsilon_5 - ye^{-\varepsilon_6}\cos\varepsilon_5 - \varepsilon_3,$$

$$\bar{t} = e^{-\varepsilon_4}t - \varepsilon_1,$$

and ε_i $(i = 1, 2, \cdots, 6)$ are parameters .

(b)

$$\begin{aligned} V &= V_5 + a_4 V_4 \\ &= y\partial_x - x\partial_y + a_4 t\partial_t + a_4\partial_w. \end{aligned}$$

The corresponding characteristic equations are

$$\frac{dx}{y} = \frac{dy}{-x} = \frac{dt}{a_4 t} = \frac{dw}{a_4},$$

leading to invariance

$$r = \sqrt{x^2 + y^2}, h = t^{-1}e^w.$$

That is the invariant solution takes the form

$$w = \ln h(r) + \ln t.$$

And the equation (4.48) reduces to

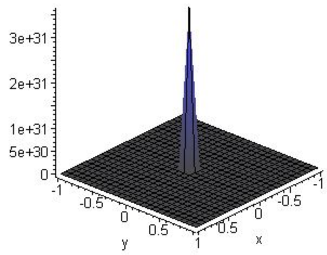
$$(h''h - (h')^2)r + h'h - h^3 r = 0.$$

Since $u(x, y, t) > 0$, we find the solution

$$h(r) = \frac{1 + (\tan(\frac{\ln r - c_2}{2c_1}))^2}{2r^2 c_1^2}.$$

Therefore, the metric

$$u(x, y, t) = h(r)t = t\frac{1 + (\tan(\frac{\ln r - c_2}{2c_1}))^2}{2r^2 c_1^2},$$



where $r = \sqrt{x^2 + y^2}$. It always has singularity at $(0, 0, t)$.

Acting by $G_1, G_2, \cdots, G_6$, one obtain more solutions of equation (4.48) by Theorem 4.1,

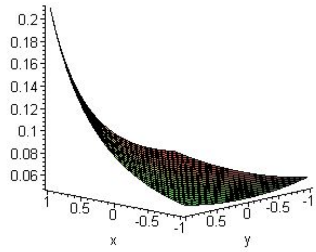
$$u(x, y, t) = e^{-2\varepsilon_6 + \varepsilon_4}\bar{t}\frac{1 + (\tan(\frac{\ln \bar{r} - c_2}{2c_1}))^2}{2\bar{r}^2 c_1^2},$$

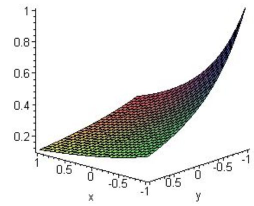
where

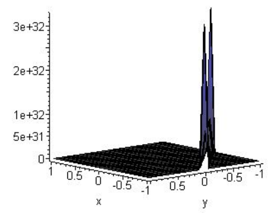
$$\begin{aligned} \bar{t} &= e^{-\varepsilon_4}t - \varepsilon_1, \\ \bar{r}^2 &= (-xe^{-\varepsilon_6}\cos\varepsilon_5 - ye^{-\varepsilon_6}\sin\varepsilon_5 - \varepsilon_2)^2 \\ &\quad + (xe^{-\varepsilon_6}\sin\varepsilon_5 - ye^{-\varepsilon_6}\cos\varepsilon_5 - \varepsilon_3)^2. \end{aligned}$$

(c1)

$$V = V_4 = t\partial_t + \partial_w,$$







,

Figure 17: case (c11) Figure 18: case (c12) $c = 4$. Figure 19: case (c12) $c = 1$.

with invariance

$$x, y, h = t^{-1}e^{w}.$$

That is the invariant solution takes the form

$$w = \ln h(x, y) + \ln t.$$

And the equation (4.48) reduces to

$$h_{xx}h - h_x^2 + h_{yy}h - h_y^2 - h^3 = 0.$$

Now look for the "traveling wave solution " of the above equation. Let $z = x + \delta y, h = h(z)$, therefore, we have

$$(1 + \delta^2)(\frac{h'}{h})' = h.$$

Denote $\lambda = 1 + \delta^2$, and since $u(x, y, t) > 0$, we have solutions

(c11)

$$h(z) = \frac{-1}{2c_1}[1 + (\tan(\frac{\sqrt{-\lambda c_1}(z + c_2)^2}{2\lambda c_1}))^2], \quad \text{if } c_1 < 0;$$

(c12)

$$h(z) = \frac{2(1 + \delta^2)}{(z + c)^2}, \quad \text{if } c_1 = 0.$$

Hence the metric $u(x, y, t) = h(z)t$, with $z = x + \delta y$, and $h(z)$ takes forms in cases (c11),(c12), respectively.

Now, we can appreciate the beautiful shape of solutions, which possess symmetry, in Figure 17, Figure 18 and Figure 19, while we set $\delta = 1$.

As before , act by $G_1, G_2, \cdots, G_6$, one obtain more solutions of equation (4.48) by Theorem 4.1,

$$u(x, y, t) = e^{-2\varepsilon_6 + \varepsilon_4}\bar{t}\frac{-1}{2c_1}[1 + (\tan(\frac{\sqrt{-\lambda c_1}(\bar{z} + c_2)^2}{2\lambda c_1}))^2], \quad \text{if } c_1 < 0;$$
$$u(x, y, t) = e^{-2\varepsilon_6 + \varepsilon_4}\bar{t}\frac{2(1 + \delta^2)}{(\bar{z} + c)^2}, \quad \text{if } c_1 = 0,$$

where

$$\begin{aligned} \bar{t} &= e^{-\varepsilon_4}t - \varepsilon_1, \\ \bar{z} &= (-xe^{-\varepsilon_6}\cos\varepsilon_5 - ye^{-\varepsilon_6}\sin\varepsilon_5 - \varepsilon_2) \\ &\quad + \delta(xe^{-\varepsilon_6}\sin\varepsilon_5 - ye^{-\varepsilon_6}\cos\varepsilon_5 - \varepsilon_3). \end{aligned}$$

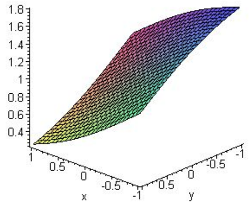


Figure 20: case (d21)

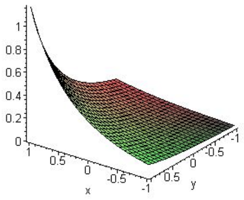


Figure 21: case (d22) $t = 1$.

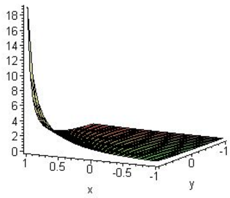


Figure 22: case (d22) $t = 0.1$.

(d2)

$$V = V_1 + V_2 = \partial_t + \partial_x$$

with invariance

$$z = x - t, y, w = w(z, y).$$

And the equation (4.48) reduces to

$$w_{zz} + w_{yy} - e^w w_z = 0.$$

Now look for the "traveling wave solution " of the above equation. Let $\xi = z + \delta y, h = e^w$, therefore, we have

$$(1 + \delta^2)(\frac{h'}{h})' = h'.$$

Denote $\lambda = 1 + \delta^2$, with some calculation, there are solutions

(d21)

$$h(\xi) = \frac{-c_1 \lambda e^{c_1(\xi + c_2)}}{1 + e^{c_1(\xi + c_2)}}, \quad \text{if } -h > c_1\lambda;$$

(d22)

$$h(\xi) = \frac{c_1 \lambda e^{c_1(\xi + c_2)}}{1 - e^{c_1(\xi + c_2)}}, \quad \text{if } -h < c_1\lambda;$$

(d23)

$$h(\xi) = \frac{-\lambda}{\xi + c}, \quad \text{if } c_1 = 0.$$

And in the calculation , we know $\lambda \frac{h'}{h} - h = c_1\lambda$. So, given $h(0), h'(0)$, thus c_1 is determined, and weather $h(0) + c_1\lambda$ is positive or not, tells which form does $h(\xi)$ takes. Furthermore, we know $h(\xi)$ has asymptote, $h(\xi) + c_1\lambda = 0$.

Therefore, the metric

$$u(x, y, t) = e^w = h(x - t + \delta y).$$

And we can easily see that $u(x, y, t)$ has no singularity in case (d21), while it has singularity at $x - t + \delta y + c_2 = 0$ in both case (d22), (d23). Furthermore, in (d22), (d23), $c_2 < 0$, since we must have $u(x, y, t) > 0$. Thus , when t is large enough, it has no singularity, for x, y are small. When t is small, it may have singularity. Here we set $c_1 = -1, c_2 = 0, \delta = 1, t = 1$, in case (d21). And similar for $t = 1, 2, 3 \ldots$ which always shows smooth. And in case (d22), set $c_1 = 1, c_2 = -2, \delta = 1$. Refer them to Figure 20, Figure 21 and Figure 22.

Acting by $G_1, G_2, \cdots, G_6$, one obtains more solutions of equation (4.48) by Theorem 4.1,

$$u(x, y, t) = e^{-2\varepsilon_6 + \varepsilon_4} h(\bar{\xi}),$$

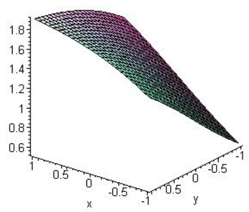


Figure 23: case (d31)

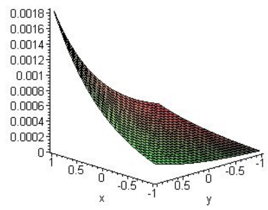


Figure 24: case (d32) $t = 1$.

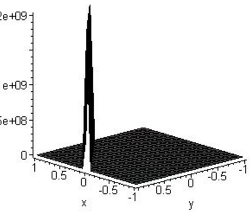


Figure 25: case (d32) $t = 9$.

while

$$\begin{aligned}\bar{\xi} &= (-xe^{-\varepsilon_6}\cos\varepsilon_5 - ye^{-\varepsilon_6}\sin\varepsilon_5 - \varepsilon_2)\\ &\quad - e^{-\varepsilon_4}t + \varepsilon_1\\ &\quad + \delta(xe^{-\varepsilon_6}\sin\varepsilon_5 - ye^{-\varepsilon_6}\cos\varepsilon_5 - \varepsilon_3),\end{aligned}$$

with $h(\xi)$ takes form in case (d21), (d22), (d23), respectively.

(d3)

$$V = V_1 - V_3 = \partial_t - \partial_y.$$

Similarly, there are solutions

(d31)

$$h(\eta) = \frac{-c_1\lambda e^{c_1(\eta+c_2)}}{1+\delta e^{c_1(\eta+c_2)}}, \quad \text{if } -\delta h > c_1\lambda;$$

(d32)

$$h(\eta) = \frac{c_1\lambda e^{c_1(\eta+c_2)}}{1-\delta e^{c_1(\eta+c_2)}}, \quad \text{if } -\delta h < c_1\lambda;$$

(d33)

$$h(\eta) = \frac{-\lambda}{\delta(\eta+c)}, \quad \text{if } c_1 = 0,$$

while $\lambda = 1+\delta^2, \eta = x + \delta(t+y)$.

And then, the metric $u(x, y, t) = h(\eta)$. For case (d32) and (d33), since $h(\eta) > 0$, the only possible chance is, $c_1 > 0, c_2 < 0$. And weather there is singularity or not, and when, depend on the absolute value of c_2, i.e. $|c_2|$. For its great similarity with case (d2), we will have no more discussion. Here we set $c_1 = -1, c_2 = 0, \delta = 1, t = 1$, in case (d31). And similar for $t = 2, 3, 4\ldots$ which always shows smooth. And in case (d32), set $c_1 = 1, c_2 = -10, \delta = 1$. Refer them to Figure 23, Figure 24 and Figure 25.

(e)

$$V = V_2 + aV_3 = \partial_x + a\partial_y.$$

Again, there are solutions

(e1)

$$h(\tau) = \frac{c_1\lambda e^{c_1(\tau+c_2)}}{1+\delta e^{c_1(\tau+c_2)}}, \quad \text{if } \delta h < c_1\lambda;$$

(e2)

$$h(\tau) = \frac{c_1\lambda e^{c_1(\tau+c_2)}}{-1+\delta e^{c_1(\tau+c_2)}}, \quad \text{if } \delta h > c_1\lambda;$$

(e3)

$$h(\tau) = \frac{-\lambda}{\delta(\eta + c)}, \quad \text{if } c_1 = 0,$$

while $\lambda = 1 + a^2, \tau = x - ay - \delta t$. And $u(x, y, t) = h(\tau)$.

References

[1] De-Xing Kong, *Hyperbolic geometric flow,* the Proceedings of ICCM 2007, Vol. II, Higher Educationial Press, Beijing, 2007, 95-110.

[2] De-Xing Kong, *Maximum principle in nonlinear hyperbolic systems and its applications,* Nonlinear Analysis, Theory, Method & Applications 32(1998), 871-880.

[3] De-Xing Kong, Kefeng Liu, *Wave character of metrics and hyperbolic geometric flow,* J. Math. Phys. 48(2007), 103508-1-103508-14.

[4] De-Xing Kong, Kefeng Liu, De-liang Xu, *The hyperbolic geometric flow on Riemann surfaces,* Comm. in Partial Differential Equation 34(2009), 553-580.

[5] De-xing Kong, Kefeng Liu, Zenggui Wang, *Hyperbolic mean curvature flow: evolution of plane curves,* Acta Mathematica Scientia, Volume 29, Issue 3, May 2009, 493-514 .

[6] Peter J. Ovler, *Applications of Lie groups to differential Equations,* GTM 107, Springer-Verla, 1993.

[7] Rozhdestvensky B. L. & Yanenko N. N. *Systems of quasylinear equations and their applications to gas dynamics, 2-nd Ed.,* Moscow, Nauka, 1978.

[8] Wen-Rong Dai, De-Xing Kong and Kefeng Liu, *Dissipative hyperbolic geometric flow,* Asian J. Math., 2008.

[9] Wen-Rong Dai, De-Xing Kong and Kefeng Liu, *Hyperbolic geometric flow (I): shorttime existence and nonlinear stability,* to appear in Pure and Applied Mathematics Quarterly: special issue dedicated to M. Atiyah's 80th birthday.

[10] Xu Chao, *Symmetries of geometric flows,* Arxiv preprint arXiv:1001.1394, 2010.