Some remarks on Kong-Liu's conjecture on Lorentzian metrics

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Abstract

For a $1+3$ dimensional Lorentzian manifold (M, g), general form of solutions of the Einsteins field equations takes one of type I, II, III(see [1]). For type I, there is a known result in [3]. In this paper, we try to find out the necessary and sufficient conditions for the Lorentzian metric to take the form of type II and III, and we show how to construct the new coordinate system.

1 Introduction

For a 1+3 dimensional Lorentzian manifold (M, g) with coordinates $\xi^\mu(\mu \in \{0,1,2,3\})$. Kong and Liu conjecture in [1] that general form of the solutions of the Einsteins field equations takes one of the following forms:

(I)

$$g_{\mu\nu} = \begin{pmatrix} * & * & * & * \\ * & 0 & 0 & 0 \\ * & 0 & * & 0 \\ * & 0 & 0 & * \end{pmatrix},$$

(II)

$$g_{\mu\nu} = \begin{pmatrix} 0 & * & * & * \\ * & * & 0 & 0 \\ * & 0 & * & 0 \\ * & 0 & 0 & * \end{pmatrix},$$

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(III)

$$g_{\mu\nu} = \begin{pmatrix} * \, 0 * * \\ 0 * 0 \, 0 \\ * \, 0 * 0 \\ * \, 0 \, 0 * \end{pmatrix}.$$

For type I, Gu has the following theorem [3].

Theorem 1.1. *In the* $1{+}3$ *dimensional Lorentzian manifold* (M, g) *with coordinate* ξ^μ*, where* $\mu \in \{0, 1, 2, 3\}$ *and* ξ^0 *corresponds to the time-like coordinate, the metric* $g_{\mu\nu}$ *can be converted into type I via a coordinate transformation, if and only if there is a smooth null vector field* $V^\mu \partial_\mu$ *on* M

$$g_{\mu\nu} V^\mu V^\nu = 0,$$

and the 1-form

$$\omega = g_{\mu\nu} V^\mu d\xi^\nu$$

is integrable.

In this paper, we consider Lorentzian metric of type II and type III, and have the following theorems.

Theorem 1.2. *In the* $1{+}3$ *dimensional Lorentzian manifold* (M, g) *with coordinate* ξ^μ*, where* $\mu \in \{0, 1, 2, 3\}$ *and* ξ^0 *corresponds to the time-like coordinate, the metric* $g_{\mu\nu}$ *can be converted into type II via a coordinate transformation, if and only if there is a smooth null vector field* $V^\mu \partial_\mu$ *on* M

$$g_{\mu\nu} V^\mu V^\nu = 0. \tag{1.1}$$

Theorem 1.3. *In the* $1{+}3$ *dimensional Lorentzian manifold* (M, g) *with coordinate* ξ^μ*, where* $\mu \in \{0, 1, 2, 3\}$ *and* ξ^0 *corresponds to the time-like coordinate, the metric* $g_{\mu\nu}$ *can be converted into type III via a coordinate transformation, if and only if there are two orthogonal smooth vector field* $A^\mu \partial_\mu$*,* $B^\nu \partial_\nu$ *on* M*,*

$$g_{\mu\nu} A^\mu B^\nu = 0, \tag{1.2}$$

with

$$g_{\mu\nu} B^\mu B^\nu < 0,$$

and

$$g_{\mu\nu} A^\mu A^\nu > 0$$

everywhere on M*. And the 1-form*

$$\omega = g_{\mu\nu} B^\mu d\xi^\nu \tag{1.3}$$

is integrable.

In the next section, we shall prove Theorems 1.2, 1.3, respectively. For more explicit explanation about some conceptions, see [2].

2 Lorentzian metric of type II and type III

Consider Lorentzian metric of type II, we now give the proof of Theorem 1.2.

Proof. For the sufficient part, Denote l the integral curve with tangent vector $V^\mu = \frac{d\xi^\mu(\tau)}{d\tau}$. Now we construct the coordinate t as

$$t = t(\xi^\nu). \tag{2.1}$$

We have

$$dt = \frac{\partial t}{\partial \xi^\mu} d\xi^\mu. \tag{2.2}$$

Taking the trajectories of the integral curve l as the t coordinate lines, then along these t lines, we have $d\xi^\mu = V^\mu d\tau$. Substituting it into (2.2), we get

$$V^\mu \partial_\mu t = \frac{dt}{d\tau} \triangleq f, \tag{2.3}$$

where $f > 0$, is any given smooth function, which acts as the scale t coordinate. Solving (2.3), we obtain the coordinate transformation (2.1). Then on the hypersurface $H = \{\xi^\mu \mid t(\xi^\mu) = c\}$, where c is any given constant, we can always find spatial coordinate (z, x, y), such that the coordinate base $(\partial_z, \partial_x, \partial_y)$ keeps orthogonal. And then the Lorentzian metric in new coordinate system (t, z, x, y) takes the following form

$$\eta_{\alpha\beta} = \begin{pmatrix} u & v & p & q \\ v & -w^2 & 0 & 0 \\ p & 0 & -a^2 & 0 \\ q & 0 & 0 & -b^2 \end{pmatrix}.$$

Since the trajectories of curve l acts as t-lines, and we have $g_{\mu\nu}V^\mu V^\nu = 0$, hence along t-lines

$$g_{\mu\nu} \frac{d\xi^\mu(t)}{dt} \frac{d\xi^\nu(t)}{dt} = 0.$$

Under coordinate transformation between (ξ^μ) and $(\delta^\mu) = (t, z, x, y), \mu \in \{0, 1, 2, 3\}$, the Lorentzain metric transforms like

$$\eta_{\alpha\beta} = g_{\mu\nu} \frac{\partial \xi^\mu}{\partial \delta^\alpha} \frac{\partial \xi^\nu}{\partial \delta^\beta}. \tag{2.4}$$

Take $\alpha = \beta = 0$, thus we get

$$\eta_{00} = g_{\mu\nu} \frac{\partial \xi^\mu}{\partial t} \frac{\partial \xi^\nu}{\partial t}.$$

Along t-lines, that is

$$u = \eta_{00} = g_{\mu\nu} \frac{d\xi^\mu(t)}{dt} \frac{d\xi^\nu(t)}{dt} = 0.$$

Thus $u = 0$.

For the necessary part, if the Lorentzian metric $g_{\mu\nu}$ can be converted into type II, that is, it takes the following form

$$g_{\mu\nu} = \begin{pmatrix} 0 & v & p & q \\ v & -w^2 & 0 & 0 \\ p & 0 & -a^2 & 0 \\ q & 0 & 0 & -b^2 \end{pmatrix}.$$

Take

$$V^\mu = (h(\xi^\nu), 0, 0, 0),$$

since $g_{00} = 0$, we get

$$g_{\mu\nu} V^\mu V^\nu = 0.$$

The proof is finished. □

Consider Lorentzian metric of type III, we now give the proof of Theorem 1.3.

Proof. For the sufficient part, define

$$d\tilde{z} = K g_{\mu\nu} B^\mu d\xi^\nu,$$

where K is a factor to make the 1-form ω become an exact differential form. In fact, the integrable condition is equivalent to the exact differential form due to (1.2).

Denote A the integral curve with tangent vector

$$A^\nu = \frac{d\xi^\nu(\tau_A)}{d\tau_A},$$

and B the integral curve with tangent vector

$$B^\nu = \frac{d\xi^\nu(\tau_B)}{d\tau_B},$$

where τ_A, τ_B are the parameters of the curve A, and curve B respectively.

Along curve A:$(\xi^\nu(\tau_A))$,

$$d\tilde{z}(\tau_A) = g_{\mu\nu} B^\mu \frac{d\xi^\nu(\tau_A)}{d\tau_A} d\tau_A = g_{\mu\nu} B^\mu A^\nu d\tau_A = 0.$$

For any given constant $\tilde{z_0}$, the hypersurface $H = \{\xi^\mu \mid \tilde{z}(\xi^\mu) = \tilde{z_0}\}$ contains the family of curve A. Moreover, $B^\mu \partial_\mu$ is orthogonal to H. Actually, along any given curve $(\xi^\mu(\tau)) \subset H$, since $d\tilde{z} = 0$, hence

$$g_{\mu\nu} B^\mu \frac{d\xi^\nu}{d\tau} = 0.$$

That is

$$(B^\mu \partial_\mu) \perp (\frac{d\xi^\nu}{d\tau}\partial_\nu).$$

Therefore, we have

$$(B^\mu \partial_\mu) \perp H.$$

The 2-dimensional surface $\{\xi^\mu \mid \tilde{z}(\xi^\mu)|_{\xi^0=constant} = \tilde{z_0}\}$ corresponds to a 2-dimensional surface S. The initial surface is $S_0 = \{\xi^\mu \mid \tilde{z}(\xi^m)|_{\xi^0=c_0} = \tilde{z_0}\}, (m = 1, 2, 3)$. By the above discussion, $(B^\mu \partial_\mu) \perp H$, therefore S is orthogonal to curve B which is generated by $B^\mu \partial_\mu$.

First of all, we construct the coordinate t as

$$t = t(\xi^\nu). \tag{2.5}$$

We have

$$dt = \frac{\partial t}{\partial \xi^\mu} d\xi^\mu. \tag{2.6}$$

Taking the trajectories of curve A, as the t coordinate line, then along these t lines, we have $d\xi^\mu = A^\mu d\tau_A$. Substituting it into (2.6), we get

$$A^\mu \partial_\mu t = \frac{dt}{d\tau_A} \triangleq g, \tag{2.7}$$

where $g > 0$, is any given smooth function, which acts as the scale t coordinate. Solving (2.7), we obtain the coordinate transformation (2.5).

Next, we construct the coordinate z as

$$z = z(\xi^\mu). \tag{2.8}$$

We have

$$dz = \frac{\partial z}{\partial \xi^\mu} d\xi^\mu. \tag{2.9}$$

Taking the trajectories of curve B, as the z coordinate lines, then along these z lines, we have $d\xi^\mu = B^\mu d\tau_B$. Substituting it into (2.9), we get

$$B^\mu \partial_\mu z = \frac{dz}{d\tau_B} \triangleq f, \tag{2.10}$$

where $f > 0$ is any given smooth function, which acts as the scale of z coordinate. Since the initial surface S_0 is orthogonal to curve B, we may add the boundary condition on surface S_0

$$z|_{S_0} = z_0, \tag{2.11}$$

where z_0 is any given constant. Since $g_{\mu\nu} B^\mu B^\nu < 0$, we may take $f = -g_{\mu\nu} B^\mu B^\nu$, and set $z_0 = -\tilde{z_0}$. Solve equation (2.10), with boundary condition (2.11), we know that $z = -\tilde{z}(\xi^\mu)$ is a solution. Since $(B^\mu \partial_\mu) \perp H = \{\xi^\mu \mid z(\xi^m) = constant\}$, hence $(B^\mu \partial_\mu) \perp S(t_0, z)$, combine with (2.11), we know that the propagating distance of $S_0 \to S$ defines the new coordinate z, so we can denote the propagating surface as $S(t_0, z_0) \to S(t_0, z)$ for clearness.

For the 2-dimensional surface $S(t_0, z_0)$, not loss generality, we can assume the parameter coordinate (x, y) are orthogonal grid. Otherwise, we can take the 2-principal curves of the surface as coordinate lines (x, y) to get orthogonal coordinates. Since z coordinate corresponds to an equidistant translation of the 2-dimensional surface $S(t_0, z_0) \to S(t_0, z)$, the (x, y) grid on the surface $S(t_0, z)$ along the curve B keeps orthogonal, and then all space-like coordinate bases $(\partial_z, \partial_x, \partial_y)$ also keep orthogonal. So the spatial coordinates (z, x, y) form an global orthogonal coordinate grid, and then the Lorentzian metric in new coordinate system (t, z, x, y) takes the following form

$$\eta_{\alpha\beta} = \begin{pmatrix} u & v & p & q \\ v & w & 0 & 0 \\ p & 0 & -a^2 & 0 \\ q & 0 & 0 & -b^2 \end{pmatrix}.$$

Since the trajectories of curve A acts as t-lines, the trajectories of curve B acts as z-lines, and we have $g_{\mu\nu} A^\mu B^\nu = 0$, hence along t-lines $\cap$ z-lines,

$$g_{\mu\nu} \frac{d\xi^\mu(t)}{dt} \frac{d\xi^\nu(z)}{dz} = 0.$$

Take $\alpha = 0, \beta = 1$ in (2.4),

$$\eta_{01} = g_{\mu\nu} \frac{\partial \xi^\mu}{\partial t} \frac{\partial \xi^\nu}{\partial z}.$$

Hence along t-lines $\cap$ z-lines, that is

$$v = \eta_{01} = g_{\mu\nu} \frac{d\xi^\mu(t)}{dt} \frac{d\xi^\nu(z)}{dz} = 0.$$

Thus $v = 0$. Similarly, $g_{\mu\nu} B^\mu B^\nu < 0$, and $g_{\mu\nu} A^\mu A^\nu > 0$, then we have $u > 0, w < 0$.

For the necessary part, if the Lorentzian metric $g_{\mu\nu}$ can be converted into type III, that is, it takes the following form

$$g_{\mu\nu} = \begin{pmatrix} u^2 & 0 & p & q \\ 0 & -w^2 & 0 & 0 \\ p & 0 & -a^2 & 0 \\ q & 0 & 0 & -b^2 \end{pmatrix}.$$

Take

$$B^\mu = (0, \frac{k}{w^2}, 0, 0),$$
$$A^\mu = (h(\xi^\nu), 0, 0, 0),$$

where k is constant, and $h(\xi^\nu)$ arbitrary smooth function. The 1-form (1.3) becomes

$$\omega = g_{\mu\nu} B^\mu d\xi^\nu = kdz,$$

which is an exact differential form. And

$$g_{\mu\nu} B^\mu A^\nu = 0,$$

$$g_{\mu\nu} B^\mu B^\nu = -\frac{k^2}{w^2} < 0,$$

$$g_{\mu\nu} A^\mu A^\nu = h^2 u^2 > 0.$$

The proof is finished. □

Remark 1. *The integrable condition of the 1-form* ω *(1.3) in theorem 1.3 is equivalent to that* ω *is an exact differential form, because the vector field* $B^\mu \partial_\mu$ *can multiply an arbitrary factor function* $K(\xi^\mu)$ *due to (1.2). Then that is to say, there is a scalar field* $z(\xi^\mu)$*, such that*

$$B_\mu = \partial_\mu z.$$

Remark 2. *For the coordinate functions*

$$t = t(\xi^\mu), x = x(\xi^\mu), y = y(\xi^\mu)$$

defines in theorem 1.3, they all satisfy the following partial differential equation

$$B^\mu \partial_\mu F(\xi^\nu) = 0. \tag{2.12}$$

Proof. Since $(B^\mu \partial_\mu) \perp H = \{\xi^\mu \mid z = constant\}$, that is z lines are orthogonal to the hypersurface H, hence along z coordinate lines we have

$$dt = \partial_\mu t d\xi^\mu = B^\mu \partial_\mu t d\tau = 0,$$
$$dx = \partial_\mu x d\xi^\mu = B^\mu \partial_\mu x d\tau = 0,$$
$$dy = \partial_\mu y d\xi^\mu = B^\mu \partial_\mu y d\tau = 0.$$

The proof is finished. □

Remark 3. *Theorem 1.2 implies that, any Lorentzain metric solving Einstein's field equations, can be converted into type II by coordinate transformations. Theorem 1.3 says that, there are some obstructions for Lorentzain metric solving Einstein's field equations to be converted into type III.*

References

[1] De-Xing Kong & Kefeng Liu, *Time-periodic solutions of the Einstein's field equations I: general framework*, Science China Mathematics 53 (2010), 1213-1230.

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[3] Ying-Qiu Gu, *A Canonical Form of the Metric in General Relativity*, arXiv:0710.5792v3, Nov. 2009.