GENERAL EXPANSION FOR PERIOD MAPPINGS OF RIEMANN SURFACES

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Abstract. In this paper, we get the full expansion for period map from the moduli space $\mathcal{M}_g$ of curves to $\mathcal{A}_g$ in Bers coordinates. This generalizes fully the famous Rauch's variational formula. As applications, we use this expansion to study its distortion problem.

1. Introduction

In this paper, we study the period mapping $\mathcal{J}$ from the moduli space $\mathcal{M}_g$ of compact Riemann surfaces with genera g to the coarse moduli space $\mathcal{A}_g$ of g-dimensional principally polarized Abelian varieties.

There are various methods to give complex orbifold structures on $\mathcal{M}_g$. These structures are biholomorphically equivalent to each other and determine a unique one, which we call the canonical complex orbifold structure. With respect to this structure on $\mathcal{M}_g$, $\mathcal{J}$ is holomorphic. Furthermore, it is the unique one such that $\mathcal{J}$ is holomorphic. In view of these, $\mathcal{J}$ plays an important role in the study of $\mathcal{M}_g$. The first basic property of $\mathcal{J}$ is its injectivity. In other words, two compact Riemann surfaces with isomorphic Jacobians must be biholomorphic to each other. This is exactly the statement of the classical Torelli's theorem[1]. Its second one is the immersive property. More precisely, the period mapping $\mathcal{J}$ is a holomorphic immersion on the complement of the hyperelliptic locus $\mathcal{M}_g - \mathcal{HE}_g$ and restricts to an immersion on $\mathcal{HE}_g$. This property is now known as the local Torelli theorem and is proved by Rauch's variational formula for $\mathcal{J}$[2].

To discuss the differential geometry of $\mathcal{M}_g$, one way is to compute the higher differentials of $\mathcal{J}$. It is presumable that $\mathcal{J}$ is very curved with respect to the invariant metric on the locally symmetric variety $\mathcal{A}_g$[3,4]. Thus a higher expansion of $\mathcal{J}$ more than the first order term seems to be very meaningful for estimates of its distortion. In our presentation, we would like to get the full expansion of $\mathcal{J}$. The idea is to deform the canonical holomorphic

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1-forms on a fixed Riemann surface. To achieve this, we employ a fundamental construction developed recently by K. Liu, X. Sun and A. Todorov. By an explicit expansion of the canonical holomorphic 1-forms, we deduce a generalization of Rauch's variational formula. Note that our method is completely different from Rauch's original one and that of Mayer's[5,6,7].

This article is organized as follows. In Section 2, we recall some background materials in moduli theory of compact Riemann surfaces. The deformation constructions of holomorphic 1-forms are arranged in the Section 3. Then we use the results of Section 3 to get the full expansion for period mapping $\mathcal{J}$ in Section 4. In particular, we also get the expansion of the period matrix along a complex curve in $\mathcal{M}_g$. Finally, as applications, we use our expansion of period mapping to study its distortion problem.

Assume all Riemann surfaces in this article have genera $g \geq 2$.

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2. Moduli space and period mapping

We construct the moduli space $\mathcal{M}_g$ of compact Riemann surfaces with genera g via Teichmuller theory. The procedure is as follows. First, we define a Teichmuller space $\mathcal{T}(X)$ associated to X by means of quasi-conformal mapping theory. Second, we prove this space admits a properly discontinuous holomorphic action by the mapping class group Mod_g. And then we can see the quotient analytic space is precisely what we want. The critical step is the construction of Teichmuller space $\mathcal{T}(X)$ of X.

Let X be a closed Riemann surface of genus $g \geq 2$. Fix a set of $2g$ simple closed curves on X which induces a symplectic homology basis of $H_1(X, \mathbb{Z})$. In other words, we choose $2g$ simple closed curves $\{A_\alpha, B_\alpha\}_{\alpha=1}^g$ on X such that their intersection matrix is the standard symplectic matrix. Consider an arbitrary pair (f, S) of a closed Riemann surface S and a quasiconformal mapping $f : X \to S$. We call the triple (X, f, S) a marked Riemann surface. Two triples (X, f, S) and (X, g, S') are said to be Teichmuller equivalent if $g \circ f^{-1} : S \to S'$ is homotopic to a biholomorphic mapping $\phi : S \to S'$. Let $[f, S]$ be the Teichmuller equivalence class of (X, f, S). The set of all these equivalence classes $[f, S]$ is denoted by $\mathcal{T}(X)$ and is called the Teichmuller space of X. The point $[id, X]$ is called the base point of $\mathcal{T}(X)$.

We introduce a topology on $\mathcal{T}(X)$ by means of the Teichmuller distance which is defined in the following:

For any two points $p_1 = [f_1, S_1]$, $p_2 = [f_2, S_2] \in \mathcal{T}(X)$, let $\mathcal{Q}(f_2 \circ f_1^{-1})$ be the set of all quasiconformal mappings of S_1 onto S_2 which are homotopic to $f_2 \circ f_1^{-1}$. The Teichmuller distance d between p_1 and p_2 is given by

$$d(p_1, p_2) = \inf_{g \in \mathcal{Q}(f_2 \circ f_1^{-1})} \log K(g),$$

where $K(g)$ is the maximal dilatation of g. This distance makes $\mathcal{T}(X)$ a complete topological space. Furthermore, different Teichmuller spaces $\mathcal{T}(X_1)$ and $\mathcal{T}(X_2)$ corresponding to different Riemann surfaces X_1 and X_2, respectively, are homeomorphic to each other by a base point translation. We call their common topological space the Teichmuller space of compact Riemann surfaces of genera g, which we denote by $\mathcal{T}_g$.

Let us describe a local holomorphic coordinate system on $\mathcal{T}_g$ around the base point. Let ω be a fixed Hermitian metric on $X = X_0$. $\mathcal{H}^{0,1}(X_0, T^{1,0}X_0)$ the space of harmonic Beltrami differentials on X_0 with respect to ω. Choose a basis $\{\mu_i\}_{i=1}^n$ $(n = 3g - 3)$ of $\mathcal{H}^{0,1}(X_0, T^{1,0}X_0)$. For sufficiently small $t = (t_1, \cdots, t_n) \in \mathbb{C}^n$, we define $\mu_t = \sum_{i=1}^n t_i \mu_i$. Each μ_t gives a quasi-conformal mapping f^{μ_t} by the existence of solution to the Beltrami equation. Write $X_t = f^{\mu_t}(X)$. In this way, we construct a family of marked Riemann surfaces $[f^{\mu_t}, X_t]$. Meanwhile, the small parameters $t = (t_1, \cdots, t_n)$ form a local holomorphic coordinate system at the base point $[id, X_0] \in \mathcal{T}_g$. These coordinates will be used to deform holomorphic 1-forms in our next section.

For the identification of Riemann's moduli space $\mathcal{M}_g$, we define a group action on the Teichmuller space $\mathcal{T}_g$. Let $Mod_g = Mod(X)$ be the set of all homotopy classes $[h]$ of quasi-conformal self-mapping of X. It forms an abstract group by the composition of maps. We call it the mapping class group Mod_g of genus g. Every element $[h]$ in Mod_g acts on $\mathcal{T}_g$ by

$$[h]([f, S]) = [f \circ h^{-1}, S]$$

for $[f, S] \in \mathcal{T}_g$. Now the moduli space of closed Riemann surfaces of genera g $\mathcal{M}_g$, i.e. the set of all biholomorphic equivalence classes of closed Riemann surfaces of genera g, is identified with the quotient space $\mathcal{T}_g/Mod_g$. With respect to above canonical complex structure on $\mathcal{T}_g$, the Mod_g-action is holomorphic and properly discontinuous. Thus by a theorem of Cartan, the quotient space $\mathcal{M}_g$ is a $3g-3$ dimensional complex analytic space. Moreover, $\mathcal{M}_g$ is a quasi-projective variety and a complex orbifold.

To each pair (f, S) we associate an element $\pi \in \mathfrak{H}_g$. Here $\mathfrak{H}_g$ is the generalized Siegel upper half plane of dimension $\frac{g(g+1)}{2}$. The details are as follows. For each (f, S), we have a symplectic homology basis $\{A_\alpha, B_\alpha\}_{\alpha=1}^g$ which is induced by the quasi-conformal mapping $f : X \to S$ from that of X. By the Hodge-Riemann bilinear relations of holomorphic 1-forms, we know that there is a unique basis $\{\theta^\alpha\}_{\alpha=1}^g$ of $H^0(S, K_S)$ such that

$$\int_{A_\alpha} \theta^\beta = \delta_{\alpha\beta}.$$

Then the period matrix $\pi = (\pi_{\alpha\beta})$, where $\pi_{\alpha\beta} = \int_{B_\alpha} \theta^\beta$, belongs to $\mathfrak{H}_g$. Actually, this matrix depends only on the Teichmuller equivalence class $[f, S]$ of (f, S) and gives a natural map, which we call the period map Π, from $\mathcal{T}_g$ to $\mathfrak{H}_g$. The basis $\{\theta^\alpha\}_{\alpha=1}^g$ is said to be a canonical basis of the space of holomorphic Abelian differentials $H^0(S, K_S)$ on S with respect to $\{A_\alpha, B_\alpha\}_{\alpha=1}^g$.

Let $Sp(g, \mathbb{Z})$ be the set of all symplectic matrices with elements in $\mathbb{Z}$. This group acts on $\mathfrak{H}_g$ also properly discontinuously. The quotient space $\mathcal{A}_g = \mathfrak{H}_g/Sp(g, \mathbb{Z})$ is a $\frac{g(g+1)}{2}$ dimensional complex analytic space. This space can be identified with the coarse moduli space of principally polarized Abelian varieties of dimension g[8]. Our previous period mapping Π descends to a map $\mathcal{J}$ from $\mathcal{M}_g$ to $\mathcal{A}_g$. This is equivalent to the following commutative diagram

$$\begin{array}{ccc} \mathcal{T}_g & \xrightarrow{\Pi} & \mathfrak{H}_g \\ \downarrow & & \downarrow \\ \mathcal{M}_g & \xrightarrow{\mathcal{J}} & \mathcal{A}_g. \end{array}$$

There is a classical result due to Torelli asserts this descending period mapping $\mathcal{J}$ is injective. In fact, it is also a holomorphic immersion outside the hyperelliptic locus $\mathcal{M}_g - \mathcal{HE}_g$ and restricts to an immersion on the hyperelliptic locus $\mathcal{HE}_g$.

3. Deformation construction of holomorphic 1-forms

In this section, we formulate the Liu-Sun-Todorov's construction of deformations of holomorphic 1-forms on a fixed Riemann surface X_0.

In above section, we point out a way to define a local holomorphic coordinate system on Teichmuller space. In the following, we will construct this local coordinate system from another point of view–the deformation theory of complex structures[9]. Fix a Riemann surface X_0 of genus $g \geq 2$. As above, choose a basis $\{\mu_i\}_{i=1}^n$ of the space of harmonic Beltrami differentials $\mathcal{H}^{0,1}(X_0, T^{1,0}X_0)$ with respect to ω. For sufficiently small $t = (t_1, \cdots, t_n) \in \mathbb{C}^n$, define $\mu_t = \sum_{i=1}^n t_i\mu_i$. Then this μ_t gives a new complex structure J_t on X_0 by putting $\Omega_t^{1,0} = (I + \mu_t \lrcorner)(\Omega_0^{1,0})$, where $\Omega_0^{1,0}$ is the holomorphic cotangent bundle on the original Riemann surface X_0 and $\lrcorner$ denotes the contraction operation. In this way, we get a small deformation $(J_t, X_0) = X_t \to \Delta_\epsilon$ of X_0, where ϵ is sufficiently small. And $t = (t_1, \cdots, t_n)$ forms a local holomorphic coordinate system at the base point $p = [id, X_0] \in \mathcal{T}_g$.

Let $\theta \in H^0(X_0, K_{X_0})$ be a global holomorphic 1-form on X_0. We want to deform it to an element θ_t in $H^0(X_t, K_{X_t})$.

Theorem 3.1. *For each $\theta \in H^0(X_0, K_{X_0})$, there exists a unique η_t which is holomorphic in t for sufficiently small $|t|$, satisfying*
(i) $H(\eta_t) = \theta$, where H is the harmonic projector on (X_0, ω),
(ii) $\theta_t = (I + \mu_t \lrcorner)\eta_t \in H^0(X_t, K_{X_t})$.

Proof. Let $G, \bar{\partial}^*, \partial, H$ be operators on X_0. $I = (i_1, \cdots, i_n) \in \mathbb{Z}_+^n$. Then we define the following forms by iterations

$$\eta_{(0,\cdots,0)} = \theta,$$

$$\eta_I = -G\bar{\partial}^*\partial(\sum_{j=1}^n \mu_j \lrcorner \eta_{(i_1,\cdots,i_j-1,\cdots,i_n)}).$$

From the estimates of Green operator $G, \bar{\partial}^*$ and ∂[9], we see that there is a constant $C = C(m, \alpha)$ depending X_0, m and α such that

$$\|\eta_I\|_{m,\alpha} \leq C\|\theta\|_{m,\alpha},$$

where $\|\cdot\|_{m,\alpha}$ denotes the Sobolev norm.

Now set

$$\eta_t = \theta - \sum_j t_j(G\bar{\partial}^*\partial(\mu_j \lrcorner \theta)) + \sum_{|I|\geq 2} t^I \eta_I.$$

Then it is a well-defined global (1,0)-form on X_0 for each sufficiently small $|t| < \varepsilon(\varepsilon$ depends on C and $\theta)$ and $H(\eta_t) = \theta$.

By the definition of complex structure on X_t,

$$\theta_t = (I + \mu_t \lrcorner)\eta_t \in A^{1,0}(X_t).$$

In order to prove $\theta_t \in H^0(X_t, K_{X_t})$, it needs only to check that $d\theta_t = 0$.

$$\begin{aligned} d\theta_t =&(\partial + \bar{\partial})\theta_t \\ =&\bar{\partial}(\eta_t) + \partial(\varphi_t \lrcorner \eta_t) \\ =&\sum_{|I|>0} t^I(\bar{\partial}\eta_I + \partial(\sum_{j=1}^n \varphi_j \lrcorner \eta_{(i_1,\cdots,i_j-1,\cdots,i_n)})) \\ =&\sum_{|I|>0} t^I(\partial(I - G\Delta)(\sum_{j=1}^n \varphi_j \wedge \eta_{(i_1,\cdots,i_j-1,\cdots,i_n)})) \\ =&\sum_{|I|>0} t^I(\partial\{H(\sum_{j=1}^n \varphi_j \wedge \eta_{(i_1,\cdots,i_j-1,\cdots,i_n)})\}) \\ =&0. \end{aligned}$$

Its uniqueness is easy. □

We want to express the deformation form θ_t in a more useful form.

Corollary 3.1. *There are smooth functions* $f^j_{(i_1,\cdots,i_j-1,\cdots,i_n)}$ *on* X_0 *such that for* $|t|<\varepsilon$,

$$\begin{aligned}\theta_t =&\theta + \sum_j t_j(H(\mu_j \dashv \theta) + df^j_{(0,\cdots,0)})\\ &+\sum_{|I|\geq 2} t^I(\sum_j H(\mu_j \dashv \eta_{(i_1,\cdots,i_j-1,\cdots,i_n)}) + df^j_{(i_1,\cdots,i_j-1,\cdots,i_n)}).\end{aligned}$$

Proof. From above theorem, we see that

$$\begin{aligned}\theta_t =&\theta + \sum_{|I|>0} t^I(I-G\bar\partial^*\partial)(\sum_{j=1}^n \mu_j \dashv \eta_{(i_1,\cdots,i_j-1,\cdots,i_n)})\\ =&\theta + \sum_{|I|>0} t^I(I-G\bar\partial^*\partial)\{\sum_{j=1}^n H(\mu_j \dashv \eta_{(i_1,\cdots,i_j-1,\cdots,i_n)}) + \bar\partial f^j_{(i_1,\cdots,i_j-1,\cdots,i_n)}\}\\ =&\theta + \sum_{|I|>0} t^I\{\sum_{j=1}^n H(\mu_j \dashv \eta_{(i_1,\cdots,i_j-1,\cdots,i_n)}) + \bar\partial f^j_{(i_1,\cdots,i_j-1,\cdots,i_n)} + \partial G\Delta f^j_{(i_1,\cdots,i_j-1,\cdots,i_n)}\}\\ =&\theta + \sum_{|I|>0} t^I\{\sum_{j=1}^n H(\mu_j \dashv \eta_{(i_1,\cdots,i_j-1,\cdots,i_n)}) + df^j_{(i_1,\cdots,i_j-1,\cdots,i_n)}\}.\end{aligned}$$

□

In the same manner, we may deform holomorphic 1-forms along a submanifold of $\mathcal{M}_g$. For simplicity, we consider the curve case. Let $S \subset \mathcal{M}_g$ be a 1-dimensional complex submanifold. For any point $p=[X_0]\in S$, we take a local holomorphic coordinate s so that $s(p)=0$. Since the Bers coordinates t_j are holomorphic, $t_j=t_j(s)$ are holomorphic functions of s. The corresponding Beltrami differentials associated to S are given by $\mu_s=\sum_j t_j(s)\mu_j$. Let $\mu_s = s\mu^{(1)} + s^2\mu^{(2)} + \cdots$ be its Taylor series expansion in s. Hence

Theorem 3.2. *For each* $\theta\in H^0(X_0,K_{X_0})$*, there exists a unique* η_s *which is holomorphic in* s *for sufficiently small* $s\in S$*, satisfying*
(i) $H(\eta_s)=\theta$*, where* H *is the harmonic projector on* (X_0,ω)*,*
(ii) $\theta_s=(I+\mu_s\dashv)\eta_s\in H^0(X_s,K_{X_s})$*.*
Moreover, θ_s *is holomorphic in* s *and has following expansion*

$$\theta_s = \theta + \sum_{k\geq 1} s^k\{\sum_{k_1+k_2=k,k_1\geq 1} H(\mu^{(k_1)}\dashv A_{k_2}) + df_k\},$$

where f_k *are smooth functions on* X_0*,* $A_0=\theta$ *and*

$$A_k = -G\bar\partial^*\partial\{\sum_{k_1+k_2=k,k_1\geq 1}\mu^{(k_1)}\dashv A_{k_2}\}.$$

4. General expansion of period mapping

Let $\Pi : \mathcal{T}_g \to \mathfrak{H}_g$ be the period mapping. Let $[id, X_0]$ be the base point of $\mathcal{T}_g$. Choose a symplectic homology basis $\{A_\alpha, B_\alpha\}_{\alpha=1}^g$ on X_0. Assume $\{\theta^\alpha\}_{\alpha=1}^g$ is a canonical basis of the space of holomorphic 1-forms on X_0. For each above local parameter $t = (t_1, \cdots, t_n)$ on $\mathcal{T}_g$, we obtain a new compact Riemann surface X_t. Before our computations, we recall the Hodge-Riemann bilinear relations for d-closed forms on compact Riemann surfaces[10].

Lemma 4.1. *Let ϕ, φ be two d-closed forms on a compact Riemann surface S, then the following relation holds*

$$\int_S \phi \wedge \varphi = \sum_\gamma \{ \int_{A_\gamma} \phi \int_{B_\gamma} \varphi - \int_{B_\gamma} \phi \int_{A_\gamma} \varphi \}.$$

On each deformed Riemann surface X_t, we construct a holomorphic 1-form θ_t^α from θ^α as in Section 3. Write $b_{\alpha\beta}(t) = \int_{A_\alpha} \theta_t^\beta$ and $\pi_{\alpha\beta}(t) = \int_{B_\alpha} \theta_t^\beta$. As a corollary of Lemma 4.1, the matrix $(b_{\alpha\beta})$ is non-singular for small $|t|$. Set $(b^{\alpha\beta}) = (b_{\alpha\beta})^{-1}$, i.e. $b^{\alpha\gamma} b_{\gamma\beta} = \delta_{\alpha\beta} = b_{\alpha\gamma} b^{\gamma\beta}$, and $\tilde{\theta}_t^\alpha = b^{\beta\alpha}\theta_t^\beta$, then $\tilde{b}_{\alpha\beta} = \int_{A_\alpha} \tilde{\theta}_t^\beta = b^{\gamma\beta} \int_{A_\alpha} \theta_t^\gamma = b^{\gamma\beta} b_{\alpha\gamma} = \delta_{\alpha\beta}$. This shows that the holomorphic 1-forms $\{\tilde{\theta}_t^\alpha\}$ form a canonical basis of $H^0(X_t, K_{X_t})$.

Lemma 4.2. $\frac{d\pi_{\alpha\beta}}{dt_k}(0) = \int_{B_\alpha} H(\mu_k \dashv \theta^\beta)$, $\frac{db_{\alpha\beta}}{dt_k}(0) = \int_{A_\alpha} H(\mu_k \dashv \theta^\beta)$. *Here, H denotes the harmonic projector on X_0.*

Proof. By definition of $\pi_{\alpha\beta}(t)$ and corollary 3.2, we have

$$\begin{aligned} \frac{d\pi_{\alpha\beta}}{dt_k}(0) &= \int_{B_\alpha} \frac{d\theta_t^\beta}{dt_k}|_{t=0}. \\ &= \int_{B_\alpha} \{H(\mu_k \dashv \theta^\beta) + df^{\beta,k}\} \\ &= \int_{B_\alpha} \{H(\mu_k \dashv \theta^\beta). \end{aligned}$$

Similarly,

$$\begin{aligned} \frac{db_{\alpha\beta}}{dt_k}(0) &= \int_{A_\alpha} \frac{d\theta_t^\beta}{dt_k}|_{t=0}. \\ &= \int_{A_\alpha} \{H(\mu_k \dashv \theta^\beta) + df^{\beta,k}\} \\ &= \int_{A_\alpha} H(\mu_k \dashv \theta^\beta). \end{aligned}$$

□

Lemma 4.3.

$$\int_{A_\alpha} H(\mu_k \dashv \theta^\beta) = \frac{\sqrt{-1}}{2} a^{\alpha\gamma} \int_{X_0} \theta^\gamma \wedge H(\mu_k \dashv \theta^\beta).$$

Proof. Set $H(\mu_k \dashv \theta^\beta) = c^\beta_{k,\gamma}\bar{\theta}^\gamma$. Then $\int_{A_\alpha} H(\mu_k \dashv \theta^\beta) = c^\beta_{k,\alpha}$. On the other hand, $\sqrt{-1}\int_X \theta^\alpha \wedge H(\mu_k \dashv \theta^\beta) = \sqrt{-1}c^\beta_{k,\gamma}\int_X \theta^\alpha \wedge \bar{\theta}^\gamma = 2c^\beta_{k,\gamma}a_{\alpha\gamma}$, where $a_{\alpha\beta} = Im\pi_{\alpha\beta}(0)$. These imply that $c^\beta_{k,\alpha} = \frac{\sqrt{-1}}{2}a^{\alpha\gamma}\int_X \theta^\gamma \wedge H(\mu_k \dashv \theta^\beta)$. □

Based on these two lemmas, we can expand the period matrix $\pi(t) = (\pi_{\alpha\beta}(t))$ as a power series of $t = (t_1, \cdots, t_n)$ at the base point of $\mathcal{T}_g$. Write $(a^{\gamma\delta}) = (Im\pi(0))^{-1}$.

Theorem 4.1. *For the period mapping $\Pi : \mathcal{T}_g \to \mathfrak{H}_g$, we have the following expansion*

$$\begin{aligned}\tilde{\pi}_{\alpha\beta}(t) =& \pi_{\alpha\beta}(0) + \int_{X_0} \theta^\alpha \wedge \mu_t \dashv \theta^\beta + \int_{X_0} \theta^\alpha \wedge \mu_t \dashv A^\beta_{t,1} \\ & - \frac{\sqrt{-1}}{2}\int_{X_0} \theta^\alpha \wedge \mu_t \dashv \theta^\gamma \cdot a^{\gamma\delta} \cdot \int_{X_0} \theta^\delta \wedge \mu_t \dashv \theta^\beta \\ & + \sum_{k\geq 3}\{\int_{X_0} \theta^\alpha \wedge \mu_t \dashv A^\beta_{t,k-1} + \sum_{\substack{1\leq k_1\leq k-1\\ 1\leq\delta\leq n}} C(\alpha,\delta,k_1,k,\mu_t)\int_{X_0} \theta^\delta \wedge \mu_t \dashv A^\beta_{t,k_1-1} \\ & + \cdots,\end{aligned}$$

where $A^\alpha_{t,0} = \theta^\alpha, A^\alpha_{t,k} = -G\bar{\partial}^*\partial(\mu_t \dashv A^\alpha_{t,k-1}), \mu_t = \sum_{j=1}^n t_j\mu_j$ and $C(\alpha,\delta,k_1,k,\mu_t)$ is a constant depending on its factors.

Proof. In our above notations, it is equivalent to check this expansion for $\tilde{\pi}_{\alpha\beta}(t)$. Since Π is holomorphic, we need only to compute all partial derivatives of $\tilde{\pi}_{\alpha\beta}(t)$ at $t = 0$.

We write $\tilde{\pi}'_{\alpha\beta} = \frac{d\tilde{\pi}_{\alpha\beta}}{dt_i}$. Then

$$\begin{aligned}
\tilde{\pi}'_{\alpha\beta}(0) =& \pi'_{\alpha\beta}(0) - \pi_{\alpha\gamma}(0)b^{\gamma\xi}(0)b'_{\xi\eta}(0)b^{\eta\beta}(0) \\
=& \int_{B_\alpha} H(\mu_i \dashv \theta^\beta) - \pi_{\alpha\xi}(0)b'_{\xi\beta}(0) \\
=& \int_{B_\alpha} H(\mu_i \dashv \theta^\beta) - \int_{B_\alpha} \theta^\xi \int_{A_\xi} H(\mu_i \dashv \theta^\beta) \\
=& \int_{B_\alpha} H(\mu_i \dashv \theta^\beta) - \sum_\xi \int_{B_\xi} \theta^\alpha \int_{A_\xi} H(\mu_i \dashv \theta^\beta) \\
=& \sum_\xi \{ \int_{A_\xi} \theta^\alpha \int_{B_\xi} H(\mu_i \dashv \theta^\beta) - \int_{B_\xi} \theta^\alpha \int_{A_\xi} H(\mu_i \dashv \theta^\beta) \} \\
=& \int_{X_0} \theta^\alpha \wedge H(\mu_i \dashv \theta^\beta) \\
=& \int_{X_0} \theta^\alpha \wedge \mu_i \dashv \theta^\beta,
\end{aligned}$$

where the last second equality comes from Lemma 4.1.

For the second derivatives, we have

$$\begin{aligned}
\frac{\partial^2 \tilde{\pi}_{\alpha\beta}}{\partial t_j \partial t_k}(0) =& \frac{\partial^2}{\partial t_j \partial t_k}(\pi_{\alpha\gamma} b^{\gamma\beta})(0) \\
=& \partial^2_{t_j t_k} \pi_{\alpha\beta}(0) - \partial_{t_j} \pi_{\alpha\gamma} \partial_{t_k} b_{\gamma\beta} - \partial_{t_k} \pi_{\alpha\gamma} \partial_{t_j} b_{\gamma\beta} + \pi_{\alpha\beta} \partial^2_{t_j t_k} b^{\gamma\beta}.
\end{aligned}$$

By Corollary 3.1, we may expand θ_t as follows

$$\theta^\alpha_t = \theta^\alpha + \sum_i t_i (H(\mu_i \dashv \theta^\alpha) + df^{\alpha,i}) + \sum_{j,k} t_j t_k (H(\mu_j \dashv A^\alpha_k) + df^{\alpha,j}_k) + \cdots,$$

where $A^\alpha_k = -G\bar{\partial}^* \partial(\mu_k \dashv \theta^\alpha)$.

Thus

$$\partial^2_{t_j t_k} \theta^\alpha(0) = \{ H(\mu_j \dashv A^\alpha_k) + df^{\alpha,j}_k + H(\mu_k \dashv A^\alpha_j) + df^{\alpha,k}_j \}$$

and

$$\partial^2_{t_j t_k} \pi_{\alpha\beta}(0) = \int_{B_\alpha} \{ H(\mu_j \dashv A^\beta_k) + H(\mu_j \dashv A^\beta_k) \}.$$

On the other hand,

$$\begin{aligned}\partial^2_{t_j t_k} b^{\gamma\beta} =& \partial_{t_j} b_{\gamma\xi} \partial_{t_k} b_{\xi\beta} - \partial^2_{t_j t_k} b_{\gamma\beta} + \partial_{t_k} b_{\gamma\eta} \partial_{t_j} b_{\eta\beta} \\ =& \int_{A_\gamma} H(\mu_j \dashv \theta^\xi) \int_{A_\xi} H(\mu_k \dashv \theta^\beta) - \int_{A_\gamma} \{H(\mu_j \dashv A^\beta_k) + H(\mu_k \dashv A^\beta_j)\} \\ &+ \int_{A_\gamma} H(\mu_k \dashv \theta^\xi) \int_{A_\xi} H(\mu_j \dashv \theta^\beta).\end{aligned}$$

Hence,

$$\begin{aligned}\frac{\partial^2 \tilde{\pi}_{\alpha\beta}}{\partial t_j \partial t_k}(0) =& \int_{B_\alpha} \{H(\mu_j \dashv A^\beta_k) + H(\mu_k \dashv A^\beta_j)\} - \int_{B_\alpha} H(\mu_j \dashv \theta^\gamma) \int_{A_\gamma} H(\mu_k \dashv \theta^\beta) \\ &- \int_{B_\alpha} H(\mu_k \dashv \theta^\gamma) \int_{A_\gamma} H(\mu_j \dashv \theta^\beta) + \int_{B_\alpha} \theta^\gamma \{ \int_{A_\gamma} H(\mu_j \dashv \theta^\xi) \int_{A_\xi} H(\mu_k \dashv \theta^\beta) \\ &- \int_{A_\gamma} \{H(\mu_j \dashv A^\beta_k) + H(\mu_k \dashv A^\beta_j)\} + \int_{A_\gamma} H(\mu_k \dashv \theta^\xi) \int_{A_\xi} H(\mu_j \dashv \theta^\beta)\} \\ =& \int_{X_0} \theta^\alpha \wedge H(\mu_j \dashv A^\beta_k) + \int_{X_0} \theta^\alpha \wedge H(\mu_k \dashv A^\beta_j) \\ &- \int_{X_0} \theta^\alpha \wedge H(\mu_k \dashv \theta^\gamma) \cdot \int_{A_\gamma} H(\mu_j \dashv \theta^\beta) \\ &- \int_{X_0} \theta^\alpha \wedge H(\mu_j \dashv \theta^\gamma) \int_{A_\gamma} H(\mu_k \dashv \theta^\beta).\end{aligned}$$

In the following, we will derive all of the higher derivatives by another method other than direct computations. Observe that the period mapping may be viewed as a composition of $t \mapsto \mu_t \mapsto \pi(\mu_t) = \pi(t)$. Without loss of generality, we assume first $\mu_t = t\mu$, $t \in \mathbb{C}$. In order to simplify the notations, we make the following conventions. Set $f^\alpha_{-1} = 0$, $H(\mu \dashv A^\alpha_{-1}) = \theta^\alpha$ and $A^\alpha_0 = \theta^\alpha$. From the construction in Section 3, we know $\frac{d^l}{dt^l}|_{t=0}\theta^\beta_t = l!\{H(\mu \dashv A^\beta_{l-1}) + df^\beta_{l-1}\}, l \geq 0$.

Lemma 4.4. $\frac{d^k}{dt^k}|_{t=0} b^{\beta\alpha}(t) = -\sum_{k_1=1}^k C^{k_1}_k \frac{d^{k_1}}{dt^{k_1}}|_{t=0} b_{\beta\delta} \frac{d^{k-k_1}}{dt^{k-k_1}}|_{t=0} b^{\delta\alpha}$.

Proof. $0 = \frac{d^k}{dt^k}(b_{\beta\delta} b^{\delta\alpha}) = \sum_{k_1=1}^k C^{k_1}_k \frac{d^{k_1}}{dt^{k_1}}|_{t=0} b_{\beta\delta} \frac{d^{k-k_1}}{dt^{k-k_1}}|_{t=0} b^{\delta\alpha} + \frac{d^k}{dt^k}|_{t=0} b^{\beta\alpha}(t)$. □

Now for $k \geq 1$

$$\begin{aligned}
\frac{d^k}{dt^k}|_{t=0}\tilde{\pi}_{\alpha\beta}(t) &= \int_{B_\alpha} \frac{d^k}{dt^k}|_{t=0}\{\theta_t^\gamma b^{\gamma\beta}(t)\} \\
&= \int_{B_\alpha} \sum_{k_1+k_2=k} C_k^{k_1} \frac{d^{k_1}}{dt^{k_1}}|_{t=0}\theta_t^\gamma \frac{d^{k_2}}{dt^{k_2}}|_{t=0} b^{\gamma\beta}(t) \\
&= k! \int_{B_\alpha} H(\mu \lrcorner A_{k-1}^\beta) + \\
&\quad \int_{B_\alpha} \sum_{\substack{k_1+k_2=k \\ k_2\geq 1}} C_k^{k_1} k_1! H(\mu \lrcorner A_{k_1-1}^\gamma)\{ \sum_{\substack{k_3+k_4=k_2 \\ k_3\geq 1}} -C_{k_2}^{k_3} \frac{d^{k_3}}{dt^{k_3}} b_{\gamma\delta} \frac{d^{k_4}}{dt^{k_4}} b^{\delta\beta}\}|_{t=0} \\
&= k! \int_{B_\alpha} H(\mu \lrcorner A_{k-1}^\beta) - k! \int_{B_\alpha} \theta^\gamma \int_{A_\gamma} H(\mu \lrcorner A_{k-1}^\beta) \\
&\quad - \int_{B_\alpha} \theta^\gamma \sum_{\substack{k_3+k_4=k \\ k_3\geq 1, k_4\geq 1}} C_k^{k_3} k_3! \int_{A_\gamma} H(\mu \lrcorner A_{k_3-1}^\delta) \frac{d^{k_4}}{dt^{k_4}}|_{t=0} b^{\delta\beta}(t) \\
&\quad + \int_{B_\alpha} \sum_{\substack{k_1+k_2=k \\ k_1\geq 1, k_2\geq 1}} C_k^{k_1} k_1! H(\mu \lrcorner A_{k_1-1}^\delta) \frac{d^{k_2}}{dt^{k_2}}|_{t=0} b^{\delta\beta}(t) \\
&= k! \int_{X_0} \theta^\alpha \wedge \mu \lrcorner A_{k-1}^\beta \\
&\quad + \sum_{\substack{k_1+k_2=k \\ k_1\geq 1, k_2\geq 1}} C_k^{k_1} k_1! \int_{X_0} (\theta^\alpha \wedge \mu \lrcorner A_{k_1-1}^\delta) \frac{d^{k_2}}{dt^{k_2}}|_{t=0} b^{\delta\beta}(t).
\end{aligned}$$

Now by Lemma 4.3 and Lemma 4.4, we can compute $\frac{d^{k_2}}{dt^{k_2}}|_{t=0} b^{\delta\beta}(t)$ in terms of integrations over X_0. This shows there are constants $C(\alpha, \delta, k_1, k, \mu)$ such that

$$\frac{1}{k!}\frac{d^k}{dt^k}|_{t=0}\tilde{\pi}_{\alpha\beta}(t) = \int_{X_0} \theta^\alpha \wedge \mu \lrcorner A_{k-1}^\beta + \sum_{\substack{1\leq k_1\leq k-1 \\ 1\leq\delta\leq n}} C(\alpha, \delta, k_1, k, \mu) \int_{X_0} \theta^\delta \wedge \mu \lrcorner A_{k_1-1}^\beta.$$

In general, the k-homogenous term in the power series expansion of $\tilde{\pi}_{\alpha\beta}(t)$ is given by

$$\int_{X_0} \theta^\alpha \wedge \mu_t \lrcorner A_{t,k-1}^\beta + \sum_{\substack{1\leq k_1\leq k-1 \\ 1\leq\delta\leq n}} C(\alpha, \delta, k_1, k, \mu_t) \int_{X_0} \theta^\delta \wedge \mu_t \lrcorner A_{t,k_1-1}^\beta.$$

In sum, we obtain

$$
\begin{aligned}
\tilde{\pi}_{\alpha\beta}(t) =& \pi_{\alpha\beta}(0) + \int_{X_0} \theta^\alpha \wedge \mu_t \dashv \theta^\beta + \int_{X_0} \theta^\alpha \wedge \mu_t \dashv A^\beta_{t,1} \\
& - \frac{\sqrt{-1}}{2} \int_{X_0} \theta^\alpha \wedge \mu_t \dashv \theta^\gamma \cdot a^{\gamma\delta} \cdot \int_{X_0} \theta^\delta \wedge \mu_t \dashv \theta^\beta \\
& + \sum_{k\geq 3} \{ \int_{X_0} \theta^\alpha \wedge \mu_t \dashv A^\beta_{t,k-1} + \sum_{\substack{1\leq k_1\leq k-1 \\ 1\leq\delta\leq n}} C(\alpha,\delta,k_1,k,\mu_t) \int_{X_0} \theta^\delta \wedge \mu_t \dashv A^\beta_{t,k_1-1} \\
& + \cdots ,
\end{aligned}
$$

The convergence is verified by the convergence of θ_t^α in Corollary 3.1. □

5. An application

In this section, we use the general expansion of period matrix to study the distorsion problem[3,4]. We will embark on the complex curve exclusively and study whether there is a totally geodesic complex curve in $\mathcal{A}_g$ which is also contained in the open Torelli locus $\mathcal{J}(\mathcal{M}_g)$. Let us first recall some more facts on the Siegel space $\mathfrak{H}_g$. Since it is a Hermitian symmetric space of non-compact type, it admits a unique (up to a positive constant) invariant metric which is Kahler. The real symplectic group

$$Sp(g,\mathbb{R}) = \{W = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GL(2g,\mathbb{R}) | A,B,C,D \in M_g(\mathbb{R}), W^t J_g W = J_g\},$$

where $J_g = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}$ and I_g is the identity $g \times g$ matrix, acts holomorphically, isometrically and transitively on $\mathfrak{H}_g$ by

$$(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, Z) = (AZ+B)(CZ+D)^{-1}.$$

$\mathfrak{H}_g$ can be realized as a bounded domain in $\mathbb{C}^{\frac{g(g+1)}{2}}$ by Cartan realization

$$Z \mapsto M = \frac{I_g + iZ}{I_g - iZ}.$$

Its image

$$D_g^{III} = \{M \in M_g(\mathbb{C}) | M^t = M, I_g - \overline{M}^t M > 0\}$$

is a bounded complex domain and admits a Bergman metric

$$\omega_b = -2i\partial\bar{\partial} \log\det(I_g - \overline{M}^t M).$$

Its pullback by Cartan realization is just the invariant metric on $\mathfrak{H}_g$ mentioned above. Hence the pullback metric is also $Sp(g,\mathbb{R})$-invariant and descends to the quotient $\mathcal{A}_g$. Its sectional curvature lies in the interval $[-1, -\frac{1}{g}]$.

Another fact we will use is the characterization of totally geodesic complex curves in $\mathfrak{H}_g$. It is a standard fact that any totally geodesic complex curve can be transformed by an element in $Sp(g,\mathbb{R})$ to one of the totally geodesic discs $\Delta_k = \{diag(z_1,\cdots,z_k,0,\cdots,0)|z_i = z, 1 \leq i \leq k \text{ and } |z| < 1\}$ after Cartan realization. Thus there are precisely g equivalent classes of totally geodesic complex curves in $\mathfrak{H}_g$ under the action of $Sp(g,\mathbb{R})$.

Definition 1.1. *A totally geodesic complex curve $X \subset \mathcal{A}_g$ is said to be of type k if and only if X is uniformed by a disc in $\mathfrak{H}_g$ equivalent to Δ_k under the action of $Sp(g,\mathbb{R})$.*

Our main result is following.

Theorem 5.1. *Let X be a totally geodesic complex curve of type k in $\mathcal{J}(\mathcal{M}_g - \mathcal{HE}_g)$. If $1 \leq k \leq g-1$, then for any point $p \in X$, there exists a nonzero local holomorphic section θ_s of Hodge bundle $E|_X$ around p which is of the form $\theta_s = \theta + df(s)$, where $f(s)$ is a smooth function on X_0.*

Proof. Let t_j be the Bers coordinates around p on $\mathcal{M}_g$ as constructed in Section 3 and s be a local holomorphic coordinate around p on X which will be determined later. Then $t_j = t_j(s)$ is a holomorphic function of s. We may expand the period matrix $\pi(s)$ by Theorem 4.1 as a power series in s along X. Assume $\mu = \mu^{(1)} = \sum_j \frac{dt_j}{ds}(0)\mu_j$ represents the tangent vector of X at p. Write

$$\pi(s) = \pi(0) + sP_1 + s^2P_2 + \cdots .$$

By assumption, X may be uniformed by a totally geodesic disc of type k, so there exists a real symplectic matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ satisfying

$$A\pi(s) + B = Z(s)(C\pi(s) + D), \tag{5.1}$$

where $Z(s) = i\frac{I_g - W(s)}{I_g + W(s)}$ and $W(s) = diag(\overbrace{z(s),\cdots,z(s)}^{k},0,\cdots,0)$. Now take $s = z$ as the local holomorphic coordinate about p on X. Set $\tilde{C} = (\tilde{c}_{\alpha\beta}) = A - iC$ and $\theta = \tilde{c}_{g\gamma}\theta^\gamma$. By comparing the homogenous terms of s on the two sides of equation (5.1), one gets

$$A\pi(0) + B = i(C\pi(0) + D), \tag{5.2}$$

$$AP_l = 2i(-1)^l \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}(C\pi(0)+D) + \sum_{j=1}^{l-1} 2i(-1)^j \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix} CP_{l-j} + iCP_l. \tag{5.3$_l$}$$

Then by $(5.3)_1$

$$\tilde{C}P_1 = \begin{pmatrix} * \\ \vec{0} \end{pmatrix}. \tag{5.3}$$

Lemma 5.1. *Each P_l is a symmetric matrix and*

$$P_l = (\int_{X_0} \theta^\alpha \wedge \{ \sum_{l_1+l_2=l, l_1\geq 1} H(\mu^{(l_1)} \dashv A^\beta_{l_2})\} +$$
$$\sum_{\substack{k_1+k_2=l, 1\leq\delta\leq n \\ 1\leq k_1\leq l-1}} C(\alpha,\delta,k_1,l,\mu^{(1)},\cdots,\mu^{(k_2)}) \int_{X_0} \theta^\delta \wedge \{ \sum_{\substack{l_3+l_4=k_1 \\ l_3\geq 1}} H(\mu^{(l_3)} \dashv A^\beta_{l_4})\}).$$

Proof. It follows immediately from the symmetry of period matrix, Theorem 3.2 and Theorem 4.1. □

So $P_1 = (\int_{X_0} \theta^\alpha \wedge \mu \dashv \theta^\beta)$,

$$\int_{X_0} \theta^\alpha \wedge \mu \dashv \theta = 0, \quad \forall \quad \alpha.$$

While X_0 is outside the hyperelliptic locus, $H(\mu \dashv \theta) = 0$. Consider its deformation θ_s as constructed in Theorem 3.2. We show that θ_s is exactly what we require. First, θ is nozero. In other words, there is a γ_0 such that $\tilde{c}_{g\gamma_0} \neq 0$. Otherwise, $\tilde{C}$ and C are of the form

$$\begin{pmatrix} * \\ \vec{0} \end{pmatrix}.$$

By equation (5.2), so are $B - iD$ and D. These imply $C\pi(0) + D$ is also of the above form and consequently not invertible. Contradiction! Second, to prove θ_s preserves the cohomology class, we need to check $\sum_{k_1+k_2=k, k_1\geq 1} H(\mu^{(k_1)} \dashv A_{k_2})$ are all zeros. We accomplish it by induction on k and Lemma 5.1. □

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