Regularity of Refinable Functions with Exponentially Decaying Masks

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Abstract

The smoothness property of refinable functions is an important issue in all multiresolution analysis and has a strong impact on applications of wavelets to image processing, geometric and numerical solutions of elliptic partial differential equations. The purpose of this paper is to characterize the smoothness properties of refinable functions with exponentially decaying masks and an isotropic dilation matrix by analyzing the spectral properties of associated transfer operators. The main results of this paper are really extensions of some results in [5], [14] and [25].

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1 Introduction

A homogeneous refinement equation with mask $a$ and a general dilation matrix $M$ is the functional equation of the form

$$\phi(x) = \sum_{\alpha \in \mathbb{Z}^s} a(\alpha) \phi(Mx - \alpha), \quad x \in \mathbb{R}^s, \quad (1.1)$$

where $\phi$ is the unknown function defined on the $s$-dimensional Euclidean space $\mathbb{R}^s$, $a$ is an exponentially decaying sequence on $\mathbb{Z}^s$ called refinement mask, and $M$ is an $s \times s$ integer matrix with $m = |\det M|$ such that $\lim_{n \to \infty} M^{-n} = 0$. The solution of (1.1) is called refinable function, and the matrix $M$ is called a dilation matrix. It is well known that refinement equations play an important role in wavelet analysis and computer graphics. Most useful wavelets in applications are generated from refinable functions.

Great efforts have been spent on the study of refinable functions when mask $a$ is a finitely supported sequence (see, e.g., [1, 5, 6, 10, 14, 26, 29] and many references therein). However, in some
cases, one need to study the refinement equations with infinitely supported masks. For example, in engineering, infinitely supported masks are needed \([9]\), the masks of various types of fractional splines in \([33]\) are infinitely supported. In particular, due to some desirable properties, infinitely supported masks with exponential decay are of interest in the area of digital signal processing in electrical engineering (see e.g., \([2, 3, 7, 12, 28]\)). To study Riesz bases of wavelets generated from multiresolution analysis, Han and Jia \([11]\) investigated the \(L_2\)-solution of refinement equation \((1.1)\) with exponentially decaying masks and a general dilation matrix. Li and Yang \([24]\) studied existence of \(L_2\)-solution of vector refinement equation with exponentially decaying masks and a general dilation matrix. In the binary case \((s = 1, M = (2))\) and mask \(a\) is an exponentially decaying sequence, Cohen and Daubechies \([3]\) studied the regularity of refinable functions by analyzing the spectral properties of transfer operators, Han \([9]\) also investigated the smoothness of refinable functions for the study of hierarchical bases in Sobolev spaces. In the case \(s > 1\) and \(M = 2I_{s \times s}\), Lorentz and Oswald \([25]\) investigated the smoothness properties of refinable functions with exponentially decaying masks. For example, infinitely supported masks are needed \([9]\), the masks of various types of fractional splines \([33]\) are infinitely supported. In particular, due to some desirable properties, infinitely supported masks with exponential decay are of interest in the area of digital signal processing in electrical engineering (see e.g., \([2, 3, 7, 12, 28]\)). To study Riesz bases of wavelets generated from multiresolution analysis, Han and Jia \([11]\) investigated the \(L_2\)-solution of refinement equation \((1.1)\) with exponentially decaying masks and a general dilation matrix. Li and Yang \([24]\) studied existence of \(L_2\)-solution of vector refinement equation with exponentially decaying masks and a general dilation matrix. In the binary case \((s = 1, M = (2))\) and mask \(a\) is an exponentially decaying sequence, Cohen and Daubechies \([3]\) studied the regularity of refinable functions by analyzing the spectral properties of transfer operators, Han \([9]\) also investigated the smoothness of refinable functions for the study of hierarchical bases in Sobolev spaces. In the case \(s > 1\) and \(M = 2I_{s \times s}\), Lorentz and Oswald \([25]\) investigated the smoothness properties of refinable functions with exponentially decaying masks.

Our goal is to investigate the smoothness of solutions of refinement equation \((1.1)\) by studying the spectral properties of associated transfer operators, and a formula for the critical Sobolev exponent is obtained, which extends some main results in \([5, 14, 25]\) to the general setting. Our characterizations were inspired by the work of Cohen and Daubechies \([3]\), Jia \([14]\), Cohen, Gröchenig, and Villemoes \([5]\) and Lorentz and Oswald \([25]\).

Before proceeding further, we introduce some notations. Let \(\mathbb{N}\) denote the set of positive integers and \(\mathbb{N}_0\) denote the set of nonnegative integers. An element \(\mu = (\mu_1, \ldots, \mu_s) \in \mathbb{N}_0^s\) is called a multi-index. For \(j = 1, \ldots, s\), let \(e_j\) be the \(j\)th coordinate unit vector in \(\mathbb{R}^s\). The norm in \(\mathbb{R}^s\) is defined by

\[
|y| := |y_1| + \cdots + |y_s|, \quad y = (y_1, \cdots, y_s) \in \mathbb{R}^s.
\]

We denote by \(\ell(\mathbb{Z}^s)\) the linear space of all (complex valued) sequences on \(\mathbb{Z}^s\), and by \(\ell_0(\mathbb{Z}^s)\) the linear space of all finitely supported sequence on \(\mathbb{Z}^s\). The difference operator \(\nabla_j\) on \(\ell(\mathbb{Z}^s)\) is defined by \(\nabla_j a := a - a(-e_j), a \in \mathbb{Z}^s\). For \(\mu = (\mu_1, \ldots, \mu_s) \in \mathbb{N}_0^s\), \(\nabla^\mu\) is the difference operator \(\nabla_1^{\mu_1} \cdots \nabla_s^{\mu_s}\), and \(D^\mu\) is the differential operator \(D_1^{\mu_1} \cdots D_s^{\mu_s}\).

As usual, for \(1 \leq p \leq \infty\), we denote by \(L_p(\mathbb{R}^s)\) the Banach space of all (complex-valued) functions \(f\) such that \(\|f\|_p < \infty\), where

\[
\|f\|_p := \left( \int_{\mathbb{R}^s} |f(x)|^p dx \right)^{1/p} \quad \text{for} \quad 1 \leq p < \infty,
\]

and \(\|f\|_\infty\) is the essential supremum of \(f\) on \(\mathbb{R}^s\). We observe that \(L_2(\mathbb{R}^s)\) is a Hilbert space with the inner product given by

\[
\langle f, g \rangle := \int_{\mathbb{R}^s} f(x) \overline{g(x)} dx.
\]

Analogously, let \(\ell_p(\mathbb{Z}^s)(1 \leq p \leq \infty)\) be the Banach space of all complex-valued sequence \(a = (a(\alpha))_{\alpha \in \mathbb{Z}^s}\) such that \(\|a\|_p < \infty\), where

\[
\|a\|_p := \left( \sum_{\alpha \in \mathbb{Z}^s} |a(\alpha)|^p \right)^{1/p} \quad \text{for} \quad 1 \leq p < \infty,
\]
and \( \|a\|_\infty \) is the supremum of \( a \) on \( \mathbb{Z}^s \).

Let \( b \) and \( c \) be two sequences on \( \mathbb{Z}^s \), the convolution of \( b \) with \( c \) is defined by
\[
b * c(\alpha) = \sum_{\beta \in \mathbb{Z}^s} b(\beta)c(\alpha - \beta), \quad \alpha \in \mathbb{Z}^s
\]
such that the sum is convergent. By the discrete version of Young’s inequality, if \( b \in \ell_1 \) and \( c \in \ell_p \) \((1 \leq p \leq \infty)\), then \( b * c \in \ell_p \), and
\[
\|b * c\|_p \leq \|b\|_1 \|c\|_p. \tag{1.2}
\]

The Fourier transform of a function \( f \) in \( L_1(\mathbb{R}^s) \) is defined by
\[
\hat{f}(\xi) := \int_{\mathbb{R}^s} f(x)e^{-ix \cdot \xi} dx, \quad \xi \in \mathbb{R}^s,
\]
where \( \cdot x \) is the inner product of two vectors \( \xi \) and \( x \) in \( \mathbb{R}^s \). The domain of the Fourier transform can be naturally extended to functions in \( L_2(\mathbb{R}^s) \) and tempered distribution. Similarly, the Fourier series of a sequence \( c \in \ell_1(\mathbb{Z}^s) \) is defined by
\[
\hat{c}(\xi) := \sum_{\alpha \in \mathbb{Z}^s} c(\alpha)e^{-i\alpha \cdot \xi}, \quad \xi \in \mathbb{R}^s.
\]
Evidently, \( \hat{c} \) is a \( 2\pi \)-periodic continuous function on \( \mathbb{R}^s \). In particular, \( \hat{c} \) is a trigonometric polynomial if \( c \) is finitely supported. We call \( \hat{c} \), the symbol of \( c \).

In order to clarify the concept of exponential decay, we introduce the weighted space \( E_{\mu} \) as follows. Suppose \( c \) is a (complex-valued) summable sequence on \( \mathbb{Z}^s \). For \( \mu \geq 0 \), define
\[
\|\hat{c}\|_{\mu} := \sum_{\alpha \in \mathbb{Z}^s} |c(\alpha)|e^{\mu|\alpha|}. \tag{1.3}
\]
Let \( E_{\mu} \) denote the Banach space of all \( 2\pi \)-periodic functions \( \hat{c} \) on \( \mathbb{R}^s \) such that \( \|\hat{c}\|_{\mu} < \infty \).

Weighted \( L_p \) spaces are defined as follows. Suppose \( f \) is a (complex-valued) measurable function on \( \mathbb{R}^s \). For \( \mu \geq 0 \) and \( 1 \leq p < \infty \), define
\[
\|f\|_{p,\mu} := \left( \int_{\mathbb{R}^s} |f(x)e^{\mu|x|}|^p dx \right)^{1/p}. \tag{1.4}
\]
For \( \mu \geq 0 \), let \( \|f\|_{\infty,\mu} \) be the essential supremum of the function \( |f(x)e^{\mu|x|} | \) on \( \mathbb{R}^s \). We use \( L_{p,\mu} \) to denote the Banach space of all measure functions \( f \) such that \( \|f\|_{p,\mu} < \infty \).

In this paper, we always assume that mask \( a \) is an exponentially decaying sequence, i.e., there exists a \( \mu > 0 \), such that \( \hat{a} \in E_{\mu} \). Denote continuous function
\[
H(\xi) := \frac{1}{m} \sum_{\alpha \in \mathbb{Z}^s} a(\alpha)e^{-i\alpha \cdot \xi}, \quad \xi \in \mathbb{R}^s. \tag{1.5}
\]

To investigate the smoothness of a refinable function associated with exponentially decaying mask \( a \) and dilation matrix \( M \), following [3], [15] and [25], we shall also assume that
\[
H(\xi) = p(\xi)q(\xi), \quad \text{with} \quad H(0) = 1, \tag{1.6}
\]
where \( p(\xi) \) is a trigonometric polynomial, \( q(\xi) \) is a \( 2\pi \)-periodic \( C^\infty \) function with exponentially decaying Fourier coefficients, and \( q(\xi) \neq 0 \), \( \forall \xi \in \mathbb{R}^s \). The particular case \( q(\xi) \equiv 1 \) covers the case of finite supported masks.

Let \( M \) be a fixed matrix with \( m = |\det M| \). Then the coset space \( \mathbb{Z}^s/M^T\mathbb{Z}^s \) consists of \( m \) elements, where \( M^T \) denotes the transpose of \( M \). Let \( \omega_k + M^T\mathbb{Z}^s, k = 0, 1, \ldots, m - 1 \) be the \( m \) distinct elements of \( \mathbb{Z}^s/M^T\mathbb{Z}^s \) with \( \omega_0 = 0 \). We denote \( \Omega := \{\omega_k, k = 0, 1, \ldots, m - 1\} \). Thus, each element \( \alpha \in \mathbb{Z}^s \) can be uniquely represented as \( \omega + M^T\varepsilon \), where \( \omega \in \Omega \) and \( \varepsilon \in \mathbb{Z}^s \). Analogously, we denote by \( \Gamma \) a complete set of representatives of the distinct cosets of \( \mathbb{Z}^s/M\mathbb{Z}^s \).

We say that mask \( a \) satisfies the basic sum rule if

\[
\sum_{\alpha \in \mathbb{Z}^s} a(\gamma + M\alpha) = \sum_{\alpha \in \mathbb{Z}^s} a(M\alpha) \quad \forall \gamma \in \Gamma.
\]

Suppose \( M \) is an \( s \times s \) matrix with its entries in \( \mathbb{C} \). We say that \( M \) is isotropic if \( M \) is similar to a diagonal matrix \( \text{diag}\{\lambda_1, \ldots, \lambda_s\} \) with \( |\lambda_1| = \cdots = |\lambda_s| \).

For \( 1 \leq p \leq \infty \), denote by \( L_p(\mathbb{R}^s) \) the linear space of all (complex-valued) functions \( f \) such that \( |f|_p < \infty \), where

\[
|f|_p := \left( \int_{[0,1]^s} \left( \sum_{\alpha \in \mathbb{Z}^s} |f(\cdot - \alpha)| \right)^p dx \right)^{1/p} \quad \text{for} \quad 1 \leq p < \infty,
\]

and \( |f|_\infty \) is the essential supremum of \( \sum_{\alpha \in \mathbb{Z}^s} |f(\cdot - \alpha)| \) on \( [0,1]^s \). Equipped with the norm \( |\cdot|_p \), \( L_p(\mathbb{R}^s) \) becomes a Banach space. There are several important subspaces of \( L_p \). For instance, if \( f \in L_p \) is compactly supported, then \( f \in L_p(1 \leq p \leq \infty) \). Also, a function \( f \in L_{p,\mu} \) with \( \mu > 0 \) is in \( L_p(1 \leq p \leq \infty) \). See [16] for more discussions of \( L_p \) spaces.

The concept of stability plays an important role in the study of the smoothness properties of refinable functions. Let \( \phi \in L_p(\mathbb{R}^s)(1 \leq p \leq \infty) \). We say that the shifts of \( \phi \) are \( \ell_p \)-stable if there there exist positive constants \( A_p \) and \( B_p \) such that for all sequences \( a \in \ell_p(\mathbb{Z}^s) \),

\[
A_p \|a\|_p \leq \left\| \sum_{\alpha \in \mathbb{Z}^s} a(\alpha)\phi(\cdot - \alpha) \right\|_p \leq B_p \|a\|_p. \tag{1.7}
\]

See [16], [19] for more detail about \( \ell_p \)-stability. It was proved by Jia and Micchelli in [16] that a function \( \phi \in L_p(\mathbb{R}^s) \) has \( \ell_p \)-stable integer translates if and only if, for any \( \xi \in \mathbb{R}^s \), there exists an element \( \beta \in \mathbb{Z}^s \) such that

\[
\hat{\phi}(\xi + 2\pi \beta) \neq 0.
\]

For \( \nu \geq 0 \), denote by \( H^\nu(\mathbb{R}^s) \) the Sobolev space of all functions \( f \in L_2(\mathbb{R}^s) \) such that

\[
\int_{\mathbb{R}^s} |\hat{f}(\xi)|^2(1 + |\xi|^2)^\nu d\xi < \infty.
\]

The critical Sobolev exponent of a function \( f \in L_2(\mathbb{R}^s) \) is defined by

\[
s_f := \sup \{\nu : f \in H^\nu(\mathbb{R}^s)\}.
\]
Sobolev spaces are closely related to Lipschitz Spaces, which is defined in terms of the modulus of continuity. For $y \in \mathbb{R}^s$, the difference operator is defined by
\[ \nabla_y f := f - f(\cdot - y), \]
where $f$ is a function defined on $\mathbb{R}^s$. The modulus of continuity of a function $f$ in $L_p(\mathbb{R}^s)$ is defined by
\[ \omega(f, h)_p := \sup_{|y| \leq h} \|\nabla_y f\|_p, \quad h \geq 0. \]
Let $k$ be a positive integer. The $k$th modulus of continuity of $f$ in $L_p(\mathbb{R}^s)$ is defined by
\[ \omega_k(f, h)_p := \sup_{|y| \leq h} \|\nabla_y^k f\|_p, \quad h \geq 0. \]
For $1 \leq p \leq \infty$, $0 < \nu \leq 1$ and a function $f \in L_p(\mathbb{R}^s)$, we say $f$ belongs to the Lipschitz Space $\text{Lip}(\nu, L_p(\mathbb{R}^s))$ if there exists a positive constant $C$ independent of $h$ such that
\[ w(f, h)_p \leq Ch^\nu \quad \forall h > 0. \]
For $\nu > 0$, we write $\nu = r + \eta$, where $r$ is an integer and $0 < \eta \leq 1$. We say that $f$ belongs to the Lipschitz Space $\text{Lip}^*(\nu, L_p(\mathbb{R}^s))$ if $D^\mu f \in \text{Lip}(\eta, L_p(\mathbb{R}^s))$ for all multi-indices $\mu$ with $|\mu| = r$. For $\nu > 0$, let $k$ be an integer greater than $\nu$. The generalized Lipschitz Space $\text{Lip}^*(\nu, L_p(\mathbb{R}^s))$ consists of all functions $f \in L_p(\mathbb{R}^s)$ such that
\[ \omega_k(f, h)_p \leq Ch^\nu \quad \forall h > 0, \]
where $C$ is a positive constant independent of $h$. If $\nu > 0$ is not an integer, then
\[ \text{Lip}(\nu, L_p(\mathbb{R}^s)) = \text{Lip}^*(\nu, L_p(\mathbb{R}^s)), \quad 1 \leq p \leq \infty. \]
See [8], [32] and [14] for more detail about Lipschitz Spaces.

Following the discussion in [14], we have
\[ \sup \{\nu : f \in H^\nu(\mathbb{R}^s)\} = \sup \{\nu : f \in \text{Lip}(\nu, L_2(\mathbb{R}^s))\} = \sup \{\nu : f \in \text{Lip}^*(\nu, L_2(\mathbb{R}^s))\}. \]

2 Some auxiliary results

In this section, we shall introduce some auxiliary results. In the following, let $T := [-\pi, \pi]^s$. We denote by $C(T)$ the space of all continuous $2\pi$-period functions on $\mathbb{R}^s$ and by $L_\infty(T)$ the space of all $2\pi$-period measurable functions $f$ on $T$ such that $\|f\|_{L_\infty(T)} < \infty$, where $\|f\|_{L_\infty(T)}$ denotes the essential supremum of $|f|$ on $T$.

Following [6] and [25], it is easy to obtain Lemma 2.1 and Lemma 2.2.

**Lemma 2.1.** Let $a$ be an exponentially decaying sequence and $H(\xi)$ be given by (1.5) with $H(0) = 1$, then the infinite product
\[ \hat{\phi}(\xi) := \prod_{j=1}^{\infty} H((MT)^{-j}\xi) \quad (2.1) \]
converges absolutely and uniformly on any compact subsets of $\mathbb{R}^s$ to a continuous function.
Lemma 2.2. For \( \mu > 0 \), let \( b \) be an exponentially decaying sequence with \( \hat{b} \in E_\mu \) and \( \hat{b}(0) = 1 \), then for all positive integers \( N \), there exist trigonometric polynomials

\[
\hat{b}_N(\xi) = \sum_{|\alpha| \leq N} b_N(\alpha)e^{-i\alpha \cdot \xi}, \quad \xi \in \mathbb{R}^s,
\]

such that

\[
\| \hat{b}(\xi) - \hat{b}_N(\xi) \|_{L_\infty(T)} \leq Ce^{-\mu N}, \quad \text{and} \quad \hat{b}_N(0) = 1,
\]

where \( C \) is a positive constant independent of \( N \) and \( |\alpha| = |\alpha_1| + \cdots + |\alpha_s| \) with \( \alpha = (\alpha_1, \ldots, \alpha_s) \in \mathbb{Z}^s \).

Proof. Let \( \hat{b}_N(\xi) \) be defined by

\[
\hat{b}_N(\xi) = \sum_{|\alpha| \leq N} b(\alpha)e^{-i\alpha \cdot \xi} + \sum_{|\alpha| > N} b(\alpha).
\]

Then we have \( \hat{b}_N(0) = 1 \) and

\[
\| \hat{b}(\xi) - \hat{b}_N(\xi) \|_\infty \leq \sum_{|\alpha| > N} 2|b(\alpha)| \leq Ce^{-\mu N},
\]

where \( C \) is independent of \( N \).

Transfer operator is an useful tool for the study of refinable functions (see e.g., [3, 5, 6, 17] and [31]). For a given \( 2\pi \)-periodic function \( u(\xi) \), the transition operator \( T_u \) acts on \( 2\pi \)-periodic functions according to

\[
(T_u f)(\xi) = \sum_{\omega_i \in \Omega} u((M^T)^{-1}(\xi + 2\pi \omega_i))f((M^T)^{-1}(\xi + 2\pi \omega_i)). \tag{2.2}
\]

We fix \( u(\xi) = |H(\xi)|^2 \), where \( H(\xi) \) is defined by (1.5) and \( H(\xi) \in E_\mu \) for some \( \mu > 0 \). It is easy to see that \( u(\xi) \in E_\mu \). In terms of the Fourier coefficients \( u_\alpha \) of \( u(\xi) = \sum_{\alpha \in \mathbb{Z}^s} u_\alpha e^{-i\alpha \cdot \xi} \), (2.2) can also be rewritten as

\[
(T_u f)_\alpha = \frac{1}{(2\pi)^s} \int_{[0,2\pi]^s} T_u f(\xi)e^{i\alpha \cdot \xi}d\xi = m \sum_{\beta \in \mathbb{Z}^s} u_{M\alpha - \beta}f_\beta, \tag{2.3}
\]

where \( f(\xi) = \sum_{\beta \in \mathbb{Z}^s} f_\beta e^{-i\beta \cdot \xi} \).

It follows from (11) that

\[
\| T_u f \|_\mu = \sum_{\alpha \in \mathbb{Z}^s} |(T_u f)_\alpha|e^{\mu|\alpha|} = m \sum_{\alpha \in \mathbb{Z}^s} \sum_{\beta \in \mathbb{Z}^s} |u_{M\alpha - \beta}f_\beta|e^{\mu|\alpha|} \leq m\| u \|_\mu \| f \|_\mu.
\]

Therefore, \( T_u \) is a bounded linear operator on \( E_\mu \). Furthermore, \( T_u \) is also a compact operator on \( E_u \).

Let \( V = \{ \tilde{v} \in E_\mu : \sum_{\alpha \in \mathbb{Z}^s} |v(\alpha)||\alpha| < \infty \text{ and } \sum_{\alpha \in \mathbb{Z}^s} v(\alpha) = 0 \} \).
Theorem 2.3. Let $T_u$ be given by (2.2) with $u$ being exponential decay. For $L \in \mathbb{N}$, define $z_L = [\sin^2(\frac{x}{2}) + \cdots + \sin^2(\frac{x}{2})]^L$. Then

$$\lim_{k \to \infty} \|T_k^k z_L\|_{L^\infty(T)}^{1/k} \leq \lim_{k \to \infty} \|T_k^k z_L\|_{\mu}^{1/k} \leq \rho(T_u|V),$$

where $\rho(T_u|V)$ denotes the spectral radius of the restriction of $T_u$ to the subspace $V$ of $E_{\mu}$.

Proof. It is easy to check that $z_L \in V$. Hence, for $k \geq 1$, we have

$$\|T_k^k z_L\|_{L^\infty(T)} \leq \|T_k^k z_L\|_{1} \leq \|T_k^k z_L\|_{1} \leq \rho(T_u|V).$$

Since

$$\lim_{k \to \infty} \|T_k^k z_L\|_{1}^{1/k} = \rho(T_u|V),$$

we obtain that

$$\lim_{k \to \infty} \|T_k^k z_L\|_{L^\infty(T)}^{1/k} \leq \lim_{k \to \infty} \|T_k^k z_L\|_{1}^{1/k} \leq \rho(T_u|V).$$

\[\square\]

Lemma 2.4. Let $T_u$ be given by (2.2) with $u$ being exponential decay. For any $f(\xi), g(\xi) \in C(T)$, and any positive integer $n$, we have the following identity

$$\int_T T_u^n f(\xi)\overline{g(\xi)}d\xi = \int_{(M^T)^nT} \prod_{k=1}^n u((M^T)^{-k}\xi)\overline{f((M^T)^{-n}\xi)\overline{g(\xi)}}d\xi.$$

To characterize the smoothness of refinable functions with masks having exponential decay, we need the following Theorem 2.5 which was established by Lorentz and Oswald in [25] for the special case $M = 2I_{s \times s}$.

Theorem 2.5. Let $u(\xi) = |H(\xi)|^2 := P(\xi)Q(\xi)$, where $H(\xi)$ is defined by (1.5) and $P(\xi) = |p(\xi)|^2, Q(\xi) = |q(\xi)|^2$. Suppose $Q_N(\xi)$ is a trigonometric polynomial sequence of approximations of $Q(\xi)$ as in Lemma 2.2 such that $\|Q(\xi) - Q_N(\xi)\|_{L^\infty(T)} \leq Ce^{-\mu N}$. Let $u_N(\xi) = P(\xi)Q_N(\xi)$. Then

$$\rho_L = \lim_{N \to \infty} \rho(T_u|V_{u_N,z_L})$$

(2.5)

and

$$\rho = \lim_{L \to \infty} \rho_L$$

(2.6)

both exist and do not depend on the specific choice of the sequence $Q_N(\xi)$, where $V_{u_N,z_L}$ is defined by

$$V_{u_N,z_L} = \text{span}\{T_u^n z_L : n \geq 0\}.$$

Proof. Since $Q(\xi)$ is a positive $2\pi$-periodic continuous function, there exists a positive constant $c$, such that $|Q(\xi)| \geq c > 0$. Consequently, for sufficiently large $N$, we have

$$\max(|1 - \frac{Q_N(\xi)}{Q(\xi)}|, |1 - \frac{Q(\xi)}{Q_N(\xi)}|) \leq C'e^{-\mu N},$$

(2.7)
where $C^*$ is a positive constant independent of $N$.

Therefore,
\[
(1 - C^*e^{-\mu N})Q_N(\xi) \leq Q(\xi) \leq (1 + C^*e^{-\mu N})Q_N(\xi),
\]
and
\[
(1 - C^*e^{-\mu N})u_N(\xi) \leq u(\xi) \leq (1 + C^*e^{-\mu N})u_N(\xi).
\]

By Lemma 2.4, we have
\[
(1 - C^*e^{-\mu N})\|T_{u_N}^k z_L\|_{L_\infty(T)}^{1/k} \leq \|T_{u_N}^k z_L\|_{L_\infty(T)}^{1/k} \leq (1 + C^*e^{-\mu N})\|T_{u_N}^k z_L\|_{L_\infty(T)}^{1/k}.
\] (2.8)

Since $u_N$ is a trigonometric polynomial, by Lemma 2.4 of [10], we have that $V_{u_N,z_L}$ is finite dimensional and
\[
\lim_{k \to \infty} \|T_{u_N}^k z_L\|_{L_\infty(T)}^{1/k} = \rho(T_{u_N}|V_{u_N,z_L}).
\]

Let $k \to \infty$ in (2.8), by Theorem 2.3, we have that
\[
(1 - C^*e^{-\mu N})\rho(T_{u_N}|V_{u_N,z_L}) \leq \lim_{k \to \infty} \|T_{u_N}^k z_L\|_{L_\infty(T)}^{1/k} \leq (1 + C^*e^{-\mu N})\rho(T_{u_N}|V_{u_N,z_L}).
\] (2.9)

Then, let $N \to \infty$ in (2.9), we obtain that
\[
\rho_L := \lim_{N \to \infty} \rho(T_{u_N}|z_L) = \lim_{k \to \infty} \|T_{u_N}^k z_L\|_{L_\infty(T)}^{1/k}
\] (2.10)
exists.

If $\tilde{u}_N(\xi)$ is another sequence of approximations of $u(\xi)$, we can easily obtain
\[
(1 - C^*e^{-\mu N})u_N(\xi) \leq \tilde{u}_N(\xi) \leq (1 + C^*e^{-\mu N})u_N(\xi).
\]

Thus,
\[
(1 - C^*e^{-\mu N})\|T_{u_N}^k z_L\|_{L_\infty(T)} \leq \|T_{u_N}^k z_L\|_{L_\infty(T)} \leq (1 + C^*e^{-\mu N})\|T_{u_N}^k z_L\|_{L_\infty(T)}.
\]

This together with (2.10) imply that $\rho_L$ is independent of the specific choice of the sequence $u_N$. Since $0 \leq z_{L+1} \leq sz_L$, (2.10) yields
\[
\rho = \lim_{L \to \infty} \rho_L \leq \rho_{L+1} \leq \rho_L \cdots \leq \rho_0.
\] (2.11)

The proof of Theorem 2.5 is complete.

### 3 Characterization of Smoothness

In this section, we give some characterizations of the smoothness property of refinable functions with mask $a$ having exponential decay and an isotropic dilation matrix $M$.

When mask $a$ is finitely supported, $H(\xi) = \frac{1}{m} \sum_{\alpha \in \mathbb{Z}^d} a(\alpha)e^{-i\alpha \cdot \xi}$ is a trigonometric polynomial. Define $S_c$ as the largest nonnegative integer such that
\[
D^\mu H(2\pi(M^T)^{-1}\omega) = 0, \quad \forall \omega \in \Omega \setminus \{0\}, \text{ and } |\mu| \leq S_c - 1.
\]
In the following, $L$ denotes an integer large than $S_c$.

It follows from [5] that $V_{H^2, Z_L} = \text{span}\{T_{H^2}^k z_L, k \geq 0, k \in \mathbb{N}_0\}$ is finite dimensional and is an invariant subspace of $T_{H^2}$.

Cohen, Gröchenig, and Villemoes [5] gave a characterization of the smoothness of refinable functions based on the spectral radius of $T_{H^2}$ restricted to $V_{H^2, Z_L}$ when mask $a$ is finitely supported. Their results were stated as following:

**Theorem 3.1.** [Cohen et al.] Let $\phi \in L_2(\mathbb{R}^s)$ be a compactly supported solution to (1.1) with mask $a$ being finitely supported and $M$ being an isotropic dilation matrix. Suppose the shifts of $\phi$ are stable. Let $\rho$ be the spectral radius of $T_{H^2}$ restricted to $V_{H^2, Z_L}$. Then the Sobolev exponent $s_\phi$ satisfies

$$s_\phi = -\frac{s}{2} \log \rho, \tag{3.1}$$

and $\phi \in H^s$ if and only if $s < s_\phi$, where $m = |\det M|$.

We also point out that similar characterization was also established by Jia in [14] with a different method. Several other researchers have considered the regularity of refinable functions in high dimensions (see [4] [18] [20] [21] [22] [23] [27] [30] and [31]).

We are in a position to establish the following characterization of smoothness of a refinable function in terms of the refinement mask.

**Theorem 3.2.** Let $M$ be an isotropic dilation matrix with $|\det M| = m$ and $\phi$ be a $L_2$-solution of refinement equation (1.1). Assume that $H(\xi)$ satisfies (1.6). Let $\rho$ be defined by (2.6). Then $s_\phi \geq -\frac{s}{2} \log \rho$. Moreover, if the shifts of $\phi$ are stable, then $s_\phi = -\frac{s}{2} \log \rho$.

**Proof.** Set $T = [-\pi, \pi]^s, G = T \setminus (M^T)^{-1} T$, then $\mathbb{R}^s = T \cup \bigcup_{n=1}^\infty (M^T)^n G$. For $\xi \in (M^T)^n T$, it is easy to see that

$$|\hat{\phi}(\xi)|^2 \leq C_1 \prod_{k=1}^n |H((M^T)^{-k}\xi)|^2 \tag{3.2}$$

with $C_1 = \max_{\xi \in T} |\hat{\phi}(\xi)|^2$. For any fixed $L \in \mathbb{N}$, since $(M^T)^{-1} T$ contains an open ball centered at the origin, we have

$$z_L(\xi) \geq C_2 > 0 \quad \text{for} \quad \xi \in G. \tag{3.3}$$

It follows from Lemma 2.4,

$$\int_{(M^T)^n G} \prod_{k=1}^n |H((M^T)^{-k}\xi)|^2 z_L((M^T)^{-n}\xi) d\xi = (T_{[H^2 z_L, 1]}^n \leq \|T_{[H^2 z_L]}^n\|_{L_\infty(T)} \cdot (3.4)$$

Thus, combine (3.2), (3.3), and (3.4), we have

$$\int_{(M^T)^n G} |\hat{\phi}(\xi)|^2 (1 + |\xi|^2)^\gamma \leq C_3 m^{\frac{2m}{s}} \int_{(M^T)^n G} |\hat{\phi}(\xi)|^2 d\xi \leq C_4 m^{\frac{2m}{s}} \|T_{[H^2 z_L]}^n\|_{L_\infty(T)}. \tag{3.5}$$

Note that,

$$\int_{\mathbb{R}^s} |\hat{\phi}(\xi)|^2 (1 + |\xi|^2)^\gamma d\xi = \int_T |\hat{\phi}(\xi)|^2 (1 + |\xi|^2)^\gamma d\xi + \sum_{n=1}^\infty \int_{(M^T)^n G} |\hat{\phi}(\xi)|^2 (1 + |\xi|^2)^\gamma d\xi. \tag{3.6}$$
Since $|\hat{\phi}(\xi)|^2(1 + |\xi|^2)^2$ is continuous on $T$, the first integer on the right-hand side of (3.6) is a finite constant $C_5$. This together with (3.5) imply that

$$
\int_{\mathbb{R}^s} |\hat{\phi}(\xi)|^2(1 + |\xi|^2)^2 d\xi \leq C_5 + C_4 \sum_{n=1}^{\infty} m^{2n/\gamma} \|T_{|H|^2}^n z_L\|_{L^\infty(T)}.
$$

By Theorem 2.3, for any $\varepsilon > 0$, there exists $C(\varepsilon) > 0$, such that for large enough $L$,

$$
\|T_{|H|^2}^n z_L\|_{L^\infty(T)} \leq C(\varepsilon)(\rho + \varepsilon)^n, \quad n \in \mathbb{N}.
$$

Consequently,

$$
\int_{\mathbb{R}^s} |\hat{\phi}(\xi)|^2(1 + |\xi|^2)^2 d\xi \leq C_5 + C_4 C(\varepsilon) \sum_{n=1}^{\infty} m^{2n/\gamma} (\rho + \varepsilon)^n.
$$

When $\gamma < -\frac{s}{2} \log_m (\rho + \varepsilon)$,

$$
\int_{\mathbb{R}^s} |\hat{\phi}(\xi)|^2(1 + |\xi|^2)^2 d\xi < \infty.
$$

Since $\varepsilon$ can be chosen arbitrary small, we conclude that $s_\phi \geq -\frac{s}{2} \log_m \rho$ and the first part of the theorem follows. In all these estimates, the positive constants $C_j$ are independent of $n$.

To prove the second part, for $H(\xi) = p(\xi)q(\xi)$, we can choose a sequence of trigonometric polynomials $q_N(\xi)$ of approximations of $q(\xi)$ as in Lemma 2.2. Since the shifts of $\phi$ are stable, it follows from Theorem 4.1 of [11] that the cascade algorithm associated mask $a$ and dilation matrix $M$ converges in $L_2(\mathbb{R}^s)$. Therefore, $a$ satisfies the basic sum rule and $\rho(T_{|H|^2}|V) < 1$. Let

$$
H_N(\xi) = p(\xi)q_N(\xi),
$$

then $H_N(\xi)$ generate a sequence of refinable functions $\phi_N(\xi)$ with compact support. Since $a$ satisfies the basic sum rule if and only if $H(2\pi M^{-1} \omega) = 0, \forall \omega \in \Omega \setminus \{0\}$. It follows that the masks $a_N$ associated with $\phi_N$ also satisfy the basic sum rule for $N$ large enough. By the choice of $H_N$, we have that

$$
\lim_{N \to \infty} T_{|H_N|^2} \to T_{|H|^2}
$$

in the $E_{\mu}$-norm, which implies that

$$
\rho(T_{|H_N|^2}|V) < 1,
$$

for $N$ large enough. Then, it follows from Theorem 4.2 of [11] that the cascade algorithms associated with masks $a_N$ and dilation matrix $M$ converge in $L_2$, which implies that $\phi_N \in L_2$.

Since

$$
\hat{\phi}(\xi) = \prod_{j=1}^{\infty} H(\frac{\xi}{2^j}) = \prod_{j=1}^{\infty} p(\frac{\xi}{2^j}) \prod_{j=1}^{\infty} q(\frac{\xi}{2^j}),
$$

and

$$
\hat{\phi}_N(\xi) = \prod_{j=1}^{\infty} H_N(\frac{\xi}{2^j}) = \prod_{j=1}^{\infty} p(\frac{\xi}{2^j}) \prod_{j=1}^{\infty} q_N(\frac{\xi}{2^j}).
$$

If the shifts of $\phi$ are stable, we obtain that there exist positive constants $A$ and $B$ such that

$$
A \leq \sum_{k \in \mathbb{Z}^s} |\hat{\phi}(\xi + 2\pi k)|^2 \leq B, \quad \forall \xi \in \mathbb{R}^s.
$$
Thus, for any $\xi \in \mathbb{R}^s$, there exists a $\alpha \in \mathbb{Z}^s$ such that $\hat{\phi}(\xi + 2\pi \alpha) \neq 0$, which implies that $\prod_{j=1}^{\infty} p(\frac{\xi + 2\pi \alpha}{\partial_j}) \neq 0$. Therefore, $\hat{\phi}_N(\xi + 2\pi \alpha) \neq 0$ for $N$ large enough. We obtain that the shifts of $\phi_N$ are also stable for sufficiently large $N$.

Assume $\phi \in H^\gamma(\mathbb{R}^s)$ for some $\gamma > -\frac{s}{2} \log_m \rho$, then

$$
\int_{\mathbb{R}^s} |\hat{\phi}(\xi)|^2 (1 + |\xi|^2)^\gamma d\xi < \infty.
$$

Fix any $\tilde{\gamma} \in (-\frac{s}{2} \log_m \rho, \gamma)$, we claim that $\phi_N \in H^{\tilde{\gamma}}(\mathbb{R}^s)$ for $N$ large enough.

For sufficiently large $N$, we have $1 + C^* e^{-\mu N} < m^{2(\gamma - \tilde{\gamma})}$, where $C^*$ is the same as in (2.7). Set $\Omega_k = (M^T)^k T \setminus (M^T)^{(k-1)} T$, it is easy to see that $\mathbb{R}^s = (\bigcup_{k=1}^{\infty} \Omega_k) \bigcup T$. It follows that

$$
\int_{\mathbb{R}^s} (1 + |\xi|^2)^{\tilde{\gamma}} |\hat{\phi}_N(\xi)|^2 d\xi
= \int_T \sum_{\alpha \in \mathbb{Z}^s} (1 + |\xi + 2\pi \alpha|^2)^{\tilde{\gamma}} \left( \prod_{k=1}^{\infty} |H_N((M^T)^{-k}(\xi + 2\pi \alpha))|^2 \right) d\xi
\leq C \sum_{k=1}^{\infty} \frac{m^{2\gamma k}}{T} \sum_{\alpha \in \Omega_k} \left( \prod_{l=1}^{k} |H_N((M^T)^{-l}(\xi + 2\pi \alpha))|^2 \right) d\xi + \int_T \sum_{\alpha \in T} (1 + |\xi + 2\pi \alpha|^2)^{\tilde{\gamma}} |\hat{\phi}_N(\xi + 2\pi \alpha)|^2 d\xi.
$$

Note that $|\hat{\phi}_N(\xi + 2\pi \alpha)|^2$ is continuous on $T$. Hence, there exists a positive constant $B_1$ such that $\int_T \sum_{\alpha \in T} (1 + |\xi + 2\pi \alpha|^2)^{\tilde{\gamma}} |\hat{\phi}_N(\xi + 2\pi \alpha)|^2 d\xi \leq B_1$.

Consequently, we deduce that

$$
\int_{\mathbb{R}^s} (1 + |\xi|^2)^{\tilde{\gamma}} |\hat{\phi}_N(\xi)|^2 d\xi
\leq C \sum_{k=1}^{\infty} \frac{m^{2\gamma k}}{T} \sum_{\alpha \in \Omega_k} \left( \prod_{l=1}^{k} |H_N((M^T)^{-l}(\xi + 2\pi \alpha))|^2 \right) d\xi + B_1
\leq C \sum_{k=1}^{\infty} \frac{m^{2\gamma k}}{T} \sum_{\alpha \in \Omega_k} \left( \prod_{l=1}^{k} |H((M^T)^{-l}(\xi + 2\pi \alpha))|^2 \right) d\xi + B_1
= C \int_T \sum_{k=1}^{\infty} \frac{m^{2\gamma k}}{T} \sum_{\alpha \in \Omega_k} \left( \prod_{l=1}^{k} |H((M^T)^{-l}(\xi + 2\pi \alpha))|^2 \right) d\xi + B_1
\leq \tilde{C} \int_{\mathbb{R}^s} |\hat{\phi}(\xi)|^2 (1 + |\xi|^2)^{\gamma} d\xi + B_1 < \infty.
$$

Therefore, $\phi_N \in H^{\tilde{\gamma}}(\mathbb{R}^s)$ for $N$ large enough. Since the shifts of $\phi_N$ are stable, it follows from Theorem 3.1 that $s_{\phi_N} = -\frac{s}{2} \log_m \rho(T_{u_N}, V_{u_N}, Z_L)$ for any $L \geq L_0$. Therefore, $s_{\phi_N} \geq \tilde{\gamma} > -\frac{s}{2} \log_m \rho$ for $N$ large enough. By the definition of $\rho$, this is impossible. Hence, $s_{\phi} \leq -\frac{s}{2} \log_m \rho$. Combined with the proof of the first part, we complete the proof of Theorem 3.2.

\[ \square \]

**Remark 3.3.** Theorem 3.2 characterizes the optimal smoothness of a refinable function with mask $a$ having exponential decay and an isotropic dilation matrix $M$, which extends Theorem 3.1 to the
case that mask $a$ is infinitely supported. Theorem 3.2 was also established in [25] for the case $M = 2I_{s \times s}$.

In some cases, when mask $a$ is exponential decay, the solution of equation (1.1) belongs to $L_p(\mathbb{R}^s)$ for $1 \leq p \leq \infty$. For example, Han [9] characterized the existence of $L_{2,\mu}$-solution of equation (1.1) with $a$ being exponential decay and $M = 2$ for the case $s = 1$, and Jia [15] also gave a characterization of the existence of the solution of refinement equation in $L_{p,\mu}(1 \leq p \leq \infty)$ by considering the convergence of the corresponding cascade algorithm associated with mask $a$ being exponential decay. Hogan [13] investigated some properties of refinement equation (1.1) under the assumptions that solution $\phi \in L_p(\mathbb{R}^s)$ for $1 \leq p \leq \infty$. In this section, we will characterize the smoothness of solution of refinement equation (1.1) in $L_p(\mathbb{R}^s)$ considering the convergence of the corresponding cascade algorithm associated with mask $a$ being exponential decay. Hogan [13] investigated some properties of refinement equation (1.1) under the assumptions that solution $\phi \in L_p(\mathbb{R}^s)$ for $1 \leq p \leq \infty$. In this section, we will characterize the smoothness of solution of refinement equation (1.1) in $L_p(\mathbb{R}^s)$ for $1 \leq p \leq \infty$. The following Theorems 3.4 and 3.5 give some characterizations of the smoothness of refinable functions in $L_2(\mathbb{R}^s)$ and in $L_p(\mathbb{R}^s)$ for $1 \leq p \leq \infty$, respectively. Our characterizations are based on a discrete version of Young’s inequality.

**Theorem 3.4.** Suppose $\nu > 0$ and $k$ is a positive integer. Let $M$ be an isotropic dilation matrix with $m = |\det M|$. Suppose $\phi \in L_2(\mathbb{R}^s)$ is the normalized solution of (1.1) with mask $a$ being exponential decay. For $n = 1, 2, \ldots$, let $a_n$ be given by

$$a_n(\alpha) = \sum_{\beta \in \mathbb{Z}^s} a(\alpha - M\beta)a_{n-1}(\beta), \quad \alpha \in \mathbb{Z}^s,$$

(3.7)

where $a_0$ is the $\delta$ sequence given by $\delta(0) = 1$ and $\delta(\alpha) = 0$ for $\alpha \in \mathbb{Z}^s \setminus \{0\}$.

If there exists a constant $C > 0$ such that

$$\|\nabla_j a_n\|_2 \leq Cm^{(1/2-\nu/s)n} \quad \forall n \in \mathbb{N} \quad \text{and} \quad j = 1, \ldots, s,$$

(3.8)

then $\phi \in \text{Lip}^*(\nu, L_2(\mathbb{R}^s))$. Conversely, if $\phi \in \text{Lip}^*(\nu, L_2(\mathbb{R}^s))$, and the shifts of $\phi$ are stable, then (3.8) holds for $k > \nu$.

**Proof.** Since $\phi \in L_2(\mathbb{R}^s)$ is the normalized solution of (1.1), it is easy to check that

$$\phi = \sum_{\alpha \in \mathbb{Z}^s} a_n(\alpha)\phi(M^n \cdot -\alpha).$$

(3.9)

Therefore,

$$\nabla_{M^{-n}e_j} \phi = \sum_{\alpha \in \mathbb{Z}^s} \nabla_j a_n(\alpha)\phi(M^n \cdot -\alpha).$$

(3.10)

By (1.2), we obtain that

$$\|\nabla_{M^{-n}e_j} \phi\|_2 = \left(\int_{\mathbb{R}^s} \left| \sum_{\alpha \in \mathbb{Z}^s} \nabla_j a_n(\alpha)\phi(M^n x - \alpha) \right|^2 dx \right)^{1/2}$$

$$= \left[ \sum_{\beta \in \mathbb{Z}^s} \int_{[\beta+[0,1]^s} \left| \sum_{\alpha \in \mathbb{Z}^s} \nabla_j a_n(\alpha)\phi(x - \alpha) \right|^2 dx \right]^{1/2}$$

$$= \left[ \sum_{\beta \in \mathbb{Z}^s} \int_{[0,1]^s} \left| \sum_{\alpha \in \mathbb{Z}^s} \nabla_j a_n(\alpha)\phi(x - \beta - \alpha) \right|^2 dx \right]^{1/2}$$

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\[
\begin{align*}
= m^{-\frac{\nu}{2}} \left( \int_{[0,1]^s} \left| \sum_{\alpha \in \mathbb{Z}^s} \nabla_j^k a_n(\alpha) \phi(x - \beta - \alpha)^2 \right| dx \right)^{\frac{1}{2}} \\
\leq m^{-\frac{\nu}{2}} \left( \int_{[0,1]^s} \| \nabla_j^k a_n \|^2 \left( \sum_{\beta \in \mathbb{Z}^s} |\phi(x - \beta)| \right)^2 dx \right)^{\frac{1}{2}} \\
\leq m^{-\frac{\nu}{2}} \| \nabla_j^k a_n \|^2 \| \phi \|_{L_2[0,1]^s}.
\end{align*}
\]

This in connection with (3.8) tell us that
\[
\| \nabla_j^k M - n e_j \phi \|^2 \leq C m^{-\frac{\nu n}{2}} \| \phi \|_{L_2[0,1]^s}. \tag{3.11}
\]

It follows from Theorem 2.1 in [13] that (3.11) is equivalent to
\[
\| \nabla_j^k \phi \|^2 \leq C |y|^\nu \quad \forall y \in \mathbb{R}^s. \tag{3.12}
\]

This shows \( \phi \in \text{Lip}^*(\nu, L_p(\mathbb{R}^s)) \), as desired.

Conversely, since the shifts of \( \phi \) are stable. It follows from (1.7) and (3.10) that
\[
m^{-\frac{\nu}{2}} \| \nabla_j^k a_n \|^2 \leq C_2 \| \nabla_j^k M - n e_j \phi \|^2, \tag{3.13}
\]

where \( C_2 \) is a constant independent of \( n \) and \( j \). If \( \phi \in \text{Lip}^*(\nu, L_2(\mathbb{R}^s)) \), then for \( k > \nu \)
\[
\| \nabla_j^k M - n e_j \phi \|^2 \leq C m^{-\frac{\nu n}{2}}. \tag{3.14}
\]

Therefore, (3.8) follows from (3.13) and (3.14) immediately. \( \square \)

**Theorem 3.5.** Let \( \nu > 0 \) and \( k \) be a positive integer. Suppose \( M = qI_{s \times s} \) and \( q \geq 2 \) is an integer. Let \( \phi \) be the normalized solution of (1.1) with mask \( a \) being exponential decay and \( \phi \in L_p(\mathbb{R}^s) \) for \( 1 \leq p \leq \infty \). For \( n = 1, 2, \ldots \), let \( a_n \) be given by (3.7). If there exists a constant \( C > 0 \) such that
\[
\| \nabla_j^k a_n \|_p \leq C q^{(s/p-\nu)n} \quad \forall n \in \mathbb{N} \quad \text{and} \quad j = 1, \ldots, s, \tag{3.15}
\]

then \( \phi \in \text{Lip}^*(\nu, L_p(\mathbb{R}^s)) \). Conversely, if \( \phi \in \text{Lip}^*(\nu, L_p(\mathbb{R}^s)) \), and the shifts of \( \phi \) are \( \ell_p \)-stable, then (3.15) holds for \( k > \nu \).

**Proof.** Since \( \phi \) is the normalized solution of (1.1), we have that
\[
\nabla_{q^{-n} e_j} \phi = \sum_{\alpha \in \mathbb{Z}^s} \nabla_j^k a_n(\alpha) \phi(q^n \cdot -\alpha). \tag{3.16}
\]
By virtue of (1.2), we have

\[
\|\nabla^k_{q-ne^j} \phi\|_p = \left( \int_{\mathbb{R}^s} | \sum_{\alpha \in \mathbb{Z}^s} \nabla^k_j a_n \phi(q^n x - \alpha)|^p dx \right)^{\frac{1}{p}}
\]

\[
= q^{-\frac{ns}{p}} \left( \sum_{\beta \in \mathbb{Z}^s} \int_{[\beta+[0,1]^s} | \sum_{\alpha \in \mathbb{Z}^s} \nabla^k_j a_n (\phi(x - \beta)|^p dx \right)^{\frac{1}{p}}
\]

\[
= q^{-\frac{ns}{p}} \left( \int_{[0,1]^s} \sum_{\beta \in \mathbb{Z}^s} | \sum_{\alpha \in \mathbb{Z}^s} \nabla^k_j a_n (\phi(x - \beta)|^p dx \right)^{\frac{1}{p}}
\]

\[
\leq q^{-\frac{ns}{p}} \left( \int_{[0,1]^s} \| \nabla^k_j a_n \|_p (\sum_{\beta \in \mathbb{Z}^s} \| \phi(x - \beta)|^p dx \right)^{\frac{1}{p}}
\]

\[
\leq q^{-\frac{ns}{p}} \| \nabla^k_j a_n \|_p \| \phi \|_{L^p[0,1]^s}.
\]

By (3.15), we obtain that

\[
\|\nabla^k_{q-ne^j} \phi\|_p \leq Cq^{-\nu n} \| \phi \|_{L^p[0,1]^s}.
\]  \(\text{(3.17)}\)

From Theorem 3.1 in \([23]\), we know that (3.17) is equivalent to

\[
\|\nabla^k_j \phi\|_p \leq C|y|^\nu \quad \forall y \in \mathbb{R}^s.
\]  \(\text{(3.18)}\)

This shows \(\phi \in Lip^*(\nu, L^p(\mathbb{R}^s))\), as desired.

Conversely, since the shifts of \(\phi\) are \(\ell^p\)-stable. It follows from (1.7) and (3.16) that

\[
q^{-\frac{ns}{p}} \| \nabla^k_j a_n \|_p \leq C_2 \nabla^k_{q-ne^j} \phi\|_p,
\]  \(\text{(3.19)}\)

where \(C_2\) is a constant independent of \(n\) and \(j\). If \(\phi \in Lip^*(\nu, L^p(\mathbb{R}^s))\), then for \(k > \nu\)

\[
\|\nabla^k_{q-ne^j} \phi\|_p \leq Cq^{-\nu n}.
\]  \(\text{(3.20)}\)

Therefore, (3.15) follows from (3.19) and (3.20). The proof of the theorem is complete. \(\blacksquare\)

**Remark 3.6.** We remark that Theorem 3.4 was established by Jia in \([14]\) for the case in which \(\phi\) is a compactly supported \(L^2\)-solution of equation (1.1) and Theorem 3.5 was obtained by Li in \([23]\) and Jia et.al in \([18]\) for the case in which \(\phi\) is a compactly supported \(L^p\)-solution of equation (1.1) for \(1 \leq p \leq \infty\).
References


