# Time-machine in Electromagnetic Field

Ming Shen     Qing-You Sun *

Center of Mathematical Sciences, Zhejiang University, Hangzhou 310027, China

**Abstract**

In this paper we investigate the time-machine problem in the electromagnetic field. Based on a metric which is a more general form of Ori's, we solve the Einstein's equations with the energy-momentum tensors for electromagnetic field and construct the time-machine solutions, which solve the time machine problem in electromagnetic field.

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## 1 Introduction

In recent years time-machine problem has been concerned a lot (see [?]-[?]). Time-machines are space-time configurations including closed timelike curves (CTCs), and allowing physical observers to return to their own past. Recently, Ori (see [?]) presented a class of curved space-time vacuum solutions which develop closed timelike curves at some particular moment, and then used these vacuum solutions to construct a time-machine model. In this model, the causality violation occurs inside an empty torus which constitutes the time-machine core and the matter field surrounding this empty torus satisfies the weak, strong and dominant energy conditions (see [?]). It is regular, asymptotically-flat and topologically-trival. More precisely, Ori described the geometry of the time-machine core by investigating the vacuum solution of Einstein's field equations

$$ds^2 = -2dzdt + dx^2 + dy^2 + [u(x, y, z) - t]dz^2. \quad (1.1)$$

The coordinates $(t, x, y)$ get all real values. $z$ is a cyclic coordinate $(0 \leq z \leq L)$, for some $L > 0$, with $z = L$ and $z = 0$ identified. $f$ is restricted in the sense $f_{xx} + f_{yy} = 0$ for the vacuum solutions. More recently, Ori(see [?]) presented a new asymptotically flat time-machine model made of vacuum and dust.

It is well known that models of the electromagnetic field play an essential role in the study of modern cosmology. The time-machine in electromagnetic field is also very important and interesting. We investigate this problem in this paper for a more general form

$$ds^2 = -2dzdt + dx^2 + dy^2 + f(t, x, y, z)dz^2. \quad (1.2)$$

*Corresponding author: qysun@cms.zju.edu.cn

It's clear that this metric include Ori's. Using the metric (1.2), we solve the Einstein's equations with the energy-momentum tensors for electromagnetic field and give an interesting example of electromagnetic field time-machine solution.

## 2 Time-machine in the electromagnetic field

Consider the Einstein's equations with the electromagnetic field energy-momentum tensor

$$G_{\mu\nu} = 8\pi T_{\mu\nu} \qquad (\mu, \nu = 0, 1, 2, 3), \tag{2.1}$$

where $G_{\mu\nu} \triangleq R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$, and $T_{\mu\nu}$ is the energy-momentum tensor. For the electromagnetic field,

$$T_{\mu\nu} = F_{\mu\lambda} F_{\nu}^{\lambda} - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}, \tag{2.2}$$

in which electromagnetic tensor $F_{\mu\nu}$ reads

$$(F_{\mu\nu}) = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix}, \tag{2.3}$$

where $E_i = E_i(t, x, y, z)(i = 1, 2, 3)$ is electric field strength and $B_i = B_i(t, x, y, z)(i = 1, 2, 3)$ is magnetic Field Strength, and $g_{\mu\nu}$ is a metric in the following form

$$(g_{\mu\nu}) = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & f(t, x, y, z) \end{pmatrix}. \tag{2.4}$$

Since $T_{\mu\nu}$ is trace-free, so the equation (**??**) becomes

$$R_{\mu\nu} = 8\pi T_{\mu\nu}. \tag{2.5}$$

Noting (**??**), (**??**) and (**??**), by a direct calculation we obtain

$$(T_{\mu\nu}) = \begin{pmatrix} T_{00} & T_{01} & T_{02} & T_{03} \\ T_{01} & T_{11} & T_{12} & T_{13} \\ T_{02} & T_{12} & T_{22} & T_{23} \\ T_{03} & T_{13} & T_{23} & T_{33} \end{pmatrix}. \tag{2.6}$$

where

$$T_{00} = -E_1^2 - E_2^2, \tag{2.7}$$

$$T_{01} = E_2 B_3 - E_1 E_3, \tag{2.8}$$

$$T_{02} = -E_1 B_3 - E_2 E_3, \tag{2.9}$$

$$T_{03} = \frac{1}{2} \left[ -E_3^2 - B_3^2 + E_1^2 f + E_2^2 f \right], \tag{2.10}$$

$$T_{11} = -E_1B_2 - E_2B_1 + \frac{1}{2}\left[-E_3^2 - B_3^2 + E_1^2 f - E_2^2 f\right], \quad (2.11)$$

$$T_{12} = E_1B_1 - E_2B_2 + E_1E_2 f, \quad (2.12)$$

$$T_{13} = B_1B_3 - E_3B_2 + E_1E_3 f, \quad (2.13)$$

$$T_{22} = E_1B_2 + E_2B_1 + \frac{1}{2}\left[-E_3^2 - B_3^2 - E_1^2 f + E_2^2 f\right], \quad (2.14)$$

$$T_{23} = E_3B_1 + B_2B_3 + E_2E_3 f, \quad (2.15)$$

$$T_{33} = -B_1^2 - B_2^2 + E_3^2 f + \left\{-E_2B_1 + E_1B_2 + \frac{1}{2}\left[-E_3^2 + B_3^2 - E_1^2 f - E_2^2 f\right]\right\} f. \quad (2.16)$$

A direct calculation gives

$$(R_{\mu\nu}) = \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{2}f_{tt} \\ 0 & 0 & 0 & -\frac{1}{2}f_{tx} \\ 0 & 0 & 0 & -\frac{1}{2}f_{ty} \\ -\frac{1}{2}f_{tt} & -\frac{1}{2}f_{tx} & -\frac{1}{2}f_{ty} & \frac{1}{2}(f_{tt} - f_{xx} - f_{yy}) \end{pmatrix}. \quad (2.17)$$

Therefore, by (**??**)-(**??**) we have the following equations

$$\begin{cases} E_1 = E_2 = E_3 = B_3 = 0, \\ f_{tt} = f_{tx} = f_{ty} = 0, \\ \dfrac{-f_{tt} + f_{xx} + f_{yy}}{2} = 8\pi(B_1^2 + B_2^2). \end{cases} \quad (2.18)$$

By the second equation of (**??**), we obtain

$$f(t,x,y,z) = h(x,y,z) - \alpha(z)t, \quad (2.19)$$

i.e.,

$$ds^2 = -2dzdt + dx^2 + dy^2 + [h(x,y,z) - \alpha(z)t]dz^2, \quad (2.20)$$

where $\alpha = \alpha(z)$ and $h = h(x,y,z)$ are two integral function. Substituting (**??**) into the third equation of (**??**) gives

$$h_{xx} + h_{yy} = 16\pi(B_1^2 + B_2^2). \quad (2.21)$$

Here we require $c \leq \alpha \leq C$ and $h > 0$, where $c$ and $C$ are two positive constants. We can immediately observe that the metric (**??**) develops CTCs at sufficiently large $t$ (see [**?**]).

The spacetime is curved for generic $h$ with

$$Rizjz = -\frac{h_{,ij}}{2},$$

and all the other components vanishing, where $i$ and $j$ stand for $x$ and $y$ respectively. It is easy to see that the metric (2.20) becomes locally flat when $h = h(z)$. This is just Misner space generalized to four dimensions in a straightforward manner (see [**?**]).

**Remark 2.1** *It's clear that when $\alpha(z) = 1$, the metric (**??**) is nothing but Ori's metric.*

The generalized covariant Maxwell equations are

$$F_{[\mu\nu;\tau]} = 0 \tag{2.22}$$

and

$$F^{\mu\nu}{}_{;\ \nu} = J^{\mu}, \tag{2.23}$$

where $J^{\mu}$ is four-dimensional vector flow. Considering the out of the electromagnetic field is vacuum, we have $F^{\mu\nu}{}_{;\ \nu} = 0$. By the antisymmetry of $F_{\mu\nu}$, the equation (**??**) is equivalent to

$$F_{\mu\nu,\tau} + F_{\nu\tau,\mu} + F_{\tau\mu,\nu} = 0, \tag{2.24}$$

and we can also get

$$F^{\mu\nu}{}_{;\ \nu} = \frac{1}{\sqrt{-g}}(\sqrt{-g}F^{\mu\nu})_{,\nu} \tag{2.25}$$

where $g = -1$ is the determinant of $(g_{\mu\nu})$. From (**??**) and (**??**), we obtain the following equations

$$\frac{\partial B_1}{\partial t} = 0, \tag{2.26}$$

$$\frac{\partial B_2}{\partial t} = 0, \tag{2.27}$$

$$\frac{\partial B_1}{\partial x} + \frac{\partial B_2}{\partial y} = 0, \tag{2.28}$$

$$\frac{\partial B_2}{\partial x} - \frac{\partial B_1}{\partial y} = 0. \tag{2.29}$$

By (**??**) and (**??**), it's easy to prove that the electromagnetic field energy-momentum tensors satisfy the conservation laws, i.e., $T^{\mu\nu}{}_{;\ \nu} = 0$.

According to (**??**)-(**??**) and (**??**), we assume

$$B_1 = \tilde{B}_1 H,\ B_2 = \tilde{B}_2 H,\ h = 16\pi\tilde{h}H^2, \tag{2.30}$$

where $\tilde{B}_1 = \tilde{B}_1(x, y)$, $\tilde{B}_2 = \tilde{B}_2(x, y)$, $\tilde{h} = \tilde{h}(x, y)$ and $H = H(z)$. Then, we have

$$\tilde{B}_1 = H_1 + H_2, \tag{2.31}$$

$$\tilde{B}_2 = iH_1 - iH_2, \tag{2.32}$$

$$\Delta\tilde{h} = 4H_1H_2, \tag{2.33}$$

where $H_1 = H_1(x + iy)$ and $H_2 = H_2(x - iy)$ are two arbitrary second-order continuously differentiable functions.

Especially, we consider the real solutions, thus

$$\tilde{B}_1 = ReV, \tag{2.34}$$

$$\tilde{B}_2 = -ImV, \tag{2.35}$$

$$\Delta\tilde{h} = |V|^2, \tag{2.36}$$

where $V = V(x+iy)$ is an analytic function.

Thus, using (**??**)-(**??**), (**??**)-(**??**) and (**??**), we can get the solutions of the Einstein-Maxwell equations

$$\begin{cases} E_1 = E_2 = E_3 = B_3 = 0, \\ B_1 = Re\{V\}H, \\ B_2 = -Im\{V\}H, \\ f = 16\pi\tilde{h}H^2 - \alpha t, \end{cases} \tag{2.37}$$

where $\tilde{h}$ and $V$ satisfy (**??**).

**Example**

Let

$$V = \exp(-x - iy), \tag{2.38}$$

$$H = \sin z. \tag{2.39}$$

$$\alpha = 1 + \sin^2 z. \tag{2.40}$$

Then we can get the following solutions of the Einstein-Maxwell equations

$$B_1 = \exp(-x)\cos y \sin z, \tag{2.41}$$

$$B_2 = \exp(-x)\sin y \sin z, \tag{2.42}$$

$$f = 16\pi\left[h_1 + h_2 + \frac{\exp(-2x)}{4}\right]\sin^2 z - (1 + \sin^2 z)t, \tag{2.43}$$

where $h_1 = h_1(x+iy)$ and $h_2 = h_2(x-iy)$ are also two second-order continuously conjugated differentiable functions.

In particular, we choose

$$h_1 = \frac{\exp(-2x - 2iy)}{8}, \tag{2.44}$$

$$h_2 = \frac{\exp(-2x + 2iy)}{8}, \tag{2.45}$$

then we obtain

$$f = 4\pi\exp(-2x)(\cos 2y + 1)\sin^2 z - (1 + \sin^2 z)t. \tag{2.46}$$

It is easy to get that

$$f \to 0 \qquad as \;\; x \to +\infty, \tag{2.47}$$

which implies that the geometry degenerates to the Misner space generalized to four dimensions in a straightforward manner.

# 3 Discussion and Summary

The intent of the energy conditions is to provide simple criteria which rule out many un-physical situations while admitting any physically reasonable situation. In fact, some possible matter tensors which are known to be physically reasonable and even realistic because they have been experimentally verified to actually fail various energy conditions. Observation of dark energy or the cosmological constant demonstrates that even the averaged strong energy condition must be false in cosmological solutions. Extending these results is an open problem (see [**?**] and [**?**]).

It follows from the equation (??), (??) and (??) that

$$(T^{\mu\nu}) = \begin{pmatrix} -B_1^2 - B_2^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3.1}$$

and

$$T_\mu^\mu = 0. \tag{3.2}$$

It's clear that the electromagnetic field energy-momentum tensor do not satisfy the energy conditions except the trace energy condition. On the other hand, comparing to Ori's model, the model we present is more applicable and natural in some sense. Thus, the solution we obtain in this paper maybe useful for the study of electromagnetic field in modern cosmology and general relativity.

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