THE LAPALACE TRANSFORM OF THE CUT-AND-JOIN EQUATION OF MARIÑO-VAFA FORMULA AND ITS APPLICATIONS

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Abstract. By the same method introduced in [9], we calculate the Laplace transform of the celebrated cut-and-join equation of Mariño-Vafa formula discovered by C. Liu, K. Liu and J. Zhou [17]. Then, we study the applications of the polynomial identity (1) obtained in theorem 1.1 of this paper. We show the proof Bouchard-Mariño conjecture for $\mathbb{C}^3$ which was given by L. Chen [5] firstly. Subsequently, we will present how to obtain series Hodge integral identities from this polynomial identity (1). In particular, the main result in [9] is one of special case in such series of Hodge integral identities. At last, we give a explicit formula for the computation of Hodge integral $\langle\tau_{b_L}\lambda_g\lambda_1\rangle_g$ where $b_L=(b_1,...,b_l)$.

Contents

1. Introduction

In a recent paper [9], Eynard, Mulase and Safnuk stated the Laplace transform of the cut-and-join equation satisfied by the partition function of Hurwitz numbers. They obtained a polynomial identity of linear Hodge integrals. As an application, they proved the Bouchard-Mariño conjecture on Hurwitz numbers. Then, Mulase and Zhang [20] stated that this polynomial identity can also be used to derive the DVV equation and λ_g-integral with the same method introduced by [14].

In 2003, C. Liu, K. Liu and J. Zhou [17] proved the celebrated Mariño-Vafa conjecture [19]. The main step in their proof is to show the generating function of Hodge integral with triple λ-classes $\mathcal{C}(\lambda, p; \tau)$ satisfies the cut-and-join equation. Combining the cut-and-join restriction from combinatorial side formula [21], they finished the proof of Mariño-Vafa formula. Moreover, the famous ELSV [7] formula for Hurwitz numbers is a large framing limit of Mariño-Vafa formula [18].

With the above motivations, we calculate the Laplace transform of the cut-and-join equation for Mariño-Vafa's case in the first part of this paper. The main result is

Theorem 1.1. *For $g \geq 1$ and $l \geq 1$, we have the following equation:*

$$
\begin{aligned}
(1)\quad & -(\tau^2+\tau)^{l-2}\sum_{b_L\geq 0}\left((l-1)(2\tau+1)\langle\tau_{b_L}\Gamma_g(\tau)\rangle_g+(\tau^2+\tau)\langle\tau_{b_L}\frac{d}{d\tau}\Gamma_g(\tau)\rangle_g\right)\hat{\Psi}_{b_L}(t_L;\tau)\\
& -(\tau^2+\tau)^{l-1}\sum_{b_L\geq 0}\langle\tau_{b_L}\Gamma_g(\tau)\rangle_g\sum_{i=1}^{l}\left(\frac{\partial}{\partial\tau}\hat{\Psi}_{b_i}(t_i;\tau)+\frac{1}{t_i\tau+1}\hat{\Psi}_{b_i+1}(t_i;\tau)\right)\hat{\Psi}_{b_{L\setminus\{i\}}}(t_{L\setminus\{i\}};\tau)\\
& =-\frac{(\tau^2+\tau)^{l-2}}{\tau+1}\sum_{1\leq i<j\leq l}\sum_{\substack{a\geq 0\\ b_{L\setminus\{i,j\}}\geq 0}}\langle\tau_a\tau_{b_{L\setminus\{i,j\}}}\Gamma_g(\tau)\rangle_g\hat{\Psi}_{b_{L\setminus\{i,j\}}}(t_{L\setminus\{i,j\}};\tau)\\
& \cdot\frac{(t_j-1)(t_i^2\tau+t_i)\hat{\Psi}_{a+1}(t_i;\tau)-(t_i-1)(t_j^2\tau+t_j)\hat{\Psi}_{a+1}(t_j;\tau)}{t_i-t_j}\\
& +\frac{(\tau^2+\tau)^{l-1}}{2}\sum_{i=1}^{l}\sum_{\substack{a_1\geq 0\\ a_2\geq 0\\ b_{L\setminus\{i\}}\geq 0}}\Big((\tau^2+\tau)\langle\tau_{a_1}\tau_{a_2}\tau_{b_{L\setminus\{i\}}}\Gamma_{g-1}(\tau)\rangle_{g-1}\\
& -\sum_{\substack{g_1+g_2=g\\ \mathcal{I}\coprod\mathcal{J}=L\setminus\{i\}}}^{stable}\langle\tau_{a_1}\tau_{b_{\mathcal{I}}}\Gamma_{g_1}(\tau)\rangle_{g_1}\langle\tau_{a_2}\tau_{b_{\mathcal{J}}}\Gamma_{g_2}(\tau)\rangle_{g_2}\Big)\prod_{n=1}^{2}\hat{\Psi}_{a_n+1}(t_i;\tau)\hat{\Psi}_{b_{L\setminus\{i\}}}(t_{L\setminus\{i\}};\tau)
\end{aligned}
$$

Where $L=\{1,2,..,l\}$ is an index set, and for any subset $I\subset L$, we denote

$$t_I=(t_i)_{i\in I}, b_I=\{b_i|i\in I\}, \tau_{b_I}=\prod_{i\in I}\tau_{b_i}, \hat{\Psi}_{b_I}(t_I,\tau)=\prod_{i\in I}\hat{\Psi}_{b_i}(t_i,\tau)$$

and $\Gamma_g(\tau)=\Lambda_g^\vee(1)\Lambda_g^\vee(-\tau-1)\Lambda_g^\vee(\tau)$, $\hat{\Psi}_n(t;\tau)=\left(\frac{(t^2-t)(t\tau+1)}{\tau+1}\frac{d}{dt}\right)^n\left(\frac{t-1}{\tau+1}\right)$ for $n\geq 0$. The last summation in the formula is taken over all partitions of g and disjoint subsets $\mathcal{I}\coprod\mathcal{J}=L$ subject to the stability condition $2g_1-1+|\mathcal{I}|>0$ and $2g_2-1+|\mathcal{J}|>0$.

We remark that theorem 1.1 is equivalent to the symmetrized cut-and-joint equation of Mariño-Vafa formula obtained by L. Chen [4]. We will present this equivalence in appendix B.

In [3], V. Bouchard and M. Mariño proposed a conjecture for the calculation of topological string amplitudes of the toric three fold $\mathbb{C}^3$ based on the recent work [2].(We will call this conjecture as Bouchard-Mariño conjecture for $\mathbb{C}^3$. The Bourchard-Mariño conjecture for Hurwitz number proved in [9] is the large framing limit of it). We calculate in section 4 that, the topological relation in Bouchard-Mariño conjecture for $\mathbb{C}^3$ is equivalent to the following identity

of Hodge integral

$$
\begin{aligned}
(2)\quad & (\tau^2+\tau)^{l-1}\sum_{b_L\geq 0}\langle\tau_{b_L}\Gamma_g(\tau)\rangle_g d\hat{\Psi}_{b_L}(t_L;\tau)\\
&=-(\tau^2+\tau)^{l-2}\sum_{i=2}^{l}\sum_{a,b_{L\setminus\{1,i\}}}\langle\tau_a\tau_{b_{L\setminus\{1,i\}}}\Gamma_g(\tau)\rangle_g P_a(t_1,t_i;\tau)dt_1dt_i d\hat{\Psi}_{b_{L\setminus\{1,i\}}}(t_{L\setminus\{1,i\}};\tau)\\
&+\sum_{\substack{a_1,a_2\geq 0\\ b_{L\setminus\{1\}}}}(\tau^2+\tau)^{l-1}\Big((\tau^2+\tau)\langle\tau_{a_1}\tau_{a_2}\tau_{b_{L\setminus\{1\}}}\Gamma_{g-1}(\tau)\rangle_{g-1}\\
&-\sum_{\substack{g_1+g_2=g\\ \mathcal{I}\bigsqcup\mathcal{J}=L\setminus\{i\}}}^{stable}\langle\tau_{a_1}\tau_{b_I}\Gamma_{g_1}(\tau)\rangle_{g_1,|I|+1}\langle\tau_{a_2}\tau_{b_J}\Gamma_{g_2}(\tau)\rangle_{g_2,|J|+1}\Big)P_{a_1,a_2}(t_1;\tau)dt_1d\hat{\Psi}_{b_L\setminus\{1\}}(t_{L\setminus\{1\}};\tau)
\end{aligned}
$$

On the other hand, thanks to the Mariño-Vafa formula, the topological amplitudes of $\mathbb{C}^3$ is completely determined by the relation (1) in theorem 1.1. In section 4, we will illustrate that the identity (2) is implied in (1). Therefore, the Bouchard-Mariño conjecture for $\mathbb{C}^3$ holds.

Furthermore, we use the result of theorem 1.1 to obtain some Hodge integral identities. Indeed, $\hat{\Psi}_b(t,\tau)$ can be written as

$$\hat{\Psi}_b(t;\tau)=\sum_{k=0}^{b}\frac{\tau^k}{(\tau+1)^{b+1}}\Psi_b^k(t)$$

and $\Psi_b^k(t)$, $0\leq k\leq b$ could be computed from the recursion relation they defined on.

At first, we consider the expansion of τ at ∞. Taking the highest level of formula (1), we get

Corollary 1.2.

$$
\begin{aligned}
&\sum_{b_L\geq 0}\langle\tau_{b_L}\Lambda_g^\vee(1)\rangle\left((2g-2+l)\Psi_{b_L}^{b_L}(t_L)+\sum_{i=1}^{l}(t_i^2-t_i)\frac{\partial}{\partial t_i}\Psi_{b_i}^{b_i}(t_i)\Psi_{b_{L\setminus\{i\}}}^{b_{L\setminus\{i\}}}(t_{L\setminus\{i\}})\right)\\
&=\sum_{1\leq i<j\leq l}\sum_{\substack{a\geq 0\\ b_{L\setminus\{i,j\}}\geq 0}}\langle\tau_a\tau_{b_{L\setminus\{i,j\}}}\Lambda_g^\vee(1)\rangle_g\Psi_{b_{L\setminus\{i,j\}}}^{b_{L\setminus\{i,j\}}}\frac{(t_j-1)t_i^2\Psi_{a+1}^{a+1}(t_i)-(t_i-1)t_j^2\Psi_{a+1}^{a+1}(t_j)}{t_i-t_j}\\
&+\frac{1}{2}\sum_{i=1}^{l}\sum_{\substack{a_1\geq 0\\ a_2\geq 0\\ b_{L\setminus\{i\}}\geq 0}}\left(\langle\tau_{a_1}\tau_{a_2}\tau_{b_{L\setminus\{i\}}}\Lambda_{g-1}^\vee(1)\rangle_{g-1}+\sum_{\substack{g_1+g_2=g\\ \mathcal{I}\cup\mathcal{J}=L\setminus\{i\}}}^{stable}\langle\tau_{a_1}\tau_{b_\mathcal{I}}\Lambda_{g_1}^\vee(1)\rangle_{g_1}\langle\tau_{a_2}\tau_{b_\mathcal{J}}\Lambda_{g_2}^\vee(1)\rangle_{g_2}\right)\\
&\times\Psi_{a_1+1}^{a_1+1}(t_i)\Psi_{a_2+1}^{a_2+1}(t_i)\Psi_{b_{L\setminus\{i\}}}^{b_{L\setminus\{i\}}}(t_{L\setminus\{i\}})
\end{aligned}
$$

We show in section 5 that $\Psi_b^b(t)$ is equal to $\hat{\xi}_b(t)$ which is defined by $\hat{\xi}_b(t)=((t^3-t^2)\frac{d}{dt})^b(t-1)$ in formula (1.2) in [9]. Hence, the main theorem 1.1 in [9] is a special case of formula (1) in this paper. Taking the sub-highest level of theorem 1.1, we also get another Hodge integral identity corollary 5.2 in section 5. Unfortunately, this identity does not contain any new information.

Next, we consider the expansion of τ at $\tau=0$. In this case, the lowest level of formula (1) is

Corollary 1.3.

$$\sum_{b_L\geq 0}\langle\tau_{b_L}\lambda_g\rangle_g\Psi^0_{b_L}(t_L)=\frac{1}{l-1}\sum_{1\leq i<j\leq l}\sum_{\substack{a\geq 0\\ b_{L\setminus\{i,j\}}\geq 0}}\langle\tau_a\tau_{b_{L\setminus\{i,j\}}}\lambda_g\rangle_g\Psi^0_{b_{L\setminus\{i,j\}}}(t_{L\setminus\{i,j\}})$$
$$\cdot\frac{(t_j-1)t_i^2\Psi^0_{a+1}(t_i)-(t_i-1)t_j^2\Psi^0_{a+1}(t_j)}{t_i-t_j}$$

We can rederive the λ_g-integral [13] from Corollary 1.3.

We also pick up the sub-lowest level of formula (1) and have

Corollary 1.4.

$$\sum_{b_L\geq 0}\langle\tau_{b_L}\lambda_g\rangle_g l\left(-(|b_L|+1)\Psi^0_{b_L}(t_L)+\sum_{j=1}\Psi^1_{b_j}(t_j)\Psi^0_{b_{L\setminus\{j\}}}(t_{L\setminus\{j\}})\right)$$
$$+\sum_{b_L\geq 0}\langle\tau_{b_L}\lambda_g\rangle_g\sum_{i=1}^{l}(t_i^2-t_i)\frac{\partial}{\partial t_i}\Psi^0_{b_i}(t_i)\Psi^0_{b_{L\setminus\{i\}}}(t_{L\setminus\{i\}})+\sum_{b_L\geq 0}\langle\tau_{b_L}\sum_{d=g-1}^{3g-3}P_d(\lambda)\rangle_g l\Psi^0_{b_L}(t_L)$$
$$=\sum_{1\leq i<j\leq l}\sum_{\substack{a\geq 0\\ b_{L\setminus\{i,j\}}\geq 0}}\langle\tau_a\tau_{b_{L\setminus\{i,j\}}}\lambda_g\rangle_g\Psi^0_{b_{L\setminus\{i,j\}}}(t_{L\setminus\{i,j\}})[\frac{(t_j-1)t_i\left(\Psi^1_{a+1}(t_i)+(t_i-|b_{L\setminus\{i,j\}}|-a-3)\Psi^0_{a+1}(t_i)\right)}{t_i-t_j}$$
$$-\frac{(t_i-1)t_j\left(\Psi^1_{a+1}(t_j)+(t_j-|b_{L\setminus\{i,j\}}|-a-3)\Psi^0_{a+1}(t_j)\right)}{t_i-t_j}]$$
$$+\sum_{1\leq i<j\leq l}\sum_{\substack{a\geq 0\\ b_{L\setminus\{i,j\}}\geq 0}}\langle\tau_a\tau_{b_{L\setminus\{i,j\}}}\lambda_g\rangle_g\sum_{r\neq i,j}\Psi^1_{b_r}(t_r)\Psi^0_{b_{L\setminus\{i,j\}}}(t_{L\setminus\{i,j\}})\frac{(t_j-1)t_i\Psi^0_{a+1}(t_i)-(t_i-1)t_j\Psi^0_{a+1}(t_j)}{t_i-t_j}$$
$$+\sum_{1\leq i<j\leq l}\sum_{\substack{a\geq 0\\ b_{L\setminus\{i,j\}}\geq 0}}\langle\tau_a\tau_{b_{L\setminus\{i,j\}}}\sum_{d=g-1}^{3g-3}P_d(\lambda)\rangle_g\Psi^0_{b_{L\setminus\{i,j\}}}(t_{L\setminus\{i,j\}})\frac{(t_j-1)t_i\Psi^0_{a+1}(t_i)-(t_i-1)t_j\Psi^0_{a+1}(t_j)}{t_i-t_j}$$
$$+\frac{1}{2}\sum_{i=1}^{l}\sum_{\substack{a_1\geq 0\\ a_2\geq 0\\ b_{L\setminus\{i\}}\geq 0}}\sum_{\substack{stable\\ g_1+g_2=g\\ |\mathcal{I}|\cup|\mathcal{J}|=L\setminus\{i\}}}\langle\tau_{a_1}\tau_{b_{\mathcal{I}}}\lambda_{g_1}\rangle_{g_1}\langle\tau_{a_2}\tau_{b_{\mathcal{J}}}\lambda_{g_2}\rangle_{g_2}\Psi^0_{a_1+1}(t_i)\Psi^0_{a_2+1}(t_i)\Psi^0_{b_{L\setminus\{i\}}}(t_{L\setminus\{i\}})$$

where $\sum_{d=g-1}^{3g-3}P_d(\lambda)=\Lambda_g^\vee(1)a_1(\lambda)$ *and* $a_1(\lambda)=\sum_{m=1}^{g}m\lambda_{g-m}\lambda_g-(-1)^g\Lambda_g^\vee(-1)\lambda_{g-1}$.

The above identity contains Hodge integral of type $\langle\tau_{b_L}\sum_{d=g-1}^{3g-3}P_d(\lambda)\rangle_g$. By some direct calculations,

$$P_{g-1}(\lambda)=\lambda_{g-1},P_g(\lambda)=g\lambda_g,P_{g+1}(\lambda)=-\lambda_g\lambda_1,\cdots,P_{3g-3}(\lambda)=(-1)^{g+1}\lambda_g\lambda_{g-1}\lambda_{g-2}$$

As an application of corollary 1.4, we get the following Hodge integral recursion.

Theorem 1.5. *If* $\sum_{i=1}^{l}b_i=2g-4+l$, *there exists a constant* $C(g,l,b_1,..,b_l)$ *related to* $g,l,b_1,..,b_l$, *such that*

$$\langle\tau_{b_L}\lambda_g\lambda_1\rangle_g=\frac{1}{l}\sum_{1\leq i<j\leq l}\langle\tau_{b_i+b_j-1}\tau_{b_{L\setminus\{i,j\}}}\lambda_g\lambda_1\rangle_g\frac{(b_i+b_j)!}{b_i!b_j!}+C(g,l,b_1,..,b_l)$$

where $C(g,l,b_1,..,b_l)$ *is a very verbose combinatoric constant which is given at Appendix B.*

The initial value $\langle\tau_{2g-3}\lambda_g\lambda_1\rangle_g = \frac{1}{12}[g(2g-3)b_g + b_1 b_{g-1}]$ has been computed by Y. Li [15]. Thus, by the recursion formula in theorem 1.5, we can compute out all the Hodge integral of type $\langle\tau_{b_L}\lambda_g\lambda_1\rangle_g$. In fact, the Hodge integrals $\langle\tau_{b_L}P_d(\lambda)\rangle_g$ appeared in Corollary 1.4 could also be calculated via the same method. But the computation will be more complicated.

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2. The Laplace Transform of Mariño-Vafa Formula

At first, we introduce some notations followed by [17]. Let

$$\Lambda_g^\vee(t) = t^g - \lambda_1 t^{g-1} + \cdots + (-1)^g\lambda_g$$

be the Chern polynomial of $\mathbb{E}^\vee$, the dual of the Hodge bundle. For a partition μ given by

$$\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{l(\mu)} > 0,$$

let $|\mu| = \sum_{i=1}^{l(\mu)}\mu_i$ and $\Gamma_g(\tau) = \Lambda_g^\vee(1)\Lambda_g^\vee(-\tau-1)\Lambda_g^\vee(\tau)$, define

$$\mathcal{C}_{g,\mu}(\tau) = -\frac{\sqrt{-1}^{|\mu|+l(\mu)}}{|Aut(\mu)|}[\tau(\tau+1)]^{l(\mu)-1}\prod_{i=1}^{l(\mu)}\frac{\prod_{a=1}^{\mu_i-1}(\mu_i\tau+a)}{(\mu_i-1)!}\int_{\overline{\mathcal{M}}_{g,l(\mu)}}\frac{\Gamma_g(\tau)}{\prod_{i=1}^{l(\mu)}(1-\mu_i\psi_i)}$$

The Deligne-Mumford stack $\overline{\mathcal{M}}_{g,l}$ is defined as the moduli space of stable curves satisfying the stability condition $2g-2+l>0$. For the unstable cases $(g,l)=(0,1)$ and $(0,2)$, we define

$$\int_{\overline{\mathcal{M}}_{0,1}}\frac{1}{1-\mu\psi} = \frac{1}{\mu^2}$$

$$\int_{\overline{\mathcal{M}}_{0,2}}\frac{1}{(1-\mu_1\psi_1)(1-\mu_2\psi_2)} = \frac{1}{\mu_1+\mu_2}$$

Thus

$$\mathcal{C}_{0,d}(\tau) = -\sqrt{-1}^{d+1}\frac{1}{\tau}\frac{\prod_{a=0}^{d-1}(d\tau+a)}{d!}d^{-2}$$

$$\mathcal{C}_{0,(\mu_1,\mu_2)}(\tau) = \frac{\sqrt{-1}^{\mu_1+\mu_2}}{|Aut(\mu_1,\mu_2)|}\frac{\tau+1}{\tau}\prod_{i=1}^{2}\frac{\prod_{a=0}^{\mu_i-1}(\mu_i\tau+a)}{\mu_i!}\frac{1}{\mu_1+\mu_2}$$

The Mariño-Vafa formula proved in [17] gives a direct combinatorial formula associated to representation of symmetric groups for the generation function of $\mathcal{C}_{g,\mu}(\tau)$. But we don't go further to this formula here.

Through a direct calculation, we have

$$\mathcal{C}_{g,\mu}(\tau) = -\frac{\sqrt{-1}^{|\mu|+l(\mu)}}{|Aut(\mu)|}[\tau(\tau+1)]^{l(\mu)-1}\sum_{\substack{b_i\geq 0\\ i=1,..,l(\mu)}}\langle\prod_{i=1}^{l(\mu)}\tau_{b_i}\Gamma_g(\tau)\rangle_g\prod_{i=1}^{l(\mu)}\frac{1}{\tau}\frac{\prod_{a=0}^{\mu_i-1}(\mu_i\tau+a)}{\mu_i!}\mu_i^{b_i} \tag{3}$$

Let $\mathcal{C}_g(\mu;\tau) = |Aut(\mu)|\mathcal{C}_{g,\mu}(\tau)$, the Laplace transform of $\mathcal{C}_{g,\mu}(\tau)$ is defined by,

$$\mathcal{C}_{g,l}(w_1,..,w_l) = \sum_{\mu\in\mathbb{N}^l}\frac{1}{\sqrt{-1}^{l+|\mu|}}\mathcal{C}_g(\mu;\tau)e^{-(\mu_1w_1+\cdots+\mu_lw_l)}$$

where we have let $l = l(\mu)$.

In order to simplify the above expression of laplace transform, we introduce some new variables. Consider the framed Lambert curve $x = y(1-y)^\tau$, and the coordinate change $x = e^{-w}$. By Lagrange inversion theorem, L. Chen [4] got,

$$y = \sum_{k\geq 1} \frac{\prod_{a=0}^{k-2}(k\tau + a)}{k!} x^k$$

Now we introduce the variable t by,

$$t = 1 + \frac{1+\tau}{\tau} \sum_{k\geq 1} \frac{\prod_{a=0}^{k-1}(k\tau + a)}{k!} e^{-kw}$$

and

$$\hat{\Psi}_n(t;\tau) = \frac{1}{\tau} \sum_{k\geq 1} \frac{\prod_{a=0}^{k-1}(k\tau + a)}{k!} k^n e^{-kw}$$

which is a polynomial with top degree $2n+1$ in variable t via the recursion relation

$$\hat{\Psi}_{n+1}(t;\tau) = (t^2 - t)\frac{t\tau + 1}{\tau + 1}\frac{d}{dt}\hat{\Psi}_n(t;\tau)$$

for $n \geq 0$ and $\hat{\Psi}_0(t;\tau) = \frac{t-1}{\tau+1}$.

Now, the laplace transform of $\mathcal{C}_{g,\mu}(\tau)$ can be written in t variable,

$$\begin{aligned} \hat{\mathcal{C}}_{g,l}(t_1,..,t_l) &= \sum_{\mu\in\mathbb{N}^l} \frac{1}{\sqrt{-1}^{l+|\mu|}} \mathcal{C}_g(\mu;\tau) e^{-(\mu_1 w_1 + \cdots + \mu_l w_l)} \\ &= -(\tau(\tau+1))^{l-1} \sum_{\substack{b_i\geq 0 \\ i=1,..,l}} \langle \prod_{i=1}^{l} \tau_{b_i} \Gamma_g(\tau) \rangle_g \prod_{i=1}^{l} \hat{\Psi}_{b_i}(t_i;\tau) \end{aligned} \tag{4}$$

3. The Laplace Transform of The Cut-and-join Equation of Mariño-Vafa Formula

Let us introduce formal variables $p = (p_1,..,p_n,..)$, and define

$$p_\mu = p_{\mu_1} \cdots p_{l(\mu)}$$

for a partition μ. Define generating functions

$$\mathcal{C}_{g,l}(\lambda, p;\tau) = \sum_{\mu, l(\mu)=l} \frac{\mathcal{C}_g(\mu;\tau)}{|Aut(\mu)|} p_\mu \lambda^{2g-2+l}$$

$$\mathcal{C}(\lambda, p;\tau) = \sum_{\substack{g\geq 0 \\ l\geq 1}} \mathcal{C}_{g,l}(\lambda, p;\tau)$$

In 2003, C. Liu, K. Liu and J. Zhou [17] proved that $\mathcal{C}(\lambda, p;\tau)$ satisfies the cut-and-join equation

$$\frac{\partial \mathcal{C}}{\partial \tau} = \frac{\sqrt{-1}}{2}\lambda \sum_{i,j\geq 1} ((i+j)p_i p_j \frac{\partial \mathcal{C}}{\partial p_{i+j}} + ijp_{i+j}\frac{\partial^2 \mathcal{C}}{\partial p_i \partial p_j} + ijp_{i+j}\frac{\partial \mathcal{C}}{\partial p_i}\frac{\partial \mathcal{C}}{\partial p_j})$$

For every choice of $g \geq 1$ and a partition μ, the coefficient of $p_\mu \lambda^{2g-2+l(\mu)}$ is

(5)

$$\frac{\partial}{\partial\tau}\left(\frac{\mathcal{C}_g(\mu;\tau)}{|Aut(\mu)|}\right)=\sqrt{-1}\sum_{i<j}(\mu_i+\mu_j)\frac{\mathcal{C}_g(\mu;\tau)}{|Aut(\mu(\hat{i},\hat{j}))|}$$
$$+\frac{\sqrt{-1}}{2}\sum_{i=1}^{l}\sum_{\alpha+\beta=\mu_i}\alpha\beta\left(\frac{\mathcal{C}_{g-1}(\mu(\alpha,\hat{i};\tau))}{|Aut(\mu(\alpha,\hat{i}))|}+\sum_{\substack{g_1+g_2=g\\ \nu_1\coprod\nu_2=\mu(\alpha,\hat{i})}}\frac{\mathcal{C}_{g_1}(\nu_1;\tau)}{|Aut(\nu_1|)}\frac{\mathcal{C}_{g_2}(\nu_2;\tau)}{|Aut(\nu_2)|}\right)$$

Let us first calculate the Laplace transform of the cut-and-join equation for the $l=1$ case which includes Proposition 3.3 in [9] as its special case.

Proposition 3.1. *For $g\geq 1$, the Laplace transform of the cut-and-join equation for the $l=1$ case is:*

$$-\sum_{b\geq 0}\langle\tau_b\frac{d}{d\tau}\Gamma_g(\tau)\rangle_g\hat{\Psi}_b(t;\tau)-\sum_{b\geq 0}\langle\tau_b\Gamma_g(\tau)\rangle_g\left(\frac{d}{d\tau}\hat{\Psi}_b(t;\tau)+\frac{1}{t\tau+1}\hat{\Psi}_{b+1}(t;\tau)\right)$$
$$=\frac{1}{2}\sum_{a_1,a_2\geq 0}\left((\tau(\tau+1))\langle\tau_{a_1}\tau_{a_2}\Gamma_{g-1}(\tau)\rangle_{g-1}-\sum_{\substack{g_1+g_2=g\\ g_1>0\\ g_2>0}}\langle\tau_{a_1}\Gamma_{g_1}(\tau)\rangle_{g_1}\langle\tau_{a_2}\Gamma_{g_2}(\tau)\rangle_{g_2}\right)\hat{\Psi}_{a_1+1}(t;\tau)\hat{\Psi}_{a_2+1}(t;\tau)$$

Proof. The cut-and-join equation for $l=1$ case is

$$\frac{\partial\mathcal{C}_g(\mu;\tau)}{\partial\tau}=\frac{\sqrt{-1}}{2}\sum_{\alpha+\beta=\mu}\alpha\beta\left(\frac{\mathcal{C}_{g-1}((\alpha,\beta);\tau)}{|Aut((\alpha,\beta))|}+\sum_{g_1+g_2=g}\mathcal{C}_{g_1}(\alpha;\tau)\mathcal{C}_{g_2}(\beta;\tau)\right)\tag{6}$$

Then, the Laplace transform of the LHS of (6) is

$$\frac{\partial}{\partial\tau}\hat{\mathcal{C}}_{g,1}(t;\tau)=\frac{\partial t}{\partial\tau}\frac{\partial}{\partial t}\hat{\mathcal{C}}_{g,1}(t;\tau)+\frac{\partial}{\partial\tau}\hat{\mathcal{C}}_{g,1}(t;\tau)\tag{7}$$

The Laplace transform of the stable part of RHS of (6) is

$$\frac{1}{2}(\tau(\tau+1))\sum_{a_1,a_2\geq 0}\langle\tau_{a_1}\tau_{a_2}\Gamma_{g-1}(\tau)\rangle_{g-1}\hat{\Psi}_{a_1+1}(t;\tau)\hat{\Psi}_{a_2+1}(t;\tau)$$
$$-\frac{1}{2}\sum_{\substack{g_1+g_2=g\\ g_1>0\\ g_2>0}}\langle\tau_{a_1}\Gamma_{g_1}(\tau)\rangle_{g_1}\langle\tau_{a_2}\Gamma_{g_2}(\tau)\rangle_{g_2}\hat{\Psi}_{a_1+1}(t;\tau)\hat{\Psi}_{a_2+1}(t;\tau)$$

The unstable term is $\mathcal{C}_0(\alpha;\tau)\mathcal{C}_g(\beta;\tau)+\mathcal{C}_g(\alpha;\tau)\mathcal{C}_0(\beta;\tau)$. Because

$$\mathcal{C}_{0,d}(\tau)=-\sqrt{-1}^{d+1}\frac{1}{\tau}\frac{\prod_{a=0}^{d-1}(d\tau+a)}{d!}\frac{1}{d^2}$$

It's Laplace transform $\hat{\mathcal{C}}_0(w;\tau)=\sum_{d\geq 1}-\frac{1}{\sqrt{-1}^{d+1}}\mathcal{C}_{0,d}(\tau)e^{-w}$ satisfies:

$$\frac{d}{dw}\hat{\mathcal{C}}_0(w;\tau)=\frac{t-1}{\tau+1}=\frac{y}{1-(\tau+1)y}$$

It is easy to calculate $\frac{d}{dw}=\frac{(1-y)y}{1-(1+\tau)y}\frac{d}{dy}$, hence, $\hat{\mathcal{C}}_0(w;\tau)=-ln(1-y)$. Moreover, remember that the framed Lambert curve is $y(1-y)^\tau=x$, where y depends on τ. Taking derivation of τ, we have the identity,

$$-ln(1-y)=\frac{\partial y}{\partial \tau}\frac{1-(\tau+1)y}{y(1-y)}$$

Therefore, the unstable part of RHS of (6) is

$$\frac{\partial y}{\partial \tau}\frac{1-(\tau+1)y}{y(1-y)}\frac{(t^2-t)(t\tau+1)}{\tau+1}\frac{\partial}{\partial t}\hat{\mathcal{C}}_{g,1}(t;\tau)=\frac{\partial y}{\partial \tau}t^2(\tau+1)\frac{\partial}{\partial t}\hat{\mathcal{C}}_{g,1}(t;\tau) \tag{8}$$

where we have used $t=\frac{1}{1-(\tau+1)y}$. we also have

$$\frac{\partial t}{\partial \tau}=t^2y+\frac{\partial y}{\partial \tau}t^2(\tau+1)=\frac{t^2-t}{\tau+1}+\frac{\partial y}{\partial \tau}t^2(\tau+1) \tag{9}$$

Hence, move the unstable part to left hand side, by (7),(8) and (9), we get

$$\left(\frac{\partial}{\partial \tau}+\frac{t^2-t}{(\tau+1)}\frac{\partial}{\partial t}\right)\hat{C}_{g,1}(t;\tau)=stable\quad part$$

which is just the Proposition 3.1. □

For the general case of l, we need to introduce two lemmas first.

Lemma 3.2. *When $(g,l)=(0,2)$, we have the following Laplace transformation formula:*

$$\begin{aligned}\hat{\mathcal{C}}_{0,2}(w_1,w_2;\tau)&=-\sum_{\alpha,\beta\geq 1}\frac{\tau+1}{\tau}\frac{1}{\alpha+\beta}\frac{\prod_{a=0}^{\alpha-1}(\alpha\tau+a)}{\alpha!}\frac{\prod_{a=0}^{\beta-1}(\beta\tau+a)}{\beta!}e^{-\alpha w_1}e^{-\beta w_2}\\&=-ln\left(\frac{y_1-y_2}{x_1-x_2}\right)-\tau\left(ln(1-y_1)+ln(1-y_2)\right)\end{aligned}$$

Proof. By definition,

$$\begin{aligned}\hat{\mathcal{C}}_{0,2}(w_1,w_2;\tau)&=\sum_{\alpha,\beta\geq 1}\frac{1}{\sqrt{-1}^{2+\alpha+\beta}}|Aut(\alpha,\beta)|\hat{\mathcal{C}}_{0,(\alpha,\beta)}\\&=-\sum_{\alpha,\beta\geq 1}\frac{\tau+1}{\tau}\frac{1}{\alpha+\beta}\frac{\prod_{a=0}^{\alpha-1}(\alpha\tau+a)}{\alpha!}\frac{\prod_{a=0}^{\beta-1}(\beta\tau+a)}{\beta!}e^{-\alpha w_1}e^{-\beta w_2}\end{aligned}$$

Thus

$$\begin{aligned}&\left(\frac{d}{dw_1}+\frac{d}{dw_2}\right)\hat{\mathcal{C}}_{0,2}(w_1,w_2;\tau)=-\tau(\tau+1)\hat{\Psi}_0(t_1;\tau)\hat{\Psi}_0(t_2;\tau)\\&=-\tau(\tau+1)\frac{y_1y_2}{(1-(\tau+1)y_1)(1-(\tau+1)y_2)}\end{aligned}$$

Because,

$$\frac{d}{dw}=x\frac{d}{dx}=\frac{(1-y)y}{1-(\tau+1)y}\frac{d}{dy}$$

Then, it is easy to get

$$\hat{\mathcal{C}}_{0,2}(w_1,w_2;\tau)=-ln\left(\frac{y_1-y_2}{x_1-x_2}\right)-\tau\left(ln(1-y_1)+ln(1-y_2)\right)$$

□

Lemma 3.3.

$$\sum_{\alpha,\beta\geq 1}\frac{1}{\tau}\frac{\prod_{a=0}^{(\alpha+\beta)-1}((\alpha+\beta)\tau+a)}{(\alpha+\beta)!}(\alpha+\beta)^{a+1}e^{-\alpha w_i}e^{-\beta w_j}$$
$$=\frac{x_i}{x_i-x_j}\hat{\Psi}_{a+1}(t_i;\tau)-\frac{x_j}{x_i-x_j}\hat{\Psi}_{a+1}(t_j;\tau)-\hat{\Psi}_{a+1}(t_i;\tau)-\hat{\Psi}_{a+1}(t_j;\tau)$$

Proof. This calculation is the same as formula (3.15) showed in paper [9].

Let $\mu=\alpha+\beta$ and $\nu=\beta$, then

$$\sum_{\alpha,\beta\geq 1}\frac{1}{\tau}\frac{\prod_{a=0}^{(\alpha+\beta)-1}((\alpha+\beta)\tau+a)}{(\alpha+\beta)!}(\alpha+\beta)^{a+1}e^{-\alpha w_i}e^{-\beta w_j}$$
$$=\sum_{\alpha,\beta\geq 0}\frac{1}{\tau}\frac{\prod_{a=0}^{(\alpha+\beta)-1}((\alpha+\beta)\tau+a)}{(\alpha+\beta)!}(\alpha+\beta)^{a+1}e^{-\alpha w_i}e^{-\beta w_j}$$
$$-\sum_{\alpha\geq 1}\frac{1}{\tau}\frac{\prod_{a=0}^{\alpha-1}(\alpha\tau+a)}{\alpha!}\alpha^{a+1}e^{-\alpha w_i}-\sum_{\beta\geq 1}\frac{1}{\tau}\frac{\prod_{a=0}^{\beta-1}(\beta\tau+a)}{\beta!}\beta^{a+1}e^{-\beta w_j}$$
$$=\sum_{\mu\geq 0}\sum_{\nu=0}^{\mu}\frac{1}{\tau}\frac{\prod_{a=0}^{\mu-1}(\mu\tau+a)}{\mu!}\mu^{a+1}e^{-(\mu-\nu)w_i}e^{-\nu w_j}-\hat{\Psi}_{a+1}(t_i;\tau)-\hat{\Psi}_{a+1}(t_j;\tau)$$
$$=\sum_{\mu\geq 0}\frac{1}{\tau}\frac{\prod_{a=0}^{\mu-1}(\mu\tau+a)}{\mu!}\mu^{a+1}\frac{x_i^{\mu+1}-x_j^{\mu+1}}{x_i-x_j}-\hat{\Psi}_{a+1}(t_i;\tau)-\hat{\Psi}_{a+1}(t_j;\tau)$$
$$=\frac{x_i}{x_i-x_j}\hat{\Psi}_{a+1}(t_i;\tau)-\frac{x_j}{x_i-x_j}\hat{\Psi}_{a+1}(t_j;\tau)-\hat{\Psi}_{a+1}(t_i;\tau)-\hat{\Psi}_{a+1}(t_j;\tau)$$

□

Now we give our main result in this paper.

Theorem 3.4. *For $g\geq 1$ and $l\geq 1$, we have the following equation:*

$$\begin{aligned}(10)\quad &-(\tau^2+\tau)^{l-2}\sum_{b_L\geq 0}\left((l-1)(2\tau+1)\langle\tau_{b_L}\Gamma_g(\tau)\rangle_g+(\tau^2+\tau)\langle\tau_{b_L}\frac{d}{d\tau}\Gamma_g(\tau)\rangle_g\right)\hat{\Psi}_{b_L}(t_L;\tau)\\
&-(\tau^2+\tau)^{l-1}\sum_{b_L\geq 0}\langle\tau_{b_L}\Gamma_g(\tau)\rangle_g\sum_{i=1}^{l}\left(\frac{\partial}{\partial\tau}\hat{\Psi}_{b_i}(t_i;\tau)+\frac{1}{t_i\tau+1}\hat{\Psi}_{b_i+1}(t_i;\tau)\right)\hat{\Psi}_{b_{L\setminus\{i\}}}(t_{L\setminus\{i\}};\tau)\\
&=-\frac{(\tau^2+\tau)^{l-2}}{\tau+1}\sum_{1\leq i<j\leq l}\sum_{\substack{a\geq 0\\ b_{L\setminus\{i,j\}}\geq 0}}\langle\tau_a\tau_{b_{L\setminus\{i,j\}}}\Gamma_g(\tau)\rangle_g\hat{\Psi}_{b_{L\setminus\{i,j\}}}(t_{L\setminus\{i,j\}};\tau)\\
&\cdot\frac{(t_j-1)(t_i^2\tau+t_i)\hat{\Psi}_{a+1}(t_i;\tau)-(t_i-1)(t_j^2\tau+t_j)\hat{\Psi}_{a+1}(t_j;\tau)}{t_i-t_j}\\
&+\frac{(\tau^2+\tau)^{l-1}}{2}\sum_{i=1}^{l}\sum_{\substack{a_1\geq 0\\ a_2\geq 0\\ b_{L\setminus\{i\}}\geq 0}}\Big((\tau^2+\tau)\langle\tau_{a_1}\tau_{a_2}\tau_{b_{L\setminus\{i\}}}\Gamma_{g-1}(\tau)\rangle_{g-1}\end{aligned}$$

$$- \sum_{\substack{g_1+g_2=g \\ \mathcal{I}\coprod\mathcal{J}=L\backslash\{i\}}}^{stable} \langle\tau_{a_1}\tau_{b_{\mathcal{I}}}\Gamma_{g_1}(\tau)\rangle_{g_1}\langle\tau_{a_2}\tau_{b_{\mathcal{J}}}\Gamma_{g_2}(\tau)\rangle_{g_2}\Bigg)\prod_{n=1}^{2}\hat{\Psi}_{a_n+1}(t_i;\tau)\hat{\Psi}_{b_{L\backslash\{i\}}}(t_{L\backslash\{i\}};\tau)$$

Proof. The Laplace transform of LHS of equation (5) is

$$\begin{aligned}(11)\qquad &\sum_{\mu\in\mathbb{N}^l}\frac{1}{\sqrt{-1}^{l(\mu)+|\mu|}}\frac{\partial}{\partial\tau}\mathcal{C}_g(\mu;\tau)e^{-(\mu_1w_1+\cdots+\mu_lw_l)}\\ &=\frac{\partial}{\partial\tau}\left(\sum_{\mu\in\mathbb{N}}\frac{1}{\sqrt{-1}^{l(\mu)+|\mu|}}\mathcal{C}_g(\mu;\tau)e^{-(\mu_1w_1+\cdots+\mu_lw_l)}\right)\\ &=\frac{\partial}{\partial\tau}\hat{\mathcal{C}}_{g,l}(t_1,..,t_l;\tau)\\ &=\sum_{i=1}^{l}\frac{\partial t_i}{\partial\tau}\frac{\partial}{\partial t_i}\hat{\mathcal{C}}_{g,l}(t_1,..,t_l;\tau)+\frac{\partial}{\partial\tau}\hat{\mathcal{C}}_{g,l}(t_1,..,t_l;\tau)\end{aligned}$$

The Laplace transform of stable geometry in the cut-term of RHS of (5) is

$$\begin{aligned}(12)\qquad &\frac{\sqrt{-1}}{2}\sum_{\mu\in\mathbb{N}^l}\frac{1}{\sqrt{-1}^{l(\mu)+|\mu|}}\sum_{i=1}^{l}\sum_{\alpha+\beta=\mu_i}\alpha\beta(\mathcal{C}_{g-1}(\mu(\alpha,\hat{i});\tau)\\ &+\sum_{\substack{g_1+g_2=g\\ \nu_1\coprod\nu_2=\mu(\alpha,\hat{i})\\ 2g_1-2+|\nu_1|>0\\ 2g_2-2+|\nu_2|>0}}\mathcal{C}_{g_1}(\nu_1;\tau)\mathcal{C}_{g_2}(\nu;\tau))e^{-(\mu_1w_1+\cdots+\mu_lw_l)}\\ &=\frac{(\tau^2+\tau)^{l-1}}{2}\sum_{i=1}^{l}\sum_{\substack{a_1\geq0\\ a_2\geq0\\ b_{L\backslash\{k\}}\geq0}}((\tau^2+\tau)\langle\tau_{a_1}\tau_{a_2}\tau_{b_{L\backslash\{k\}}}\Gamma_{g-1}(\tau)\rangle_{g-1}\\ &-\sum_{\substack{g_1+g_2=g\\ \mathcal{I}\coprod\mathcal{J}=\{1,..,\hat{i},..l\}\\ 2g_1-1+|\mathcal{I}|>0\\ 2g_2-1+|\mathcal{J}|>0}}\langle\tau_{a_1}\tau_{b_{\mathcal{I}}}\Gamma_{g_1}(\tau)\rangle_{g_1}\langle\tau_{a_2}\tau_{b_{\mathcal{J}}}\Gamma_{g_2}(\tau)\rangle_{g_2})\prod_{n=1}^{2}\hat{\Psi}_{a_n+1}(t_i;\tau)\hat{\Psi}_{b_{L\backslash\{i\}}}(t_{L\backslash\{i\}};\tau)\end{aligned}$$

The unstable geometry in the cut-term of RHS has two terms for $l\geq2$.

$$U_1=\frac{\sqrt{-1}}{2}\sum_{i=1}^{l}\sum_{\alpha+\beta=\mu_i}\alpha\beta\left(\mathcal{C}_0(\alpha;\tau)\cdot\frac{\mathcal{C}_g(\mu_{\hat{i}},\beta;\tau)}{|Aut(\mu_{\hat{i}})|}+\frac{\mathcal{C}_g(\mu_{\hat{i}},\alpha;\tau)}{|Aut(\mu_{\hat{i}},\alpha)|}\cdot\mathcal{C}_0(\beta;\tau)\right)$$

$$U_2=\frac{\sqrt{-1}}{2}\sum_{i=1}^{l}\sum_{\alpha+\beta=\mu_i}\alpha\beta\sum_{j\neq i}\left(\frac{\mathcal{C}_0(\mu_j,\alpha;\tau)}{|Aut(\mu_j,\alpha)|}\cdot\frac{\mathcal{C}_g(\mu_{\hat{i},\hat{j}},\beta;\tau)}{|Aut(\mu_{\hat{i},\hat{j}})|}+\frac{\mathcal{C}_g(\mu_{\hat{i},\hat{j}},\alpha;\tau)}{|Aut(\mu_{\hat{i},\hat{j}},\alpha)|}\cdot\mathcal{C}_0(\beta;\tau)\right)$$

As we have calculated in the proof of proposition 3.1, the Laplace transform of U_1 is

$$(13)\qquad \sum_{i=1}^{l}\frac{\partial y_l}{\partial\tau}t_i^2(\tau+1)\frac{\partial}{\partial t_i}\hat{\mathcal{C}}_{g,l}(t_1,..,t_l;\tau)$$

Moving the formula (13) to left hand side, it will cancel the first term of formula (11), i.e.

$$
\begin{aligned}
(14)\quad (11)-(13) &= \left(\frac{\partial}{\partial\tau}+\sum_{i=1}^{l}\frac{t_i^2-t_i}{\tau+1}\frac{\partial}{\partial t_i}\right)\hat{\mathcal{C}}_{g,l}(t_1,..,t_l;\tau)\\
&= -(\tau^2+\tau)^{l-2}\sum_{b_L\geq 0}\left((l-1)(2\tau+1)\langle\tau_{b_L}\Gamma_g(\tau)\rangle_g+(\tau^2+\tau)\langle\tau_{b_L}\frac{d}{d\tau}\Gamma_g(\tau)\rangle_g\right)\hat{\Psi}_{b_L}(t_L;\tau)\\
&-(\tau^2+\tau)^{l-1}\sum_{b_L\geq 0}\langle\tau_{b_L}\Gamma_g(\tau)\rangle_g\sum_{i=1}^{l}\left(\frac{\partial}{\partial\tau}\hat{\Psi}_{b_i}(t_i;\tau)+\frac{1}{t_i\tau+1}\hat{\Psi}_{b_i+1}(t_i;\tau)\right)\hat{\Psi}_{b_{L\setminus\{i\}}}(t_{L\setminus\{i\}};\tau)
\end{aligned}
$$

The Laplace transform of U_2 equals to

(15)

$$
\begin{aligned}
&\sum_{\mu\in\mathcal{P}_l}\frac{1}{\sqrt{-1}^{l+|\mu|}}U_2e^{-(\mu_1w_1+\cdots+\mu_lw_l)}\\
&=-\sum_{i=1}^{l}\sum_{j\neq i}\sum_{\substack{\mu_j\geq 1\\ \alpha\geq 1}}\frac{1}{\sqrt{-1}^{2+\mu_j+\alpha}}\alpha\mathcal{C}_0(\mu_j,\alpha;\tau)e^{-\mu_jw_j}e^{-\alpha w_i}\\
&\times\sum_{\substack{\beta\geq 0\\ \mu_k\geq 0\\ k\neq i,j}}\frac{1}{\sqrt{-1}^{l+|\mu|-\alpha-1}}\beta\mathcal{C}_g(\mu_{\hat{i},\hat{j}},\beta;\tau)e^{-\beta w_i}e^{-\sum_{k\neq i,j}\mu_kw_k}\\
&=\sum_{i=1}^{l}\sum_{j\neq i}\left(\frac{\partial}{\partial w_i}\hat{\mathcal{C}}_{0,2}(w_i,w_j;\tau)(\tau^2+\tau)^{l-2}\sum_{\substack{a\geq 0\\ b_k\geq 0,k\neq i,j}}\langle\tau_a\prod_{k\neq i,j}\tau_{b_k}\Gamma_g(\tau)\rangle_g\hat{\Psi}_{a+1}(t_i;\tau)\prod_{k\neq i,j}\hat{\Psi}_{b_k}(t_k;\tau)\right)\\
&=\sum_{i=1}^{l}\sum_{j\neq i}\left(-\frac{y_i}{y_i-y_j}(1+\frac{\tau y_j}{1-(\tau+1)y_i})+\frac{x_i}{x_i-x_j}\right)\\
&\cdot(\tau^2+\tau)^{l-2}\sum_{\substack{a\geq 0\\ b_k\geq 0,k\neq i,j}}\langle\tau_a\prod_{k\neq i,j}\tau_{b_k}\Gamma_g(\tau)\rangle_g\hat{\Psi}_{a+1}(t_i;\tau)\prod_{k\neq i,j}\hat{\Psi}_{b_k}(t_k;\tau)\\
&=(\tau^2+\tau)^{l-2}\sum_{1\leq i<j\leq l}\sum_{\substack{a\geq 0\\ b_k\geq 0,k\neq i,j}}\langle\tau_a\prod_{k\neq i,j}\tau_{b_k}\Gamma_g(\tau)\rangle_g\prod_{k\neq i,j}\hat{\Psi}_{b_k}(t_k;\tau)(\frac{x_i}{x_i-x_j}\hat{\Psi}_{a+1}(t_i;\tau)\\
&-\frac{x_j}{x_i-x_j}\hat{\Psi}_{a+1}(t_j;\tau)-\frac{y_i}{y_i-y_j}(1+\frac{\tau y_j}{1-(\tau+1)y_i})\hat{\Psi}_{a+1}(t_i;\tau)+\frac{y_j}{y_i-y_j}(1+\frac{\tau y_i}{1-(\tau+1)y_j})\hat{\Psi}_{a+1}(t_j;\tau))
\end{aligned}
$$

where we have used the lemma 3.3.

At last, we need to calculate the Laplace transform of join term in RHS. It's Laplace transform is

(16)

$$
\sqrt{-1}\sum_{\mu\in\mathbb{N}^l}\sum_{i<j}\frac{1}{\sqrt{-1}^{l(\mu)+|\mu|}}(\mu_i+\mu_j)\mathcal{C}_g(\mu(\hat{i},\hat{j});\tau)e^{-\mu_iw_i}e^{-\mu_jw_j}e^{-\sum_{k\neq i,j}\mu_kw_k}
$$

$$= -(\tau^2+\tau)^{l-2}\sum_{\mu\in\mathbb{N}^l}\sum_{i<j}\sum_{\substack{a\geq 0\\ b_{L\setminus\{i,j\}}\geq 0}}\langle\tau_a\tau_{b_{L\setminus\{i,j\}}}\Gamma_g(\tau)\rangle_g\prod_{k\neq i,j}\frac{1}{\tau}\frac{\prod_{a=0}^{\mu_k-1}(\mu_k\tau+a)}{\mu_k!}\mu_k^{b_k}e^{-\sum_{k\neq i,j}\mu_k w_k}$$

$$\cdot\frac{1}{\tau}\frac{\prod_{a=0}^{\mu_i+\mu_j-1}((\mu_i+\mu_j)\tau+a)}{(\mu_i+\mu_j)!}(\mu_i+\mu_j)^{a+1}e^{-\mu_i w_i}e^{-\mu_j w_j}$$

$$= -(\tau^2+\tau)^{l-2}\sum_{1\leq i<j\leq l}\sum_{\substack{a\geq 0\\ b_{L\setminus\{i,j\}}\geq 0}}\langle\tau_a\tau_{b_{L\setminus\{i,j\}}}\Gamma_g(\tau)\rangle_g\hat{\Psi}_{b_{L\setminus\{i,j\}}}(t_{L\setminus\{i,j\}};\tau)\left(\frac{x_i}{x_i-x_j}\hat{\Psi}_{a+1}(t_i;\tau)\right.$$

$$\left.-\frac{x_j}{x_i-x_j}\hat{\Psi}_{a+1}(t_j;\tau)-\hat{\Psi}_{a+1}(t_i;\tau)-\hat{\Psi}_{a+1}(t_j;\tau)\right)$$

Combining (15) and (16), we get

$$(10)+(11) = -\frac{(\tau^2+\tau)^{l-2}}{\tau+1}\sum_{1\leq i<j\leq l}\sum_{\substack{a\geq 0\\ b_{L\setminus\{i,j\}}\geq 0}}\langle\tau_a\tau_{b_{L\setminus\{i,j\}}}\Gamma_g(\tau)\rangle_g\hat{\Psi}_{b_{L\setminus\{i,j\}}}(t_{L\setminus\{i,j\}};\tau) \tag{17}$$

$$\cdot\frac{(t_j-1)(t_i^2\tau+t_i)\hat{\Psi}_{a+1}(t_i;\tau)-(t_i-1)(t_j^2\tau+t_j)\hat{\Psi}_{a+1}(t_j;\tau)}{t_i-t_j}$$

Collecting the remainder equations (12),(14) and (17), we obtain the equation (10) of theorem 3.4. □

4. Application I: Proof of the Bouchard-Mariño conjecture for $\mathbb{C}^3$

Motivated by Eynard and his collaborators' series works on Matrix model [12, 10, 8, 11] , Bourchard, Klemm, Mariño and Pasquetti propose a new approach to compute the topological string amplitudes for local Calabi-Yau manifolds [2]. Based on such proposal, Bouchard and Mariño calculated the string amplitude for toric three fold $\mathbb{C}^3$ [3]. In this section, we will follow the work of [9] and [5] to illustrate that Bouchard-Marño's approach is implied by theorem 1.1.

Following the work of [9], in subsection 4.1, we illustrate Bourchard-Mariño's proposal of calculation the string amplitude of $\mathbb{C}^3$(which is also called Bouchard-Mariño conjecture for $\mathbb{C}^3$). By some residue computations, we obtain a equivalent description of Bourchard-Marño's approach in subsection 4.2. The calculation in subsections 4.3 and 4.4 to prove the Bouchard-Mariño conjecture for $\mathbb{C}^3$ is firstly done by L. Chen [5]

4.1. Bouchard-Mariño conjecture for $\mathbb{C}^3$.

Here we consider the spectral curve

$$C : y(1-y)^\tau = x \tag{18}$$

and the variable $t = \frac{1}{1-(\tau+1)y}$.

It is easy to see that, variables x, y, t have the relations that

$$x\frac{d}{dx} = \frac{(1-y)y}{1-(\tau+1)y}\frac{d}{dy} = \left(\frac{t(t-1)(t\tau+1)}{\tau+1}\right)\frac{d}{dt}$$

We defined in section 2 that,

$$\hat{\Psi}_n(t;\tau) = \left(\left(\frac{t(t-1)(t\tau+1)}{\tau+1}\right)\frac{d}{dt}\right)^n\frac{t-1}{\tau+1}$$

for $n \geq 0$.

In the following state, we need to introduce the differential,

$$\Psi_n(t;\tau) = d\hat{\Psi}(t;\tau)$$

which can be formulated in variables x, y as

$$\Psi_n(y;\tau) = dy\frac{1-(\tau+1)y}{y(1-y)}\left(\frac{y(1-y)}{1-(\tau+1)y}\frac{d}{dy}\right)^{n+1}\frac{1}{(\tau+1)(1-(\tau+1)y)} \tag{19}$$

$$\Psi_n(x;\tau) = \frac{1}{\tau}\sum_{k\geq 1}\frac{\prod_{a=0}^{k-1}(k\tau+a)}{k!}k^{n+1}x^{k-1}dx \tag{20}$$

respectively.

In [3], in order to formulate their conjecture, Bourchard and Mariño introduce

$$W_g(x_1,..,x_l;\tau) = \sum_{\mu,l(\mu)=l} z_\mu W_{g,\mu}(\tau)\frac{1}{|Aut(\mu)|}\sum_{\sigma\in S_l}\prod_{i=1}^{l}x_{\sigma(i)}^{\mu_i-1} \tag{21}$$

where $W_{g,\mu}(\tau) = -\frac{1}{\sqrt{-1}^{g+l}}\mathcal{C}_{g,\mu}(\tau)$.

By (3),(19), (20) and (21), we have

$$W_g(x_1,..,x_l;\tau)dx_1\cdots dx_l = (-1)^{g+l}(\tau^2+\tau)^{l-1}\sum_{b_L\geq 0}\langle\tau_{b_L}\Gamma_g(\tau)\rangle_g\Psi_{b_L}(y_L;\tau)$$

Definition 4.1. Let us call the symmetric polynomial differential form

$$d^{\otimes l}\hat{\mathcal{C}}_{g,l}(t_1,..,t_l;\tau) = -(\tau(\tau+1))^{l-1}\sum_{b_L\geq 0}\langle\tau_{b_L}\Gamma_g(\tau)\rangle_g\Psi_{b_L}(t_L;\tau)$$

on C^l the Mariño-Vafa differential of type (g,l).

Thus

$$W_g(x_1,..,x_l;\tau)dx_1\cdots dx_l = (-1)^{g+l-1}d^{\otimes l}\hat{\mathcal{C}}_{g,l}(t_1,..,t_l;\tau)$$

$t = \frac{1}{1-(\tau+1)y}$, it then follows that $y = \frac{t-1}{(\tau+1)t}$. Substituting to the spectral curve (18), we have $\frac{t-1}{t}\left(\frac{t\tau+1}{t}\right)^\tau\frac{1}{(\tau+1)^{\tau+1}} = x$. Let $s(t)$ be the solution of function equation

$$\frac{t-1}{t}\left(\frac{t\tau+1}{t}\right)^\tau = \frac{s(t)-1}{s(t)}\left(\frac{s(t)\tau+1}{s(t)}\right)^\tau \tag{22}$$

From (22), we have

$$ln\left(\frac{t\tau+1}{t}\right) - ln\left(\frac{s(t)\tau+1}{s(t)}\right) = \frac{1}{\tau}\left(ln\left(\frac{s(t)-1}{s(t)}\right) - ln\left(\frac{t-1}{t}\right)\right) \tag{23}$$

It is clear that the spectral curve $C: y(1-y)^\tau = x$ has only one ramification point of the x-projection, which we denote by ν. It is given by $y(\nu) = \frac{1}{\tau+1}$. Then, we can find two points q and $\overline{q}$ on the curve such that $x(q) = x(\overline{q})$ near the ramification point ν. Let us write

$$y(q) = \frac{1}{\tau+1} - z, \quad y(\overline{q}) = \frac{1}{\tau+1} - P(z)$$

with

$$P(z) = -z + \mathcal{O}(z^2)$$

From functional equation $y(q)(1-y(q))^\tau = y(\overline{q})(1-y(\overline{q}))^\tau$, we can solve exact $P(z)$ as a power series in z,

$$P(z) = -z - \frac{2(-1+\tau^2)}{3\tau}z^2 - \frac{4(-1+\tau^2)^2}{9\tau^2}z^3 - \frac{2(1+\tau)^3(-22+57\tau-57\tau^2+22\tau^3)}{135\tau^3}z^4 + \cdots$$

Note that $P(z)$ is an involution $P(P(z)) = z$.

In terms of the t-coordinate, the involution $P(z)$ corresponds to $s(t)$ in (22). We have,

$$\begin{cases} t(q) = \dfrac{1}{1-(\tau+1)y(q)} = \dfrac{1}{\tau+1}\dfrac{1}{z} = t \\ t(\overline{q}) = \dfrac{1}{1-(\tau+1)y(\overline{q})} = \dfrac{1}{\tau+1}\dfrac{1}{P(z)} = s(t) \end{cases} \tag{24}$$

It follows that

$$\frac{(\tau+1)dt}{(t^2-t)(t\tau+1)} = \frac{(\tau+1)ds(t)}{s(t)^2(s(t)-1)} = \frac{dx}{x}$$

Using the coordinate t of spectral curve C, the Bergman kernel is defined by

$$B(t_1,t_2) = \frac{dt_1 dt_2}{(t_1-t_2)}$$

Define a 1-form on C by,

$$dE(q,\overline{q};t_2) = \frac{1}{2}\int_q^{\overline{q}} B(\cdot,t_2) = \frac{1}{2}\left(\frac{1}{t_1-t_2} - \frac{1}{s(t_1)-t_2}\right)dt_2$$

where the integral is taken with respect to the first variable of $B(t_1,t_2)$ along any path from q to $\overline{q}$. The natural holomorphic symplectic form on $\mathbb{C}\times\mathbb{C}$ is given by $\Omega = dln(1-y)\wedge ln(x)$.

Again, let us introduce another 1-form on the curve C by,

$$\begin{aligned} \omega(q,\overline{q}) = \int_q^{\overline{q}} \Omega(\cdot,x) &= \left(ln(1-y(\overline{q})) - ln(1-y(q))\right)\frac{dx}{x} \\ &= \frac{1}{\tau}(lny(q) - lny(\overline{q}))\frac{dx}{x} \\ &= \frac{1}{\tau}\left(ln\left(1-\frac{1}{t}\right) - ln\left(1-\frac{1}{s(t)}\right)\right)\frac{(\tau+1)}{t^2-t}\frac{dt}{(t\tau+1)} \end{aligned}$$

The kernel operator is defined as the quotient

$$K(t_1,t_2) = \frac{dE(q,\overline{q},t_2)}{\omega(q,\overline{q})} = \frac{\tau}{2(\tau+1)}\frac{(s(t_1)-t_1)(t_1^2-t_1)(t_1\tau+1)}{(t_1-t_2)(s(t_1)-t_2)(ln(1-\frac{1}{t_1})-ln(1-\frac{1}{s(t_1)}))}\frac{1}{dt_1}dt_2$$

It is clear that $K(s(t_1),t_2) = K(t_1,t_2)$.

Definition 4.2. The topological recursion formula is an inductive mechanism of defining a symmetric l-form $W_{g,l}(t_1,..,t_l;\tau)$ on C^l for (g,l) subject to $2g-2+l>0$ by

$$\begin{aligned} W_{g,l+1}(t_0,t_L;\tau) = \frac{1}{2\pi i}\oint_{\gamma_\infty} \Big[& K(t,t_0;\tau)\left(W_{g-1,l+2}(t,s(t),t_L;\tau) + \right. \\ & \sum_{i=1}^{l}\left(W_{g,l}(t,t_{L\setminus\{i\}})B(s(t),t_i) + W_{g,l}(s(t),t_{L\setminus\{i\}};\tau)B(t,t_i)\right) \\ & \left. + \sum_{g_1+g_2=g,\mathcal{I}\cup\mathcal{J}=L}^{stable} W_{g_1,|\mathcal{I}|+1}(t,t_{\mathcal{I}};\tau)W_{g_2,|\mathcal{J}|+1}(s(t),t_{\mathcal{J}};\tau)\right)\Big] \end{aligned}$$

where $t_I = (t_i)_{i\in I}$ for a subset $I\subset L$, and the last sum is taken over all partitions of g and disjoint decompositions $I\coprod J = L$ subject to the stability condition $2g_i - 1 + |I| > 0$ and $2g_2 - 1 + |J| > 0$. The integration is taken with respect to dt on the contour γ_∞, which is a

positively oriented loop in the complex t-plane of large radius such that $|t| > max(|t_0|, s(t_0))$ for $t \in \gamma_\infty$.

Remark 4.3. The original topological recursion formula for $W_{g,l}(t_1, .., t_l; \tau)$ was formulated by taken the residue at $z = 0$ [3]. But here, for simplicity, we have changed the coordinate from z to t by $t = \frac{1}{(\tau+1)z}$. Hence, we get the contour integral in definition 4.2.

Now, we can formulate the Bouchard-Mariño conjecture for $\mathbb{C}^3$.

Conjecture 4.4. *For every g and l subject to the stability condition $2g-2+l > 0$, the topological recursion formula with the initial condition*

$$\begin{cases} W_{0,3}(t_1, t_2, t_3; \tau) = d^{\otimes 3}\hat{\mathcal{C}}_{0,3}(t_1, t_2, t_3; \tau) = -\dfrac{\tau^2}{(\tau+1)} dt_1 dt_2 dt_3 \\ W_{1,1}(t_1; \tau) = -d\hat{\mathcal{C}}_{1,1}(t_1; \tau) = \dfrac{1}{24}((1+\tau+\tau^2)\Psi_0(t_1; \tau) - \tau(\tau+1)\Psi_1(t_1; \tau)) \end{cases} \tag{25}$$

gives the Mariño-Vafa differential with a signature.

$$W_{g,l}(t_1, .., t_l; \tau) = (-1)^{g+l-1}\hat{\mathcal{C}}_{g,l}(t_1, .., t_l; \tau) \tag{26}$$

4.2. **Residue calculation.** In this subsection, we calculate the residues(contour integrals) appearing definition 4.2. In fact, we have to calculate the following two types of residues:

i)$R_{a,b}(t; \tau) = \frac{1}{2\pi i} \oint_{\gamma_\infty} K(t', t; \tau)\Psi_a(t'; \tau)\Psi_b(s(t'); \tau)$

ii)$R_n(t, t_i; \tau) = \frac{1}{2\pi i} \oint_{\gamma_\infty} K(t', t; \tau) \left(\Psi_n(t'; \tau)B(s(t'), t_i) + \Psi_n(s(t'); \tau)B(t', t_i)\right)$

We need one lemma first,

Lemma 4.5. *In the z-coordinate, the kernel $K(t'(z), t; \tau)$ has the expansion,*

$$K = \left(\frac{\tau^2}{2(\tau+1)^3} z^{-1} - \frac{(\tau-1)}{2(\tau+1)^2} + \frac{3\tau^2 t^2 + 2\tau(1-\tau)t - 4\tau}{6(\tau+1)} z \right.$$
$$\left. + \left(\frac{\tau - \tau^2}{6} t^2 + \frac{\tau^2 - 2\tau + 1}{9} t + \frac{(\tau+1)}{18}\right) z^2 + \cdots \right) dt \frac{1}{dz}$$

In particular, the coefficients of z in above expansion are polynomials of t.

Proof. Substituting $t' = \frac{1}{(\tau+1)z}$ and $s(t') = \frac{1}{(\tau+1)s(z)}$ to $K(t', t; \tau)$ and calculate directly, we obtain lemma 4.5. □

Definition 4.6. For a laurent series $\sum_{n\in\mathbb{Z}} a_n t^n$, we denote

$$\left[\sum_{n\in\mathbb{Z}} a_n t^n\right]_+ = \sum_{n>0} a_n t^n$$

Now, we have two theorems for the calculation of $R_{a,b}(t; \tau)$ and $R_n(t, t_i; \tau)$.

Theorem 4.7.

$$R_{a,b}(t; \tau) = \frac{1}{2\pi i} \oint_{\gamma_\infty} K(t', t; \tau)\Psi_a(t'; \tau)\Psi_b(s(t'); \tau)$$
$$= \frac{\tau}{2} \left(\frac{1}{ln\left(1 - \frac{1}{t}\right) - ln\left(1 - \frac{1}{s(t)}\right)} \left(\Psi_a(t; \tau)\hat{\Psi}_{b+1}(s(t); \tau) + \hat{\Psi}_{a+1}(s(t); \tau)\Psi_b(t; \tau)\right) \right)_+$$

Proof. We note that, in term of the original z-coordinate of [3], the residue $R_{a,b}(t;\tau)$ is the coefficient of z^{-1} in $K(t',t;\tau)\Psi_a(t';\tau)\Psi_b(s(t');\tau)$, after expanding it in the Laurent series in z. $\Psi_n(t';\tau)$ is a polynomial in $t'=\frac{1}{(\tau+1)z}$, thus the contribution to the z^{-1} term in the expression is a polynomial in t by lemma 4.5. Hence $R_{a,b}(t;\tau)$ is a polynomial in t.

Let us write $\Psi_n(t';\tau)=f_n(t';\tau)dt'$. Let γ_ϵ be a positively oriented loop, such that function

$$\frac{(t'^2-t')(t'\tau)+1}{ln\left(1-\frac{1}{t'}\right)-ln\left(1-\frac{1}{s(t')}\right)}s'(t')f_a(t';\tau)f_b(s(t');\tau) \tag{27}$$

has no singularity on and outside loop γ_ϵ. On the compact set γ_ϵ we have a bound M_ϵ,

$$|\ \frac{(t'^2-t')(t'\tau)+1}{ln\left(1-\frac{1}{t'}\right)-ln\left(1-\frac{1}{s(t')}\right)}s'(t')f_a(t';\tau)f_b(s(t');\tau)\ |< M_\epsilon$$

Choose $|t|>>1$. Then,

$$\begin{aligned}
&|\ \frac{1}{2\pi i}\oint_{\gamma_\epsilon}K(t',t;\tau)\Psi(t';\tau)\Psi(s(t');\tau)\ |\\
=&|\ \frac{1}{2\pi i}\oint_{\gamma_\epsilon}\frac{\tau}{2(\tau+1)}\left(\frac{1}{t'-t}-\frac{1}{s(t')-t}\right)\frac{(t'^2-t')(t'\tau+1)}{ln\left(1-\frac{1}{t'}\right)-ln\left(1-\frac{1}{s(t')}\right)}s'(t')f_a(t';\tau)f_b(s(t');\tau)dt'\ |\ dt\\
<&\frac{\tau M_\epsilon}{4\pi(\tau+1)}\oint_{\gamma_\epsilon}|\ \frac{1}{t'-t}-\frac{1}{s(t')-t}\ |\ dt'dt\sim\frac{\tau M_\epsilon}{4\pi(\tau+1)|t|}dt
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\frac{1}{2\pi i}\oint_{\gamma_\infty}K(t',t;\tau)\Psi_a(t';\tau)\Psi_b(s(t');\tau)&=\frac{1}{2\pi i}\oint_{\gamma_\infty-\gamma_\epsilon}K(t',t;\tau)\Psi_a(t';\tau)\Psi_b(s(t');\tau)+O(t^{-1})\\
&=\left[\frac{1}{2\pi i}\oint_{\gamma_\infty-\gamma_\epsilon}K(t',t;\tau)\Psi_a(t';\tau)\Psi_b(s(t');\tau)\right]_+
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{2\pi i}\oint_{\gamma_\infty-\gamma_\epsilon}K(t',t;\tau)\Psi_a(t';\tau)\Psi_b(s(t');\tau)\\
&=\frac{\tau}{4(\tau+1)\pi i}\oint_{\gamma_\infty-\gamma_\epsilon}\left(\frac{1}{t'-t}-\frac{1}{s(t')-t}\right)\frac{(t'^2-t')(t'\tau+1)}{ln\left(1-\frac{1}{t'}\right)-ln\left(1-\frac{1}{s(t')}\right)}s'(t')f_a(t';\tau)f_b(s(t');\tau)dt'dt\\
&=\frac{\tau}{2(\tau+1)}\frac{(t^2-t)(t\tau+1)}{ln\left(1-\frac{1}{t}\right)-ln\left(1-\frac{1}{s(t')}\right)}s'(t')f_a(t;\tau)f_b(s(t);\tau)dt\\
&+\frac{\tau}{2(\tau+1)}\frac{(s(t)^2-s(t))(s(t)\tau+1)}{ln\left(1-\frac{1}{t}\right)-ln\left(1-\frac{1}{s(t)}\right)}f_a(s(t);\tau)f_b(t;\tau)dt\\
&=\frac{\tau}{2(\tau+1)}\frac{(t^2-t)(t\tau+1)}{ln\left(1-\frac{1}{t}\right)-ln\left(1-\frac{1}{s(t)}\right)}s'(t)\left(f_a(t;\tau)f_b(s(t);\tau)+f_a(s(t);\tau)f_b(t;\tau)\right)dt\\
&=\frac{\tau}{2}\frac{1}{ln\left(1-\frac{1}{t}\right)-ln\left(1-\frac{1}{s(t)}\right)}\left(\Psi_a(t;\tau)\hat{\Psi}_{b+1}(s(t);\tau)+\Psi_{a+1}(s(t);\tau)\hat{\Psi}_b(t;\tau)\right)
\end{aligned}$$

□

Similarly, with the same proof as showed above, we have

Theorem 4.8.

$$R_n(t,t_i;\tau)=\frac{1}{2\pi i}\oint_{\gamma_\infty}K(t',t;\tau)\left(\Psi_n(t';\tau)B(s(t'),t_i)+\Psi_n(s(t');\tau)B(t';t_i)\right)$$
$$=\tau\left[\frac{1}{ln\left(1-\frac{1}{t}\right)-ln\left(1-\frac{1}{s(t)}\right)}\left(\hat{\Psi}_{n+1}(t;\tau)B(s(t),t_i)+\hat{\Psi}_{n+1}(s(t);\tau)B(t,t_i)\right)\right]_+$$

Let us define polynomials $P_{a,b}(t;\tau)$ and $P_n(t,t_i;\tau)$ by

$$P_{a,b}(t;\tau)dt=\frac{\tau(\tau+1)}{2}\left[\frac{1}{ln\left(1-\frac{1}{t}\right)-ln\left(1-\frac{1}{s(t)}\right)}\left(\hat{\Psi}_{a+1}(t;\tau)\hat{\Psi}_{b+1}(s(t);\tau)\right.\right.$$
$$\left.\left.+\hat{\Psi}_{a+1}(s(t);\tau)\hat{\Psi}_{b+1}(t;\tau)\right)\frac{dt}{(t^2-t)(t\tau+1)}\right]_+ \tag{28}$$

$$P_n(t,t_i;\tau)dtdt_i=\tau d_{t_i}\left[\frac{1}{ln\left(1-\frac{1}{t}\right)-ln\left(1-\frac{1}{s(t)}\right)}\left(\frac{\hat{\Psi}_{n+1}(t;\tau)ds(t)}{s(t)-t_i}+\frac{\hat{\Psi}_{n+1}(s(t);\tau)dt}{t-t_i}\right)\right]_+ \tag{29}$$

With the above notations, the Bourchard-Mariño conjecture for $\mathbb{C}^3$ is equivalent to

(30)

$$(\tau^2+\tau)^{l-1}\sum_{b_L\geq 0}\langle\tau_{b_L}\Gamma_g(\tau)\rangle_g d\hat{\Psi}_{b_L}(t_L;\tau)$$
$$=-(\tau^2+\tau)^{l-2}\sum_{i=2}^{l}\sum_{a,b_{L\setminus\{1,i\}}}\langle\tau_a\tau_{b_{L\setminus\{1,i\}}}\Gamma_g(\tau)\rangle_g P_a(t_1,t_i;\tau)dt_1dt_id\hat{\Psi}_{b_{L\setminus\{1,i\}}}(t_{L\setminus\{1,i\}};\tau)$$
$$+\sum_{\substack{a_1,a_2\geq 0\\ b_{L\setminus\{1\}}}}(\tau^2+\tau)^{l-1}\left((\tau^2+\tau)\langle\tau_{a_1}\tau_{a_2}\tau_{b_{L\setminus\{1\}}}\Gamma_{g-1}(\tau)\rangle_{g-1}\right.$$
$$\left.-\sum_{\substack{g_1+g_2=g\\ \mathcal{I}\bigsqcup\mathcal{J}=L\setminus\{i\}}}^{stable}\langle\tau_{a_1}\tau_{b_I}\Gamma_{g_1}(\tau)\rangle_{g_1,|I|+1}\langle\tau_{a_2}\tau_{b_J}\Gamma_{g_2}(\tau)\rangle_{g_2,|J|+1}\right)P_{a_1,a_2}(t_1;\tau)dt_1d\hat{\Psi}_{b_L\setminus\{1\}}(t_{L\setminus\{1\}};\tau)$$

4.3. **Calculation in new variable v.** Now, let us introduce a new variable v by $x=e^{-w}=e^{-\frac{1}{2}v^2}$. As $y(1-y)^\tau=x$ and $t=\frac{1}{1-(\tau+1)y}$, we have

$$\left(\frac{t-1}{(\tau+1)t}\right)\left(1-\frac{t-1}{(\tau+1)t}\right)^\tau=e^{-\frac{1}{2}v^2}$$

Thus

$$v^2=-2\left(ln\left(1-\frac{1}{t}\right)+\tau ln\left(1+\frac{1}{\tau t}\right)+\tau ln(\tau)-(\tau+1)ln(\tau+1)\right)$$
$$=2\left(\sum_{n\geq 2}\frac{1}{n}\left((-1)^n+\frac{1}{\tau^{n-1}}\right)\frac{1}{t^n}+(\tau+1)ln(\tau+1)-\tau ln(\tau)\right)$$

Solving the above functional equation,

$$v = \frac{1}{t}\left(\sqrt{\frac{\tau+1}{\tau}} + \sum_{k\geq 1} a_k(\tau)\frac{1}{t^k}\right) \tag{31}$$

and

$$-v = \frac{1}{s(t)}\left(\sqrt{\frac{\tau+1}{\tau}} + \sum_{k\geq 1} a_k(\tau)\frac{1}{s(t)^k}\right) \tag{32}$$

Taking the inverse of (31), we have

$$\frac{1}{t} = v\left(\sqrt{\frac{\tau}{\tau+1}} + \sum_{k\geq 1} b_k(\tau)v^k\right) \tag{33}$$

and

$$\frac{1}{s(t)} = (-v)\left(\sqrt{\frac{\tau}{\tau+1}} + \sum_{k\geq 1} b_k(\tau)(-v)^k\right) \tag{34}$$

Lemma 4.9. *Let*

$$\eta_{-1}(v;\tau) = \frac{1}{2}\left(\hat{\Psi}_{-1}(t;\tau) - \hat{\Psi}_{-1}(s(t);\tau)\right) = -\frac{1}{2}\left(ln\left(\frac{t\tau+1}{t\tau}\right) - ln\left(\frac{s(t)\tau+1}{s(t)\tau}\right)\right)$$

and define $\eta_{n+1}(v;\tau) = -\frac{1}{v}\frac{d}{dv}\eta_n(v;\tau)$ *when* $n \geq -1$, *then*

$$\eta_n(v;\tau) = \frac{1}{2}\left(\hat{\Psi}_n(t;\tau) - \hat{\Psi}_n(s(t);\tau)\right)$$

Moreover, there exists formal series $F_n(w;\tau)$ *such that* $\eta_n(v;\tau) = \hat{\Psi}_n(t;\tau) + F_n(w;\tau)$.

Proof. Let $\hat{\Psi}_{-1}(t;\tau) = -ln\left(\frac{t\tau+1}{t\tau}\right)$ be the solution of differential equation

$$\frac{(t\tau+1)(t^2-t)}{\tau+1}\frac{d}{dt}\hat{\Psi}_{-1}(t;\tau) = \hat{\Psi}_0(t;\tau) = \frac{t-1}{\tau+1}$$

then

$$\eta_{-1}(v;\tau) = \frac{1}{2}\left(\hat{\Psi}_{-1}(t;\tau) - \hat{\Psi}_{-1}(s(t);\tau)\right) = -\frac{1}{2}\left(ln\left(\frac{t\tau+1}{t\tau}\right) - ln\left(\frac{s(t)\tau+1}{s(t)\tau}\right)\right)$$

By definition,

$$x\frac{d}{dx} = -\frac{d}{dw} = -\frac{1}{v}\frac{d}{dv} = \frac{(t\tau+1)(t^2-t)}{\tau+1}\frac{d}{dt} = \frac{(s(t)\tau+1)(s(t)^2-s(t))}{\tau+1}\frac{d}{ds(t)} \tag{35}$$

Therefore

$$\eta_n(v;\tau) = \frac{1}{2}\left(\hat{\Psi}_n(t;\tau) - \hat{\Psi}(s(t);\tau)\right)$$

Moreover,

$$\begin{aligned}&\eta_{-1}(v;\tau) - \hat{\Psi}_{-1}(t;\tau)\\ &= -\frac{1}{2}\left(\hat{\Psi}_{-1}(t;\tau) + \hat{\Psi}_{-1}(s(t);\tau)\right)\\ &= \frac{1}{2}\left(ln\left(1+\frac{1}{t\tau}\right) + ln\left(1+\frac{1}{s(t)\tau}\right)\right)\end{aligned} \tag{36}$$

$$= \frac{1}{2}\left(\sum_{n\geq 1}\frac{(-1)^{n-1}}{n}\frac{1}{\tau^n}\left(\frac{1}{t^n}+\frac{1}{s(t)^n}\right)\right)$$

By (33) and (34), we note that right hand side of (36) is a series of variable $w=\frac{1}{2}v^2$, we write it as $F_{-1}(w;\tau)$. (36) is equivalent to

$$\eta_{-1}(v;\tau)=\hat{\Psi}_{-1}(t;\tau)+F_{-1}(w;\tau)$$

Let us define

$$F_n(w;\tau)=\left(-\frac{d}{dw}\right)^{n+1}F_{-1}(w;\tau) \tag{37}$$

then, we have

$$\eta_n(v;\tau)=\hat{\Psi}_n(t;\tau)+F_n(w;\tau)$$

□

Corollary 4.10.

$$P_{a,b}(t;\tau)dt=\frac{1}{2}\left(\frac{\eta_{a+1}(v;\tau)\eta_{b+1}(v;\tau)}{\eta_{-1}(v;\tau)}vdv|_{v=v(t)}\right)_+$$

Proof. By (23) and (28), we have

$$-\frac{1}{2}\frac{1}{ln\left(1+\frac{1}{\tau t}\right)-ln\left(1+\frac{1}{\tau s(t)}\right)}\left(\hat{\Psi}_{a+1}(t;\tau)\hat{\Psi}_{b+1}(s(t);\tau)+\hat{\Psi}_{a+1}(s(t);\tau)\hat{\Psi}_{b+1}(t;\tau)\right)\frac{(\tau+1)dt}{(t^2-t)(t\tau+1)}$$

$$=-\frac{1}{2\eta_{-1}(v;\tau)}\frac{(\tau+1)dt}{(t^2-t)(t\tau+1)}\left(\frac{(\hat{\Psi}_{a+1}(t;\tau)-\hat{\Psi}_{a+1}(s(t);\tau))(\hat{\Psi}_{b+1}(t;\tau)-\hat{\Psi}_{b+1}(s(t);\tau))}{4}\right.$$

$$\left.-\frac{(\hat{\Psi}_{a+1}(t;\tau)+\hat{\Psi}_{a+1}(s(t);\tau))(\hat{\Psi}_{b+1}(t;\tau)+\hat{\Psi}_{b+1}(s(t);\tau))}{4}\right)$$

$$=\frac{vdv}{2\eta_{-1}(v;\tau)}\left(\eta_{a+1}(v;\tau)\eta_{b+1}(v;\tau)-F_{a+1}(w;\tau)F_{b+1}(w;\tau)\right)$$

Noticing that

$$\left(\frac{F_{a+1}(w;\tau)F_{b+1}(w;\tau)vdv}{\eta_{-1}(v;\tau)}|_{v=v(t)}\right)_+=0$$

we complete the proof. □

4.4. **Proof of the Bouchard-Mariño conjecture for $\mathbb{C}^3$.** In the proof of the theorem 3.4, we know that LHS of (10) is equal to

$$\frac{d}{d\tau}\hat{\mathcal{C}}_{g,l}(t_1,t_2,..,t_l;\tau)-\sum_{i=1}^{l}\frac{\partial y_i}{\partial\tau}t_i^2(\tau+1)\frac{\partial}{\partial t_i}\hat{\mathcal{C}}_{g,l}(t_1,t_2,..,t_l;\tau)$$

$$=\frac{\partial}{\partial\tau}\hat{\mathcal{C}}_{g,l}(t_1,t_2,..,t_l;\tau)+\sum_{i=1}^{l}\frac{\partial v_i}{\partial\tau}\frac{\partial}{\partial v_i}\hat{\mathcal{C}}_{g,l}(v_1,v_2,..v_l;\tau)-\sum_{i=1}^{l}\frac{\partial y_i}{\partial\tau}t_i^2(\tau+1)\frac{\partial}{\partial t_i}\hat{\mathcal{C}}_{g,l}(t_1,t_2,..t_l;\tau)$$

Recall the following relationship of x,y,t,v.

$$x=e^{-\frac{1}{2}v^2},y(1-y)^\tau=x,t=\frac{1}{1-(\tau+1)y}$$

Thus

$$\frac{\partial v_1}{\partial \tau}=0, \frac{\partial y}{\partial \tau}=\frac{y(1-y)ln(1-y)}{y(\tau+1)-1}, \frac{(t^2-t)(t\tau+1)}{\tau+1}\frac{d}{dt}=-\frac{1}{v}\frac{d}{dv}$$

After some simple calculation,

$$\text{LHS of (10)}=\frac{\partial}{\partial \tau}\hat{\mathcal{C}}_{g,l}(t_1,t_2,..,t_l;\tau)+\sum_{i=1}^{l} ln(1-y_i)\left(-\frac{1}{v_i}\frac{d}{dv_i}\right)\hat{\mathcal{C}}_{g,l}(t_1,t_2,..,t_l;\tau)$$

The direct image of a function $f(v)$ on C via the projection $\pi: C\to\mathbb{C}$ is:

$$\pi_* f=f(v)+f(-v)$$

Thus,

$$\pi_*(\hat{C}_{g,l}(t_1,t_2,..,t_l;\tau))=-(\tau^2+\tau)^{l-1}\sum_{b_L\geq 0}\langle\tau_{b_L}\Gamma_g(\tau)\rangle_g\left(\hat{\Psi}_{b_1}(t_1;\tau)+\hat{\Psi}_{b_1}(s(t_1);\tau)\right)\hat{\Psi}_{b\backslash\{1\}}(t_{L\backslash\{1\}};\tau)$$

By the definition of $F_b(w;\tau)$,

$$\begin{aligned}\pi_*\left(\frac{\partial}{\partial\tau}\hat{C}_{g,l}(t_1,t_2,..,t_l;\tau)\right)&=\frac{\partial}{\partial\tau}\left(\pi_*\hat{\mathcal{C}}_{g,l}(t_1,t_2,..,t_l;\tau)\right)\\&=\frac{\partial}{\partial\tau}\left((\tau^2+\tau)^{l-1}\sum_{b_L\geq 0}\langle\tau_{b_L}\Gamma_g(\tau)\rangle_g F_{b_1}(w_1;\tau)\hat{\Psi}_{b_{L\backslash\{1\}}}(t_{L\backslash\{1\}})\right)\end{aligned}$$

is regular in w_1. We will use $\mathcal{O}(w_1)$ to denote some function regular in w_1.

Consider the direct sum

$$\pi_*\left(\sum_{i=1}^{l} ln(1-y_i)\left(-\frac{1}{v_1}\frac{d}{dv_1}\right)\hat{\mathcal{C}}_{g,l}(t_1,t_2,..,t_l;\tau)\right)$$

when $i=1$,

$$\begin{aligned}&\pi_*\left(ln(1-y_1)\left(-\frac{1}{v_1}\frac{d}{dv_1}\right)\hat{\mathcal{C}}_{g,l}(t_1,t_2,..,t_l;\tau)\right)\\&=-(\tau^2+\tau)^{l-1}\sum_{b_L\geq 0}\langle\tau_{b_L}\Gamma_g(\tau)\rangle_g\left[ln\left(\frac{t_1\tau+1}{(\tau+1)t_1}\right)\left(-\frac{1}{v_1}\frac{d}{dv_1}\right)(\eta_{b_1}(v_1;\tau)-F_{b_1}(w_1;\tau))\right.\\&\left.+ln\left(\frac{s(t_1)\tau+1}{(\tau+1)s(t_1)}\right)\left(-\frac{1}{v_1}\frac{d}{dv_1}\right)(\eta_{b_1}(v_1;\tau)-F_{b_1}(w_1;\tau))\right]\hat{\Psi}_{b\backslash\{1\}}(t_{b\backslash\{1\}})\\&=-(\tau^2+\tau)^{l-1}\sum_{b_L\geq 0}\langle\tau_{b_L}\Gamma_g(\tau)\rangle_g\left[\left(ln\left(\frac{t_1\tau+1}{(\tau+1)t_1}\right)-ln\left(\frac{s(t_1)\tau+1}{(\tau+1)s(t_1)}\right)\right)\eta_{b_1}(v_1;\tau)\right.\\&\left.+\left(ln\left(\frac{t_1\tau+1}{(\tau+1)t_1}\right)-ln\left(\frac{s(t_1)\tau+1}{(\tau+1)s(t_1)}\right)\right)F_{b_1}(w_1;\tau)\right]\hat{\Psi}_{b\backslash\{1\}}(t_{b\backslash\{1\}})\\&=(\tau^2+\tau)^{l-1}\sum_{b_L\geq 0}\langle\tau_{b_L}\Gamma_g(\tau)\rangle_g 2\eta_{-1}(v_1;\tau)\eta_{b_1+1}(v_1;\tau)\hat{\Psi}_{b_L\backslash\{1\}}(t_{L\backslash\{1\}};\tau)+\mathcal{O}(w_1)\end{aligned}$$

and it is easy to see that

$$\pi_*\left(\sum_{i=2}^{l} ln(1-y_i)\left(-\frac{1}{v_1}\frac{d}{dv_1}\right)\hat{\mathcal{C}}_{g,l}(t_1,t_2,..,t_l;\tau)\right)=\mathcal{O}(w_1)$$

Thus

$$\pi_*(\text{LHS of (10)}) = (\tau^2+\tau)^{l-1}\sum_{b_L\geq 0}\langle\tau_{b_L}\Gamma_g(\tau)\rangle_g 2\eta_{-1}(v_1;\tau)\eta_{b_1+1}(v_1;\tau)\hat{\Psi}_{b_L\setminus\{1\}}(t_{L\setminus\{1\}};\tau)+\mathcal{O}(w_1)$$

$$\begin{aligned}
&(\tau^2+\tau)^{l-1}\sum_{b_L\geq 0}\langle\tau_{b_L}\Gamma_g(\tau)\rangle_g d\hat{\Psi}_{b_L}(t_L;\tau)\\
&=(\tau^2+\tau)^{l-1}\sum_{b_L\geq 0}\langle\tau_{b_L}\Gamma_g(\tau)\rangle_g dv_1\frac{d}{dv_1}\left(\hat{\Psi}_{b_1}(t_1;\tau)\right)d\hat{\Psi}_{b_L\setminus\{1\}}(t_{L\setminus\{1\}};\tau)\\
&=(-v_1dv_1)d_2\cdots d_l\left((\tau^2+\tau)^{l-1}\sum_{b_L\geq 0}\langle\tau_{b_L}\Gamma_g(\tau)\rangle_g\left(-\frac{1}{v_1}\frac{d}{dv_1}\right)(\eta_{b_1}(v_1;\tau)-F_{b_1}(w_1;\tau))\right)\hat{\Psi}_{b_L\setminus\{1\}}(t_{L\setminus\{1\}})\\
&=\frac{-v_1dv_1}{2\eta_{-1}(v_1;\tau)}d_2\cdots d_l\left((\tau^2+\tau)^{l-1}\sum_{b_L\geq 0}\langle\tau_{b_L}\Gamma_g(\tau)\rangle_g 2\eta_{-1}(v_1;\tau)\left(\eta_{b_1}(v_1;\tau)-F_{b_1}(w_1;\tau)\right)\right)\hat{\Psi}_{b_L\setminus\{1\}}(t_{L\setminus\{1\}})\\
&=\frac{-v_1dv_1}{2\eta_{-1}(v_1;\tau)}d_2\cdots d_l\left(\pi_*(LHS)-2\eta_{-1}(v_1;\tau)F_{b_1}(w_1;\tau)(\tau^2+\tau)^{l-1}\sum_{b_L\geq 0}\langle\tau_{b_L}\Gamma_g(\tau)\rangle_g\hat{\Psi}_{b_L\setminus\{1\}}(t_{L\setminus\{1\}})\right.\\
&\left.-\mathcal{O}(w_1)\right)
\end{aligned}$$

It is important to note that

$$\left(\frac{-v_1dv_1\mathcal{O}(w_1)}{2\eta_{-1}(v_1;\tau)}|_{v_1=v_1(t_1)}\right)_+=0$$

and

$$\left(-v_1dv_1F_{b_1+1}(w_1;\tau)|_{v_1=v_1(t_1)}\right)_+=0$$

Thus,

$$(\tau^2+\tau)^{l-1}\sum_{b_L\geq 0}\langle\tau_{b_L}\Gamma_g(\tau)\rangle_g d\hat{\Psi}_{b_L}(t_L;\tau)=\left(-\frac{v_1dv_1}{2\eta_{-1}(v_1;\tau)}d_2\cdots d_l\left(\pi_*(\text{LHS of (10)})\right)\right)\tag{38}$$

Then we need to calculate $\left(\frac{-v_1dv_1}{2\eta_{-1}(v_1;\tau)}d_2\cdots d_l\pi_*(T_1)\right)_+$

Let us write $T_1=T_{11}+T_{12}$, where

$$T_{11}=-\frac{(\tau^2+\tau)^{l-2}}{(\tau+1)}\sum_{2\leq j\leq l}\sum_{\substack{a\geq 0\\ b_{L\setminus\{1,j\}}\geq 0}}\langle\tau_a\tau_{b_{L\setminus\{1,j\}}}\Gamma_g(\tau)\rangle_g\hat{\Psi}_{b_{L\setminus\{1,j\}}}(t_{L\setminus\{1,j\}};\tau)$$
$$\frac{(t_j-1)(t_1^2\tau+t_1)\hat{\Psi}_{a+1}(t_1;\tau)-(t_1-1)(t_j^2\tau+t_j)\hat{\Psi}_{a+1}(t_j;\tau)}{t_1-t_j}$$

$$T_{12}=-\frac{(\tau^2+\tau)^{l-2}}{(\tau+1)}\sum_{2\leq i<j\leq l}\sum_{\substack{a\geq 0\\ b_{L\setminus\{i,j\}}\geq 0}}\langle\tau_a\tau_{b_{L\setminus\{i,j\}}}\Gamma_g(\tau)\rangle_g\hat{\Psi}_{b_{L\setminus\{i,j\}}}(t_{L\setminus\{i,j\}};\tau)$$
$$\frac{(t_j-1)(t_i^2\tau+t_i)\hat{\Psi}_{a+1}(t_i;\tau)-(t_i-1)(t_j^2\tau+t_j)\hat{\Psi}_{a+1}(t_j;\tau)}{t_i-t_j}$$

By the definition,

$$\left(\frac{-v_1dv_1}{2\eta_{-1}(v_1;\tau)}\pi_*T_{12}\right)_+=0 \tag{39}$$

we have that

$$\begin{aligned}&\frac{-v_1dv_1}{2\eta_{-1}(v_1;\tau)}d_j\left(\frac{(t_j-1)(t_1^2\tau+t_1)\hat{\Psi}_{a+1}(t_1;\tau)}{t_1-t_j}\right)\\&=\frac{1}{2\eta_{-1}(v_1;\tau)}d_j\left(\frac{(\tau+1)dt_1}{(t_1^2-t_1)(t_1\tau+1)}\frac{(t_j-1)(t_1^2\tau+t_1)}{t_1-t_j}\hat{\Psi}_{a+1}(t_1;\tau)\right)\\&=\frac{\tau+1}{2\eta_{-1}(v_1;\tau)}\frac{\hat{\Psi}_{a+1}(t_1;\tau)dt_1dt_j}{(t_1-t_j)^2}\\&=\frac{\tau+1}{2\eta_{-1}(v_1;\tau)}\hat{\Psi}_{a+1}(t_1;\tau)B(t_1,t_j;\tau)\end{aligned} \tag{40}$$

Similarly,

$$\begin{aligned}&\frac{-v_1dv_1}{2\eta_{-1}(v_1;\tau)}d_j\left(\frac{(t_j-1)(s(t_1)^2\tau+s(t_1))\hat{\Psi}_{a+1}(s(t_1);\tau)}{s(t_1)-t_j}\right)\\&=\frac{\tau+1}{2\eta_{-1}(v_1;\tau)}\hat{\Psi}_{a+1}(s(t_1);\tau)B(s(t_1),t_j;\tau)\end{aligned} \tag{41}$$

Moreover, by the expansion,

$$\frac{t_1-1}{t_1-t_j}+\frac{s(t_1)-1}{s(t_1)-t_j}=\sum_{k\geq 0}\left(\frac{t_j^k}{t_1^k}+\frac{t_j^k}{t_1^{k+1}}+\frac{t_j^k}{s(t_1)^k}+\frac{t_j^k}{s(t_1)^{k+1}}\right)$$

we have

$$\left(\frac{-v_1dv_1}{2\eta_{-1}(v_1;\tau)}\left(\frac{t_1-1}{t_1-t_j}+\frac{s(t_1)-1}{s(t_1)-t_j}\right)|_{v_1=v_1(t_1)}\right)_+=0 \tag{42}$$

Therefore

$$\begin{aligned}&\left(\frac{-v_1dv_1}{2\eta_{-1}(v_1;\tau)}d_2\cdots d_l\pi_*T_1\right)_+ \quad \text{by (39)}\\&=\left(\frac{-v_1dv_1}{2\eta_{-1}(v_1;\tau)}d_2\cdots d_l\pi_*T_{11}\right)_+ \quad \text{by (40), (41), (42)}\\&=-(\tau^2+\tau)^{l-2}\sum_{2\leq j\leq l}\sum_{\substack{a\geq 0\\ b_{L\setminus\{1,j\}}}}\langle\tau_a\tau_{b_{L\setminus\{1,j\}}}\Gamma_g(\tau)\rangle_g d\hat{\Psi}_{b_{L\setminus\{1,j\}}}(t_{L\setminus\{1,j\}};\tau)\\&\left(\frac{\hat{\Psi}_{a+1}(t_1;\tau)B(t_1,t_j;\tau)+\hat{\Psi}_{a+1}(s(t_1);\tau)B(s(t_1),t_j;\tau)}{2\eta_{-1}(v_1;\tau)}\right)_+ \quad \text{by (28)}\\&=-(\tau^2+\tau)^{l-2}\sum_{2\leq j\leq l}\sum_{\substack{a\geq 0\\ b_{L\setminus\{1,j\}}}}\langle\tau_a\tau_{b_{L\setminus\{1,j\}}}\Gamma_g(\tau)\rangle_g P_{a+1}(t_1,t_j;\tau)dt_1dt_jd\hat{\Psi}_{b_{L\setminus\{1,j\}}}(t_{L\setminus\{1,j\}};\tau)\end{aligned} \tag{43}$$

Finally, we need to calculate $\left(\frac{-v_1dv_1}{2\eta_{-1}(v_1;\tau)}d_2\cdots d_l\pi_*(T_2+T_3)\right)_+$

Because

$$\left(\frac{-v_1dv_1}{2\eta_{-1}(v_1;\tau)}\pi_*\left(\hat{\Psi}_{a_1+1}(t_1;\tau)\hat{\Psi}_{a_2+1}(t_1;\tau)\right)\right)_+$$

$$
\begin{aligned}
&= \left(\frac{-v_1dv_1}{2\eta_{-1}(v_1;\tau)}\left(\hat{\Psi}_{a_1+1}(t_1;\tau)\hat{\Psi}_{a_2+1}(t_1;\tau)+\hat{\Psi}_{a_1+1}(s(t_1);\tau)\hat{\Psi}_{a_2+1}(s(t_1);\tau)\right)\right)_+\\
&= \left(\frac{-v_1dv_1}{2\eta_{-1}(v_1;\tau)}\left(2\eta_{a_1+1}(v_1;\tau)\eta_{a_2+1}(v_1;\tau)+2F_{a_1+1}(w_1;\tau)F_{a_2+1}(w_1;\tau)\right)\right)_+\\
&= -2P_{a_1,a_2}(t_1;\tau)dt_1
\end{aligned}
$$

In the last "=", we have used Corollary 4.10 and $\left(\frac{F_{a_1+1}(w_1;\tau)F_{a_2+1}(w_1;\tau)v_1dv_1}{\eta_{-1}(v_1;\tau)}|_{v_1=v_1(t_1)}\right)_+=0$.

We have

(44)

$$
\begin{aligned}
\left(\frac{-v_1dv_1}{2\eta_{-1}(v_1;\tau)}d_2\cdots d_l\pi_*(T_2+T_3)\right)_+ = (\tau^2+\tau)^{l-1}\sum_{\substack{a_1\geq 0\\ a_2\geq 0\\ b_{L\setminus\{i\}}\geq 0}}\Big((\tau^2+\tau)\langle\tau_{a_1}\tau_{a_2}\tau_{b_{L\setminus\{i\}}}\Gamma_{g-1}(\tau)\rangle_{g-1}\\
-\sum_{\substack{g_1+g_2=g\\ \mathcal{I}\bigcup\mathcal{J}=L\setminus\{i\}}}^{stable}\langle\tau_{a_1}\tau_{b_{\mathcal{I}}}\Gamma_{g_1}(\tau)\rangle_{g_1}\langle\tau_{a_2}\tau_{b_{\mathcal{J}}}\Gamma_{g_2}(\tau)\rangle_{g_2}\Big)P_{a_1,a_2}(t_1;\tau)dt_1d\hat{\Psi}_{b_{L\setminus\{1\}}}(t_{L\setminus\{1\}};\tau)
\end{aligned}
$$

Hence, by equations (38), (43), (44) and (10), we have proved the identity (30), i.e the Bouchard-Marino conjecture for $\mathbb{C}^3$ case.

Remark 4.11. At the same time as L.Chen published his proof [5], J. Zhou also proved the Bouchard-Mariño conjecture for $\mathbb{C}^3$ [22]. He formulated this new recursion relation of conjecture and equation (1) of theorem 1.1 in the coordinate v. Then, he regarded equation (1) as meromorphic functions in v_1, took the principal parts and took only the even powers in v_1 would get the recursion relation for this conjecture. J. Zhou has also applied his method to formulate and prove this new recursion relation for topological vertex [23] based on the work [16].

5. Application II: Derivation of Some Hodge integral identities

In this section, we will use the main theorem 1.1 to obtain some Hodge integral identities.

5.1. **Preliminary calculations.** For convenience, let us recall the formula (1) in theorem 1.1 at first: for $g\geq 1$ and $l\geq 1$, Then,

(45)

$$
\begin{aligned}
LHS :=& -(\tau^2+\tau)^{l-2}\sum_{b_L\geq 0}\left((l-1)(2\tau+1)\langle\tau_{b_L}\Gamma_g(\tau)\rangle_g+(\tau^2+\tau)\langle\tau_{b_L}\frac{d}{d\tau}\Gamma_g(\tau)\rangle_g\right)\hat{\Psi}_{b_L}(t_L;\tau)\\
&-(\tau^2+\tau)^{l-1}\sum_{b_L\geq 0}\langle\tau_{b_L}\Gamma_g(\tau)\rangle_g\sum_{i=1}^{l}\left(\frac{\partial}{\partial\tau}\hat{\Psi}_{b_i}(t_i;\tau)+\frac{1}{t_i\tau+1}\hat{\Psi}_{b_i+1}(t_i;\tau)\right)\hat{\Psi}_{b_{L\setminus\{i\}}}(t_{L\setminus\{i\}};\tau)\\
=& T_1+T_2+T_3
\end{aligned}
$$

where

$$
\begin{aligned}
T_1 :=& -\frac{(\tau^2+\tau)^{l-2}}{\tau+1}\sum_{1\leq i<j\leq l}\sum_{\substack{a\geq 0\\ b_{L\setminus\{i,j\}}\geq 0}}\langle\tau_a\tau_{b_{L\setminus\{i,j\}}}\Gamma_g(\tau)\rangle_g\hat{\Psi}_{b_{L\setminus\{i,j\}}}(t_{L\setminus\{i,j\}};\tau)\\
&\cdot\frac{(t_j-1)(t_i^2\tau+t_i)\hat{\Psi}_{a+1}(t_i;\tau)-(t_i-1)(t_j^2\tau+t_j)\hat{\Psi}_{a+1}(t_j;\tau)}{t_i-t_j}
\end{aligned}
$$

$$T_2 := \frac{(\tau^2+\tau)^l}{2}\sum_{i=1}^{l}\sum_{\substack{a_1\geq 0\\ a_2\geq 0\\ b_{L\setminus\{i\}}\geq 0}}\langle \tau_{a_1}\tau_{a_2}\tau_{b_{L\setminus\{i\}}}\Gamma_{g-1}(\tau)\rangle_{g-1}\prod_{n=1}^{2}\hat{\Psi}_{a_n+1}(t_i;\tau)\hat{\Psi}_{b_{L\setminus\{i\}}}(t_{L\setminus\{i\}};\tau)$$

$$T_3 := -\frac{(\tau^2+\tau)^{l-1}}{2}\sum_{i=1}^{l}\sum_{\substack{a_1\geq 0\\ a_2\geq 0\\ b_{L\setminus\{i\}}}}\sum_{\substack{g_1+g_2=g\\ \mathcal{I}\coprod\mathcal{J}=L\setminus\{i\}\\ 2g_1-1+|\mathcal{I}|>0\\ 2g_2-1+|\mathcal{J}|>0}}\langle \tau_{a_1}\tau_{b_{\mathcal{I}}}\Gamma_{g_1}(\tau)\rangle_{g_1}\langle \tau_{a_2}\tau_{b_{\mathcal{J}}}\Gamma_{g_2}(\tau)\rangle_{g_2}\prod_{n=1}^{2}\hat{\Psi}_{a_n+1}(t_i;\tau)\hat{\Psi}_{b_{L\setminus\{i\}}}(t_{L\setminus\{i\}};\tau)$$

$$\hat{\Psi}_n(t;\tau) = \left(\frac{(t^2-t)(t\tau+1)}{\tau+1}\frac{d}{dt}\right)^n\left(\frac{t-1}{\tau+1}\right), n\geq 0$$

and

$$\Gamma_g(\tau) = \Lambda_g^\vee(1)\Lambda_g^\vee(-\tau-1)\Lambda_g^\vee(\tau)$$

Let us denote the τ expansion of $\hat{\Psi}_n(t;\tau)$ and $\Gamma_g(\tau)$ as follow,

$$\hat{\Psi}_b(t;\tau) = \sum_{k=0}^{b}\frac{\tau^k}{(\tau+1)^{b+1}}\Psi_b^k(t) \tag{46}$$

and

$$\Gamma_g(\tau) = \sum_{m=0}^{2g}\Lambda_g^\vee(1)a_m(\lambda)\tau^m \tag{47}$$

By the definition of $\Lambda_g^\vee(t) = \sum_{j=0}^{g}(-1)^{g-j}\lambda_{g-j}t^j$ and Mumford's relation $\Lambda_g^\vee(t)\Lambda_g^\vee(-t) = (-1)^g t^{2g}$, we can compute out the coefficients $a_m(\lambda)$ in (47). For example,

$$\begin{aligned} &a_{2g}(\lambda) = (-1)^g, a_{2g-1}(\lambda) = (-1)^g g, \cdots \\ &a_1(\lambda) = \sum_{m=1}^{g} m\lambda_{g-m}\lambda_g - (-1)^g\lambda_{g-1}\Lambda_g^\vee(-1) \\ &a_0(\lambda) = (-1)^g\lambda_g\Lambda_g^\vee(-1) \end{aligned} \tag{48}$$

We also need to show the expansion form of (46) and how to calculate the coefficients $\Psi_b^k(t)$. By definition $\hat{\Psi}_{b+1}(t,\tau) = (t^2-t)(\frac{y\tau+1}{\tau+1})\frac{d}{dy}\Psi_b(y,\tau)$, $\hat{\Psi}_0(t,\tau) = \frac{t-1}{\tau+1}$, we have the expansion form (46). Moreover, we have

$$\Psi_{b+1}^k(t) = (t^3-t^2)\frac{d}{dt}\Psi_b^{k-1}(t) + (t^2-t)\frac{d}{dt}\Psi_b^k(t), k = 0,..,b+1 \tag{49}$$

Therefore, through the recursion formula (49), all the $\Psi_b^k(t)$ can be calculated.

As an illustration, we calculate some cases which will be used in the following discussion.

$$\Psi_{b+1}^{b+1}(t) = (t^3-t^2)\frac{d}{dt}\Psi_b^b(t) \tag{50}$$

$$\Psi_{b+1}^{b}(t) = (t^3-t^2)\frac{d}{dt}\Psi_b^{b-1}(t) + (t^2-t)\frac{d}{dt}\Psi_b^b(t) \tag{51}$$

$$\Psi_{b+1}^{1}(t) = (t^3-t^2)\frac{d}{dt}\Psi_b^{0}(t) + (t^2-t)\frac{d}{dt}\Psi_b^1(t) \tag{52}$$

$$\Psi^0_{b+1}(t) = (t^2 - t)\frac{d}{dt}\Psi^0_b(t) \tag{53}$$

It is clear that, the recursion relation for $\Psi^b_b(t)$ in (50) is just same for the definition of $\hat{\xi}_b(t)$ in [9], and they share the same initial. thus

$$\Psi^b_b(t) = \hat{\xi}_b(t) \tag{54}$$

By (50), we write $\Psi^b_b(t)$ as

$$\Psi^b_b(t) = \sum_{i=b+1}^{2b+1} f^b(b,i)t^i \tag{55}$$

then all the $f^b(b,i)$ could be calculated from the recursion (50). But we can explicitly write down $f^b(b,2b+1) = (2b-1)!!$, $f^b(b,b+1) = (-1)^b b!$ which are the coefficients used in the proof of DVV equation and λ_g integral respectively[20, 6, 14].

We can also let

$$\Psi^{b-1}_b(t) = \sum_{i=b}^{2b} f^{b-1}(b,i)t^i \tag{56}$$

then by (51) we have

$$f^b(b+1,k) = (k-2)f^{b-1}(b,k-2) - (k-1)f^{b-1}(b,k-1) + (k-1)f^b(b,k-1) - kf^b(b,k) \tag{57}$$

Thus, all $f^b(b+1,k)$ can be calculated from (57). For example,

$$f^b(b+1,2b+2) = 2bf^{b-1}(b,2b) + (2b+1)f^b(b,2b+1) = 2bf^{b-1}(b,2b) + (2b+1)!!$$

Hence

$$f^{b-1}(b,2b) = \sum_{k=0}^{b-1} \frac{(2b-2)!!(2k+1)!!}{(2k)!!} \tag{58}$$

Similarly,

$$f^b(b+1,b+1) = -bf^{b+1}(b,b) - (b+1)f^b(b,b+1) = -bf^b(b,b) + (-1)^{b+1}(b+1)!$$

Thus

$$f^{b-1}(b,b) = (-1)^b\frac{(b+1)!}{2} \tag{59}$$

On the other side, from (53), we can let

$$\Psi^0_b(t) = \sum_{i=1}^{b+1} f^0_b(b,i)y^i$$

Then

$$f^0_{b+1}(b+1,i) = (i-1)f^0_b(b,i-1) - if^0_b(b,i)$$

Thus,

$$f^0_b(b,b+1) = b! \tag{60}$$

Then, from (52), we can let

$$\Psi^1_b(t) = \sum_{j=2}^{b+2} f^1(b,j)t^j \tag{61}$$

and

$$f^1(b+1,k)=(k-2)f^0(b,k-2)-(k-1)f^0(b,k-1)+(k-1)f^1(b,k-1)-kf^1(b,k)$$

Thus

$$f^1(b+1,b+3)=(b+1)f^0(b,b+1)+(b+2)f^1(b,b+2) \tag{62}$$

by initial value, $f^1(1,3)=1$ and $f^0(b,b+1)=b!$, we get

$$f^1(b,b+2)=(b+1)!\sum_{k=2}^{b+1}\frac{1}{k}. \tag{63}$$

Now, let us substitute (46) and (47) to (45), we have

$$\begin{aligned} LHS=&-\sum_{b_L\geq 0}\sum_{0\leq k_L\leq b_L}\sum_{m=0}^{2g}\langle\tau_{b_L}\Lambda_g^\vee(1)a_m(\lambda)\rangle_g\Psi_{b_L}^{k_L}(t_L)\\ &\times\frac{(l-2+m+|k_L|-|b_L|)\tau^{|k_L|+l+m-1}+(l-1+m+|k_L|)\tau^{|k_L|+l+m-2}}{(\tau+1)^{|b_L|+2}}\\ &-\sum_{b_L\geq 0}\sum_{0\leq k_L\leq b_L}\sum_{m=0}^{2g}\langle\tau_{b_L}\Lambda_g^\vee a_m(\lambda)\rangle_g\sum_{i=1}^{l}(t_i^2-t_i)\frac{\partial}{\partial t_i}\Psi_{b_i}^{k_i}(t_i)\Psi_{b_{L\backslash\{i\}}}^{b_{L\backslash\{i\}}}(t_{L\backslash\{i\}})\frac{\tau^{|b_L|+m+l-2}}{(\tau+1)^{|b_L|+2}} \end{aligned} \tag{64}$$

$$\begin{aligned} T_1=&-\sum_{1\leq i<j\leq l}\sum_{\substack{a\geq 0\\ b_{L\backslash\{i,j\}}\geq 0}}\sum_{m=0}^{2g}\sum_{\substack{0\leq k\leq a+1\\ 0\leq k_{L\backslash\{i,j\}}\leq b_{L\backslash\{i,j\}}}}\langle\tau_a\tau_{b_{L\backslash\{i,j\}}}\Lambda_g^\vee(1)a_m(\lambda)\rangle_g\Psi_{b_{L\backslash\{i,j\}}}^{b_{L\backslash\{i,j\}}}(t_{L\backslash\{i,j\}})\\ &\times\frac{(t_j-1)t_i^2\Psi_{a+1}^k(t_i)-(t_i-1)t_j^2\Psi_{a+1}^k(t_j)}{t_i-t_j}\frac{\tau^{|k_{L\backslash\{i,j\}}|+k+l+m-1}}{(\tau+1)^{|b_{L\backslash\{i,j\}}|+a+3}}\\ &-\sum_{1\leq i<j\leq l}\sum_{\substack{a\geq 0\\ b_{L\backslash\{i,j\}}\geq 0}}\sum_{m=0}^{2g}\sum_{\substack{0\leq k\leq a+1\\ 0\leq k_{L\backslash\{i,j\}}\leq b_{L\backslash\{i,j\}}}}\langle\tau_a\tau_{b_{L\backslash\{i,j\}}}\Lambda_g^\vee(1)a_m(\lambda)\rangle_g\Psi_{b_{L\backslash\{i,j\}}}^{b_{L\backslash\{i,j\}}}(t_{L\backslash\{i,j\}})\\ &\times\frac{(t_j-1)t_i\Psi_{a+1}^k(t_i)-(t_i-1)t_j\Psi_{a+1}^k(t_j)}{t_i-t_j}\frac{\tau^{|k_{L\backslash\{i,j\}}|+k+l+m-2}}{(\tau+1)^{|b_{L\backslash\{i,j\}}|+a+3}} \end{aligned} \tag{65}$$

$$\begin{aligned} T_2=&\frac{1}{2}\sum_{i=1}^{l}\sum_{\substack{a_1\geq 0\\ a_2\geq 0\\ b_{L\backslash\{i\}}\geq 0}}\sum_{m=0}^{2(g-1)}\sum_{\substack{0\leq n_1\leq a_1+1\\ 0\leq n_2\leq a_2+1\\ 0\leq k_{L\backslash\{i\}}\leq b_{L\backslash\{i\}}}}\langle\tau_{a_1}\tau_{a_2}\tau_{b_{L\backslash\{i,j\}}}\Lambda_{g-1}^\vee(1)a_m(\lambda)\rangle_{g-1}\\ &\times\Psi_{a_1+1}^{n_1}(t_i)\Psi_{a_2+1}^{n_2}(t_i)\Psi_{b_{L\backslash\{i\}}}^{b_{L\backslash\{i\}}}(t_{L\backslash\{i\}})\frac{\tau^{|k_{L\backslash\{i\}}|+n_1+n_2+l+m}}{(\tau+1)^{a_1+a_2+|k_{L\backslash\{i\}}|+3}} \end{aligned} \tag{66}$$

$$T_3=-\frac{1}{2}\sum_{i=1}^{l}\sum_{\substack{a_1\geq 0\\ a_2\geq 0\\ b_{L\backslash\{i\}}\geq 0}}\sum_{\substack{g_1+g_2=g\\ \mathcal{I}\cup\mathcal{J}=L\backslash\{i\}\\ 2g_1-1+|\mathcal{I}|>0\\ 2g_2-1+|J|>0}}\sum_{\substack{0\leq m_1\leq 2g_1\\ 0\leq m_2\leq 2g_2}}\sum_{\substack{0\leq n_1\leq a_1+1\\ 0\leq n_2\leq a_2+1\\ 0\leq k_{L\{i\}}\leq b_{L\backslash\{i\}}}}\langle\tau_{a_1}\tau_{b_{\mathcal{I}}}\Lambda_{g_1}^\vee(1)a_{m_1}(\lambda)\rangle_{g_1} \tag{67}$$

$$\times \langle \tau_{a_2}\tau_{b_{\mathcal{J}}}\Lambda^{\vee}_{g_2}(1)a_{m_2}(\lambda)\rangle_{g_2}\Psi^{n_1}_{a_1+1}(t_i)\Psi^{n_2}_{a_2+1}(t_i)\Psi^{k_{L\setminus\{i\}}}_{b_{L\setminus\{i\}}}(t_{L\setminus\{i\}})\frac{\tau^{|k_{L\setminus\{i\}}|+n_1+n_2+l+m_1+m_2-1}}{(\tau+1)^{a_1+a_2+|k_{L\setminus\{i\}}|+4}}$$

Where we have used the notation $|b_L| = \sum_{i=1}^{l} b_i$ in above formulas. We note that (64), (65), (66) and (67) are all the functions of τ. If we expand them as the series of τ at different points, we will get some identities of Hodge integrals after recollecting the coefficients of τ at both sides of (64) = (65) + (66) + (67).

5.2. **Expansion of τ at ∞.** For $b \geq 0$, expanding at $\tau = \infty$,

$$\frac{\tau^a}{(\tau+1)^b} = \sum_{h\geq 0}\binom{-b}{h}\tau^{a-b-i}$$

for example, the term containing τ in (35),

$$\frac{\tau^{|k_L|+l+m-1}}{(\tau+1)^{|b_L|+2}} = \sum_{h\geq 0}\binom{-|b_L|-2}{h}\tau^{l+m-3+|k_L|-|b_L|-h}$$

After this expansion, it is easy to see that the largest degree of τ is $2g+l-3$ in the formulas (64), (65), (66), (67). If $F(t,\tau) \in Q[t][[\tau]]$, we will use the notation $[\tau^k]F(t,\tau)$ to mean the coefficient of τ^k in $F(t,\tau)$. By $a_{2g} = (-1)^g$ in (48) , after some calculations,

(68)

$$[\tau^{2g+l-3}]LHS = (-1)^{g+1}\sum_{b_L\geq 0}\langle\tau_{b_L}\Lambda^{\vee}_g(1)\rangle\left((2g-2+l)\Psi^{b_L}_{b_L}(t_L) + \sum_{i=1}^{l}(t_i^2-t_i)\frac{\partial}{\partial t}\Psi^{b_i}_{b_i}(t_i)\Psi^{b_{L\setminus\{i\}}}_{b_{L\setminus\{i\}}}(t_{L\setminus\{i\}})\right)$$

(69)

$$[\tau^{2g+l-3}]T_1 = (-1)^{g+1}\sum_{1\leq i<j\leq l}\sum_{\substack{a\geq 0\\ b_{L\setminus\{i,j\}}\geq 0}}\langle\tau_a\tau_{b_{L\setminus\{i,j\}}}\Lambda^{\vee}_g(1)\rangle_g\Psi^{b_{L\setminus\{i,j\}}}_{b_{L\setminus\{i,j\}}}(t_{L\setminus\{i,j\}})$$
$$\frac{(t_j-1)t_i^2\Psi^{a+1}_{a+1}(t_i) - (t_i-1)t_j^2\Psi^{a+1}_{a+1}(t_j)}{t_i-t_j}$$

(70)

$$[\tau^{2g+l-3}]T_2 = (-1)^{g+1}\frac{1}{2}\sum_{i=1}^{l}\sum_{\substack{a_1\geq 0\\ a_2\geq 0\\ b_{L\setminus\{i\}}\geq 0}}\langle\tau_{a_1}\tau_{a_2}\tau_{b_{L\setminus\{i\}}}\Lambda^{\vee}_{g-1}(1)\rangle_{g-1}\Psi^{a_1+1}_{a_1+1}(t_i)\Psi^{a_2+1}_{a_2+1}(t_i)\Psi^{b_{L\setminus\{i\}}}_{b_{L\setminus\{i\}}}(t_{L\setminus\{i\}})$$

(71)

$$[\tau^{2g+l-3}]T_3 = (-1)^{g+1}\frac{1}{2}\sum_{i=1}^{l}\sum_{\substack{a_1\geq 0\\ a_2\geq 0\\ b_{L\setminus\{i\}}\geq 0}}\sum_{\substack{g_1+g_2=g\\ \mathcal{I}\cup\mathcal{J}=L\setminus\{i\}}}\langle\tau_{a_1}\tau_{b_{\mathcal{I}}}\Lambda^{\vee}_{g_1}(1)\rangle_{g_1}\langle\tau_{a_2}\tau_{b_{\mathcal{J}}}\Lambda^{\vee}_{g_2}(1)\rangle_{g_2}\prod_{n=1}^{2}\Psi^{a_n+1}_{a_n+1}(t_i)\Psi^{b_{L\setminus\{i\}}}_{b_{L\setminus\{i\}}}(t_{L\setminus\{i\}})$$

As showed in formula (54), $\Psi^b_b(t) = \hat{\xi}_b(t)$, we have got the following corollary from (68) = (69) + (70) + (71),

Corollary 5.1.

(72)

$$\sum_{b_L\geq 0}\langle\tau_{b_L}\Lambda_g^\vee(1)\rangle\left((2g-2+l)\hat{\xi}_{b_L}(t_L)+\sum_{i=1}^{l}(t_i^2-t_i)\frac{\partial}{\partial t}\hat{\xi}_{b_i}(t_i)\hat{\xi}_{b_{L\backslash\{i\}}}(t_{L\backslash\{i\}})\right)$$
$$=\sum_{1\leq i<j\leq l}\sum_{\substack{a\geq 0\\ b_{L\backslash\{i,j\}}\geq 0}}\langle\tau_a\tau_{b_{L\backslash\{i,j\}}}\Lambda_g^\vee(1)\rangle_g\hat{\xi}_{b_{L\backslash\{i,j\}}}\frac{(t_j-1)t_i^2\hat{\xi}_{a+1}(t_i)-(t_i-1)t_j^2\hat{\xi}_{a+1}(t_j)}{t_i-t_j}$$
$$+\frac{1}{2}\sum_{i=1}^{l}\sum_{\substack{a_1\geq 0\\ a_2\geq 0\\ b_{L\backslash\{i\}}\geq 0}}\left(\langle\tau_{a_1}\tau_{a_2}\tau_{b_{L\backslash\{i\}}}\Lambda_{g-1}^\vee(1)\rangle_{g-1}+\sum_{\substack{g_1+g_2=g\\ \mathcal{I}\cup\mathcal{J}=L\backslash\{i\}}}^{stable}\langle\tau_{a_1}\tau_{b_{\mathcal{I}}}\Lambda_{g_1}^\vee(1)\rangle_{g_1}\langle\tau_{a_2}\tau_{b_{\mathcal{J}}}\Lambda_{g_2}^\vee(1)\rangle_{g_2}\right)$$
$$\times\hat{\xi}_{a_1+1}(t_i)\hat{\xi}_{a_2+1}(t_i)\hat{\xi}_{b_{L\backslash\{i\}}}(t_{L\backslash\{i\}})$$

which is just the main theorem 1.1 showed in paper [9]. Formula (72) has been applied by many people to derive DVV equation [6, 20], λ_g-conjecture [14, 20] and λ_{g-1} Hodge integral recursion [24].

Obviously, in this procedure, we can obtain a series of such corollaries. For example, calculating out the coefficients of τ^{2g+l-4} by $a_{2g}=(-1)^g$, $a_{2g-1}(\lambda)=(-1)^g g$ in (48) we have

(73)

$$[\tau^{2g+l-4}]LHS$$
$$=(-1)^{g+1}(2g-3+l)\sum_{b_L\geq 0}\langle\tau_b\Lambda_g^\vee(1)\rangle_g[(g-1-|b_L|)\Psi_{b_L}^{b_L}(t_L)+\sum_{i=1}^{l}\Psi_{b_i}^{b_i-1}(t_i)\Psi_{b_{L\backslash\{i\}}}^{b_{L\backslash\{i\}}}(t_{L\backslash\{i\}})]$$
$$+(-1)^{g+1}\sum_{b_L\geq 0}\langle\tau_{b_L}\Lambda_g^\vee(1)\rangle_g[\sum_{i=1}^{l}(t_i^2-t_i)\frac{\partial}{\partial t_i}\left((g-|b_L|-2)\Psi_{b_i}^{b_i}(t_i)+\Psi_{b_i}^{b_i-1}(t_i)\right)(\Psi_{b_{L\backslash\{i\}}}^{b_{L\backslash\{i\}}}(t_{L\backslash\{i\}})$$
$$+\sum_{j\neq i}\Psi_{b_j}^{b_j-1}(t_j)\Psi_{b_{L\backslash\{i,j\}}}^{b_{L\backslash\{i,j\}}}(t_{L\backslash\{i,j\}})]$$

(74)
$$[\tau^{2g+l-4}]T_1=(-1)^{g+1}\sum_{1\leq i<j\leq l}\sum_{\substack{a\geq 0\\ b_{L\backslash\{i,j\}}\geq 0}}\langle\tau_a\tau_{b_{L\backslash\{i,j\}}}\Lambda_g^\vee(1)\rangle_g\Psi_{b_{L\backslash\{i,j\}}}^{b_{L\backslash\{i,j\}}}(t_{L\backslash\{i,j\}})$$
$$(\frac{(t_j-1)t_i^2[\Psi_{a+1}^a(t_i)+(g-a-|b_{L\backslash\{i,j\}}|-3+\frac{1}{t_i})\Psi_{a+1}^{a+1}(t_i)]}{t_i-t_j}$$
$$-\frac{(t_i-1)t_j^2[\Psi_{a+1}^a(t_j)+(g-a-|b_{L\backslash\{i,j\}}|-3+\frac{1}{t_j})\Psi_{a+1}^{a+1}(t_j)]}{t_i-t_j})$$

(75)

$$[\tau^{2g+l-4}]T_2=(-1)^{g+1}\frac{1}{2}\sum_{i=1}^{l}\sum_{\substack{a_1\geq 0\\ a_2\geq 0\\ b_{L\backslash\{i\}}\geq 0}}\langle\tau_{a_1}\tau_{a_2}\tau_{b_{L\backslash\{i\}}}\Lambda_{g-1}^\vee(1)\rangle_{g-1}[(g-a_1-a_2-|b_{L\backslash\{i,j\}}|-4)$$

$$\cdot \Psi_{a_1+1}^{a_1+1}(t_i)\Psi_{a_2+1}^{a_2+1}(t_i)+\Psi_{a_1}^{a_1+1}(t_i)\Psi_{a_2+1}^{a_2+1}(t_i)+\Psi_{a_1+1}^{a_1+1}(t_i)\Psi_{a_2+1}^{a_2}(t_i)]\Psi_{b_{L\setminus\{i\}}}^{b_{L\setminus\{i\}}}(t_{L\setminus\{i\}})$$
$$+(-1)^{g+1}\frac{1}{2}\sum_{i=1}^{l}\sum_{\substack{a_1\geq 0\\ a_2\geq 0\\ b_{L\setminus\{i\}}\geq 0}}\langle\tau_{a_1}\tau_{a_2}\tau_{b_{L\setminus\{i\}}}\Lambda_{g-1}^{\vee}(1)\rangle_{g-1}\Psi_{a_1}^{a_1+1}(t_i)\Psi_{a_2+1}^{a_2+1}(t_i)\sum_{j\neq i}\Psi_{b_j}^{b_j-1}(t_j)\Psi_{b_{L\setminus\{i,j\}}}^{b_{L\setminus\{i,j\}}}(t_{L\setminus\{i,j\}})$$

(76)

$$[\tau^{2g+l-4}]T_3$$
$$=(-1)^{g+1}\frac{1}{2}\sum_{i=1}^{l}\sum_{\substack{a_1\geq 0\\ a_2\geq 0\\ b_{L\setminus\{i\}}\geq 0}}\sum_{\substack{g_1+g_2=g\\ \mathcal{I}\cup\mathcal{J}=L\setminus\{i\}}}^{stable}\langle\tau_{a_1}\tau_{b_{\mathcal{I}}}\Lambda_{g_1}^{\vee}(1)\rangle_{g_1}\langle\tau_{a_2}\tau_{b_{\mathcal{J}}}\Lambda_{g_2}^{\vee}(1)\rangle_{g_2}[(g-a_1-a_2-|b_{L\setminus\{i,j\}}|-4)$$
$$\cdot \Psi_{a_1+1}^{a_1+1}(t_i)\Psi_{a_2+1}^{a_2+1}(t_i)+\Psi_{a_1}^{a_1+1}(t_i)\Psi_{a_2+1}^{a_2+1}(t_i)+\Psi_{a_1+1}^{a_1+1}(t_i)\Psi_{a_2+1}^{a_2}(t_i)]\Psi_{b_{L\setminus\{i\}}}^{b_{L\setminus\{i\}}}(t_{L\setminus\{i\}})$$
$$+(-1)^{g+1}\frac{1}{2}\sum_{i=1}^{l}\sum_{\substack{a_1\geq 0\\ a_2\geq 0\\ b_{L\setminus\{i\}}\geq 0}}\sum_{\substack{g_1+g_2=g\\ \mathcal{I}\cup\mathcal{J}=L\setminus\{i\}}}^{stable}\langle\tau_{a_1}\tau_{b_{\mathcal{I}}}\Lambda_{g_1}^{\vee}(1)\rangle_{g_1}$$
$$\langle\tau_{a_2}\tau_{b_{\mathcal{J}}}\Lambda_{g_2}^{\vee}(1)\rangle_{g_2}\Psi_{a_1}^{a_1+1}(t_i)\Psi_{a_2+1}^{a_2+1}(t_i)\sum_{j\neq i}\Psi_{b_j}^{b_j-1}(t_j)\Psi_{b_{L\setminus\{i,j\}}}^{b_{L\setminus\{i,j\}}}(t_{L\setminus\{i,j\}})$$

Thus, by (73) = (74) + (75) + (76), we get a Hodge integral identity. However, unfortunately, the only Hodge integral contained in this identity is $\langle\tau_{b_L}\Lambda_g^{\vee}(1)\rangle_g$ because only the terms $a_{2g}(\lambda)=(-1)^g$ and $a_{2g-1}(\lambda)=(-1)^g g$ was involved in the above calculation which contains no more new information.

In order to get the Hodge integral identity with more than one λ-class, we need to compare the coefficients of τ^{2g+l-5} with the same procedure, we will arrive at a Hodge integral identity which involves $\langle\tau_{b_L}\Lambda_g^{\vee}(1)\rangle_g$ and $\langle\tau_{b_L}\Lambda_g^{\vee}(1)\lambda_1\rangle_g$, because $a_{2g-2}(\lambda)=(-1)^g\left(\frac{g(g-1)}{2}-\lambda_1\right)$.

From such Hodge integral identity, we can calculate all the Hodge integral of type $\langle\tau_{b_L}\lambda_k\lambda_1\rangle_g$, $k=1,..,g$. However, more terms will appear in the calculation of coefficients of τ^{2g+l-5} which make the computation very complicated. so far, we have not written down the explicit formula to calculate the Hodge integral of type $\langle\tau_{b_L}\lambda_k\lambda_1\rangle_g$. But when $k=g$, in the following subsection 5.4, we will give an explicit formula for $\langle\tau_{b_L}\lambda_g\lambda_1\rangle_g$.

5.3. **Expansion of τ at 0.** Now, let us consider the expansion of $\frac{\tau^a}{(\tau+1)^b}$ at $\tau=0$, we have

$$\frac{\tau^a}{(\tau+1)^b}=\sum_{i\geq 0}\binom{-b}{i}\tau^{a+i}$$

After this expansion, it is easy to show that the lowest degree of τ in (64) is $l-2$. In fact, by $a_0(\lambda)=(-1)^g\lambda_g\Lambda_g^{\vee}(-1)$ show in (48) and $\Lambda_g^{\vee}(1)\Lambda_g^{\vee}(-1)=(-1)^g$, we have

$$[\tau^{l-2}]LHS=-(l-1)\sum_{b_L\geq 0}\langle\tau_{b_L}\lambda_g\rangle_g\Psi_{b_L}^0(t_L) \tag{77}$$

(78)

$$[\tau^{l-2}]T_1 = -\sum_{1\le i<j\le l}\sum_{\substack{a\ge 0\\ b_{L\setminus\{i,j\}}\ge 0}}\langle\tau_a\tau_{b_{L\setminus\{i,j\}}}\lambda_g\rangle_g\Psi^0_{b_{L\setminus\{i,j\}}}(t_{L\setminus\{i,j\}})\frac{(t_j-1)t_i^2\Psi^0_{a+1}(t_i)-(t_i-1)t_j^2\Psi^0_{a+1}(t_j)}{t_i-t_j}$$

$$[\tau^{l-2}]T_2 = [\tau^{l-2}]T_3 = 0 \tag{79}$$

Thus, we get

Corollary 5.2.

$$\sum_{b_L\ge 0}\langle\tau_{b_L}\lambda_g\rangle_g\Psi^0_{b_L}(t_L) = \frac{1}{l-1}\sum_{1\le i<j\le l}\sum_{\substack{a\ge 0\\ b_{L\setminus\{i,j\}}\ge 0}}\langle\tau_a\tau_{b_{L\setminus\{i,j\}}}\lambda_g\rangle_g\Psi^0_{b_{L\setminus\{i,j\}}}(t_{L\setminus\{i,j\}}) \cdot\frac{(t_j-1)t_i^2\Psi^0_{a+1}(t_i)-(t_i-1)t_j^2\Psi^0_{a+1}(t_j)}{t_i-t_j} \tag{80}$$

Here, we introduce two notations. Let $g(x_1,..,x_l)=\sum_{i_k\ge 0}a_{i_1i_2\cdots i_l}x_1^{i_1}\cdots x_l^{i_l}\in Q[x_1,..,x_l]$,

$$F_d(g(x_1,\cdots,x_l)) = \sum_{\sum_{k=1}^l i_k=d}a_{i_1i_2\cdots i_l}x_1^{i_1}\cdots x_l^{i_l}$$

and

$$[x_1^{j_1}\cdots x_l^{j_l}]g(x_1,\cdots,x_l) = a_{j_1j_2\cdots j_l}.$$

Now, let us use these two operator to corollary 5.2. By (60), we have

$$F_{2g-3+2l}(LHS \text{ of } (80)) = (l-1)\sum_{b_L\ge 0}\langle\tau_{b_L}\lambda\rangle_g b_L!y_L^{b_L+1}$$

$$F_{2g-3+2l}(RHS \text{ of } (80)) = \sum_{1\le i<j\le l}\sum_{\substack{a\ge 0\\ b_L\ge 0}}\langle\tau_{b_{L\setminus\{i,j\}}}\lambda_g\rangle_g(a+1)!b_{L\setminus\{i,j\}}!t_{L\setminus\{i,j\}}^{b_{L\setminus\{i,j\}}+1}\frac{t_jt_i^{a+4}-t_it_j^{a+4}}{t_i-t_j}$$

Then,

$$[t_L^{b_L+1}]F_{2g-3+2l}(LHS \text{ of } (80)) = (l-1)\langle\tau_{b_L}\lambda\rangle_g b_L!$$

$$[t_L^{b_L+1}]F_{2g-3+2l}(RHS \text{ of } (80)) = \sum_{1\le i<j\le l}\langle\tau_{b_i+b_j-1}\tau_{b_{L\setminus\{i,j\}}}\lambda_g\rangle_g(b_i+b_j)!b_{L\setminus\{i,j\}}!$$

Hence, we have

$$\langle\tau_{b_L}\lambda\rangle_g = \frac{1}{l-1}\sum_{1\le i<j\le l}\langle\tau_{b_i+b_j-1}\tau_{b_{L\setminus\{i,j\}}}\lambda_g\rangle_g\frac{(b_i+b_j)!}{b_i!b_j!}$$

Induction on l, we get the λ_g integral,

$$\langle\tau_{b_L}\lambda\rangle_g = \binom{2g-3+l}{b_L}c_g$$

Now, let us calculate the next degree of τ. By $a_0(\lambda)=(-1)^g\lambda_g\Lambda_g^\vee(-1)$ showed in (48) and $\Lambda_g^\vee(1)\Lambda_g^\vee(-1)=(-1)^g$, we have

(81)
$$[\tau^{l-1}]LHS=-\sum_{b_L\geq 0}\langle\tau_{b_L}\lambda_g\rangle_g l\left(-(|b_L|+1)\Psi^0_{b_L}(t_L)+\sum_{j=1}\Psi^1_{b_j}(t_j)\Psi^0_{b_{L\setminus\{j\}}}(t_{L\setminus\{j\}})\right)$$
$$-\sum_{b_L\geq 0}\langle\tau_{b_L}\lambda_g\rangle_g\sum_{i=1}^{l}(t_i^2-t_i)\frac{\partial}{\partial t_i}\Psi^0_{b_i}(t_i)\Psi^0_{b_{L\setminus\{i\}}}(t_{L\setminus\{i\}})-\sum_{b_L\geq 0}\langle\tau_{b_L}\Lambda_g^\vee(1)a_1(\lambda)\rangle_g l\Psi^0_{b_L}(t_L)$$

(82)
$$[\tau^{l-1}]T_1$$
$$=-\sum_{1\leq i<j\leq l}\sum_{\substack{a\geq 0\\ b_{L\setminus\{i,j\}}\geq 0}}\langle\tau_a\tau_{b_{L\setminus\{i,j\}}}\lambda_g\rangle_g\Psi^0_{b_{L\setminus\{i,j\}}}(t_{L\setminus\{i,j\}})[\frac{(t_j-1)t_i\left(\Psi^1_{a+1}(t_i)+(t_i-|b_{L\setminus\{i,j\}}|-a-3)\Psi^0_{a+1}(t_i)\right)}{t_i-t_j}$$
$$-\frac{(t_i-1)t_j\left(\Psi^1_{a+1}(t_j)+(t_j-|b_{L\setminus\{i,j\}}|-a-3)\Psi^0_{a+1}(t_j)\right)}{t_i-t_j}]$$
$$-\sum_{1\leq i<j\leq l}\sum_{\substack{a\geq 0\\ b_{L\setminus\{i,j\}}\geq 0}}\langle\tau_a\tau_{b_{L\setminus\{i,j\}}}\lambda_g\rangle_g\sum_{r\neq i,j}\Psi^1_{b_r}(t_r)\Psi^0_{b_{L\setminus\{i,j\}}}(t_{L\setminus\{i,j\}})\frac{(t_j-1)t_i\Psi^0_{a+1}(t_i)-(t_i-1)t_j\Psi^0_{a+1}(t_j)}{t_i-t_j}$$
$$-\sum_{1\leq i<j\leq l}\sum_{\substack{a\geq 0\\ b_{L\setminus\{i,j\}}\geq 0}}\langle\tau_a\tau_{b_{L\setminus\{i,j\}}}\Lambda_g^\vee(1)a_1(\lambda)\rangle_g\Psi^0_{b_{L\setminus\{i,j\}}}(t_{L\setminus\{i,j\}})\frac{(t_j-1)t_i\Psi^0_{a+1}(t_i)-(t_i-1)t_j\Psi^0_{a+1}(t_j)}{t_i-t_j}$$

(83)
$$[\tau^{l-1}]T_2=0$$

(84)
$$[\tau^{l-1}]T_3=-\frac{1}{2}\sum_{i=1}^{l}\sum_{\substack{a_1\geq 0\\ a_2\geq 0\\ b_{L\setminus\{i\}}\geq 0}}\sum_{\substack{stable\\ g_1+g_2=g\\ |\mathcal{I}|\cup|\mathcal{J}|=L\setminus\{i\}}}\langle\tau_{a_1}\tau_{b_{\mathcal{I}}}\lambda_{g_1}\rangle_{g_1}\langle\tau_{a_2}\tau_{b_{\mathcal{J}}}\lambda_{g_2}\rangle_{g_2}\Psi^0_{a_1+1}(t_i)\Psi^0_{a_2+1}(t_i)\Psi^0_{b_{L\setminus\{i\}}}(t_{L\setminus\{i\}})$$

Recall formula (48), $a_1(\lambda)=\sum_{m=1}^{g}m\lambda_{g-m}\lambda_g-(-1)^g\Lambda_g^\vee(-1)\lambda_{g-1}$ and Mumford's relation $\Lambda_g^\vee(1)\Lambda_g^\vee(-1)=(-1)^g$. We can write

(85)
$$\Lambda_g^\vee(1)a_1(\lambda)=\sum_{d=g-1}^{3g-3}P_d(\lambda)$$

where $P_d(\lambda)$ is some combinatoric of λ-class with degree d. For example

(86)
$$P_{g-1}(\lambda)=\lambda_{g-1},P_g(\lambda)=g\lambda_g,P_{g+1}(\lambda)=-\lambda_g\lambda_1,\cdots,P_{3g-3}(\lambda)=(-1)^{g+1}\lambda_g\lambda_{g-1}\lambda_{g-2}$$

By the above calculation, substituting (85) to the identity (81) = (82) + (84), we have

Corollary 5.3.

(87)

$$\sum_{b_L\geq 0}\langle\tau_{b_L}\lambda_g\rangle_g l\left(-(|b_L|+1)\Psi^0_{b_L}(t_L)+\sum_{j=1}\Psi^1_{b_j}(t_j)\Psi^0_{b_{L\setminus\{j\}}}(t_{L\setminus\{j\}})\right)$$

$$+\sum_{b_L\geq 0}\langle\tau_{b_L}\lambda_g\rangle_g\sum_{i=1}^{l}(t_i^2-t_i)\frac{\partial}{\partial t_i}\Psi^0_{b_i}(t_i)\Psi^0_{b_{L\setminus\{i\}}}(t_{L\setminus\{i\}})+\sum_{b_L\geq 0}\langle\tau_{b_L}\sum_{d=g-1}^{3g-3}P_d(\lambda)\rangle_g l\Psi^0_{b_L}(t_L)$$

$$=\sum_{1\leq i<j\leq l}\sum_{\substack{a\geq 0\\ b_{L\setminus\{i,j\}}\geq 0}}\langle\tau_a\tau_{b_{L\setminus\{i,j\}}}\lambda_g\rangle_g\Psi^0_{b_{L\setminus\{i,j\}}}(t_{L\setminus\{i,j\}})[\frac{(t_j-1)t_i\left(\Psi^1_{a+1}(t_i)+(t_i-|b_{L\setminus\{i,j\}}|-a-3)\Psi^0_{a+1}(t_i)\right)}{t_i-t_j}$$

$$-\frac{(t_i-1)t_j\left(\Psi^1_{a+1}(t_j)+(t_j-|b_{L\setminus\{i,j\}}|-a-3)\Psi^0_{a+1}(t_j)\right)}{t_i-t_j}]$$

$$+\sum_{1\leq i<j\leq l}\sum_{\substack{a\geq 0\\ b_{L\setminus\{i,j\}}\geq 0}}\langle\tau_a\tau_{b_{L\setminus\{i,j\}}}\lambda_g\rangle_g\sum_{r\neq i,j}\Psi^1_{b_r}(t_r)\Psi^0_{b_{L\setminus\{i,j\}}}(t_{L\setminus\{i,j\}})\frac{(t_j-1)t_i\Psi^0_{a+1}(t_i)-(t_i-1)t_j\Psi^0_{a+1}(t_j)}{t_i-t_j}$$

$$+\sum_{1\leq i<j\leq l}\sum_{\substack{a\geq 0\\ b_{L\setminus\{i,j\}}\geq 0}}\langle\tau_a\tau_{b_{L\setminus\{i,j\}}}\sum_{d=g-1}^{3g-3}P_d(\lambda)\rangle_g\Psi^0_{b_{L\setminus\{i,j\}}}(t_{L\setminus\{i,j\}})\frac{(t_j-1)t_i\Psi^0_{a+1}(t_i)-(t_i-1)t_j\Psi^0_{a+1}(t_j)}{t_i-t_j}$$

$$+\frac{1}{2}\sum_{i=1}^{l}\sum_{\substack{a_1\geq 0\\ a_2\geq 0\\ b_{L\setminus\{i\}}\geq 0}}\sum_{\substack{stable\\ g_1+g_2=g\\ |\mathcal{I}|\cup|\mathcal{J}|=L\setminus\{i\}}}\langle\tau_{a_1}\tau_{b_{\mathcal{I}}}\lambda_{g_1}\rangle_{g_1}\langle\tau_{a_2}\tau_{b_{\mathcal{J}}}\lambda_{g_2}\rangle_{g_2}\Psi^0_{a_1+1}(t_i)\Psi^0_{a_2+1}(t_i)\Psi^0_{b_{L\setminus\{i\}}}(t_{L\setminus\{i\}})$$

Since $\langle\tau_{b_L}\lambda_g\rangle_g=\binom{2g-3+l}{b_L}c_g$, thus all the Hodge integral of type

$$\langle\tau_{b_L}\sum_{d=g-1}^{3g-3}P_d(\lambda)\rangle_g \tag{88}$$

can be calculated from formula (87) in corollary 4.3.

5.4. **Explicit expression for Hodge integral $\langle\tau_{b_L}\lambda_g\lambda_1\rangle_g$.** Although corollary 4.3 says that all the Hodge integral (88) can be computed, it is not easy to write down their explicit formula. In this subsection, we will show how to obtain an explicit formula for $\langle\tau_{b_L}\lambda_g\lambda_1\rangle_g$.

Both sides of (87) belong to $Q[t_1,..,t_l]$. We have known that $\langle\tau_{b_L}\lambda_g\rangle_g=\binom{2g-3+l}{b_L}c_g$, and $\langle\tau_{b_L}\lambda_{g-1}\rangle_g$ can also be calculated easily from a recursion formula showed in [24]. Thus, we have

Theorem 5.4. *If $\sum_{i=1}^{l}b_i=2g-4+l$, there exists a constant $C(g,l,b_1,..,b_l)$ related to $g,l,b_1,..,b_l$, such that*

$$\langle\tau_{b_L}\lambda_g\lambda_1\rangle_g=\frac{1}{l}\sum_{1\leq i<j\leq l}\langle\tau_{b_i+b_j-1}\tau_{b_{L\setminus\{i,j\}}}\lambda_g\lambda_1\rangle_g\frac{(b_i+b_j)!}{b_i!b_j!}+C(g,l,b_1,..b_l) \tag{89}$$

$C(g,l,b_1,..,b_l)$ is a very complicated combinatoric constant which is given at Appendix B.

Proof. Taking all the terms with degree $2g+2l-4$ in formula (87). we have

$$F_{2g+2l-4}(LHS\text{ of }(87))=-l\sum_{b_L\geq 0}\langle\tau_{b_L}\lambda_g\lambda_1\rangle_g b_L!t_L^{b_L+1}+G_1(t_1,..,t_l) \tag{90}$$

$$F_{2g+2l-4}(RHS \text{ of } (87)) = -\sum_{1\le i<j\le l}\sum_{\substack{a\ge 0\\ b_{L\setminus\{i,j\}}\ge 0}} \langle \tau_a \tau_{b_{L\setminus\{i,j\}}}\lambda_g\lambda_1\rangle_g \tag{91}$$

$$\cdot\, b_{L\setminus\{i,j\}}!(a+1)!\sum_{m=0}^{a+1} t_{L\setminus\{i,j\}}^{b_{L\setminus\{i,j\}}+1} t_i^{a+2-m} t_j^{m+1} + G_2(t_1,..,t_l)$$

where $G_i(t_1,..,t_l)=\sum_{|d_L|=2g+2l-4} C_i(g,l,d_1,..,d_l)t_L^{d_L}\in\mathbb{Q}[t_1,..t_l]$, $C_i(g,l,d_1,..,d_l)$ is a combinatoric constant related to $g,l,d_1,..,d_l$. Thus, we have

$$\sum_{b_L\ge 0}\langle\tau_{b_L}\lambda_g\lambda_1\rangle_g b_L! t_L^{b_L+1} = \frac{1}{l}\sum_{1\le i<j\le l}\sum_{\substack{a\ge 0\\ b_{L\setminus\{i,j\}}\ge 0}} \langle \tau_a \tau_{b_{L\setminus\{i,j\}}}\lambda_g\lambda_1\rangle_g \tag{92}$$

$$\cdot\, b_{L\setminus\{i,j\}}!(a+1)!\sum_{m=0}^{a+1} t_{L\setminus\{i,j\}}^{b_{L\setminus\{i,j\}}+1} t_i^{a+2-m} t_j^{m+1} + G(t_1,..,t_l)$$

where $G(t_1,..,t_l)=\sum_{|d_L|=2g+2l-4}\hat{C}(g,l,b_1,..,b_l)t_L^{d_L}=\frac{1}{l}(G_1(t_1,..t_l)-G_2(t_1,..,t_l))$

After taking the coefficients of (92),

$$\langle\tau_{b_L}\lambda_g\lambda_1\rangle_g = \frac{1}{l}\sum_{1\le i<j\le l}\langle\tau_{b_i+b_j-1}\tau_{b_{L\setminus\{i,j\}}}\lambda_g\lambda_1\rangle_g\frac{(b_i+b_j)!}{b_i!b_j!}+\frac{\hat{C}(g,l,b_1+1,..,b_l+1)}{b_L!} \tag{93}$$

Let $C(g,l,b_1,..,b_l)=\frac{\hat{C}(g,l,b_1+1,..,b_l+1)}{b_L!}$, thus, the theorem is proved. □

We know that the initial value $\langle\tau_{2g-3}\lambda_g\lambda_1\rangle_g=\frac{1}{12}[g(2g-3)b_g+b_1b_{g-1}]$ has been computed by Y. Li [15]. Therefore, from the recursion formula (62), we can compute out all the Hodge integral of type $\langle\tau_{b_L}\lambda_g\lambda_1\rangle_g$.

6. Conclusion

The original cut-and-join equation of Mariño-Vafa formula is a complicated formula about the partition which can be changed to a polynomial identity with the help of the Laplace transform in this paper or the symmetrization method in [4]. This polynomial identity has some applications because it is manageable. We derived some corollaries about the Hodge integral identities. In fact, we have found an algorithm to calculate the Hodge integral appearing in Mariño-Vafa formula.

Step1, expanding the τ at some special point, and collect the corresponding level of τ in theorem 1.1.

Step2, taking certain degrees of t_L, we will get the corresponding Hodge integral appearing in Mariño-Vafa formula.

In this paper, we only calculate four cases using step1 and calculate out a new Hodge integral using Step2 from the result of last case after doing step1. We don't consider other cases here, because the computation will be more complicated. We hope that this algorithm can be made to a computer program.

We note that, the recursion formulas to calculate $\langle\tau_{b_L}\lambda_{g-1}\rangle_g$ [24] and $\langle\tau_{b_L}\lambda_g\lambda_1\rangle_g$ in theorem 1.5 have the similar recursion structure. For a given g, Hodge integral with l-point will reduce to 1-point after only l times recursions. Thus, it is a very effective recursion relation. In fact, many Hodge integral recursions will have this type structure if they are calculated out by above algorithm. Thus it is interesting to consider the following combinatoric problem.

Let $Q_l(b_L)=Q_l(b_1,b_2,..,b_l)$ be a symmetric function on $(b_1,b_2,..,b_l)\in(\mathbb{Z}^+)^l$ which is defined by the following recursion relation.

For given g, k,

$$Q_1(b) = \begin{cases} constant, & b = 3g-2-k, \\ 0, & others. \end{cases} \tag{94}$$

$$Q_l(b_L) = \begin{cases} \sum_{1\le i<j\le l} Q_{l-1}(b_i+b_j-1, b_{L\setminus\{i,j\}})A_l(b_L) + B_l(b_L) & \sum_{i=1}^l b_i = 3g-3+l-k, \\ 0, & others. \end{cases} \tag{95}$$

Where $A_l(b_L) = A_l(b_1,..,b_l)$ and $B_l(b_L) = B_l(b_1,..,b_l)$ are some fixed functions defined on $(b_1, b_2, .., b_l) \in (\mathbb{Z}^+)^l$.

It is interesting to study the properties of $Q_l(b_L)$ defined above. We hope that this combinatoric structure of Hodge integral will be studied further in future.

7. Appendix

7.1. **Appendix A.** In this appendix, we will show how to derive our main result in theorem 3.4 from the symmetrized cut-and-join equation of the Mariño-Vafa formula [4].

when $b \ge 0$, Let

$$\Psi_b(y;\tau) = ((y^2-y)(\frac{y\tau+1}{\tau+1})\frac{d}{dy})^b(\frac{y-1}{\tau+1}) = \sum_{k=0}^{b} \frac{\tau^k}{(\tau+1)^{b+1}}\psi_b^k(y)$$

Then the symmetrized partition function of Mariño-Vafa formula can be written as

$$W_{g,n}(y_1,..,y_n;\tau) = -(\tau(\tau+1))^{n-1} \sum_{\substack{b_i\ge 0 \\ i=1,..,n}} \langle \prod_{i=1}^n \tau_{b_i}\Gamma_g(\tau)\rangle_g \prod_{i=1}^n \Psi_{b_i}(y_i;\tau) \tag{96}$$

The main result of [4], i.e Theorem 3 in that paper, says

Theorem 7.1. *The symmetrized generating series $W_{g,n}(y_1,..,y_n;\tau)$ is a polynomial in the y_i variables of total degree $6g-6+3n$ and satisfies the symmetrized cut–and-join equation.*

$$(\frac{\partial}{\partial\tau} + \sum_{l=1}^n \frac{y_l^2-y_l}{\tau+1}\frac{\partial}{\partial y_l})W_{g,n}(y_1,..,y_n;\tau) = T_1+T_2+T_3$$

where

$$T_1 = sym_{1,1}^y \frac{y_1(y_2-1)}{y_1-y_2}(\frac{y_1\tau+1}{\tau+1})D_{y_1}^\tau W_{g,n-1}(y_1,y_3,..,y_n;\tau)$$

$$T_2 = -\frac{1}{2}\sum_{l=1}^n D_{y_l}^\tau D_{y_{n+1}}^\tau W_{g-1,n+1}(y_1,..,y_n,y_{n+1};\tau)|_{y_{n+1}=y_l}$$

$$T_3 = -\frac{1}{2} \sum_{\substack{g_1+g_2=g \\ n_1+n_2=n+1 \\ 2g_1-2+n_1>0 \\ 2g_2-2+n_2>0}} sym_{1,n_1-1}^y D_{y_1}^\tau W_{g_1,n_1}(y_1,..,y_{n_1};\tau) D_{y_1}^\tau W_{g_2,n_2}(y_1,y_{n_1+1},..,y_n;\tau)$$

We need to explain the notations appear in above theorem. Where $D_y^\tau = (y^2-y)(\frac{y\tau+1}{\tau+1})\frac{\partial}{\partial y}$. For $i,j\ge 0$, $i+j\le n$, let $sym_{i,j}^x$ be the mapping, applied to a series in $x_1,..,x_n$, given by

$$sym_{i,j}^x f(x_1,..,x_n) = \sum_{\mathcal{R},\mathcal{S},\mathcal{T}} f(\mathbf{x}_\mathcal{R}, \mathbf{x}_\mathcal{S}, \mathbf{x}_\mathcal{T})$$

where the sum is over all ordered partitions $(\mathcal{R},\mathcal{S},\mathcal{T})$ *of* $\{1,2,..,n\}$, $\mathcal{R}=\{x_{r_1},..,x_{r_i}\}$, $\mathcal{S}=\{x_{s_1},..,x_{s_j}\}$, $\mathcal{T}=\{x_{t_1},..,x_{t_{n-i-j}}\}$ *and* $(\mathbf{x}_{\mathcal{R}},\mathbf{x}_{\mathcal{S}},\mathbf{x}_{\mathcal{T}})=(x_{r_1},..,x_{r_i},x_{s_1},..,x_{s_j},x_{t_1},..,x_{t_{n-i-j}})$, *where* $r_1<\cdots<r_i$, $s_1<\cdots<s_j$, *and* $t_1<\cdots<t_{n-i-j}$.

Then, by equation (96), we have

$$LHS=-(\tau^2+\tau)^{n-2}\sum_{\substack{b_i\geq 0\\ i=1,..,n}}\left((n-1)(2\tau+1)\langle\prod_{i=1}^n\tau_{b_i}\Gamma_g(\tau)\rangle_g+(\tau^2+\tau)\langle\prod_{i=1}^n\tau_{b_i}\frac{d}{d\tau}\Gamma_g(\tau)\rangle_g\right)\prod_{i=1}^n\Psi_{b_i}(y_i;\tau)$$

$$-(\tau^2+\tau)^{n-1}\sum_{\substack{b_i\geq 0\\ i=1,2,..,n}}\langle\prod_{i=1}^n\tau_{b_i}\Gamma_g(\tau)\rangle_g\sum_{l=1}^n(\frac{\partial}{\partial\tau}\Psi_{b_l}(y_l;\tau)+\frac{1}{y_l\tau+1}\Psi_{b_l+1}(y_l;\tau))\prod_{\substack{k=1\\ k\neq l}}^n\Psi_{b_k}(y_k;\tau)$$

$$T_1=-\frac{(\tau^2+\tau)^{n-2}}{\tau+1}\sum_{1\leq i<j\leq n}\sum_{\substack{a\geq 0\\ b_k\geq 0\\ k\neq i,j}}\langle\tau_a\prod_{\substack{k=1\\ k\neq i,j}}^n\tau_{b_k}\Gamma_g(\tau)\rangle_g\prod_{\substack{k=1\\ k\neq i,j}}^n\Psi_{b_k}(y_k;\tau)$$
$$\times\frac{(y_j-1)(y_i^2\tau+y_i)\Psi_{a+1}(y_i;\tau)-(y_i-1)(y_j^2\tau+y_j)\Psi_{a+1}(y_j;\tau)}{y_i-y_j}$$

$$T_2=\frac{1}{2}(\tau^2+\tau)^n\sum_{i=1}^n\sum_{\substack{a_1\geq 0\\ a_2\geq 0\\ b_k\geq 0,k\neq i}}\langle\tau_{a_1}\tau_{a_2}\prod_{\substack{k=1\\ k\neq i}}^n\tau_{b_k}\Gamma_{g-1}(\tau)\rangle_{g-1}\Psi_{a_1+1}(y_i;\tau)\Psi_{a_2+1}(y_i;\tau)\prod_{\substack{k=1\\ k\neq i}}^n\Psi_{b_k}(y_k;\tau)$$

$$T_3=-\frac{(\tau^2+\tau)^{n-1}}{2}\sum_{i=1}^n\sum_{\substack{a_1\geq 0\\ a_2\geq 0\\ b_k\geq 0,k\neq i}}\sum_{\substack{g_1+g_2=g\\ \mathcal{I}\coprod\mathcal{J}=\{1,..,\hat{i},..n\}\\ 2g_1-1+|\mathcal{I}|>0\\ 2g_2-1+|\mathcal{J}|>0}}\langle\tau_{a_1}\prod_{k\in\mathcal{I}}\tau_{b_k}\Gamma_{g_1}(\tau)\rangle_{g_1}$$
$$\times\langle\tau_{a_2}\prod_{k\in\mathcal{J}}\tau_{b_k}\Gamma_{g_2}(\tau)\rangle_{g_2}\Psi_{a_1}(y_i;\tau)\Psi_{a_2}(y_i;\tau)\prod_{\substack{k=1\\ k\neq i}}^n\Psi_{b_i}(y_i;\tau)$$

Then, by theorem 5.1, we arrive our main results in section 3, i.e. theorem 3.4.

7.2. **Appendix B.** In this appendix, we will calculate the constant $C(g,l,b_1,..,b_l)$ which appears in theorem 1.5. The computation is verbose and boring.

Let us write the formula (77) of Corollary 5.3 as

(97)
$$L_1+L_2+L_3+L_4+L_5+L_6+L_7$$
$$=R_1+R_2+R_3+R_4+R_5+R_6+R_7+R_8+R_9+R_{10}+R_{11}+R_{12}+R_{13}+R_{14}+R_{15}+R_{16}$$

where

$$L_1=\sum_{b_L\geq 0}\langle\tau_{b_L}\lambda_g\rangle_g l(g-|b_L|-1)\Psi^0_{b_L}(t_L)$$

$$L_2=\sum_{b_L\geq 0}\langle\tau_{b_L}\lambda_g\rangle_g l\sum_{j=1}^l\Psi^1_{b_j}(t_j)\Psi^0_{b_{L\setminus\{j\}}}(t_{L\setminus\{j\}})$$

$$L_3 = \sum_{b_L\geq 0}\langle \tau_{b_L}\lambda_g\rangle_g \sum_{j=1}^{l} t_j^2\frac{\partial}{\partial t_j}\Psi^0_{b_j}(t_j)\Psi^0_{b_{L\setminus\{j\}}}(t_{L\setminus\{j\}})$$

$$L_4 = -\sum_{b_L\geq 0}\langle \tau_{b_L}\lambda_g\rangle_g \sum_{j=1}^{l} t_j\frac{\partial}{\partial t_j}\Psi^0_{b_j}(t_j)\Psi^0_{b_{L\setminus\{j\}}}(t_{L\setminus\{j\}})$$

$$L_5 = -\sum_{b_L\geq 0}\langle \tau_{b_L}\lambda_{g-1}\rangle_g l\Psi^0_{b_L}(t_L)$$

$$L_6 = -\sum_{b_L\geq 0}\langle \tau_{b_L}\lambda_g\lambda_1\rangle_g l\Psi^0_{b_L}(t_L)$$

$$L_7 = \sum_{b_L\geq 0}\langle \tau_{b_L}\sum_{d=g+2}^{3g-3} P_d(\lambda)\rangle_g l\Psi^0_{b_L}(t_L)$$

$$R_1 = \sum_{1\leq i<j\leq l}\sum_{\substack{a\geq 0\\ b_{L\setminus\{i,j\}}\geq 0}}\langle \tau_a\tau_{b_{L\setminus\{i,j\}}}\lambda_g\rangle_g\Psi^0_{b_{L\setminus\{i,j\}}}(t_{L\setminus\{i,j\}})\frac{t_jt_i\Psi^1_{a+1}(t_i)-t_it_j\Psi^1_{a+1}(t_j)}{t_i-t_j}$$

$$R_2 = \sum_{1\leq i<j\leq l}\sum_{\substack{a\geq 0\\ b_{L\setminus\{i,j\}}\geq 0}}\langle \tau_a\tau_{b_{L\setminus\{i,j\}}}\lambda_g\rangle_g\Psi^0_{b_{L\setminus\{i,j\}}}(t_{L\setminus\{i,j\}})\frac{-t_i\Psi^1_{a+1}(t_i)+t_j\Psi^1_{a+1}(t_j)}{t_i-t_j}$$

$$R_3 = \sum_{1\leq i<j\leq l}\sum_{\substack{a\geq 0\\ b_{L\setminus\{i,j\}}\geq 0}}\langle \tau_a\tau_{b_{L\setminus\{i,j\}}}\lambda_g\rangle_g\Psi^0_{b_{L\setminus\{i,j\}}}(t_{L\setminus\{i,j\}})\frac{t_jt_i^2\Psi^0_{a+1}(t_i)-t_it_j^2\Psi^0_{a+1}(t_j)}{t_i-t_j}$$

$$R_4 = \sum_{1\leq i<j\leq l}\sum_{\substack{a\geq 0\\ b_{L\setminus\{i,j\}}\geq 0}}\langle \tau_a\tau_{b_{L\setminus\{i,j\}}}\lambda_g\rangle_g\Psi^0_{b_{L\setminus\{i,j\}}}(t_{L\setminus\{i,j\}})\frac{-t_i^2\Psi^0_{a+1}(t_i)+t_j^2\Psi^0_{a+1}(t_j)}{t_i-t_j}$$

$$R_5 = \sum_{1\leq i<j\leq l}\sum_{\substack{a\geq 0\\ b_{L\setminus\{i,j\}}\geq 0}}\langle \tau_a\tau_{b_{L\setminus\{i,j\}}}\lambda_g\rangle_g\Psi^0_{b_{L\setminus\{i,j\}}}(t_{L\setminus\{i,j\}})(|b_{L\setminus\{i,j\}}|+a+3)\frac{-t_jt_i\Psi^0_{a+1}(t_i)+t_it_j\Psi^0_{a+1}(t_j)}{t_i-t_j}$$

$$R_6 = \sum_{1\leq i<j\leq l}\sum_{\substack{a\geq 0\\ b_{L\setminus\{i,j\}}\geq 0}}\langle \tau_a\tau_{b_{L\setminus\{i,j\}}}\lambda_g\rangle_g\Psi^0_{b_{L\setminus\{i,j\}}}(t_{L\setminus\{i,j\}})(|b_{L\setminus\{i,j\}}|+a+3)\frac{t_i\Psi^0_{a+1}(t_i)-t_j\Psi^0_{a+1}(t_j)}{t_i-t_j}$$

$$R_7 = \sum_{1\leq i<j\leq l}\sum_{\substack{a\geq 0\\ b_{L\setminus\{i,j\}}\geq 0}}\langle \tau_a\tau_{b_{L\setminus\{i,j\}}}\lambda_g\rangle_g\sum_{r\in L\setminus\{i,j\}}\Psi^1_{b_r}(t_r)\Psi^0_{b_{L\setminus\{i,j,r\}}}(t_{L\setminus\{i,j,r\}})\frac{t_jt_i\Psi^0_{a+1}(t_i)-t_it_j\Psi^0_{a+1}(t_j)}{t_i-t_j}$$

$$R_8 = \sum_{1\leq i<j\leq l}\sum_{\substack{a\geq 0\\ b_{L\setminus\{i,j\}}\geq 0}}\langle \tau_a\tau_{b_{L\setminus\{i,j\}}}\lambda_g\rangle_g\sum_{r\in L\setminus\{i,j\}}\Psi^1_{b_r}(t_r)\Psi^0_{b_{L\setminus\{i,j,r\}}}(t_{L\setminus\{i,j,r\}})\frac{-t_i\Psi^0_{a+1}(t_i)+t_j\Psi^0_{a+1}(t_j)}{t_i-t_j}$$

$$R_9 = -\sum_{1\le i<j\le l}\sum_{\substack{a\ge 0\\ b_{L\setminus\{i,j\}}\ge 0}} \langle \tau_a \tau_{b_{L\setminus\{i,j\}}} \lambda_{g-1}\rangle_g \Psi^0_{b_{L\setminus\{i,j\}}}(t_{L\setminus\{i,j\}}) \frac{t_j t_i \Psi^0_{a+1}(t_i) - t_i t_j \Psi^0_{a+1}(t_j)}{t_i - t_j}$$

$$R_{10} = -\sum_{1\le i<j\le l}\sum_{\substack{a\ge 0\\ b_{L\setminus\{i,j\}}\ge 0}} \langle \tau_a \tau_{b_{L\setminus\{i,j\}}} \lambda_{g-1}\rangle_g \Psi^0_{b_{L\setminus\{i,j\}}}(t_{L\setminus\{i,j\}}) \frac{-t_i \Psi^0_{a+1}(t_i) + t_j \Psi^0_{a+1}(t_j)}{t_i - t_j}$$

$$R_{11} = \sum_{1\le i<j\le l}\sum_{\substack{a\ge 0\\ b_{L\setminus\{i,j\}}\ge 0}} \langle \tau_a \tau_{b_{L\setminus\{i,j\}}} \lambda_{g}\rangle_g g \Psi^0_{b_{L\setminus\{i,j\}}}(t_{L\setminus\{i,j\}}) \frac{t_j t_i \Psi^0_{a+1}(t_i) - t_i t_j \Psi^0_{a+1}(t_j)}{t_i - t_j}$$

$$R_{12} = \sum_{1\le i<j\le l}\sum_{\substack{a\ge 0\\ b_{L\setminus\{i,j\}}\ge 0}} \langle \tau_a \tau_{b_{L\setminus\{i,j\}}} \lambda_{g}\rangle_g g \Psi^0_{b_{L\setminus\{i,j\}}}(t_{L\setminus\{i,j\}}) \frac{-t_i \Psi^0_{a+1}(t_i) + t_j \Psi^0_{a+1}(t_j)}{t_i - t_j}$$

$$R_{13} = -\sum_{1\le i<j\le l}\sum_{\substack{a\ge 0\\ b_{L\setminus\{i,j\}}\ge 0}} \langle \tau_a \tau_{b_{L\setminus\{i,j\}}} \lambda_g\lambda_1\rangle_g \Psi^0_{b_{L\setminus\{i,j\}}}(t_{L\setminus\{i,j\}}) \frac{t_j t_i \Psi^0_{a+1}(t_i) - t_i t_j \Psi^0_{a+1}(t_j)}{t_i - t_j}$$

$$R_{14} = -\sum_{1\le i<j\le l}\sum_{\substack{a\ge 0\\ b_{L\setminus\{i,j\}}\ge 0}} \langle \tau_a \tau_{b_{L\setminus\{i,j\}}} \lambda_g\lambda_1\rangle_g \Psi^0_{b_{L\setminus\{i,j\}}}(t_{L\setminus\{i,j\}}) \frac{-t_i \Psi^0_{a+1}(t_i) + t_j \Psi^0_{a+1}(t_j)}{t_i - t_j}$$

$$R_{15} = \sum_{1\le i<j\le l}\sum_{\substack{a\ge 0\\ b_{L\setminus\{i,j\}}\ge 0}} \langle \tau_a \tau_{b_{L\setminus\{i,j\}}} \sum_{d=g+2}^{3g-3} P_d(\lambda)\rangle_g \Psi^0_{b_{L\setminus\{i,j\}}}(t_{L\setminus\{i,j\}}) \frac{(t_j-1) t_i \Psi^0_{a+1}(t_i) - (t_i-1) t_j \Psi^0_{a+1}(t_j)}{t_i - t_j}$$

$$R_{16} = \frac{1}{2}\sum_{j=1}^{l}\sum_{\substack{a_1\ge 0\\ a_2\ge 0\\ b_{L\setminus\{j\}}\ge 0}}\sum_{\substack{g_1+g_2=g\\ \mathcal{I}\cup\mathcal{J}=L\setminus\{j\}}}^{stable} \langle \tau_{a_1}\tau_{b_{\mathcal{I}}}\lambda_{g_1}\rangle_{g_1} \langle \tau_{a_2}\tau_{b_{\mathcal{J}}}\lambda_{g_2}\rangle_{g_2} \Psi^0_{a_1+1}(t_j)\Psi^0_{a_2+1}(t_j)\Psi^0_{b_{L\setminus\{j\}}}(t_{L\setminus\{j\}})$$

Then,

(98)

$$F_{2g+2l-4}(L_1) = \sum_{|b_L|=2g-3+l} \langle \tau_{b_L}\lambda_g\rangle_g l(2-g-l) \sum_{j=1}^{l} f^0(b_j,b_j) t_j^{b_j} f^0(b_{L\setminus\{j\}}, b_{L\setminus\{j\}}+1) t_{L\setminus\{j\}}^{b_{L\setminus\{j\}}+1}$$

(99)
$$\begin{aligned} F_{2g+2l-4}(L_2) = \sum_{|b_L|=2g-3+l} \langle \tau_{b_L}\lambda_g\rangle_g l \Bigg(& \sum_{j=1}^{l} f^1(b_j,b_j+1) t_j^{b_j+1} \sum_{k\in L\setminus\{j\}} f^0(b_k,b_k) t_k^{b_k} \\ & f^0(b_{L\setminus\{j,k\}}, b_{L\setminus\{j,k\}}+1) t_{L\setminus\{j,k\}}^{b_{L\setminus\{j,k\}}+1} + \sum_{j=1}^{l} f^1(b_j,b_j) t_j^{b_j} f^0(b_{L\setminus\{j\}}, b_{L\setminus\{j\}}+1) t_{L\setminus\{j\}}^{b_{L\setminus\{j\}}+1} \\ & + \sum_{j=1}^{l} f^1(b_j,b_j+2) t_j^{b_j+2} \sum_{k\in L\setminus\{j\}} f^0(b_k,b_k-1) t_k^{b_k-1} f^0(b_{L\setminus\{j,k\}}, b_{L\setminus\{j,k\}}+1) t_{L\setminus\{j,k\}}^{b_{L\setminus\{j,k\}}+1} \end{aligned}$$

$$+\sum_{j=1}^{l} f^1(b_j,b_j+2)t_j^{b_j+2}\sum_{k_1\in L\setminus\{j\}} f^0(b_{k_1},b_{k_1})t_{k_1}^{b_{k_1}}\sum_{k_2\in L\setminus\{j,k_1\}} f^0(b_{k_2},b_{k_2})t_{k_2}^{b_{k_2}}$$
$$f^0(b_{L\setminus\{j,k_1,k_2\}},b_{L\setminus\{j,k_1,k_2\}}+1)t_{L\setminus\{j,k_1,k_2\}}^{b_{L\setminus\{j,k_1,k_2\}}+1}\Big)$$

(100)

$$F_{2g+2l-4}(L_3)=\sum_{|b_L|=2g-3+l}\langle\tau_{b_L}\lambda_g\rangle_g\left(\sum_{j=1}^{l} b_j f^0(b_j,b_j)t_j^{b_j+1}\sum_{k\in L\setminus\{j\}} f^0(b_k,b_k)t_k^{b_k}\right.$$
$$f^0(b_{L\setminus\{j,k\}},b_{L\setminus\{j,k\}}+1)t_{L\setminus\{j,k\}}^{b_{L\setminus\{j,k\}}+1}+\sum_{j=1}^{l}(b_j-1)f^0(b_j,b_j-1)t_j^{b_j}f^0(b_{L\setminus\{j\}},b_{L\setminus\{j\}}+1)t_{L\setminus\{j\}}^{b_{L\setminus\{j\}}+1}$$
$$+\sum_{j=1}^{l}(b_j+1)f^0(b_j,b_j+1)t_j^{b_j+2}\sum_{k\in L\setminus\{j\}} f^0(b_k,b_k-1)t_k^{b_k-1}f^0(b_{L\setminus\{j,k\}},b_{L\setminus\{j,k\}}+1)t_{L\setminus\{j,k\}}^{b_{L\setminus\{j,k\}}+1}$$
$$+\sum_{j=1}^{l}(b_j+1)f^0(b_j,b_j+1)t_j^{b_j+2}\sum_{k_1\in L\setminus\{j\}} f^0(b_{k_1},b_{k_1})t_{k_1}^{b_{k_1}}\sum_{k_2\in L\setminus\{j,k_1\}} f^0(b_{k_2},b_{k_2})t_{k_2}^{b_{k_2}}$$
$$\left.f^0(b_{L\setminus\{j,k_1,k_2\}},b_{L\setminus\{j,k_1,k_2\}}+1)t_{L\setminus\{j,k_1,k_2\}}^{b_{L\setminus\{j,k_1,k_2\}}+1}\right)$$

(101)
$$F_{2g+2l-4}(L_4)=-\sum_{|b_L|=2g-3+l}\langle\tau_{b_L}\lambda_g\rangle_g\left(\sum_{j=1}^{l} b_j f^0(b_j,b_j)t_j^{b_j}f^0(b_{L\setminus\{j\}},b_{L\setminus\{j\}}+1)t_{L\setminus\{j\}}^{b_{L\setminus\{j\}}+1}\right.$$
$$\left.+\sum_{j=1}^{l}(b_j+1)f^0(b_j,b_j+1)t_j^{b_j+1}\sum_{k\in L\setminus\{j\}} f^0(b_k,b_k)t_k^{b_k}f^0(b_{L\setminus\{j,k\}},b_{L\setminus\{j,k\}}+1)t_{L\setminus\{j,k\}}^{b_{L\setminus\{j,k\}}+1}\right)$$

(102)

$$F_{2g+2l-4}(L_5)=-\sum_{|b_L|=2g-2+l}\langle\tau_{b_L}\lambda_{g-1}\rangle_g l\left(\sum_{j=1}^{l} f^0(b_j,b_j-1)t_j^{b_j-1}f^0(b_{L\setminus\{j\}},b_{L\setminus\{j\}}+1)t_{L\setminus\{j\}}^{b_{L\setminus\{j\}}+1}\right.$$
$$\left.+\sum_{j=1}^{l} f^0(b_j,b_j)t_j^{b_j}\sum_{k\in L\setminus\{j\}} f^0(b_k,b_k)t_k^{b_k}f^0(b_{L\setminus\{j,k\}},b_{L\setminus\{j,k\}}+1)t_{L\setminus\{j,k\}}^{b_{L\setminus\{j,k\}}+1}\right)$$

(103)
$$F_{2g+2l-4}(L_6)=-\sum_{|b_L|=2g-4+l}\langle\tau_{b_L}\lambda_g\lambda_1\rangle_g l f^0(b_L,b_L+1)t_L^{b_L+1}$$

(104)
$$F_{2g+2l-4}(L_7)=0$$

(105)

$$F_{2g+2l-4}(R_1)=\sum_{1\le i<j\le l}\ \sum_{a+|b_{L\setminus\{i,j\}}|=2g+l-4}\langle\tau_a\tau_{b_{L\setminus\{i,j\}}}\lambda_g\rangle_g\left(f^0(b_{L\setminus\{i,j\}},b_{L\setminus\{i,j\}}+1)t_{L\setminus\{i,j\}}^{b_{L\setminus\{i,j\}}+1}\right.$$

$$f^1(a+1,a+1)\sum_{k\geq 0}^{a} t_i^{k+1}t_j^{a+1-k}+\sum_{r\in L\setminus\{i,j\}} f^0(b_r,b_r)t_r^{b_r}f^0(b_{L\setminus\{i,j,r\}},b_{L\setminus\{i,j,r\}}+1)t_{L\setminus\{i,j,r\}}^{b_{L\setminus\{i,j,r\}}+1}$$

$$f^1(a+1,a+2)\sum_{k\geq 0}^{a+1} t_i^{k+1}t_j^{a+2-k}+\sum_{r\in L\setminus\{i,j\}} f^0(b_r,b_r-1)t_r^{b_r-1}f^0(b_{L\setminus\{i,j,r\}},b_{L\setminus\{i,j,r\}}+1)t_{L\setminus\{i,j,r\}}^{b_{L\setminus\{i,j,r\}}+1}$$

$$f^1(a+1,a+3)\sum_{k\geq 0}^{a+2} t_i^{k+1}t_j^{a+3-k}+\sum_{r_1\in L\in L\setminus\{i,j\}} f^0(b_{r_1},b_{r_1})t_{r_1}^{b_{r_1}}\sum_{r_2\in L\setminus\{i,j,r_1\}} f^0(b_{r_2},b_{r_2})t_{r_2}^{b_{r_2}}$$

$$f^0(b_{L\setminus\{i,j,r_1,r_2\}},b_{L\setminus\{i,j,r_1,r_2\}}+1)t_{L\setminus\{i,j,r_1,r_2\}}^{b_{L\setminus\{i,j,r_1,r_2\}}+1}f^1(a+1,a+3)\sum_{k\geq 0}^{a+2} t_i^{k+1}t_j^{a+3-k}\Bigg)$$

(106)

$$F_{2g+2l-4}(R_2)=-\sum_{1\leq i<j\leq l}\sum_{a+|b_{L\setminus\{i,j\}}|=2g+l-4}\langle\tau_a\tau_{b_{L\setminus\{i,j\}}}\lambda_g\rangle_g\Big(f^0(b_{L\setminus\{i,j\}},b_{L\setminus\{i,j\}}+1)t_{L\setminus\{i,j\}}^{b_{L\setminus\{i,j\}}+1}$$

$$f^1(a+1,a+2)\sum_{k\geq 0}^{a+2} t_i^{k}t_j^{a+2-k}+\sum_{r\in L\setminus\{i,j\}} f^0(b_r,b_r)t_r^{b_r}f^0(b_{L\setminus\{i,j,r\}},b_{L\setminus\{i,j,r\}}+1)t_{L\setminus\{i,j,r\}}^{b_{L\setminus\{i,j,r\}}+1}$$

$$f^1(a+1,a+3)\sum_{k\geq 0}^{a+3} t_i^{k}t_j^{a+3-k}\Bigg)$$

(107)

$$F_{2g+2l-4}(R_3)=\sum_{1\leq i<j\leq l}\sum_{a+|b_{L\setminus\{i,j\}}|=2g+l-4}\langle\tau_a\tau_{b_{L\setminus\{i,j\}}}\lambda_g\rangle_g\Big(f^0(b_{L\setminus\{i,j\}},b_{L\setminus\{i,j\}}+1)t_{L\setminus\{i,j\}}^{b_{L\setminus\{i,j\}}+1}$$

$$f^0(a+1,a)\sum_{k\geq 0}^{a} t_i^{k+1}t_j^{a+1-k}+\sum_{r\in L\setminus\{i,j\}} f^0(b_r,b_r)t_r^{b_r}f^0(b_{L\setminus\{i,j,r\}},b_{L\setminus\{i,j,r\}}+1)t_{L\setminus\{i,j,r\}}^{b_{L\setminus\{i,j,r\}}+1}$$

$$f^0(a+1,a+1)\sum_{k\geq 0}^{a+1} t_i^{k+1}t_j^{a+2-k}+\sum_{r\in L\setminus\{i,j\}} f^0(b_r,b_r-1)t_r^{b_r-1}f^0(b_{L\setminus\{i,j,r\}},b_{L\setminus\{i,j,r\}}+1)t_{L\setminus\{i,j,r\}}^{b_{L\setminus\{i,j,r\}}+1}$$

$$f^0(a+1,a+2)\sum_{k\geq 0}^{a+2} t_i^{k+1}t_j^{a+3-k}+\sum_{r_1\in L\in L\setminus\{i,j\}} f^0(b_{r_1},b_{r_1})t_{r_1}^{b_{r_1}}\sum_{r_2\in L\setminus\{i,j,r_1\}} f^0(b_{r_2},b_{r_2})t_{r_2}^{b_{r_2}}$$

$$f^0(b_{L\setminus\{i,j,r_1,r_2\}},b_{L\setminus\{i,j,r_1,r_2\}}+1)t_{L\setminus\{i,j,r_1,r_2\}}^{b_{L\setminus\{i,j,r_1,r_2\}}+1}f^0(a+1,a+2)\sum_{k\geq 0}^{a+2} t_i^{k+1}t_j^{a+3-k}\Bigg)$$

(108)

$$F_{2g+2l-4}(R_4)=-\sum_{1\leq i<j\leq l}\sum_{a+|b_{L\setminus\{i,j\}}|=2g+l-4}\langle\tau_a\tau_{b_{L\setminus\{i,j\}}}\lambda_g\rangle_g\Big(f^0(b_{L\setminus\{i,j\}},b_{L\setminus\{i,j\}}+1)t_{L\setminus\{i,j\}}^{b_{L\setminus\{i,j\}}+1}$$

$$f^0(a+1,a+1)\sum_{k\geq 0}^{a+2} t_i^k t_j^{a+2-k} + \sum_{r\in L\setminus\{i,j\}} f^0(b_r,b_r)t_r^{b_r} f^0(b_{L\setminus\{i,j,r\}}, b_{L\setminus\{i,j,r\}}+1)t_{L\setminus\{i,j,r\}}^{b_{L\setminus\{i,j,r\}}+1}$$

$$f^0(a+1,a+2)\sum_{k\geq 0}^{a+3} t_i^k t_j^{a+3-k}\Bigg)$$

(109)

$$F_{2g+2l-4}(R_5) = -\sum_{1\leq i<j\leq l}\ \sum_{a+|b_{L\setminus\{i,j\}}|=2g+l-4} \langle \tau_a \tau_{b_{L\setminus\{i,j\}}}\lambda_g\rangle_g \Big((2g+l-1)f^0(b_{L\setminus\{i,j\}}, b_{L\setminus\{i,j\}}+1)t_{L\setminus\{i,j\}}^{b_{L\setminus\{i,j\}}+1}$$

$$f^0(a+1,a+1)\sum_{k\geq 0}^{a} t_i^{k+1} t_j^{a+1-k} + (2g+l-1)\sum_{r\in L\setminus\{i,j\}} f^0(b_r,b_r)t_r^{b_r} f^0(b_{L\setminus\{i,j,r\}}, b_{L\setminus\{i,j,r\}}+1)t_{L\setminus\{i,j,r\}}^{b_{L\setminus\{i,j,r\}}+1}$$

$$f^0(a+1,a+2)\sum_{k\geq 0}^{a+1} t_i^{k+1} t_j^{a+2-k}\Bigg)$$

(110)

$$F_{2g+2l-4}(R_6) = \sum_{1\leq i<j\leq l}\ \sum_{a+|b_{L\setminus\{i,j\}}|=2g+l-4} \langle \tau_a \tau_{b_{L\setminus\{i,j\}}}\lambda_g\rangle_g \Big((2g+l-1)f^0(b_{L\setminus\{i,j\}}, b_{L\setminus\{i,j\}}+1)t_{L\setminus\{i,j\}}^{b_{L\setminus\{i,j\}}+1}$$

$$f^0(a+1,a+2)\sum_{k\geq 0}^{a+2} t_i^k t_j^{a+2-k}\Bigg)$$

(111)

$$F_{2g+2l-4}(R_7) = \sum_{1\leq i<j\leq l}\ \sum_{a+|b_{L\setminus\{i,j\}}|=2g+l-4} \langle \tau_a \tau_{b_{L\setminus\{i,j\}}}\lambda_g\rangle_g \Bigg(\sum_{r\in L\setminus\{i,j\}} f^1(b_r,b_r+2)t_r^{b_r+2}$$

$$f^0(b_{L\setminus\{i,j,r\}}, b_{L\setminus\{i,j,r\}}+1)t_{L\setminus\{i,j,r\}}^{b_{L\setminus\{i,j,r\}}+1} f^0(a+1,a)\sum_{k\geq 0}^{a-1} t_i^{k+1} t_j^{a-k} + \sum_{r\in L\setminus\{i,j\}} f^1(b_r,b_r+2)t_r^{b_r+2}$$

$$\sum_{s\in L\setminus\{i,j,r\}} f^0(b_s,b_s-1)t_s^{b_s-1} f^0(b_{L\setminus\{i,j,r,s\}}, b_{L\setminus\{i,j,r,s\}}+1)t_{L\setminus\{i,j,r,s\}}^{b_{L\setminus\{i,j,r,s\}}+1} f^0(a+1,a+2)\sum_{k\geq 0}^{a+1} t_i^{k+1} t_j^{a+2-k}$$

$$+\sum_{r\in L\setminus\{i,j\}} f^1(b_r,b_r+2)t_r^{b_r+2} \sum_{s_1\in L\setminus\{i,j,r\}} f^0(b_{s_1},b_{s_1})t_{s_1}^{b_{s_1}} \sum_{s_2\in L\setminus\{i,j,r_1,s_1\}} f^0(b_{s_2},b_{s_2})t_{s_2}^{b_{s_2}}$$

$$f^0(b_{L\setminus\{i,j,r,s_1,s_2\}}, b_{L\setminus\{i,j,r,s_1,s_2\}}+1)t_{L\setminus\{i,j,r,s_1,s_2\}}^{b_{L\setminus\{i,j,r,s_1,s_2\}}+1} f^0(a+1,a+2)\sum_{k\geq 0}^{a+1} t_i^{k+1} t_j^{a+2-k} + \sum_{r\in L\setminus\{i,j\}} f^1(b_r,b_r)t_r^{b_r}$$

$$f^0(b_{L\setminus\{i,j,r\}}, b_{L\setminus\{i,j,r\}}+1)t_{L\setminus\{i,j,r\}}^{b_{L\setminus\{i,j,r\}}+1} f^0(a+1,a+2)\sum_{k\geq 0}^{a+1} t_i^{k+1} t_j^{a+2-k} + \sum_{r_1\in L\setminus\{i,j\}} f^1(b_{r_1},b_{r_1}+1)t_{r_1}^{b_{r_1}+1}$$

$$\sum_{r_2\in L\setminus\{i,j,r_1\}} f^0(b_{r_2},b_{r_2})t_{r_2}^{b_{r_2}} f^0(b_{L\setminus\{i,j,r_1,r_2\}}, b_{L\setminus\{i,j,r_1,r_2\}}+1)t_{L\setminus\{i,j,r_1,r_2\}}^{b_{L\setminus\{i,j,r_1,r_2\}}+1} f^0(a+1,a+2)$$

$$\sum_{k\geq 0}^{a+1} t_i^{k+1}t_j^{a+2-k} + \sum_{r\in\{i,j\}} f^1(b_r,b_r+1)t_r^{b_r+1} f^0(b_{L\setminus\{i,j,r\}},b_{L\setminus\{i,j,r\}}+1)t_{L\setminus\{i,j,r\}}^{b_{L\setminus\{i,j,r\}}+1} f^0(a+1,a+1)$$

$$\sum_{k\geq 0}^{a} t_i^{k+1}t_j^{a+1-k} + \sum_{r_1\in L\setminus\{i,j\}} f^1(b_{r_1},b_{r_1}+2)t_{r_1}^{b_{r_1}+2} \sum_{r_2\in L\setminus\{i,j,r_1\}} f^0(b_{r_2},b_{r_2})t_{r_2}^{b_{r_2}}$$

$$f^0(b_{L\setminus\{i,j,r_1,r_2\}},b_{L\setminus\{i,j,r_1,r_2\}}+1)t_{L\setminus\{i,j,r_1,r_2\}}^{b_{L\setminus\{i,j,r_1,r_2\}}+1} f^0(a+1,a+1)\sum_{k\geq 0}^{a} t_i^{k+1}t_j^{a+1-k}\Bigg)$$

(112)

$$F_{2g+2l-4}(R_8) = -\sum_{1\leq i<j\leq l}\ \sum_{a+|b_{L\setminus\{i,j\}}|=2g+l-4} \langle \tau_a\tau_{b_{L\setminus\{i,j\}}}\lambda_g\rangle_g \Bigg(\sum_{r\in L\setminus\{i,j\}} f^1(b_r,b_r+2)t_r^{b_r+2}$$

$$f^0(b_{L\setminus\{i,j,r\}},b_{L\setminus\{i,j,r\}}+1)t_{L\setminus\{i,j,r\}}^{b_{L\setminus\{i,j,r\}}+1} f^0(a+1,a+1)\sum_{k\geq 0}^{a+1} t_i^k t_j^{a+1-k} + \sum_{r\in L\setminus\{i,j\}} f^1(b_r,b_r+1)t_r^{b_r+1}$$

$$f^0(b_{L\setminus\{i,j,r\}},b_{L\setminus\{i,j,r\}}+1)t_{L\setminus\{i,j,r\}}^{b_{L\setminus\{i,j,r\}}+1} f^0(a+1,a+2)\sum_{k\geq 0}^{a+2} t_i^k t_j^{a+2-k} + \sum_{r\in L\setminus\{i,j\}} f^1(b_r,b_r+2)t_r^{b_r+2}$$

$$\sum_{s\in L\setminus\{i,j,r\}} f^0(b_s,b_s)t_s^{b_s} f^0(b_{L\setminus\{i,j,r,s\}},b_{L\setminus\{i,j,r,s\}}+1)t_{L\setminus\{i,j,r,s\}}^{b_{L\setminus\{i,j,r,s\}}+1} f^0(a+1,a+2)\sum_{k\geq 0}^{a+2} t_i^k t_j^{a+2-k}\Bigg)$$

(113)

$$F_{2g+2l-4}(R_9) = -\sum_{1\leq i<j\leq l}\ \sum_{a+|b_{L\setminus\{i,j\}}|=2g+l-3} \langle \tau_a\tau_{b_{L\setminus\{i,j\}}}\lambda_{g-1}\rangle_g \Big(f^0(b_{L\setminus\{i,j\}},b_{L\setminus\{i,j\}}+1)t_{L\setminus\{i,j\}}^{b_{L\setminus\{i,j\}}+1}$$

$$f^0(a+1,a)\sum_{k\geq 0}^{a-1} t_i^{k+1}t_j^{a-k} + \sum_{r\in L\setminus\{i,j\}} f^0(b_r,b_r)t_r^{b_r} f^0(b_{L\setminus\{i,j,r\}},b_{L\setminus\{i,j,r\}}+1)t_{L\setminus\{i,j,r\}}^{b_{L\setminus\{i,j,r\}}+1}$$

$$f^0(a+1,a+1)\sum_{k\geq 0}^{a} t_i^{k+1}t_j^{a+1-k} + \sum_{r\in L\setminus\{i,j\}} f^0(b_r,b_r-1)t_r^{b_r-1} f^0(b_{L\setminus\{i,j,r\}},b_{L\setminus\{i,j,r\}}+1)t_{L\setminus\{i,j,r\}}^{b_{L\setminus\{i,j,r\}}+1}$$

$$f^0(a+1,a+2)\sum_{k\geq 0}^{a+1} t_i^{k+1}t_j^{a+2-k} + \sum_{r_1\in L\in L\setminus\{i,j\}} f^0(b_{r_1},b_{r_1})t_{r_1}^{b_{r_1}} \sum_{r_2\in L\setminus\{i,j,r_1\}} f^0(b_{r_2},b_{r_2})t_{r_2}^{b_{r_2}}$$

$$f^0(b_{L\setminus\{i,j,r_1,r_2\}},b_{L\setminus\{i,j,r_1,r_2\}}+1)t_{L\setminus\{i,j,r_1,r_2\}}^{b_{L\setminus\{i,j,r_1,r_2\}}+1} f^0(a+1,a+2)\sum_{k\geq 0}^{a+1} t_i^{k+1}t_j^{a+2-k}\Bigg)$$

(114)

$$F_{2g+2l-4}(R_{10}) = \sum_{1\leq i<j\leq l}\ \sum_{a+|b_{L\setminus\{i,j\}}|=2g+l-3} \langle \tau_a\tau_{b_{L\setminus\{i,j\}}}\lambda_{g-1}\rangle_g \Big(f^0(b_{L\setminus\{i,j\}},b_{L\setminus\{i,j\}}+1)t_{L\setminus\{i,j\}}^{b_{L\setminus\{i,j\}}+1}$$

$$f^0(a+1,a+1)\sum_{k\geq 0}^{a+1} t_i^k t_j^{a+1-k} + \sum_{r\in L\setminus\{i,j\}} f^0(b_r,b_r)t_r^{b_r} f^0(b_{L\setminus\{i,j,r\}}, b_{L\setminus\{i,j,r\}}+1)t_{L\setminus\{i,j,r\}}^{b_{L\setminus\{i,j,r\}}+1}$$

$$f^0(a+1,a+2)\sum_{k\geq 0}^{a+2} t_i^k t_j^{a+2-k}\Bigg)$$

(115)

$$F_{2g+2l-4}(R_{11}) = \sum_{1\leq i<j\leq l}\ \sum_{a+|b_{L\setminus\{i,j\}}|=2g+l-4} \langle \tau_a \tau_{b_{L\setminus\{i,j\}}}\lambda_g\rangle_g g\Big(f^0(b_{L\setminus\{i,j\}}, b_{L\setminus\{i,j\}}+1)t_{L\setminus\{i,j\}}^{b_{L\setminus\{i,j\}}+1}$$

$$f^0(a+1,a+1)\sum_{k\geq 0}^{a} t_i^{k+1} t_j^{a+1-k} + \sum_{r\in L\setminus\{i,j\}} f^0(b_r,b_r)t_r^{b_r} f^0(b_{L\setminus\{i,j,r\}}, b_{L\setminus\{i,j,r\}}+1)t_{L\setminus\{i,j,r\}}^{b_{L\setminus\{i,j,r\}}+1}$$

$$f^0(a+1,a+2)\sum_{k\geq 0}^{a+1} t_i^k t_j^{a+2-k}\Bigg)$$

(116)

$$F_{2g+2l-4}(R_{12}) = -\sum_{1\leq i<j\leq l}\ \sum_{a+|b_{L\setminus\{i,j\}}|=2g+l-4} \langle \tau_a \tau_{b_{L\setminus\{i,j\}}}\lambda_g\rangle_g g\Big(f^0(b_{L\setminus\{i,j\}}, b_{L\setminus\{i,j\}}+1)t_{L\setminus\{i,j\}}^{b_{L\setminus\{i,j\}}+1}$$

$$f^0(a+1,a+2)\sum_{k\geq 0}^{a+2} t_i^k t_j^{a+2-k}\Bigg)$$

(117)

$$F_{2g+2l-4}(R_{13}) = -\sum_{1\leq i<j\leq l}\ \sum_{a+|b_{L\setminus\{i,j\}}|=2g+l-5} \langle \tau_a \tau_{b_{L\setminus\{i,j\}}}\lambda_g\lambda_1\rangle_g \Big(f^0(b_{L\setminus\{i,j\}}, b_{L\setminus\{i,j\}}+1)t_{L\setminus\{i,j\}}^{b_{L\setminus\{i,j\}}+1}$$

$$f^0(a+1,a+2)\sum_{k\geq 0}^{a+1} t_i^{k+1} t_j^{a+2-k}\Bigg)$$

(118)
$$F_{2g+2l-4}(R_{14}) = F_{2g+2l-4}(R_{15}) = 0$$

(119)
$$F_{2g+2l-4}(R_{16}) = \frac{1}{2}\sum_{j=1}^{l}\ \sum_{\substack{g_1+g_2=g\\ \mathcal{I}\cup\mathcal{J}=L\setminus\{j\}}}^{stable}\ \sum_{\substack{a_1+|b_{\mathcal{I}}|=2g_1-2+|\mathcal{I}|\\ a_2+|b_{\mathcal{J}}|=2g_2-2+|\mathcal{J}|}} \langle \tau_{a_1}\tau_{b_{\mathcal{I}}}\lambda_{g_1}\rangle_{g_1}\langle \tau_{a_2}\tau_{b_{\mathcal{J}}}\lambda_{g_2}\rangle_{g_2}$$
$$f^0(a_1+1,a_1+2)f^0(a_2+1,a_2+2)t_j^{a_1+a_2+4} f^0(b_{L\setminus\{j\}}, b_{L\setminus\{j\}}+1)t_{L\setminus\{j\}}^{b_{L\setminus\{j\}}+1}$$

(120)

$$[t_L^{b_L+1}]F_{2g+2l-4}(L_1) = \sum_{j=1}^{l}\langle \tau_{b_j+1}\tau_{b_{L\setminus\{j\}}}\lambda_g\rangle_g l(2-g-l)f^0(b_j+1,b_j+1)f^0(b_{L\setminus\{j\}}, b_{L\setminus\{j\}}+1)$$

(121)

$$[t_L^{b_L+1}]F_{2g+2l-4}(L_2)=\sum_{j=1}^{l}\sum_{k\in L\backslash\{j\}}\langle\tau_{b_j}\tau_{b_k+1}\tau_{b_{L\backslash\{j,k\}}}\lambda_g\rangle_g lf^1(b_j,b_j+1)f^0(b_k+1,b_k+1)$$
$$f^0(b_{L\backslash\{j,k\}},b_{L\backslash\{j,k\}}+1)+\sum_{j=1}^{l}\langle\tau_{b_j+1}\tau_{b_{L\backslash\{j\}}}\lambda_g\rangle_g lf^1(b_j-1,b_j+1)f^0(b_{L\backslash\{j\}},b_{L\backslash\{j\}}+1)$$
$$+\sum_{j=1}^{l}\sum_{k\in L\backslash\{j\}}\langle\tau_{b_j-1}\tau_{b_k+2}\tau_{b_{L\backslash\{j,k\}}}\lambda_g\rangle_g lf^1(b_j-1,b_j+1)f^0(b_k+2,b_k+1)f^0(b_{L\backslash\{j,k\}},b_{L\backslash\{j,k\}}+1)$$
$$+\sum_{j=1}^{l}\sum_{k_1\in L\backslash\{j\}}\sum_{k_2\in L\backslash\{j,k_1\}}\langle\tau_{b_j-1}\tau_{b_{k_1}+1}\tau_{b_{k_2}+1}\tau_{b_{L\backslash\{j,k_1,k_2\}}}\lambda_g\rangle_g lf^1(b_j-1,b_j+1)$$
$$f^0(b_{k_1}+1,b_{k_1}+1)f^0(b_{k_2}+1,b_{k_2}+1)f^0(b_{L\backslash\{j,k_1,k_2\}},b_{L\backslash\{j,k_1,k_2\}}+1)$$

(122)

$$[t_L^{b_L+1}]F_{2g+2l-4}(L_3)=\sum_{j=1}^{l}\sum_{k\in L\backslash\{j\}}\langle\tau_{b_j}\tau_{b_k+1}\tau_{b_{L\backslash\{j,k\}}}\lambda_g\rangle_g b_j f^0(b_j,b_j)f^0(b_k+1,b_k+1)$$
$$f^0(b_{L\backslash\{j,k\}},b_{L\backslash\{j,k\}}+1)+\sum_{j=1}^{l}\langle\tau_{b_j+1}\tau_{b_{L\backslash\{j\}}}\lambda_g\rangle_g b_j f^0(b_j+1,b_j)f^0(b_{L\backslash\{j\}},b_{L\backslash\{j\}}+1)$$
$$+\sum_{j=1}^{l}\sum_{k\in L\backslash\{j\}}\langle\tau_{b_j-1}\tau_{b_k+2}\tau_{b_{L\backslash\{j,k\}}}\lambda_g\rangle_g b_j f^0(b_j-1,b_j)f^0(b_k+2,b_k+1)f^0(b_{L\backslash\{j,k\}},b_{L\backslash\{j,k\}}+1)$$
$$+\sum_{j=1}^{l}\sum_{k_1\in L\backslash\{j\}}\sum_{k_2\in L\backslash\{j,k_1\}}\langle\tau_{b_j-1}\tau_{b_{k_1}+1}\tau_{b_{k_2}+1}\tau_{b_{L\backslash\{j,k_1,k_2\}}}\lambda_g\rangle_g b_j f^0(b_j-1,b_j)$$
$$f^0(b_{k_1}+1,b_{k_1}+1)f^0(b_{k_2}+1,b_{k_2}+1)f^0(b_{L\backslash\{j,k_1,k_2\}},b_{L\backslash\{j,k_1,k_2\}}+1)$$

(123)

$$[t_L^{b_L+1}]F_{2g+2l-4}(L_4)=-\sum_{j=1}^{l}\langle\tau_{b_j+1}\tau_{b_{L\backslash\{j\}}}\lambda_g\rangle_g(b_j+1)f^0(b_j+1,b_j+1)f^0(b_{L\backslash\{j\}},b_{L\backslash\{j\}}+1)$$
$$-\sum_{j=1}^{l}\sum_{k\in L\backslash\{j\}}\langle\tau_{b_j}\tau_{b_k+1}\tau_{b_{L\backslash\{j,k\}}}\lambda_g\rangle_g(b_j+1)f^0(b_j,b_j+1)f^0(b_k+1,b_k+1)f^0(b_{L\backslash\{j,k\}},b_{L\backslash\{j,k\}}+1)$$

(124)

$$[t_L^{b_L+1}]F_{2g+2l-4}(L_5)=-\sum_{j=1}^{l}\langle\tau_{b_j+2}\tau_{b_{L\backslash\{j\}}}\lambda_{g-1}\rangle_g lf^0(b_j+2,b_j+1)f^0(b_{L\backslash\{j\}},b_{L\backslash\{j\}}+1)$$
$$-\sum_{j=1}^{l}\sum_{k\in L\backslash\{j\}}\langle\tau_{b_j+1}\tau_{b_k+1}\tau_{b_{L\backslash\{j,k\}}}\lambda_{g-1}\rangle_g lf^0(b_j+1,b_j+1)f^0(b_k+1,b_k+1)f^0(b_{L\backslash\{j,k\}},b_{L\backslash\{j,k\}}+1)$$

$$[t_L^{b_L+1}]F_{2g+2l-4}(L_6) = -\langle \tau_{b_L}\lambda_g\lambda_1\rangle_g l f^0(b_L, b_L+1) \tag{125}$$

on the right hand side,

(126)

$$[t_L^{b_L+1}]F_{2g+2l-4}(R_1) = \sum_{1\le i<j\le l}\langle \tau_{b_i+b_j}\tau_{b_{L\setminus\{i,j\}}}\lambda_g\rangle_g f^0(b_{L\setminus\{i,j\}}, b_{L\setminus\{i,j\}}+1) f^1(b_i+b_j+1, b_i+b_j+1)$$
$$+\sum_{1\le i<j\le l}\sum_{r\in L\setminus\{i,j\}}\langle \tau_{b_i+b_j-1}\tau_{b_r+1}\tau_{b_{L\setminus\{i,j,r\}}}\lambda_g\rangle_g f^1(b_i+b_j, b_i+b_j+1) f^0(b_{L\setminus\{i,j,r\}}, b_{L\setminus\{i,j,r\}}+1)$$
$$f^0(b_r+1, b_r+1) + \sum_{1\le i<j\le l}\sum_{r\in L\setminus\{i,j\}}\langle \tau_{b_i+b_j-2}\tau_{b_r+2}\tau_{b_{L\setminus\{i,j,r\}}}\lambda_g\rangle_g f^1(b_i+b_j-1, b_i+b_j+1) f^0(b_r+2, b_r+1)$$
$$f^0(b_{L\setminus\{i,j,r\}}, b_{L\setminus\{i,j,r\}}+1) + \sum_{1\le i<j\le l}\sum_{r_1\in L\setminus\{i,j\}}\sum_{r_2\in L\setminus\{i,j,r_1\}}\langle \tau_{b_i+b_j-2}\tau_{b_{r_1}+1}\tau_{b_{r_2}+1}\tau_{b_{L\setminus\{i,j,r_1,r_2\}}}\lambda_g\rangle_g$$
$$f^0(b_{r_1}+1, b_{r_1}+1) f^0(b_{r_2}+1, b_{r_2}+1) f^0(b_{L\setminus\{i,j,r_1,r_2\}}, b_{L\setminus\{i,j,r_1,r_2\}}+1) f^1(b_i+b_j-1, b_i+b_j+1)$$

(127)

$$[t_L^{b_L+1}]F_{2g+2l-4}(R_2) = -\sum_{1\le i<j\le l}\langle \tau_{b_i+b_j}\tau_{b_{L\setminus\{i,j\}}}\lambda_g\rangle_g f^0(b_{L\setminus\{i,j\}}, b_{L\setminus\{i,j\}}+1) f^1(b_i+b_j+1, b_i+b_j+2)$$
$$-\sum_{1\le i<j\le l}\sum_{r\in L\setminus\{i,j\}}\langle \tau_{b_i+b_j-1}\tau_{b_r+1}\tau_{b_{L\setminus\{i,j,r\}}}\lambda_g\rangle_g f^0(b_r+1, b_r+1) f^0(b_{L\setminus\{i,j,r\}}, b_{L\setminus\{i,j,r\}}+1)$$
$$f^1(b_i+b_j+1, b_i+b_j+3)$$

(128)

$$[t_L^{b_L+1}]F_{2g+2l-4}(R_3) = \sum_{1\le i<j\le l}\langle \tau_{b_i+b_j}\tau_{b_{L\setminus\{i,j\}}}\lambda_g\rangle_g f^0(b_{L\setminus\{i,j\}}, b_{L\setminus\{i,j\}}+1) f^0(b_i+b_j+1, b_i+b_j)$$
$$+\sum_{1\le i<j\le l}\sum_{r\in L\setminus\{i,j\}}\langle \tau_{b_i+b_j-1}\tau_{b_r+1}\tau_{b_{L\setminus\{i,j,r\}}}\lambda_g\rangle_g f^0(b_i+b_j, b_i+b_j) f^0(b_{L\setminus\{i,j,r\}}, b_{L\setminus\{i,j,r\}}+1)$$
$$f^0(b_r+1, b_r+1) + \sum_{1\le i<j\le l}\sum_{r\in L\setminus\{i,j\}}\langle \tau_{b_i+b_j-2}\tau_{b_r+2}\tau_{b_{L\setminus\{i,j,r\}}}\lambda_g\rangle_g f^0(b_i+b_j-1, b_i+b_j) f^0(b_r+2, b_r+1)$$
$$f^0(b_{L\setminus\{i,j,r\}}, b_{L\setminus\{i,j,r\}}+1) + \sum_{1\le i<j\le l}\sum_{r_1\in L\setminus\{i,j\}}\sum_{r_2\in L\setminus\{i,j,r_1\}}\langle \tau_{b_i+b_j-2}\tau_{b_{r_1}+1}\tau_{b_{r_2}+1}\tau_{b_{L\setminus\{i,j,r_1,r_2\}}}\lambda_g\rangle_g$$
$$f^0(b_{r_1}+1, b_{r_1}+1) f^0(b_{r_2}+1, b_{r_2}+1) f^0(b_{L\setminus\{i,j,r_1,r_2\}}, b_{L\setminus\{i,j,r_1,r_2\}}+1) f^0(b_i+b_j-1, b_i+b_j)$$

(129)

$$[t_L^{b_L+1}](F_{2g+2l-4}(R_4)+F_{2g+2l-4}(R_5)) = -\sum_{1\le i<j\le l}\langle \tau_{b_i+b_j}\tau_{b_{L\setminus\{i,j\}}}\lambda_g\rangle_g(2g+l)$$
$$f^0(b_{L\setminus\{i,j\}}, b_{L\setminus\{i,j\}}+1) f^0(b_i+b_j+1, b_i+b_j+1) - \sum_{1\le i<j\le l}\sum_{r\in L\setminus\{i,j\}}\langle \tau_{b_i+b_j-1}\tau_{b_r+1}\tau_{b_{L\setminus\{i,j,r\}}}\lambda_g\rangle_g$$
$$(2g+l) f^0(b_r+1, b_r+1) f^0(b_{L\setminus\{i,j,r\}}, b_{L\setminus\{i,j,r\}}+1) f^0(b_i+b_j, b_i+b_j+1)$$

$$(130)\qquad [t_L^{b_L+1}]F_{2g+2l-4}(R_6)=\sum_{1\le i<j\le l}\langle\tau_{b_i+b_j}\tau_{b_{L\setminus\{i,j\}}}\lambda_g\rangle_g(2g+l-1)f^0(b_{L\setminus\{i,j\}},b_{L\setminus\{i,j\}}+1)$$
$$f^0(b_i+b_j+1,b_i+b_j+2)$$

(131)
$$[t_L^{b_L+1}]F_{2g+2l-4}(R_7)=\sum_{1\le i<j\le l}\sum_{r\in L\setminus\{i,j\}}\langle\tau_{b_i+b_j+1}\tau_{b_r-1}\tau_{b_{L\setminus\{i,j,r\}}}\lambda_g\rangle_g f^1(b_r-1,b_r+1)f^0(b_{L\setminus\{i,j,r\}},b_{L\setminus\{i,j,r\}}+1)$$
$$f^0(b_i+b_j+2,b_i+b_j+1)+\sum_{1\le i<j\le l}\sum_{r\in L\setminus\{i,j\}}\sum_{s\in L\setminus\{i,j,r\}}\langle\tau_{b_i+b_j-1}\tau_{b_r-1}\tau_{b_s+2}\tau_{b_{L\setminus\{i,j,r,s\}}}\lambda_g\rangle_g$$
$$f^1(b_r-1,b_r+1)f^0(b_s+2,b_s+1)f^0(b_{L\setminus\{i,j,r,s\}},b_{L\setminus\{i,j,r,s\}}+1)f^0(b_i+b_j,b_i+b_j+1)$$
$$+\sum_{1\le i<j\le l}\sum_{r\in L\setminus\{i,j\}}\sum_{s_1\in L\setminus\{i,j,r\}}\sum_{s_2\in L\setminus\{i,j,r,s_1\}}\langle\tau_{b_i+b_j-1}\tau_{b_r-1}\tau_{b_{s_1}+1}\tau_{b_{s_2}+1}\tau_{b_{L\setminus\{i,j,r,s_1,s_2\}}}\lambda_g\rangle_g f^1(b_r-1,b_r+1)$$
$$f^0(b_{s_1}+1,b_{s_1}+1)f^0(b_{s_2}+1,b_{s_2}+1)f^0(b_{L\setminus\{i,j,r,s_1,s_2\}},b_{L\setminus\{i,j,r,s_1,s_2\}}+1)f^0(b_i+b_j,b_i+b_j+1)$$
$$+\sum_{1\le i<j\le l}\sum_{r\in L\setminus\{i,j\}}\langle\tau_{b_i+b_j-1}\tau_{b_r+1}\tau_{b_{L\setminus\{i,j,r\}}}\lambda_g\rangle_g f^1(b_r+1,b_r+1)f^0(b_{L\setminus\{i,j,r\}},b_{L\setminus\{i,j,r\}}+1)$$
$$f^0(b_i+b_j,b_i+b_j+1)+\sum_{1\le i<j\le l}\sum_{r\in L\setminus\{i,j\}}\sum_{s\in L\setminus\{i,j,r\}}\langle\tau_{b_i+b_j-1}\tau_{b_r}\tau_{b_s+1}\tau_{b_{L\setminus\{i,j,r,s\}}}\lambda_g\rangle_g f^1(b_r,b_r+1)$$
$$f^0(b_s+1,b_s+1)f^0(b_{L\setminus\{i,j,r,s\}},b_{L\setminus\{i,j,r,s\}}+1)f^0(b_i+b_j,b_i+b_j+1)+\sum_{1\le i<j\le l}\sum_{r\in L\setminus\{i,j\}}\langle\tau_{b_i+b_j}\tau_{b_r}\tau_{b_{L\setminus\{i,j,r\}}}\lambda_g\rangle_g$$
$$f^1(b_r,b_r+1)f^0(b_{L\setminus\{i,j,r\}},b_{L\setminus\{i,j,r\}}+1)f^0(b_i+b_j+1,b_i+b_j+1)$$
$$+\sum_{1\le i<j\le l}\sum_{r\in L\setminus\{i,j\}}\sum_{s\in L\setminus\{i,j,r\}}\langle\tau_{b_i+b_j}\tau_{b_r-1}\tau_{b_s+1}\tau_{b_{L\setminus\{i,j,r,s\}}}\lambda_g\rangle_g f^1(b_r-1,b_r+1)f^0(b_s+1,b_s+1)$$
$$f^0(b_{L\setminus\{i,j,r,s\}},b_{L\setminus\{i,j,r,s\}}+1)f^0(b_i+b_j+1,b_i+b_j+1)$$

(132)
$$[t_L^{b_L+1}]F_{2g+2l-4}(R_8)=-\sum_{1\le i<j\le l}\sum_{r\in L\setminus\{i,j\}}\langle\tau_{b_i+b_j+1}\tau_{b_r-1}\tau_{b_{L\setminus\{i,j,r\}}}\lambda_g\rangle_g f^1(b_r-1,b_r+1)$$
$$f^0(b_{L\setminus\{i,j,r\}},b_{L\setminus\{i,j,r\}}+1)f^0(b_i+b_j+2,b_i+b_j+2)-\sum_{1\le i<j\le l}\sum_{r\in L\setminus\{i,j\}}\langle\tau_{b_i+b_j}\tau_{b_r}\tau_{b_{L\setminus\{i,j,r\}}}\lambda_g\rangle_g$$
$$f^1(b_r,b_r+1)f^0(b_{L\setminus\{i,j,r\}},b_{L\setminus\{i,j,r\}}+1)f^0(b_i+b_j+1,b_i+b_j+2)$$
$$-\sum_{1\le i<j\le l}\sum_{r\in L\setminus\{i,j\}}\sum_{s\in L\setminus\{i,j,r\}}\langle\tau_{b_i+b_j}\tau_{b_r-1}\tau_{b_s+1}\tau_{b_{L\setminus\{i,j,r,s\}}}\lambda_g\rangle_g f^1(b_r-1,b_r+1)f^0(b_s+1,b_s+1)$$
$$f^0(b_{L\setminus\{i,j,r,s\}},b_{L\setminus\{i,j,r,s\}}+1)f^0(b_i+b_j+1,b_i+b_j+2)$$

(133)
$$[t_L^{b_L+1}]F_{2g+2l-4}(R_9)=-\sum_{1\le i<j\le l}\langle\tau_{b_i+b_j+1}\tau_{b_{L\setminus\{i,j\}}}\lambda_{g-1}\rangle_g f^0(b_{L\setminus\{i,j\}},b_{L\setminus\{i,j\}}+1)f^0(b_i+b_j+2,b_i+b_j+1)$$

$$-\sum_{1\le i<j\le l}\sum_{r\in L\setminus\{i,j\}}\langle\tau_{b_i+b_j}\tau_{b_r+1}\tau_{b_{L\setminus\{i,j,r\}}}\lambda_{g-1}\rangle_g f^0(b_r+1,b_r+1)f^0(b_{L\setminus\{i,j,r\}},b_{L\setminus\{i,j,r\}}+1)$$
$$f^0(b_i+b_j+1,b_i+b_j+1)-\sum_{1\le i<j\le l}\sum_{r\in L\setminus\{i,j\}}\sum_{s\in L\setminus\{i,j,r\}}\langle\tau_{b_i+b_j-1}\tau_{b_r+1}\tau_{b_s+1}\tau_{b_{L\setminus\{i,j,r,s\}}}\lambda_{g-1}\rangle_g$$
$$f^0(b_r+1,b_r+1)f^0(b_s+1,b_s+1)f^0(b_{L\setminus\{i,j,r,s\}},b_{L\setminus\{i,j,r,s\}}+1)f^0(b_i+b_j,b_i+b_j+1)$$
$$-\sum_{1\le i<j\le l}\sum_{r\in L\setminus\{i,j\}}\langle\tau_{b_i+b_j-1}\tau_{b_r+2}\tau_{b_{L\setminus\{i,j,r\}}}\lambda_{g-1}\rangle_g f^0(b_r+2,b_r+1)f^0(b_{L\setminus\{i,j,r\}},b_{L\setminus\{i,j,r\}}+1)$$
$$f^0(b_i+b_j,b_i+b_j+1)$$

(134)
$$[t_L^{b_L+1}]F_{2g+2l-4}(R_{10})=\sum_{1\le i<j\le l}\langle\tau_{b_i+b_j+1}\tau_{b_{L\setminus\{i,j\}}}\lambda_{g-1}\rangle_g f^0(b_{L\setminus\{i,j\}},b_{L\setminus\{i,j\}}+1)f^0(b_i+b_j+2,b_i+b_j+2)$$
$$+\sum_{1\le i<j\le l}\sum_{r\in L\setminus\{i,j\}}\langle\tau_{b_i+b_j}\tau_{b_r+1}\tau_{b_{L\setminus\{i,j,r\}}}\lambda_{g-1}\rangle_g f^0(b_r+1,b_r+1)f^0(b_{L\setminus\{i,j,r\}},b_{L\setminus\{i,j,r\}}+1)$$
$$f^0(b_i+b_j+1,b_i+b_j+2)$$

(135)
$$[t_L^{b_L+1}]F_{2g+2l-4}(R_{11})=\sum_{1\le i<j\le l}\langle\tau_{b_i+b_j}\tau_{b_{L\setminus\{i,j\}}}\lambda_g\rangle_g g f^0(b_{L\setminus\{i,j\}},b_{L\setminus\{i,j\}}+1)f^0(b_i+b_j+1,b_i+b_j+1)$$
$$+\sum_{1\le i<j\le l}\sum_{r\in L\setminus\{i,j\}}\langle\tau_{b_i+b_j-1}\tau_{b_r+1}\tau_{b_{L\setminus\{i,j,r\}}}\lambda_g\rangle_g g f^0(b_r+1,b_r+1)f^0(b_{L\setminus\{i,j,r\}},b_{L\setminus\{i,j,r\}}+1)$$
$$f^0(b_i+b_j,b_i+b_j+1)$$

(136)
$$[t_L^{b_L+1}]F_{2g+2l-4}(R_{12})=-\sum_{1\le i<j\le l}\langle\tau_{b_i+b_j}\tau_{b_{L\setminus\{i,j\}}}\lambda_g\rangle_g g f^0(b_{L\setminus\{i,j\}},b_{L\setminus\{i,j\}}+1)f^0(b_i+b_j+1,b_i+b_j+2)$$

(137)
$$[t_L^{b_L+1}]F_{2g+2l-4}(R_{13})=-\sum_{1\le i<j\le l}\langle\tau_{b_i+b_j-1}\tau_{b_{L\setminus\{i,j\}}}\lambda_g\lambda_1\rangle_g f^0(b_{L\setminus\{i,j\}},b_{L\setminus\{i,j\}}+1)f^0(b_i+b_j,b_i+b_j+1)$$

(138)
$$[t_L^{b_L+1}]F_{2g+2l-4}(R_{16})=\frac{1}{2}\sum_{j=1}^{l}\sum_{a_1+a_2=b_j-3}\sum_{\substack{g_1+g_2=g\\ \mathcal{I}\cup\mathcal{J}=L\setminus\{j\}}}^{stable}\langle\tau_{a_1}\tau_{b_{\mathcal{I}}}\lambda_{g_1}\rangle_{g_1}\langle\tau_{a_2}\tau_{b_{\mathcal{J}}}\lambda_{g_2}\rangle_{g_2}$$
$$f^0(a_1+1,a_1+2)f^0(a_2+1,a_2+2)f^0(b_{L\setminus\{j\}},b_{L\setminus\{j\}}+1)$$

Therefore, from the equation (97), we get the identity

(139) $(120)+(121)+(122)+(123)+(124)+(125)=(126)+(127)$
$+(128)+(129)+(130)+(131)+(132)+(133)+(134)+(135)+136)+(137)+(138)$

We note that, three types of the Hodge integrals appear in identity (139), $\langle\tau_{b_L}\lambda_g\rangle_g$, $\langle\tau_{b_L}\lambda_{g-1}\rangle_g$ and $\langle\tau_{b_L}\lambda_g\lambda_1\rangle_g$.

By now, we have known the closed formula for λ_g-integral [13] and a recursion formula for λ_{g-1}-integral [24].

$$\int_{\overline{\mathcal{M}}_{g,l}} \psi_1^{d_1} \dots \psi_l^{d_l} \lambda_g = \binom{2g-3+l}{b_1,..,b_l} c_g, \tag{140}$$

where $\sum_{i=1}^{l} b_i = 2g-3+l, b_1,..,b_l \geq 0$ and c_g is a constant that depends only on g.

$$\langle \tau_{b_1} \cdots \tau_{b_l} \lambda_{g-1} \rangle_g = \frac{1}{l} \sum_{1\leq i<j\leq l} \langle \tau_{b_i+b_j-1} \prod_{k\neq i,j} \tau_{b_k} \lambda_{g-1} \rangle_g \frac{(b_i+b_j)!}{b_i! b_j!} + C_{g,l}(b_1,..,b_l) \tag{141}$$

where $C_{g,n}(b_1,..,b_l)$ is a constant related to $b_1,..,b_l$ and g,l.

On the other hand, the constants $f^0(b,k)$ and $f^1(b,k)$ are available by their definition and the recursions (52), (53). For example, $f^0(b,b+1) = b!$, $f^1(b,b+2) = (b+1)! \sum_{k=2}^{b+1} \frac{1}{k}$. Thus, solving the equation (139), we will get a formula for the computation of Hodge integral $\langle \tau_{b_L} \lambda_g \lambda_1 \rangle_g$.

We observe that only the terms (125) and (137) contain the Hodge integral of type $\langle \tau_{b_L} \lambda_g \lambda_1 \rangle_g$. Moving and combining the corresponding terms, we have

$$\langle \tau_{b_L} \lambda_g \lambda_1 \rangle_g = \frac{1}{l} \sum_{1\leq i<j\leq l} \langle \tau_{b_i+b_j-1} \tau_{b_{L\setminus\{i,j\}}} \lambda_g \lambda_1 \rangle_g \frac{(b_i+b_j)!}{b_i! b_j!} + C(g,l,b_1,..,b_l) \tag{142}$$

Where

$$\begin{aligned} C(g,l,b_1,..,b_l) = -\frac{1}{l \prod_{i=1}^{l} b_i!} \big(&(126)+(127)+(128)+(129)+(130)+(131)+(132) \\ &+(133)+(134)+(135)+(136)+(138)-(120)-(121)-(122)-(123)-(124) \big) \end{aligned} \tag{143}$$

Formula (143) contains many terms, we hope to make a computer program to calculate $C(g,l,b_1,..,b_l)$.

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