AMS/IP Studies in Advanced Mathematics

Recent Development on the Geometry of the Teichmüller and Moduli Spaces of Riemann Surfaces and Polarized Calabi-Yau Manifolds

Kefeng Liu, Xiaofeng Sun, Shing-Tung Yau

Contents

1. Introduction

The moduli space $\mathcal{M}_{g,k}$ of Riemann surfaces of genus g with k punctures plays an important role in many area of mathematics and theoretical physics. In this article we first survey some of our recent works on the geometry of this moduli space. In the following we assume $g \geq 2$ and $k = 0$ to simplify notations. All the results in this paper work for the general case when $3g - 3 + k > 0$. We will focus on the Kähler metrics on the moduli and Teichmüller spaces, especially the Weil-Petersson metric, the Ricci, the perturbed Ricci, and the Kähler-Einstein metrics.

We will review certain new geometric properties we found and proved for these metrics, such as the bounded geometry, the goodness and their naturalness under restriction to boundary divisors. The algebro-geometric corollaries such as the stability of the logarithmic cotangent bundles and the infinitesimal rigidity of the moduli spaces will also be briefly discussed. Similar to our previous survey articles [**15, 14**], we will briefly describe the basic ideas of our proofs, the details of the proofs will be published soon, see [**16, 17**].

After introducing the definition of Weil-Petersson metric in Section 2, we discuss the fundamental curvature formula of Wolpert for the Weil-Petersson metric. For the reader's convenience we also briefly give a proof of the negativity of the Riemannian curvature of the Weil-Petersson metric. In Section 3 we discuss the Ricci and the perturbed Ricci metrics and their curvature formulas. In Section 4 we describe the asymptotics of these metrics and their curvatures which are important for our understanding of their bounded geometry. In Section 5 we briefly discuss the equivalence of all of the complete metrics on Teichmüller spaces to the Ricci and the perturbed Ricci metrics, which is a simple corollary of our understanding of these two new metrics. In Section 6 we discuss the goodness of the Weil-Petersson metric, the Ricci, the perturbed Ricci metric and the Kähler-Einstein metric. To prove the goodness we need much more subtle estimates on the connection and the curvatures of these metrics. Section 7 contains discussions of the dual Nakano negativity of the logarithmic tangent bundle of the moduli space and the naturalness of the Ricci and the perturbed Ricci metrics. In Section 8 we discuss the Kähler-Ricci flow and the Kähler-Einstein metric on the moduli space. There are many interesting corollaries from our understanding of the geometry of the moduli spaces. In Section 9 we discuss the stability of the logarithmic cotangent bundle, the L^2 cohomology and the infinitesimal rigidity of the moduli spaces as well as the Gauss-Bonnet theorem on the moduli space.

The Teichmüller and moduli spaces of polarized Calabi-Yau (CY) manifolds and Hyper-Kähler manifolds are also important in mathematics and high energy physics. In Section 10 we will describe our recent joint work with A. Todorov on the proof of global Torelli theorem of the Teichmüller space of polarized CY manifolds and Hyper-Kähler manifolds. As applications we will describe the construction of a global holomorphic flat connection on the Teichmüller space of CY manifolds and the existence of Kähler-Einstein metrics on the Hodge completion of such Teichmüller spaces.

2. The Weil-Petersson Metric and Its Curvature

Let $\mathcal{M}_g$ be the moduli space of Riemann surfaces of genus g where $g \geq 2$. It is well known that the $\mathcal{M}_g$ is a complex orbifold. The Teichmüller space $\mathcal{T}_g$, as the space parameterizing marked Riemann surfaces, is a smooth contractible pseudo-convex domain and can be embedded into the Euclidean space of the same dimension.

Remark 2.1. *Since $\mathcal{M}_g$ is only an orbifold, in the following when we work near a point $p \in \mathcal{M}_g$ which is an orbifold point, we always work on a local manifold cover of $\mathcal{M}_g$ around p. An alternative way is to add a level structure on the moduli space so that it becomes smooth* [**29**]. *All the following results are still valid. In particular, when we use the universal curve over the moduli space, we always mean the universal curve over the local manifold cover. When we deal with global properties of the moduli space, we can use the moduli space with a level structure such that it becomes smooth. We take quotients after we derive the estimates. We can also work on the Teichmüller space which is smooth.*

For any point $p \in \mathcal{M}_g$ we let X_p be the corresponding Riemann surface. By the Kodaira-Spencer theory we have the identification

$$T_p^{1,0}\mathcal{M}_g \cong H^1\left(X_p, T_{X_p}^{1,0}\right).$$

It follows from Serre duality that

$$\Omega_p^{1,0}\mathcal{M}_g \cong H^0\left(X_p, K_{X_p}^2\right).$$

By the Riemann-Roch theorem we know that the dimension $\dim_{\mathbb{C}} \mathcal{M}_g = n = 3g-3$.

The Weil-Petersson (WP) metric is the first known Kähler metric on $\mathcal{M}_g$. Ahlfors showed that the WP metric is Kähler and its holomorphic sectional curvature is bounded above by a negative constant which only depends on the genus g. Royden conjectured that the Ricci curvature of the WP metric is also bounded above by a negative constant. This conjecture was proved by Wolpert [**33**].

Now we briefly describe the WP metric and its curvature formula. Please see the works [**39**], [**37**] of Wolpert for detailed description and various aspects of the WP metric.

Let $\pi : \mathfrak{X} \to \mathcal{M}_g$ be the universal family over the moduli space. For any point $s \in \mathcal{M}_g$ we let $X_s = \pi^{-1}(s)$ be the corresponding smooth Riemann surface. Since the Euler characteristic $\chi(X_s) = 2-2g < 0$, by the uniformization theorem we know that each fiber X_s is equipped with a unique Kähler-Einstein metric λ. In the following we will always use the Kähler-Einstein metric λ on X_s. Let z be any holomorphic coordinate on X_s. We have

$$\partial_z \partial_{\overline{z}} \log \lambda = \lambda.$$

Now we fix a point $s \in \mathcal{M}_g$ and let $(U, s_1, \cdots, s_n)$ be any holomorphic coordinate chart on $\mathcal{M}_g$ around s. In the following we will denote by ∂_i and ∂_z the local vector fields $\frac{\partial}{\partial s_i}$ and $\frac{\partial}{\partial z}$ respectively. By the Kodaira-Spencer theory and the Hodge theory we have the identification

$$T_s^{1,0}\mathcal{M}_g \cong \check{H}^1\left(X_s, T_{X_s}^{1,0}\right) \cong \mathbb{H}^{0,1}\left(X_s, T_{X_s}^{1,0}\right)$$

where the right side of the above formula is the space of harmonic Beltrami differentials. In fact we can explicitly construct the above identification. We let

$$a_i = a_i(z,s) = -\lambda^{-1}\partial_i \partial_{\bar z} \log \lambda$$

and let

$$v_i = \frac{\partial}{\partial s_i} + a_i \frac{\partial}{\partial z}.$$

The vector field v_i is a smooth vector field on $\pi^{-1}(U)$ and is called the harmonic lift of $\frac{\partial}{\partial s_i}$. If we let $B_i = \overline{\partial}_F v_i \in A^{0,1}\left(X_s, T^{1,0}_{X_s}\right)$ then B_i is harmonic and the map $\frac{\partial}{\partial s_i} \mapsto B_i$ is precisely the Kodaira-Spencer map. Here ∂_F is the operator in the fiber direction. In local coordinates if we let $B_i = A_i d\bar z \otimes \partial_z$ then $A_i = \partial_{\bar z} a_i$. Furthermore, it was proved by Schumacher that if η is any relative $(1,1)$-form on $\mathfrak{X}$ then

$$\frac{\partial}{\partial s_i}\int_{X_s} \eta = \int_{X_s} L_{v_i}\eta. \tag{2.1}$$

We note that although A_i is a local smooth function on X_s, the product

$$A_i \overline{A}_j = B_i \cdot \overline{B}_j \in C^\infty(X_s)$$

is globally defined. We let

$$f_{i\bar j} = A_i \overline{A}_j \in C^\infty(X_s).$$

The Weil-Petersson metric on $\mathcal{M}_g$ is given by

$$h_{i\bar j}(s) = \int_{X_s} B_i \cdot \overline{B}_j \, dv = \int_{X_s} f_{i\bar j} \, dv$$

where $dv = \frac{\sqrt{-1}}{2}\lambda dz \wedge d\bar z$ is the volume form on X_s with respect to the Kähler-Einstein metric.

Now we describe the curvature formula of the WP metric. We let $\Box = -\lambda^{-1}\partial_z\partial_{\bar z}$ be the Hodge-Laplacian acting on $C^\infty(X_s)$. It is clear that the operator $\Box + 1$ has no kernel and thus is invertible. We let

$$e_{i\bar j} = (\Box + 1)^{-1}\left(f_{i\bar j}\right) \in C^\infty(X_s).$$

The following curvature formula is due to Wolpert. See [**12**] for the detailed proof.

Proposition 2.1. *Let $R_{i\bar j k\bar l}$ be the curvature of the WP metric. Then*

$$R_{i\bar j k\bar l} = -\int_{X_s}\left(e_{i\bar j} f_{k\bar l} + e_{i\bar l} f_{k\bar j}\right) dv. \tag{2.2}$$

The curvature of the WP metric has very strong negativity property. In fact we shall see in Section 7 that the WP metric is dual Nakano negative. We collect the negativity property of the WP metric in the following proposition.

Proposition 2.2. *The bisectional curvature of the WP metric on the moduli space $\mathcal{M}_g$ is negative. The holomorphic sectional and Ricci curvatures of the WP metric are bounded above by negative constants. Furthermore, the Riemannian sectional curvature of the WP metric is also negative.*

Proof. These results are well known, see [**33**]. Here we give a short proof of the negativity of the Riemannian sectional curvature of the WP metric for the reader's convenience. The proof follows from expressing the Riemannian sectional curvature in term of complex curvature tensors and using the curvature formula (2.2).

In general, let (X^n, g, J) be a Kähler manifold. For any point $p \in X$ and two orthonormal real tangent vectors $u, v \in T_p^{\mathbb{R}} X$, we let $X = \frac{1}{2}(u - iJu)$ and $Y = \frac{1}{2}(v - iJv)$ and we know that $X, Y \in T_p^{1,0} X$. We can choose holomorphic local coordinate $s = (s_1, \cdots, s_n)$ around p such that $X = \frac{\partial}{\partial s_1}$. If $v = span_{\mathbb{R}}\{u, Ju\}$, since v is orthogonal to u and its length is 1, we know $v = \pm Ju$. In this case we have

$$R(u, v, u, v) = R(u, Ju, u, Ju) = 4R_{1\bar{1}1\bar{1}}.$$

Thus the Riemannian sectional curvature and the holomorphic sectional curvature have the same sign.

If v is not contained in the real plane spanned by u and Ju we can choose the coordinate s such that $X = \frac{\partial}{\partial s_1}$ and $Y = \frac{\partial}{\partial s_2}$. In this case a direct computation shows that

$$R(u, v, u, v) = 2\left(R_{1\bar{1}2\bar{2}} - Re\left(R_{1\bar{2}1\bar{2}}\right)\right). \tag{2.3}$$

Now we fix a point $p \in \mathcal{M}_g$ and let $u, v \in T_p^{\mathbb{R}} \mathcal{M}_g$. Let X, Y be the corresponding $(1,0)$-vectors. Since we know that the holomorphic sectional curvature of the WP metric is strictly negative, we assume $v \notin span_{\mathbb{R}}\{u, Ju\}$ and thus we can choose holomorphic local coordinates $s = (s_1, \cdots, s_n)$ around p such that $X = \frac{\partial}{\partial s_1}(p)$ and $Y = \frac{\partial}{\partial s_2}(p)$. By formulas (2.3) and (2.2) we have

$$\begin{aligned} R(u, v, u, v) &= -2\left(\int_{X_p} \left(e_{1\bar{1}} f_{2\bar{2}} + e_{1\bar{2}} f_{2\bar{1}} - 2Re(e_{1\bar{2}} f_{1\bar{2}})\right) dv\right) \\ &= -2\left(\int_{X_p} \left(e_{1\bar{1}} f_{2\bar{2}} + e_{1\bar{2}} f_{2\bar{1}} - e_{1\bar{2}} f_{1\bar{2}} - e_{2\bar{1}} f_{2\bar{1}}\right) dv\right). \end{aligned} \tag{2.4}$$

To prove the proposition we only need to show that

$$\int_{X_p} e_{1\bar{2}} f_{2\bar{1}}\, dv \geq \int_{X_p} Re\left(e_{1\bar{2}} f_{1\bar{2}}\right) dv \tag{2.5}$$

and

$$\int_{X_p} e_{1\bar{1}} f_{2\bar{2}}\, dv \geq \int_{X_p} e_{1\bar{2}} f_{2\bar{1}}\, dv \tag{2.6}$$

and both equalities cannot hold simultaneously.

To prove inequality (2.5) we let $\alpha = Re(e_{1\bar{2}})$ and $\beta = Im(e_{1\bar{2}})$. Then we know

$$\int_{X_p} e_{1\bar{2}} f_{2\bar{1}}\ dv = \int_{X_p} (\alpha(\Box+1)\alpha + \beta(\Box+1)\beta)\, dv$$

and

$$\int_{X_p} Re\,(e_{1\bar{2}} f_{1\bar{2}})\, dv = \int_{X_p} (\alpha(\Box+1)\alpha - \beta(\Box+1)\beta)\, dv.$$

Thus formula (2.5) reduces to

$$\int_{X_p} \beta(\Box+1)\beta\ dv \geq 0.$$

However, we know

$$\int_{X_p} \beta(\Box+1)\beta\ dv = \int_{X_p} \left(\|\nabla'\beta\|^2 + \beta^2\right) dv \geq 0$$

and the equality holds if and only if $\beta = 0$. If this is the case then we know that $e_{1\bar{2}}$ is a real value function and $f_{1\bar{2}}$ is real valued too. Since $f_{1\bar{1}} = A_1\overline{A}_1$ and $f_{1\bar{2}} = A_1\overline{A}_2$ and $f_{1\bar{1}}$ is real-valued we know that there is a function $f \in C^\infty(X_p \setminus S, \mathbb{R})$ such that $A_2 = f(z)A_1$ on $X_p \setminus S$. Here S is the set of zeros of A_1. Since both A_1 and A_2 are harmonic, we know that $\overline{\partial}^* A_1 = \overline{\partial}^* A_2 = 0$. These reduce to $\partial_z(\lambda A_1) = \partial_z(\lambda A_2) = 0$ locally. It follows that $\partial_z f\,|_{X_p\setminus S} = 0$. Since f is real-valued we know that f must be a constant. But A_1 and A_2 are linearly independent which is a contradiction. So the strict inequality (2.5) always holds.

Now we prove formula (2.6). Let $G(z,w)$ be the Green's function of the operator $\Box+1$ and let $T = (\Box+1)^{-1}$. By the maximum principle we know that T maps positive functions to positive functions. This implies that the Green's function G is nonnegative. Since $G(z,w) = G(w,z)$ is symmetric we know that

(2.7)

$$\begin{aligned}\int_{X_p} e_{1\bar{1}} f_{2\bar{2}}\ dv =& \int_{X_p\times X_p} G(z,w) f_{1\bar{1}}(w) f_{2\bar{2}}(z)\ dv(w)dv(z)\\ =& \frac{1}{2}\int_{X_p\times X_p} G(z,w)\left(f_{1\bar{1}}(w)f_{2\bar{2}}(z) + f_{1\bar{1}}(z)f_{2\bar{2}}(w)\right) dv(w)dv(z).\end{aligned}$$

Similarly we have

(2.8)

$$\begin{aligned}\int_{X_p} e_{1\bar{2}} f_{2\bar{1}}\ dv =& \int_{X_p\times X_p} G(z,w) f_{1\bar{2}}(w) f_{2\bar{1}}(z)\ dv(w)dv(z)\\ =& \frac{1}{2}\int_{X_p\times X_p} G(z,w)\left(f_{1\bar{2}}(w)f_{2\bar{1}}(z) + f_{1\bar{2}}(z)f_{2\bar{1}}(w)\right) dv(w)dv(z).\end{aligned}$$

Formula (2.6) follows from the fact that

$$\begin{aligned} & f_{1\bar{1}}(w)f_{2\bar{2}}(z)+f_{1\bar{1}}(z)f_{2\bar{2}}(w)-f_{1\bar{2}}(w)f_{2\bar{1}}(z)-f_{1\bar{2}}(z)f_{2\bar{1}}(w) \\ = & \left|A_1(z)A_2(w)-A_1(w)A_2(z)\right|^2 \geq 0. \end{aligned}$$

□

Although the WP metric has very strong negativity properties, as we shall see in Section 4, the WP metric is not complete and its curvatures have no lower bound and this is very restrictive.

3. The Ricci and Perturbed Ricci Metrics

In [**12**] and [**13**] we studied two new Kähler metrics: the Ricci metric ω_τ and the perturbed Ricci metric $\omega_{\widetilde{\tau}}$ on the moduli space $\mathcal{M}_g$. These new Kähler metrics are complete and have bounded geometry and thus have many important applications. We now describe these new metrics.

Since the Ricci curvature of the WP metric has negative upper bound, we define the Ricci metric

$$\omega_\tau = -Ric\left(\omega_{_{WP}}\right).$$

We also define the perturbed Ricci metric to be a linear combination of the Ricci metric and the WP metric

$$\omega_{\widetilde{\tau}} = \omega_\tau + C\omega_{_{WP}}$$

where C is a positive constants. In local coordinates we have $\tau_{i\bar{j}} = -h^{k\bar{l}}R_{i\bar{j}k\bar{l}}$ and $\widetilde{\tau}_{i\bar{j}} = \tau_{i\bar{j}} + Ch_{i\bar{j}}$ where $R_{i\bar{j}k\bar{l}}$ is the curvature of the WP metric.

Similar to curvature formula (2.2) of the WP metric we can establish integral formulae for the curvature of the Ricci and perturbed Ricci metrics. These curvature formulae are crucial in estimating the asymptotics of these metrics and their curvature. To establish these formulae, we need to introduce some operators. We let

$$P : C^\infty(X_s) \to A^{1,0}\left(T^{0,1}_{X_s}\right)$$

be the operator defined by

$$f \mapsto \partial\left(\omega_{_{KE}}^{-1} \lrcorner \partial f\right).$$

In local coordinate we have $P(f) = \partial_z\left(\lambda^{-1}\partial_z f\right) dz \otimes \partial_{\bar{z}}$. For each $1 \leq k \leq n$ we let

$$\xi_k : C^\infty(X_s) \to C^\infty(X_s)$$

be the operator defined by

$$f \mapsto \overline{\partial}^*\left(B_k \lrcorner \partial f\right) = -B_k \cdot P(f).$$

In the local coordinate we have $\xi_k(f) = -\lambda^{-1}\partial_z\left(A_k \partial_z f\right)$. Finally for any $1 \leq k, l \leq n$ we define the operator

$$Q_{k\bar{l}} : C^\infty(X_s) \to C^\infty(X_s)$$

by

$$Q_{k\bar{l}}(f)=\overline{P}\left(e_{k\bar{l}}\right)P(f)-2f_{k\bar{l}}\Box f+\lambda^{-1}\partial_z f_{k\bar{l}}\partial_{\bar{z}}f.$$

These operators are commutators of various classical operators on X_s. See [**12**] for details. Now we recall the curvature formulae of the Ricci and perturbed Ricci metrics established in [**12**]. For convenience, we introduce the symmetrization operator.

Definition 3.1. *Let U be any quantity which depends on indices $i,k,\alpha,\overline{j},\overline{l},\overline{\beta}$. The symmetrization operator σ_1 is defined by taking the summation of all orders of the triple (i,k,α). That is*

$$\begin{aligned}\sigma_1(U(i,k,\alpha,\overline{j},\overline{l},\overline{\beta}))=&U(i,k,\alpha,\overline{j},\overline{l},\overline{\beta})+U(i,\alpha,k,\overline{j},\overline{l},\overline{\beta})+U(k,i,\alpha,\overline{j},\overline{l},\overline{\beta})\\&+U(k,\alpha,i,\overline{j},\overline{l},\overline{\beta})+U(\alpha,i,k,\overline{j},\overline{l},\overline{\beta})+U(\alpha,k,i,\overline{j},\overline{l},\overline{\beta}).\end{aligned}$$

Similarly, σ_2 is the symmetrization operator of $\overline{j}$ and $\overline{\beta}$ and $\widetilde{\sigma_1}$ is the symmetrization operator of $\overline{j}$, $\overline{l}$ and $\overline{\beta}$.

Now we can state the curvature formulae. We let $T=(\Box+1)^{-1}$ be the operator in the fiber direction.

Theorem 3.1. *Let $s_1,\cdots,s_n$ be local holomorphic coordinates at $s\in\mathcal{M}_g$ and let $\widetilde{R}_{i\bar{j}k\bar{l}}$ be the curvature of the Ricci metric. Then at s, we have*

(3.1)

$$\begin{aligned}\widetilde{R}_{i\bar{j}k\bar{l}}=&-h^{\alpha\overline{\beta}}\left\{\sigma_1\sigma_2\int_{X_s}\left\{T(\xi_k(e_{i\bar{j}}))\overline{\xi}_l(e_{\alpha\overline{\beta}})+T(\xi_k(e_{i\bar{j}}))\overline{\xi}_\beta(e_{\alpha\bar{l}})\right\}\,dv\right\}\\&-h^{\alpha\overline{\beta}}\left\{\sigma_1\int_{X_s}Q_{k\bar{l}}(e_{i\bar{j}})e_{\alpha\overline{\beta}}\,dv\right\}\\&+\tau^{p\bar{q}}h^{\alpha\overline{\beta}}h^{\gamma\overline{\delta}}\left\{\sigma_1\int_{X_s}\xi_k(e_{i\bar{q}})e_{\alpha\overline{\beta}}\,dv\right\}\left\{\widetilde{\sigma}_1\int_{X_s}\overline{\xi}_l(e_{p\bar{j}})e_{\gamma\overline{\delta}})\,dv\right\}\\&+\tau_{p\bar{j}}h^{p\bar{q}}R_{i\bar{q}k\bar{l}}.\end{aligned}$$

Theorem 3.2. *Let $\widetilde{\tau}_{i\bar{j}}=\tau_{i\bar{j}}+Ch_{i\bar{j}}$ where τ and h are the Ricci and WP metrics respectively where $C>0$ is a constant. Let $P_{i\bar{j}k\bar{l}}$ be the curvature of the perturbed Ricci metric. Then we have*

(3.2)

$$\begin{aligned}P_{i\bar{j}k\bar{l}}=&-h^{\alpha\overline{\beta}}\left\{\sigma_1\sigma_2\int_{X_s}\left\{T(\xi_k(e_{i\bar{j}}))\overline{\xi}_l(e_{\alpha\overline{\beta}})+T(\xi_k(e_{i\bar{j}}))\overline{\xi}_\beta(e_{\alpha\bar{l}})\right\}\,dv\right\}\\&-h^{\alpha\overline{\beta}}\left\{\sigma_1\int_{X_s}Q_{k\bar{l}}(e_{i\bar{j}})e_{\alpha\overline{\beta}}\,dv\right\}\\&+\widetilde{\tau}^{p\bar{q}}h^{\alpha\overline{\beta}}h^{\gamma\overline{\delta}}\left\{\sigma_1\int_{X_s}\xi_k(e_{i\bar{q}})e_{\alpha\overline{\beta}}\,dv\right\}\left\{\widetilde{\sigma}_1\int_{X_s}\overline{\xi}_l(e_{p\bar{j}})e_{\gamma\overline{\delta}})\,dv\right\}\\&+\tau_{p\bar{j}}h^{p\bar{q}}R_{i\bar{q}k\bar{l}}+CR_{i\bar{j}k\bar{l}}.\end{aligned}$$

In [**12**] and [**13**] we proved various properties of these new metrics. Here we collect the important ones.

Theorem 3.3. *The Ricci and perturbed Ricci metrics are complete Kähler metrics on $\mathcal{M}_g$. Furthermore we have*

- *These two metrics have bounded curvature.*
- *The injectivity radius of the Teichmüller space $\mathcal{T}_g$ equipped with any of these two metrics is bounded from below.*
- *These metrics have Poincaré growth and thus the moduli space has finite volume when equipped with any of these metrics.*
- *The perturbed Ricci metric has negatively pinched holomorphic sectional and Ricci curvatures when we choose the constant C to be large enough.*

The Ricci metric is also cohomologous to the Kähler-Einstein metric on $\mathcal{M}_g$ in the sense of currents and hence can be used as the background metric to estimate the Kähler-Einstein metric. We will discuss this in Section 5.

4. Asymptotics

Since the moduli space $\mathcal{M}_g$ is noncompact, it is important to understand the asymptotic behavior of the canonical metrics in order to study their global properties. We first describe the local pinching coordinates near the boundary of the moduli space by using the plumbing construction of Wolpert.

Let $\mathcal{M}_g$ be the moduli space of Riemann surfaces of genus $g \geq 2$ and let $\overline{\mathcal{M}}_g$ be its Deligne-Mumford compactification [**5**]. Each point $y \in \overline{\mathcal{M}}_g \setminus \mathcal{M}_g$ corresponds to a stable nodal surface X_y. A point $p \in X_y$ is a node if there is a neighborhood of p which is isometric to the germ $\{(u,v) \mid uv = 0,\ |u|,|v| < 1\} \subset \mathbb{C}^2$.

We first recall the rs-coordinate on a Riemann surface defined by Wolpert in [**35**]. There are two cases: the puncture case and the short geodesic case. For the puncture case, we have a nodal surface X and a node $p \in X$. Let a, b be two punctures which are glued together to form p.

Definition 4.1. *A local coordinate chart (U,u) near a is called rs-coordinate if $u(a) = 0$ where u maps U to the punctured disc $0 < |u| < c$ with $c > 0$, and the restriction to U of the Kähler-Einstein metric on X can be written as $\frac{1}{2|u|^2(\log|u|)^2}|du|^2$. The rs-coordinate (V,v) near b is defined in a similar way.*

For the short geodesic case, we have a closed surface X, a closed geodesic $\gamma \subset X$ with length $l < c_*$ where c_* is the collar constant.

Definition 4.2. *A local coordinate chart (U,z) is called rs-coordinate at γ if $\gamma \subset U$ where z maps U to the annulus $c^{-1}|t|^{\frac{1}{2}} < |z| < c|t|^{\frac{1}{2}}$, and the Kähler-Einstein metric on X can be written as $\frac{1}{2}(\frac{\pi}{\log|t|}\frac{1}{|z|}\csc\frac{\pi\log|z|}{\log|t|})^2|dz|^2$.*

By Keen's collar theorem [**9**], we have the following lemma:

Lemma 4.1. *Let X be a closed surface and let γ be a closed geodesic on X such that the length l of γ satisfies $l < c_*$. Then there is a collar Ω on X with holomorphic coordinate z defined on Ω such that*

(1) *z maps Ω to the annulus $\frac{1}{c}e^{-\frac{2\pi^2}{l}} < |z| < c$ for $c > 0$;*
(2) *the Kähler-Einstein metric on X restricted to Ω is given by*

$$(\frac{1}{2}u^2 r^{-2}\csc^2\tau)|dz|^2 \tag{4.1}$$

where $u = \frac{l}{2\pi}$, $r = |z|$ and $\tau = u\log r$;

(3) *the geodesic γ is given by the equation $|z| = e^{-\frac{\pi^2}{l}}$.*

We call such a collar Ω a genuine collar.

We notice that the constant c in the above lemma has a lower bound such that the area of Ω is bounded from below. Also, the coordinate z in the above lemma is an rs-coordinate. In the following, we will keep the notations u, r and τ.

Now we describe the local manifold cover of $\overline{\mathcal{M}}_g$ near the boundary. We take the construction of Wolpert [**35**]. Let $X_{0,0}$ be a stable nodal surface corresponding to a codimension m boundary point and let $p_1, \cdots, p_m$ be the nodes of $X_{0.0}$. The smooth part $X_0 = X_{0,0} \setminus \{p_1, \cdots, p_m\}$ is a union of punctured Riemann surfaces. Fix the rs-coordinate charts (U_i, η_i) and (V_i, ζ_i) at p_i for $i = 1, \cdots, m$ such that all the U_i and V_i are mutually disjoint.

Now pick an open set $U_0 \subset X_0$ such that the intersection of each connected component of X_0 and U_0 is a nonempty relatively compact set and the intersection $U_0 \cap (U_i \cup V_i)$ is empty for all i. We pick Beltrami differentials $\nu_{m+1}, \cdots, \nu_n$ which are supported in U_0 and span the tangent space at X_0 of the deformation space of X_0. For $s = (s_{m+1}, \cdots, s_n)$, let $\nu(s) = \sum_{i=m+1}^n s_i\nu_i$. We assume $|s| = (\sum |s_i|^2)^{\frac{1}{2}}$ small enough such that $|\nu(s)| < 1$. The nodal surface $X_{0,s}$ is obtained by solving the Beltrami equation $\overline{\partial}w = \nu(s)\partial w$. Since $\nu(s)$ is supported in U_0, (U_i, η_i) and (V_i, ζ_i) are still holomorphic coordinates for $X_{0,s}$. Note that they are no longer rs-coordinates. By the theory of Ahlfors and Bers [**1**] and Wolpert [**35**] we can assume that there are constants $\delta, c > 0$ such that when $|s| < \delta$, η_i and ζ_i are holomorphic coordinates on $X_{0,s}$ with $0 < |\eta_i| < c$ and $0 < |\zeta_i| < c$.

Now we assume $t = (t_1, \cdots, t_m)$ has small norm. We do the plumbing construction on $X_{0,s}$ to obtain $X_{t,s}$ in the following way. We remove from $X_{0,s}$ the discs $0 < |\eta_i| \le \frac{|t_i|}{c}$ and $0 < |\zeta_i| \le \frac{|t_i|}{c}$ for each $i = 1, \cdots, m$, and identify $\frac{|t_i|}{c} < |\eta_i| < c$ with $\frac{|t_i|}{c} < |\zeta_i| < c$ by the rule $\eta_i\zeta_i = t_i$. This defines the surface $X_{t,s}$. The tuple $(t_1, \cdots, t_m, s_{m+1}, \cdots, s_n)$ are the local pinching coordinates for the manifold cover of $\overline{\mathcal{M}}_g$. We call the coordinates η_i (or ζ_i) the plumbing coordinates on $X_{t,s}$ and the collar defined by $\frac{|t_i|}{c} < |\eta_i| < c$ the plumbing collar.

REMARK 4.1. *¿From the estimate of Wolpert* [**34**], [**35**] *on the length of short geodesic, we have* $u_i = \frac{l_i}{2\pi} \sim -\frac{\pi}{\log|t_i|}$.

Let $(t,s) = (t_1,\cdots,t_m,s_{m+1},\cdots,s_n)$ be the pinching coordinates near $X_{0,0}$. For $|(t,s)| < \delta$, let Ω_c^j be the j-th genuine collar on $X_{t,s}$ which contains a short geodesic γ_j with length l_j. Let $u_j = \frac{l_j}{2\pi}$, $u_0 = \sum_{j=1}^m u_j + \sum_{j=m+1}^n |s_j|$, $r_j = |z_j|$ and $\tau_j = u_j \log r_j$ where z_j is the properly normalized rs-coordinate on Ω_c^j such that

$$\Omega_c^j = \{z_j \mid c^{-1} e^{-\frac{2\pi^2}{l_j}} < |z_j| < c\}.$$

¿From the above argument, we know that the Kähler-Einstein metric λ on $X_{t,s}$, restrict to the collar Ω_c^j, is given by

$$\lambda = \frac{1}{2} u_j^2 r_j^{-2} \csc^2 \tau_j. \tag{4.2}$$

For convenience, we let $\Omega_c = \cup_{j=1}^m \Omega_c^j$ and $R_c = X_{t,s} \setminus \Omega_c$. In the following, we may change the constant c finitely many times, clearly this will not affect the estimates.

To estimate the WP, Ricci and perturbed Ricci metrics and their curvatures, we first need to to find all the harmonic Beltrami differentials $B_1,\cdots,B_n$ corresponding to the tangent vectors $\frac{\partial}{\partial t_1},\cdots,\frac{\partial}{\partial s_n}$. In [**21**], Masur constructed $3g-3$ regular holomorphic quadratic differentials $\psi_1,\cdots,\psi_n$ on the plumbing collars by using the plumbing coordinate η_j. These quadratic differentials correspond to the cotangent vectors $dt_1,\cdots,ds_n$.

However, it is more convenient to estimate the curvature if we use the rs-coordinate on $X_{t,s}$ since we have the accurate form of the Kähler-Einstein metric λ in this coordinate. In [**32**], Trapani used the graft metric constructed by Wolpert [**35**] to estimate the difference between the plumbing coordinate and rs-coordinate and described the holomorphic quadratic differentials constructed by Masur in the rs-coordinate. We collect Trapani's results (Lemma 6.2-6.5, [**32**]) in the following theorem:

THEOREM 4.1. *Let* (t,s) *be the pinching coordinates on* $\overline{\mathcal{M}}_g$ *near* $X_{0,0}$ *which corresponds to a codimension* m *boundary point of* $\overline{\mathcal{M}}_g$. *Then there exist constants* $M,\delta > 0$ *and* $1 > c > 0$ *such that if* $|(t,s)| < \delta$, *then the* j*-th plumbing collar on* $X_{t,s}$ *contains the genuine collar* Ω_c^j. *Furthermore, one can choose rs-coordinate* z_j *on the collar* Ω_c^j *such that the holomorphic quadratic differentials* $\psi_1,\cdots,\psi_n$ *corresponding to the cotangent vectors* $dt_1,\cdots,ds_n$ *have the form* $\psi_i = \varphi_i(z_j)dz_j^2$ *on the genuine collar* Ω_c^j *for* $1 \le j \le m$, *where*

(1) $\varphi_i(z_j) = \frac{1}{z_j^2}(q_i^j(z_j) + \beta_i^j)$ *if* $i \ge m+1$;
(2) $\varphi_i(z_j) = (-\frac{t_j}{\pi})\frac{1}{z_j^2}(q_j(z_j) + \beta_j)$ *if* $i = j$;
(3) $\varphi_i(z_j) = (-\frac{t_i}{\pi})\frac{1}{z_j^2}(q_i^j(z_j) + \beta_i^j)$ *if* $1 \le i \le m$ *and* $i \ne j$.

Here β_i^j and β_j are functions of (t,s), q_i^j and q_j are functions of (t,s,z_j) given by

$$q_i^j(z_j)=\sum_{k<0}\alpha_{ik}^j(t,s)t_j^{-k}z_j^k+\sum_{k>0}\alpha_{ik}^j(t,s)z_j^k$$

and

$$q_j(z_j)=\sum_{k<0}\alpha_{jk}(t,s)t_j^{-k}z_j^k+\sum_{k>0}\alpha_{jk}(t,s)z_j^k$$

such that

(1) $\sum_{k<0}|\alpha_{ik}^j|c^{-k}\leq M$ *and* $\sum_{k>0}|\alpha_{ik}^j|c^k\leq M$ *if* $i\neq j$*;*
(2) $\sum_{k<0}|\alpha_{jk}|c^{-k}\leq M$ *and* $\sum_{k>0}|\alpha_{jk}|c^k\leq M$*;*
(3) $|\beta_i^j|=O(|t_j|^{\frac{1}{2}-\epsilon})$ *with* $\epsilon<\frac{1}{2}$ *if* $i\neq j$*;*
(4) $|\beta_j|=(1+O(u_0))$*.*

An immediate consequence is the precise asymptotics of the WP metric which was computed in [**12**]. These asymptotic estimates were also given by Wolpert in [**36**].

THEOREM 4.2. *Let (t,s) be the pinching coordinates and let h be the WP metric. Then*

(1) $h^{i\bar{i}}=2u_i^{-3}|t_i|^2(1+O(u_0))$ *and* $h_{i\bar{i}}=\frac{1}{2}\frac{u_i^3}{|t_i|^2}(1+O(u_0))$ *for* $1\leq i\leq m$*;*
(2) $h^{i\bar{j}}=O(|t_it_j|)$ *and* $h_{i\bar{j}}=O\left(\frac{u_i^3u_j^3}{|t_it_j|}\right)$*, if* $1\leq i,j\leq m$ *and* $i\neq j$*;*
(3) $h^{i\bar{j}}=O(1)$ *and* $h_{i\bar{j}}=O(1)$*, if* $m+1\leq i,j\leq n$*;*
(4) $h^{i\bar{j}}=O(|t_i|)$ *and* $h_{i\bar{j}}=O\left(\frac{u_i^3}{|t_i|}\right)$ *if* $i\leq m<j$*;*
(5) $h^{i\bar{j}}=O(|t_j|)$ *and* $h_{i\bar{j}}=O\left(\frac{u_j^3}{|t_j|}\right)$ *if* $j\leq m<i$*.*

By using the asymptotics of the WP metric and the fact that

$$B_i=\lambda^{-1}\sum_{j=1}^n h_{i\bar{j}}\overline{\psi}_j$$

we can derive the expansion of the harmonic Beltrami differentials corresponding to $\frac{\partial}{\partial t_i}$ and $\frac{\partial}{\partial s_j}$.

THEOREM 4.3. *For c small, on the genuine collar Ω_c^j, the coefficient functions A_i of the harmonic Beltrami differentials have the form:*

(1) $A_i=\frac{z_j}{\bar{z}_j}\sin^2\tau_j\left(\overline{p_i^j(z_j)}+\overline{b_i^j}\right)$ *if* $i\neq j$*;*
(2) $A_j=\frac{z_j}{\bar{z}_j}\sin^2\tau_j\left(\overline{p_j(z_j)}+\overline{b_j}\right)$

where

(1) $p_i^j(z_j)=\sum_{k\leq -1}a_{ik}^j\rho_j^{-k}z_j^k+\sum_{k\geq 1}a_{ik}^jz_j^k$ *if* $i\neq j$*;*
(2) $p_j(z_j)=\sum_{k\leq -1}a_{jk}\rho_j^{-k}z_j^k+\sum_{k\geq 1}a_{jk}z_j^k$*.*

In the above expressions, $\rho_j = e^{-\frac{2\pi^2}{l_j}}$ *and the coefficients satisfy the following conditions:*

(1) $\sum_{k\le -1} |a_{ik}^j| c^{-k} = O\left(u_j^{-2}\right)$ *and* $\sum_{k\ge 1} |a_{ik}^j| c^{k} = O\left(u_j^{-2}\right)$ *if* $i \ge m+1$*;*

(2) $\sum_{k\le -1} |a_{ik}^j| c^{-k} = O\left(\frac{u_i^3 u_j^{-2}}{|t_i|}\right)$ *and* $\sum_{k\ge 1} |a_{ik}^j| c^{k} = O\left(\frac{u_i^3 u_j^{-2}}{|t_i|}\right)$ *if* $i \le m$ *and* $i \ne j$*;*

(3) $\sum_{k\le -1} |a_{jk}| c^{-k} = O\left(\frac{u_j}{|t_j|}\right)$ *and* $\sum_{k\ge 1} |a_{jk}| c^{k} = O\left(\frac{u_j}{|t_j|}\right)$*;*

(4) $|b_i^j| = O(u_j)$ *if* $i \ge m+1$*;*

(5) $|b_i^j| = O\left(u_j\right) O\left(\frac{u_i^3}{|t_i|}\right)$ *if* $i \le m$ *and* $i \ne j$*;*

(6) $b_j = -\frac{u_j}{\pi \bar{t}_j}(1 + O(u_0))$.

By a detailed study of the curvature of the WP metric we derived the precise asymptotics of the Ricci metric in [**12**].

THEOREM 4.4. *Let* (t, s) *be the pinching coordinates. Then we have*

(1) $\tau_{i\bar{i}} = \frac{3}{4\pi^2} \frac{u_i^2}{|t_i|^2}(1 + O(u_0))$ *and* $\tau^{i\bar{i}} = \frac{4\pi^2}{3} \frac{|t_i|^2}{u_i^2}(1 + O(u_0))$*, if* $i \le m$*;*

(2) $\tau_{i\bar{j}} = O\left(\frac{u_i^2 u_j^2}{|t_i t_j|}(u_i + u_j)\right)$ *and* $\tau^{i\bar{j}} = O(|t_i t_j|)$*, if* $i, j \le m$ *and* $i \ne j$*;*

(3) $\tau_{i\bar{j}} = O\left(\frac{u_i^2}{|t_i|}\right)$ *and* $\tau^{i\bar{j}} = O(|t_i|)$*, if* $i \le m$ *and* $j \ge m+1$*;*

(4) $\tau_{i\bar{j}} = O\left(\frac{u_j^2}{|t_j|}\right)$ *and* $\tau^{i\bar{j}} = O(|t_j|)$*, if* $j \le m$ *and* $i \ge m+1$*;*

(5) $\tau_{i\bar{j}} = O(1)$*, if* $i, j \ge m+1$.

In [**12**] we also derived the asymptotics of the curvature of the Ricci metric.

THEOREM 4.5. *Let* $X_0 \in \overline{\mathcal{M}_g} \setminus \mathcal{M}_g$ *be a codimension* m *point and let* $(t_1, \cdots, t_m, s_{m+1}, \cdots, s_n)$ *be the pinching coordinates at* X_0 *where* $t_1, \cdots, t_m$ *correspond to the degeneration directions. Then the holomorphic sectional curvature is negative in the degeneration directions and is bounded in the non-degeneration directions. More precisely, there exists* $\delta > 0$ *such that, if* $|(t, s)| < \delta$*, then*

$$\widetilde{R}_{i\bar{i}i\bar{i}} = -\frac{3u_i^4}{8\pi^4 |t_i|^4}(1 + O(u_0)) \tag{4.3}$$

if $i \le m$ *and*

$$\left| \widetilde{R}_{i\bar{i}i\bar{i}} \right| = O(1) \tag{4.4}$$

if $i \ge m+1$*. Here* $\widetilde{R}$ *is the curvature of the Ricci metric.*

Furthermore, on $\mathcal{M}_g$*, the holomorphic sectional curvature, the bisectional curvature and the Ricci curvature of the Ricci metric are bounded from above and below.*

In [**16**] and [**17**] we derived more precise estimates of the curvature of the Ricci and perturbed Ricci metrics which we will discuss in Section 6.

5. Canonical Metrics and Equivalence

In addition to the WP, Ricci and perturbed Ricci metrics on the moduli space, there are several other canonical metrics on $\mathcal{M}_g$. These include the Teichmüller metric, the Kobayashi metric, the Carathéodory metric, the Kähler-Einstein metric, the induced Bergman metric, the McMullen metric and the asymptotic Poincaré metric.

Firstly, on any complex manifold there are two famous Finsler metrics: the Carathéodory and Kobayashi metrics. Now we describe these metrics.

Let X be a complex manifold and of dimension n. let Δ_R be the disk in $\mathbb{C}$ with radius R. Let $\Delta = \Delta_1$ and let ρ be the Poincaré metric on Δ. Let $p \in X$ be a point and let $v \in T_pX$ be a holomorphic tangent vector. Let $\mathrm{Hol}(X, \Delta_R)$ and $\mathrm{Hol}(\Delta_R, X)$ be the spaces of holomorphic maps from X to Δ_R and from Δ_R to X respectively. The Carathéodory norm of the vector v is defined to be

$$\|v\|_C = \sup_{f \in \mathrm{Hol}(X,\Delta)} \|f_* v\|_{\Delta,\rho}$$

and the Kobayashi norm of v is defined to be

$$\|v\|_K = \inf_{f \in \mathrm{Hol}(\Delta_R, X),\ f(0)=p,\ f'(0)=v} \frac{2}{R}.$$

It is well known that the Carathéodory metric is bounded from above by the Kobayashi metric after proper normalization. The first known metric on the Teichmüller space $\mathcal{T}_g$ is the Teichmüller metric which is also an Finsler metric. Royden showed that, on $\mathcal{T}_g$, the Teichmüller metric coincides with the Kobayashi metric. Generalizations and proofs of Royden's theorem can be found in [**20**].

Now we look at the Kähler metrics. Firstly, since the Teichmüller space $\mathcal{T}_g$ is a pseudo-convex domain, by the work of Cheng and Yau [**4**] and the later work of Yau, there exist a unique complete Kähler-Einstein metric on $\mathcal{T}_g$ whose Ricci curvature is -1.

There is also a canonical Bergman metric on $\mathcal{T}_g$ which we describe now. In general, let X be any complex manifold, let K_X be the canonical bundle of X and let W be the space of L^2 holomorphic sections of K_X in the sense that if $\sigma \in W$, then

$$\|\sigma\|_{L^2}^2 = \int_X (\sqrt{-1})^{n^2} \sigma \wedge \overline{\sigma} < \infty.$$

The inner product on W is defined to be

$$(\sigma, \rho) = \int_X (\sqrt{-1})^{n^2} \sigma \wedge \overline{\rho}$$

for all $\sigma, \rho \in W$. Let $\sigma_1, \sigma_2, \cdots$ be an orthonormal basis of W. The Bergman kernel form is the non-negative (n,n)-form

$$B_X = \sum_{j=1}^{\infty} (\sqrt{-1})^{n^2} \sigma_j \wedge \overline{\sigma}_j.$$

With a choice of local coordinates $z_i, \cdots, z_n$, we have

$$B_X = BE_X(z, \overline{z})(\sqrt{-1})^{n^2} dz_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_1 \wedge \cdots \wedge d\overline{z}_n$$

where $BE_X(z, \overline{z})$ is called the Bergman kernel function. If the Bergman kernel B_X is positive, one can define the Bergman metric

$$B_{i\overline{j}} = \frac{\partial^2 \log BE_X(z, \overline{z})}{\partial z_i \partial \overline{z}_j}.$$

The Bergman metric is well-defined and is nondegenerate if the elements in W separate points and the first jet of X.

It is easy to see that both the Kähler-Einstein metric and the Bergman metric on the Teichmüller space $\mathcal{T}_g$ are invariant under the action of the mapping class group and thus descend down to the moduli space.

Remark 5.1. *We note that the induced Bergman metric on $\mathcal{M}_g$ is different from the Bergman metric on $\mathcal{M}_g$.*

In [**22**] McMullen introduced another Kähler metric $g_{1/l}$ on $\mathcal{M}_g$ which is equivalent to the Teichmüller metric. Let $Log: \mathbb{R}_+ \to [0, \infty)$ be a smooth function such that

(1) $Log(x) = \log x$ if $x \geq 2$;
(2) $Log(x) = 0$ if $x \leq 1$.

For suitable choices of small constants $\delta, \epsilon > 0$, the Kähler form of the McMullen metric $g_{1/l}$ is

$$\omega_{1/l} = \omega_{WP} - i\delta \sum_{l_\gamma(X) < \epsilon} \partial\overline{\partial} Log \frac{\epsilon}{l_\gamma}$$

where the sum is taken over primitive short geodesics γ on X.

Finally, since $\mathcal{M}_g$ is quasi-projective, there exists a non-canonical asymptotic Poincaré metric ω_P on $\mathcal{M}_g$. In general, Let $\overline{M}$ be a compact projective manifold of dimension m. Let $Y \subset \overline{M}$ be a divisor of normal crossings and let $M = \overline{M} \setminus Y$. Cover $\overline{M}$ by coordinate charts $U_1, \cdots, U_p, \cdots, U_q$ such that $(\overline{U}_{p+1} \cup \cdots \cup \overline{U}_q) \cap Y = \emptyset$. We also assume that, for each $1 \leq \alpha \leq p$, there is a constant n_α such that $U_\alpha \setminus Y = (\Delta^*)^{n_\alpha} \times \Delta^{m-n_\alpha}$ and on U_α, Y is given by $z_1^\alpha \cdots z_{n_\alpha}^\alpha = 0$. Here Δ is the disk of radius $\frac{1}{2}$ and Δ^* is the punctured disk of radius $\frac{1}{2}$. Let $\{\eta_i\}_{1 \leq i \leq q}$ be the partition of unity subordinate to the cover $\{U_i\}_{1 \leq i \leq q}$. Let ω be a Kähler metric on $\overline{M}$ and let C be a positive constant. Then for C large, the Kähler form

$$\omega_P = C\omega + \sum_{i=1}^{p} \sqrt{-1} \partial\overline{\partial} \left(\eta_i \log\log \frac{1}{\left| z_1^i \cdots z_{n_i}^i \right|} \right)$$

defines a complete metric on M with finite volume since on each U_i with $1 \leq i \leq p$, ω_p is bounded from above and below by the local Poincaré metric on U_i. We call this metric the asymptotic Poincaré metric.

In 2004 we proved in [**12**] that all complete metrics on the moduli space are equivalent. The proof is based on asymptotic analysis of these metrics and Yau's Schwarz Lemma. It is an easy corollaries of our understanding of the Ricci and the perturbed Ricci metrics. In July 2004 we learned from the announcement of S.-K. Yeung in Hong Kong University where he announced he could prove a small and easy part of our results about the equivalences of some of these metrics by using a bounded pluri-subharmonic function. We received a hard copy of Yeung's paper in November 2004 where he used a method similar to ours in [**12**] to compare the Bergman, the Kobayashi and the Carathéodory metric. It should be interesting to see how one can use the bounded psh function to derive these equivalences.

We recall that two metrics on $\mathcal{M}_g$ are equivalent if one metric is bounded from above and below by positive constant multiples of the other metric.

Theorem 5.1. *On the moduli space $\mathcal{M}_g$ the Ricci metric, the perturbed Ricci metric, the Kähler-Einstein metric, the induced Bergman metric, the McMullen metric, the asymptotic Poincaré metric, the Carathéodory metric and the Teichmüller-Kobayashi metric are equivalent.*

The equivalence of several of these metrics hold in more general setting. In 2004 we defined the holomorphic homogeneous regular manifolds in [**12**] which generalized the idea of Morrey.

Definition 5.1. *A complex manifold X of dimension n is called holomorphic homogeneous regular if there are positive constants $r < R$ such that for each point $p \in X$ there is a holomorphic map $f_p : X \to \mathbb{C}^n$ which satisfies*

(1) $f_p(p) = 0$;
(2) $f_p : X \to f_p(X)$ *is a biholomorphism;*
(3) $B_r \subset f_p(X) \subset B_R$ *where B_r and B_R are Euclidean balls with center 0 in $\mathbb{C}^n$.*

In 2009 Yeung [**40**] used the above definition without appropriate reference which he called domain with uniform squeezing property. It follows from the restriction properties of canonical metrics and Yau's Schwarz Lemma that

Theorem 5.2. *Let X be a holomorphic homogeneous regular manifold. Then the Kobayashi metric, the Bergman metric and the Carathéodory metric on X are equivalent.*

Remark 5.2. *It follows from the Bers embedding theorem that the Teichmüller space of genus g Riemann surfaces is a holomorphic homogeneous regular manifold if we choose $r = 2$ and $R = 6$ in Definition 5.1.*

6. Goodness of Canonical Metrics

In his work [**23**], Mumford defined the goodness condition to study the currents of Chern forms defined by a singular Hermitian metric on a holomorphic bundle over a quasi-projective manifold where he generalized the Hirzebruch's proportionality theorem to noncompact case. The goodness condition is a growth condition of the Hermitian metric near the compactification divisor of the base manifold. The major property of a good metric is that the currents of its Chern forms define the Chern classes of the bundle. Namely the Chern-Weil theory works in this noncompact case.

Beyond the case of homogeneous bundles over symmetric spaces discussed by Mumford in [**23**], several natural bundles over moduli spaces of Riemann surfaces give beautiful and useful examples. In [**35**], Wolpert showed that the metric induced by the hyperbolic metric on the relative dualizing sheaf over the universal curve of moduli space of hyperbolic Riemann surfaces is good. Later it was shown by Trapani [**32**] that the metric induced by the WP metric on the determinant line bundle of the logarithmic cotangent bundle of the Deligne-Mumford moduli space is good. In both cases, the bundles involved are line bundles in which cases it is easier to estimate the connection and curvature. Other than these, very few examples of natural good metrics are known.

The goodness of the WP metric has been a long standing open problem. In this section we describe our work in [**16**] which gives a positive answer to this problem.

We first recall the definition of good metrics and their basic properties described in [**23**]. Let $\overline{X}$ be a projective manifold of complex dimension $\dim_{\mathbb{C}} \overline{X} = n$. Let $D \subset \overline{X}$ be a divisor of normal crossing and let $X = \overline{X} \setminus D$ be a Zariski open manifold. We let Δ_r be the open disk in $\mathbb{C}$ with radius r, let $\Delta = \Delta_1$, $\Delta_r^* = \Delta_r \setminus \{0\}$ and $\Delta^* = \Delta \setminus \{0\}$. For each point $p \in D$ we can find a coordinate chart $(U, z_1, \cdots, z_n)$ around p in $\overline{X}$ such that $U \cong \Delta^n$ and $V = U \cap X \cong (\Delta^*)^m \times \Delta^{n-m}$. We assume that $U \cap D$ is defined by the equation $z_1 \cdots z_k = 0$. We let $U(r) \cong \Delta_r^n$ for $0 < r < 1$ and let $V(r) = U(r) \cap X$.

On the chart V of X we can define a local Poincaré metric:

$$\omega_{loc} = \frac{\sqrt{-1}}{2} \sum_{i=1}^{k} \frac{dz_i \wedge d\overline{z}_i}{2|z_i|^2 \left(\log |z_i|\right)^2} + \frac{\sqrt{-1}}{2} \sum_{i=k+1}^{n} dz_i \wedge d\overline{z}_i. \tag{6.1}$$

Now we cover $D \subset \overline{X}$ by such coordinate charts $U_1, \cdots, U_q$ and let $V_i = U_i \cap X$. We choose coordinates $z_1^i, \cdots, z_n^i$ such that $D \cap U_i$ is given by $z_1^i \cdots z_{m_i}^i = 0$.

A Kähler metric ω_g on X has Poincaré growth if for each $1 \leq i \leq q$ there are constants $0 \leq r_i \leq 1$ and $0 \leq c_i < C_i$ such that $\omega_g \mid_{V_i(r_i)}$ is equivalent to the local Poincaré metric ω_{loc}^i:

$$c_i \omega_{loc}^i \leq \omega_g \mid_{V_i(r_i)} \leq C_i \omega_{loc}^i.$$

In [**23**] Mumford defined differential forms with Poincaré growth:

Definition 6.1. *Let $\eta \in A^p(X)$ be a smooth p-form. Then η has Poincaré growth if for each $1 \leq i \leq q$ there exists a constant $c_i > 0$ such that for each point $s \in V_i\left(\frac{1}{2}\right)$ and tangent vectors $t_1, \cdots, t_p \in T_sX$ one has*

$$|\eta(t_1, \cdots, t_p)|^2 \leq c_i \prod_{j=1}^{p} \omega^i_{loc}(t_j, t_j).$$

The p-form η is good if and only if both η and $d\eta$ have Poincaré growth.

Remark 6.1. *It is easy to see that the above definition does not depend on the choice of the cover $(U_1, \cdots, U_q)$ but it does depend on the compactification $\overline{X}$ of X.*

The above definition is local. We now give a global formulation.

Lemma 6.1. *Let ω_g be a Kähler metric on X with Poincaré growth. Then a p-form $\eta \in A^p(X)$ has Poincaré growth if and only if $\|\eta\|_g < \infty$ where $\|\eta\|_g$ is the C^0 norm of η with respect to the metric g. Furthermore, the fact that η has Poincaré growth is independent of the choice of g. It follows that if $\eta_1 \in A^p(X)$ and $\eta_2 \in A^q(X)$ have Poincaré growth, then $\eta_1 \wedge \eta_2$ also has Poincaré growth.*

Now we collect the basic properties of forms with Poincaré growth as described in [**23**].

Lemma 6.2. *Let $\eta \in A^p(X)$ be a form with Poincaré growth. Then η defines a p-current on $\overline{X}$. Furthermore, if η is good then $d[\eta] = [d\eta]$.*

Now we consider a holomorphic vector bundle $\overline{E}$ of rank r over $\overline{X}$. Let $E = \overline{E}\mid_X$ and let h be a Hermitian metric on E. According to [**23**] we have

Definition 6.2. *The Hermitian metric h is good if for any point $x \in D$, assume $x \in U_i$ for some i, and any basis $e_1, \cdots, e_r$ of $\overline{E}\mid_{U_i\left(\frac{1}{2}\right)}$, if we let $h_{\alpha\overline{\beta}} = h(e_\alpha, e_\beta)$ then there exist positive constants c_i, d_i such that*

(1) $\left|h_{\alpha\overline{\beta}}\right|, (\det h)^{-1} \leq c_i \left(\sum_{j=1}^{m_i} \log|z_j|\right)^{2d_i}$;
(2) *the* 1*-forms* $\left(\partial h \cdot h^{-1}\right)_{\alpha\beta}$ *are good on* $V_i\left(\frac{1}{2}\right)$.

Remark 6.2. *A simple computation shows that the goodness of h is independent of the choice of the cover of D. Furthermore, to check whether a metric h is good or not by using the above definition, we only need to check the above two conditions for one choice of the basis $e_1, \cdots, e_r$.*

The most important features of a good metric are

Theorem 6.1. *Let h be a Hermitian metric on E. Then there is at most one extension of E to $\overline{X}$ for which h is good. Furthermore, if h is a good metric on E, then the Chern forms $c_k(E, h)$ are good and the current $[c_k(E, h)] = c_k(\overline{E}) \in H^{2k}(\overline{X})$.*

See [**23**] for details. This theorem allows us to compute the Chern classes by using Chern forms of a singular good metric.

Now we look at a special choice of the bundle E. In the following we let $\overline{E} = T_{\overline{X}}(-\log D)$ to be the logarithmic tangent bundle and let $E = \overline{E} \mid_X$. Let U be one of the charts U_i described above and assume $D \cap U$ is given by $z_1 \cdots z_m = 0$. Let $V = V_i = U_i \cap X$. In this case a local frame of $\overline{E}$ restricting to V is given by

$$e_1 = z_1 \frac{\partial}{\partial z_1}, \cdots, e_m = z_m \frac{\partial}{\partial z_m},\ e_{m+1} = \frac{\partial}{\partial z_{m+1}}, \cdots, e_n = \frac{\partial}{\partial z_n}.$$

Let g be any Kähler metric on X. It induces a Hermitian metric $\widetilde{g}$ on E. In local coordinate $z = (z_1, \cdots, z_n)$ we have

$$\widetilde{g}_{i\overline{j}} = \begin{cases} z_i \overline{z}_j g_{i\overline{j}} & \text{if} \quad i, j \leq m \\ z_i g_{i\overline{j}} & \text{if} \quad i \leq m < j \\ \overline{z}_j g_{i\overline{j}} & \text{if} \quad j \leq m < i \\ g_{i\overline{j}} & \text{if} \quad i, j > m. \end{cases} \tag{6.2}$$

In the following we denote by ∂_i the partial derivative $\frac{\partial}{\partial z_i}$. Let

$$\Gamma^p_{ik} = g^{p\overline{q}} \partial_i g_{k\overline{q}}$$

be the Christoffel symbol of the Kähler metric g and let

$$R^p_{ik\overline{l}} = g^{p\overline{j}} R_{i\overline{j}k\overline{l}} = g^{p\overline{j}} \left(-\partial_k \partial_{\overline{l}} g_{i\overline{j}} + g^{s\overline{t}} \partial_k g_{i\overline{t}} \partial_{\overline{l}} g_{s\overline{j}} \right)$$

be the curvature of g. We define

$$D^k_i = \begin{cases} \frac{z_i}{z_k} & \text{if} \quad i, k \leq m \\ \frac{1}{z_k} & \text{if} \quad k \leq m < i \\ z_i & \text{if} \quad i \leq m < k \\ 1 & \text{if} \quad i, k > m \end{cases} \tag{6.3}$$

and we let

$$\Lambda_i = \begin{cases} \frac{-1}{|z_i| \log |z_i|} & \text{if} \quad i \leq m \\ 1 & \text{if} \quad i > m \end{cases}. \tag{6.4}$$

Now we give an equivalent local condition of the metric $\widetilde{g}$ on E induced by the Kähler metric g to be good. We have

Proposition 6.1. *The metric $\widetilde{g}$ on E induced by g is good on $V\left(\frac{1}{2}\right)$ if and only if*

(6.5)

$$|\widetilde{g}_{i\bar{j}}|,\ |z_1\cdots z_m|^{-2}\deg(g)\le c\left(\sum_{i=1}^{m}\log|z_i|\right)^{2d}\quad \text{for some constants } c,d>0$$

$$\left|D_i^k\Gamma_{ip}^k\right|=O(\Lambda_p)\quad \text{for all } 1\le i,k,p\le n \text{ except } i=k=p$$

$$\left|\frac{1}{t_i}+\Gamma_{ii}^i\right|=O(\Lambda_i)\quad \text{if } i\le m$$

$$\left|D_i^k R_{ip\bar{q}}^k\right|=O(\Lambda_p\Lambda_q).$$

In [**16**] we showed the goodness of the WP, Ricci and perturbed Ricci metrics.

Theorem 6.2. *Let $\mathcal{M}_g$ be the moduli space of genus g Riemann surfaces. We assume $g\ge 2$. Let $\overline{\mathcal{M}}_g$ be the Deligne-Mumford compactification of $\mathcal{M}_g$ and let $D=\overline{\mathcal{M}}_g\setminus\mathcal{M}_g$ be the compactification divisor which is a normal crossing divisor. Let $\overline{E}=T_{\overline{\mathcal{M}}_g}(-\log D)$ and let $E=\overline{E}\mid_{\mathcal{M}_g}$. Let $\hat{h}$, $\hat{\tau}$ and $\hat{\widetilde{\tau}}$ be the metrics on E induced by the WP, Ricci and perturbed Ricci metrics respectively. Then $\hat{h}$, $\hat{\tau}$ and $\hat{\widetilde{\tau}}$ are good in the sense of Mumford.*

This theorem is based on very accurate estimates of the connection and curvature forms of these metrics. One of the difficulties is to estimate the Gauss-Manin connection of the fiberwise Kähler-Einstein metric where we use the compound graft metric construction of Wolpert together with maximum principle.

7. Negativity and Naturalness

In Section 2 we have seen various negative properties of the WP metric. In fact, we showed in [**16**] that the WP metric is dual Nakano negative. This means the complex curvature operator of the dual metric of the WP metric is positive. We first recall the precise definition of dual Nakano negativity of a Hermitian metric.

Let (E,h) be a Hermitian holomorphic vector bundle of rank m over a complex manifold M of dimension n. Let $e_1,\cdots,e_m$ be a local holomorphic frame of E and let $z_1,\cdots,z_n$ be local holomorphic coordinates on M. The Hermitian metric h has expression $h_{i\bar{j}}=h\left(e_i,e_j\right)$ locally.

The curvature of E is given by

$$P_{i\bar{j}\alpha\bar{\beta}}=-\partial_\alpha\partial_{\bar{\beta}}h_{i\bar{j}}+h^{p\bar{q}}\partial_\alpha h_{i\bar{q}}\partial_{\bar{\beta}}h_{p\bar{j}}.$$

Definition 7.1. *The Hermitian vector bundle (E,h) is Nakano positive if the curvature P defines a Hermitian metric on the bundle $E\otimes T_M^{1,0}$.*

Namely, $P_{i\bar{j}\alpha\bar{\beta}}C^{i\alpha}\overline{C^{j\beta}} > 0$ *for all* $m \times n$ *nonzero matrices* C*. The bundle* (E,h) *is Nakano semi-positive if* $P_{i\bar{j}\alpha\bar{\beta}}C^{i\alpha}\overline{C^{j\beta}} \geq 0$*. The bundle is dual Nakano (semi-)negative if the dual bundle with dual metric* (E^*,h^*) *is Nakano (semi-)positive.*

We have proved the following theorem in [**16**]

THEOREM 7.1. *Let* $\mathcal{M}_g$ *be the moduli space of Riemann surfaces of genus* g *where* $g \geq 2$*. Let* h *be the WP metric on* $\mathcal{M}_g$*. Then the holomorphic tangent bundle* $T^{1,0}\mathcal{M}_g$ *equipped with the WP metric* h *is dual Nakano negative.*

The dual Nakano negativity is the strongest negativity property of the WP metric.

Now we look at the naturalness of the canonical metrics on the moduli space. We let $\mathcal{M}_g$ be the moduli space of genus g curves where $g \geq 2$ and let $\overline{\mathcal{M}}_g$ be its Deligne-Mumford compactification. We fix a point $p \in \overline{\mathcal{M}}_g \setminus \mathcal{M}_g$ of codimension m and let $X = X_p$ be the corresponding stable nodal curve. The moduli space $\mathcal{M}(X)$ of the nodal surface X is naturally embedded into $\overline{\mathcal{M}}_g$. Furthermore, since each element Y in $\mathcal{M}(X)$ corresponds to a hyperbolic Riemann surface when we remove the nodes from Y, the complement can be uniformized by the upper half plane and thus there is a unique complete Kähler-Einstein metric on Y whose Ricci curvature is -1. We note that the moduli space $\mathcal{M}(X)$ can be viewed as an irreducible component of the intersection of m compactification divisors.

By the discussion in Section 2 there is a natural WP metric $\hat{h}$ on $\mathcal{M}(X)$. The curvature formula (2.2) is still valid for this WP metric and it is easy to see that the Ricci curvature of the WP metric $\hat{h}$ is negative. We can take $\hat{\tau} = -Ric\left(\omega_{\hat{h}}\right)$ to be the Kähler form of a Kähler metric on $\mathcal{M}(X)$. This is the Ricci metric $\hat{\tau}$ on $\mathcal{M}(X)$.

In [**21**] Masur showed that the WP metric h on $\mathcal{M}_g$ extends to $\overline{\mathcal{M}}_g$ and its restriction to $\mathcal{M}(X)$ via the natural embedding $\mathcal{M}(X) \hookrightarrow \overline{\mathcal{M}}_g$ coincides with the WP metric $\hat{h}$ on $\mathcal{M}(X)$. This implies the WP metric is natural. In [**38**] Wolpert showed that the WP Levi-Civita connection restricted to directions which are almost tangential to the compactification divisors limits to the lower dimensional WP Levi-Civita connection. In [**16**] we proved the naturalness of the Ricci metric.

THEOREM 7.2. *The Ricci metric on* $\mathcal{M}_g$ *extends to* $\overline{\mathcal{M}}_g$ *in non-degenerating directions. Furthermore, the restriction of the extension of* τ *to* $\mathcal{M}(X)$ *coincides with* $\hat{\tau}$*, the Ricci metric on* $\mathcal{M}(X)$*.*

8. The Kähler-Ricci Flow and Kähler-Einstein Metric on the Moduli Space

The existence of the Kähler-Einstein metric on the Teichmüller space was based on the work of Cheng-Yau since the Teichmüller space is pseudoconvex. By the uniqueness we know that the Kähler-Einstein metric is invariant under the action of the mapping class group and thus is also the Kähler-Einstein metric on the moduli space. It follows from the later work of Yau that the Kähler-Einstein metric is complete. However, the detailed properties of the Kähler-Einstein metric remain unknown.

In [**13**] we proved the strongly bounded geometry property of the Kähler-Einstein metric. We showed

THEOREM 8.1. *The Kähler-Einstein metric on the Teichmüller space $\mathcal{T}_g$ has strongly bounded geometry. Namely, the curvature and its covariant derivatives of the Kähler-Einstein metric are bounded and the injectivity radius of the Kähler-Einstein metric is bounded from below.*

This theorem was proved in two steps. Firstly, we deform the Ricci metric via the Kähler-Ricci flow

$$\begin{cases} \frac{\partial g_{i\bar{j}}}{\partial t} = -(R_{i\bar{j}} + g_{i\bar{j}}) \\ g_{i\bar{j}}(0) = \tau_{i\bar{j}} \end{cases} \tag{8.1}$$

Let $h = g(s)$ be the deformed metric at time $s \ll 1$. By the work of Shi [**28**] we know that the metric h is equivalent to the initial metric τ and is cohomologous to τ in the sense of currents. Thus h is complete and has Poincaré growth. Furthermore, the curvature and covariant derivatives of h are bounded.

We then use the metric h as a background metric to derive a priori estimates for the Kähler-Einstein metric by using the Monge-Amperé equation

$$\frac{\det\left(h_{i\bar{j}} + u_{i\bar{j}}\right)}{\det h_{i\bar{j}}} = e^{u+F}$$

where F is the Ricci potential of the metric h. If we denote by g the Kähler-Einstein metric and let

$$S = g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} u_{;i\bar{q}k} u_{;\bar{j}p\bar{l}}$$

and

$$V = g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} g^{m\bar{n}} u_{;i\bar{q}k\bar{n}} u_{;\bar{j}p\bar{l}m} + g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} g^{m\bar{n}} u_{;i\bar{n}kp} u_{;\bar{j}m\bar{l}\bar{q}}$$

to be the third and fourth order quantities respectively. We have

$$\begin{aligned} \Delta' \left[(S+\kappa)V\right] \geq & C_1 \left[(S+\kappa)V\right]^2 - C_2 \left[(S+\kappa)V\right]^{\frac{3}{2}} - C_3 \left[(S+\kappa)V\right] \\ & - C_4 \left[(S+\kappa)V\right]^{\frac{1}{2}} \end{aligned}$$

where Δ' is the Laplace operator of the Kähler-Einstein metric g and $C_1 > 0$.

It follows from the mean value inequality that S is bounded. Furthermore, by the above estimate and the maximum principle we know V is bounded. In fact this method works for all higher order derivatives of u and we deduce that the Kähler-Einstein metric has strongly bounded geometry.

The Kähler-Ricci flow and the goodness are closely tied together. Firstly, since the most important feature of a Mumford good metric is that the

Chern-Weil theory still holds, we say metrics with this property are intrinsic good. In [**17**] we showed

THEOREM 8.2. *Let $\overline{X}$ be a projective manifold with* $\dim_{\mathbb{C}} \overline{X} = n$. *Let $D \subset \overline{X}$ be a divisor with normal crossings, let $X = \overline{X} \setminus D$, let $\overline{E} = T_{\overline{X}}(-\log D)$ and let $E = \overline{E}\mid_X$.*

Let ω_g be a Kähler metric on X with bounded curvature and Poincaré growth. Assume $Ric(\omega_g) + \omega_g = \partial\overline{\partial} f$ where f is a bounded smooth function. Then

- *There exists a unique Kähler-Einstein metric $\omega_{_{KE}}$ on X with Poincaré growth.*
- *The curvature and covariant derivatives of curvature of the Kähler-Einstein metric are bounded.*
- *If ω_g is intrinsic good, then $\omega_{_{KE}}$ is intrinsic good. Furthermore, all metrics along the paths of continuity and Kähler-Ricci flow are intrinsic good.*

9. Applications

In this last section we briefly look at some geometric applications of the canonical metrics. The first application of the control of the Kähler-Einstein metric is the stability of the logarithmic cotangent bundle of the Deligne-Mumford moduli space. In [**13**] we proved

THEOREM 9.1. *Let $\overline{E} = T^*_{\overline{\mathcal{M}}_g}(\log D)$ be the logarithmic cotangent bundle. Then $c_1(\overline{E})$ is positive and $\overline{E}$ is slope stable with respect to the polarization $c_1(\overline{E})$.*

An immediate consequence of the intrinsic goodness of the Kähler-Einstein metric is the Chern number inequality. We have

THEOREM 9.2. *Let $\overline{E} = T_{\overline{\mathcal{M}}_g}(-\log D)$ be the logarithmic tangent bundle of the moduli space. Then*

$$c_1(\overline{E})^2 \leq \frac{6g-4}{3g-3} c_2(\overline{E}).$$

An immediate consequence of the dual Nakano negativity and the goodness of the WP metric is the positivity of the Chern numbers of this bundle. We have

THEOREM 9.3. *The Chern numbers of the logarithmic cotangent bundle $T^*_{\overline{\mathcal{M}}_g}(\log D)$ of the moduli spaces of Riemann surfaces are all positive.*

The dual Nakano negativity of a Hermitian metric on a bundle over a compact manifold gives strong vanishing theorems by using Bochner techniques. However, in our case the base variety $\mathcal{M}_g$ is only quasi-projective. Thus we can only describe vanishing theorems of the L^2 cohomology. In [**26**], Saper showed that the L^2 cohomology of the moduli space equipped with the Weil-Petersson metric can be identified with the ordinary cohomology of the Deligne-Mumford moduli space. Our situation is more subtle since the natural object to be considered in our case is the tangent bundle valued L^2 cohomology. Parallel to Saper's work, we proved in [**17**]

THEOREM 9.4. *We have the following natural isomorphism*

$$H^*_{(2)}\left((\mathcal{M}_g, \omega_\tau), (T_{\mathcal{M}_g}, \omega_{WP})\right) \cong H^*\left(\overline{\mathcal{M}}_g, T_{\overline{\mathcal{M}}_g}(-\log D)\right).$$

Now we combine the above result with the dual Nakano negativity of the WP metric. In [**17**] we proved the following Nakano-type vanishing theorem

THEOREM 9.5. *The L^2 cohomology groups vanish:*

$$H^{0,q}_{(2)}\left((\mathcal{M}_g, \omega_\tau), (T_{\mathcal{M}_g}, \omega_{WP})\right) = 0$$

unless $q = 3g-3$.

As a direct corollary we have

COROLLARY 9.1. *The pair $\left(\overline{\mathcal{M}}_g, D\right)$ is infinitesimally rigid.*

Another important application of the properties of the Ricci, perturbed Ricci and Kähler-Einstein metrics is the Gauss-Bonnet theorem on the noncompact moduli space. Together with L. Ji, in [**8**] we showed

THEOREM 9.6. *The Gauss-Bonnet theorem holds on the moduli space equipped with the Ricci, perturbed Ricci or Kähler-Einstein metrics:*

$$\int_{\mathcal{M}_g} c_n(\omega_\tau) = \int_{\mathcal{M}_g} c_n(\omega_{\widetilde{\tau}}) = \int_{\mathcal{M}_g} c_n(\omega_{KE}) = \chi(\mathcal{M}_g) = \frac{B_{2g}}{4g(g-1)}.$$

Here $\chi(\mathcal{M}_g)$ is the orbifold Euler characteristic of $\mathcal{M}_g$ and $n = 3g-3$.

The explicit topological computation of the Euler characteristic of the moduli space is due to Harer-Zagier [**7**]. See also the work of Penner [**24**].

As an application of the Mumford goodness of the WP metric and the Ricci metric we have

THEOREM 9.7.

$$\chi(T_{\overline{\mathcal{M}}_g}(-\log D)) = \int_{\mathcal{M}_g} c_n(\omega_\tau) = \int_{\mathcal{M}_g} c_n(\omega_{WP}) = \frac{B_{2g}}{4g(g-1)}$$

where $n = 3g-3$.

It is very hard to prove the Gauss-Bonnet theorem for the WP metric directly since the WP metric is incomplete and its curvature is not bounded. The proof is based substantially on the Mumford goodness of the WP metric.

By using the goodness of canonical metrics this theorem also gives an explicit expression of the top log Chern number of the moduli space.

THEOREM 9.8.

$$\chi(\overline{\mathcal{M}}_g, T_{\overline{\mathcal{M}}_g}(-\log D)) = \chi(\mathcal{M}_g) = \frac{B_{2g}}{4g(g-1)}.$$

10. Global Torelli Theorem of the Teichmüller Spaces of Polarized Calabi-Yau Manifolds (Joint with Andrey Todorov)

The geometry of the Teichmüller and moduli spaces of polarized Calabi-Yau (CY) manifolds are the central objects in geometry and string theory. One of the most important question in understanding the geometry of the Teichmüller and moduli space of polarized CY manifolds is the global Torelli problem which asks whether the variation of polarized Hodge structures determines the marked polarized Calabi-Yau structure. In the rest of this article, after briefly discussing the deformation theory of CY manifolds and the geometry of period domain, we will describe the global Torelli theorem of the Teichmüller spaces of polarized CY manifolds and its proof. See [**11**] for details.

Let M be a Calabi-Yau manifold of dimension $\dim_{\mathbb{C}} M = n$. Here we assume $n \geq 3$. Let L be an ample line bundle over M. By definition we assume that the canonical bundle K_M is trivial. Let X be the underlying real $2n$-dimensional manifold. We know that there is a nowhere vanishing holomorphic $(n,0)$-form on M which is unique up to scaling. The Teichmüller space of (M,L) is the connected, simply connected, reduced and irreducible manifold parameterizing triples $(M, L, (\gamma_1, \cdots, \gamma_{b_n}))$ where M is a CY manifold, L is the polarization and $(\gamma_1, \cdots, \gamma_{b_n})$ is a basis of the middle homology group $H_n(X,\mathbb{Z})/tor$. Such triples are called marked polarized CY manifolds.

10.1. Deformation Theory of Polarized Calabi-Yau Manifolds. We first recall the deformation of complex structures on a given smooth manifold. Let X be a smooth manifold of dimension $\dim_{\mathbb{R}} X = 2n$ and let J_0 be an integrable complex structure on X. We denote by $M_0 = (X, J_0)$ the corresponding complex manifold.

Let $\varphi \in A^{0,1}\left(M_0, T^{1,0}_{M_0}\right)$ be a Beltrami differential. We can view φ as a map

$$\varphi : \Omega^{1,0}(M_0) \to \Omega^{0,1}(M_0).$$

By using φ we define a new almost complex structure J_φ in the following way. For a point $p \in M_0$ we pick a local holomorphic coordinate chart $(U, z_1, \cdots, z_n)$ around p. Let

$$\Omega^{1,0}_{\varphi}(p) = \operatorname{span}_{\mathbb{C}}\{dz_1 + \varphi(dz_1), \cdots, dz_n + \varphi(dz_n)\} \tag{10.1}$$

and

$$\Omega^{0,1}_{\varphi}(p) = \operatorname{span}_{\mathbb{C}}\{d\overline{z}_1 + \overline{\varphi}(d\overline{z}_1), \cdots, d\overline{z}_n + \overline{\varphi}(d\overline{z}_n)\}$$

be the eigenspaces of J_φ with respect to the eigenvalue $\sqrt{-1}$ and $-\sqrt{-1}$ respectively.

The almost complex structure J_φ is integrable if and only if

$$\overline{\partial}\varphi = \frac{1}{2}[\varphi, \varphi] \tag{10.2}$$

where $\overline{\partial}$ is the operator on M_0.

It was proved in [**31**] and [**30**] that the local deformation of a polarized CY manifold is unobstructed.

THEOREM 10.1. *The universal deformation space of a polarized CY manifold is smooth.*

The operation of contracting with Ω_0 plays an important role in converting bundle valued differential forms into ordinary differential forms. The following lemma is the key step in the proof of local Torelli theorem.

LEMMA 10.1. *Let (M,L) be a polarized CY n-fold and let ω_g be the unique CY metric in the class $[L]$. We pick a nowhere vanishing holomorphic $(n,0)$-form Ω_0 such that*

$$\left(\frac{\sqrt{-1}}{2}\right)^n (-1)^{\frac{n(n-1)}{2}} \Omega_0 \wedge \overline{\Omega}_0 = \omega_g^n. \tag{10.3}$$

Then the map $\iota : A^{0,1}\left(M, T_M^{1,0}\right) \to A^{n-1,1}(M)$ given by $\iota(\varphi) = \varphi \lrcorner \Omega_0$ is an isometry with respect to the natural Hermitian inner product on both spaces induced by ω_g. Furthermore, ι preserves the Hodge decomposition.

In [**31**] the existence of flat coordinates was established and the flat coordinates played an important role in string theory [**3**]. Here we recall this construction. Let $\mathfrak{X}$ be the universal family over $\mathcal{T}$ and let π be the projection map. For each $p \in \mathcal{T}$ we let $M_p = (X, J_p)$ be the corresponding CY manifold. In the following we always use the unique CY metric on M_p in the polarization class $[L]$.

By the Kodaira-Spencer theory and Hodge theory, we have the following identification

$$T_p^{1,0}\mathcal{T} \cong \mathbb{H}^{0,1}\left(M_p, T_{M_p}^{1,0}\right)$$

where we use $\mathbb{H}$ to denote the corresponding space of harmonic forms. We have the following expansion of the Beltrami differentials:

THEOREM 10.2. *Let $\varphi_1, \cdots, \varphi_N \in \mathbb{H}^{0,1}\left(M_p, T_{M_p}^{1,0}\right)$ be a basis. Then there is a unique power series*

$$\varphi(\tau) = \sum_{i=1}^N \tau_i \varphi_i + \sum_{|I| \geq 2} \tau^I \varphi_I \tag{10.4}$$

which converges for $|\tau| < \varepsilon$. Here $I = (i_1, \cdots, i_N)$ is a multi-index, $\tau^I = \tau_1^{i_1} \cdots \tau_N^{i_N}$ and $\varphi_I \in A^{0,1}\left(M_p, T_{M_p}^{1,0}\right)$. Furthermore, if Ω is a nowhere vanishing holomorphic $(n,0)$-form, then the family of Beltrami differentials $\varphi(\tau)$ satisfy the following conditions:

$$\begin{aligned}
&\overline{\partial}_{M_p} \varphi(\tau) = \frac{1}{2}[\varphi(\tau), \varphi(\tau)] \\
&\overline{\partial}_{M_p}^* \varphi(\tau) = 0 \\
&\varphi_I \lrcorner \Omega = \partial_{M_p} \psi_I
\end{aligned} \tag{10.5}$$

for each $|I| \geq 2$ where $\psi_I \in A^{n-2,1}(M_p)$. Furthermore, by shrinking ε we can pick each ψ_I appropriately such that $\sum_{|I|\geq 2} \tau^I \psi_I$ converges for $|\tau| < \varepsilon$.

The coordinates constructed in the above theorem are just the flat coordinates described in [**3**]. They are unique up to affine transformation and they are also the normal coordinates of the Weil-Petersson metric at p. In fact the first equation in (10.5) is the obstruction equation and the second equation is the Kuranishi gauge which fixes the gauge in the fiber. The last equation which characterized the flat coordinates around a point in the Teichmüller space is known as the Todorov gauge.

From Theorem 10.2 the local Torelli theorem and the Griffiths transversality follow immediately. However, Theorem 10.2 contains more information.

By using the local deformation theory, in [**31**] Todorov constructed a canonical local holomorphic section of the line bundle $H^{n,0} = F^n$ over any flat coordinate chart $U \subset \mathcal{T}$ in the form level. This canonical section plays a crucial role in the proof of the global Torelli theorem.

We first consider the general construction of holomorphic $(n,0)$-forms in [**31**].

Lemma 10.2. *Let $M_0 = (X, J_0)$ be a CY manifold where J_0 is the complex structure on X. Let $\varphi \in A^{0,1}\left(M_0, T^{1,0}_{M_0}\right)$ be a Beltrami differential on M_0 which define an integrable complex structure J_φ and let $M_\varphi = (X, J_\varphi)$ be the CY manifold whose underlying differentiable manifold is X. Let Ω_0 be a nowhere vanishing holomorphic $(n,0)$-form on M_0 and let*

$$\Omega_\varphi = \sum_{k=0}^{n} \frac{1}{k!} (\wedge^k \varphi \lrcorner \Omega_0). \tag{10.6}$$

Then Ω_φ is a well-defined smooth $(n,0)$-form on M_φ. It is holomorphic with respect to the complex structure J_φ if and only if $\partial(\varphi \lrcorner \Omega_0) = 0$. Here ∂ is the operator with respect to the complex structure J_0.

By combining Lemma 10.2 and Theorem 10.2 we define the canonical family

$$\Omega^c = \Omega^c(\tau) = \sum_{k=0}^{n} \frac{1}{k!} \left(\wedge^k \varphi(\tau) \lrcorner \Omega_0\right) \tag{10.7}$$

and we have

Corollary 10.1. *Let $\Omega^c(\tau)$ be a canonical family defined by (10.7) where $\varphi(\tau)$ is defined as in (10.5). Then we have the expansion*

$$\begin{aligned} [\Omega^c(\tau)] =& [\Omega_0] + \sum_{i=1}^{N} \tau_i [\varphi_i \lrcorner \Omega_0] \\ & + \frac{1}{2} \sum_{i,j} \tau_i \tau_j \left[\mathbb{H}(\varphi_i \wedge \varphi_j \lrcorner \Omega_0)\right] + \Xi(\tau) \end{aligned} \tag{10.8}$$

where $\Xi(\tau) \subset \bigoplus_{k=2}^{n} H^{n-k,k}(M_p)$ *and* $\Xi(\tau) = O(|\tau|^3)$.

The most important application of the cohomological expansion (10.8) is the invariance of the CY Kähler forms. The theorem plays a central role in the proof of the global Torelli theorem. This theorem was implicitly proved in [**2**]. Please see [**11**] for a simple and self-contained proof.

THEOREM 10.3. *For each point* $p \in \mathcal{T}$, *let* ω_p *be the Kähler form of the unique CY metric on* M_p *in the polarization class* $[L]$. *Then* ω_p *is invariant. Namely,*

$$\nabla^{GM} \omega_p = 0.$$

Furthermore, since $\mathcal{T}$ *is simply connected, we know that* ω_p *is a constant section of the trivial bundle* $A^2(X, \mathbb{C})$ *over* $\mathcal{T}$.

10.2. The Teichmüller Space of Polarized Calabi-Yau Manifolds. Now we recall the construction of the universal family of marked polarized CY manifolds and the Teichmüller space. See [**18**] for details. Let M be a CY manifold of dimension $\dim_{\mathbb{C}} M = n \geq 3$. Let L be an ample line bundle over M. We call a tuple $(M, L, \gamma_1, \cdots, \gamma_{h^n})$ a marked polarized CY manifold if M is a CY manifold, L is a polarization of M and $\{\gamma_1, \cdots, \gamma_{h^n}\}$ is a basis of $H_n(M, \mathbb{Z})/tor$.

REMARK 10.1. *To simplify notations we assume in this section that a CY manifold* M *of dimension* n *is simply connected and* $h^{k,0}(M) = 0$ *for* $1 \leq k \leq n-1$. *All the results in this section hold when these conditions are removed. This is due to the fact that we fix a polarization.*

Since the Teichmüller space of M with fixed marking and polarization is constructed via GIT quotient, we need the following results about group actions.

THEOREM 10.4. *Let* $(M, L, (\gamma_1, \cdots, \gamma_{b_n}))$ *be a marked polarized CY manifold and let* $\pi : \mathfrak{X} \to \mathcal{K}$ *be the Kuranishi family of* M. *We let* $p \in \mathcal{K}$ *such that* $M = \pi^{-1}(p)$. *If* G *is a group of holomorphic automorphisms of* M *which preserve the polarization* L *and act trivially on* $H_n(M, \mathbb{Z})$, *then for any* $q \in \mathcal{K}$ *the group* G *acts on* $M_q = \pi^{-1}(q)$ *as holomorphic automorphisms.*

Now we recall the construction of the Teichmüller space. We first note that there is a constant $m_0 > 0$ which only depends on n such that for any polarized CY manifold (M, L) of dimension n, the line bundle L^m is very ample for any $m \geq m_0$. We replace L by L^{m_0} and we still denote it by L.

Let $N_m = h^0(M, L^m)$. It follows from the Kodaira embedding theorem that M is embedded into $\mathbb{P}^{N_m - 1}$ by the holomorphic sections of L^m. Let $\mathcal{H}_L$ be the component of the Hilbert scheme which contains M and parameterizes smooth CY varieties embedded in $\mathbb{P}^{N_m-1}$ with Hilbert polynomial

$$P(m) = h^0\left(M, L^m\right).$$

We know that $\mathcal{H}_L$ is a smooth quasi-projective variety and there exists a universal family $\mathcal{X}_L \to \mathcal{H}_L$ of pairs $(M, (\sigma_0, \cdots, \sigma_{N_m}))$ where $(\sigma_0, \cdots, \sigma_{N_m})$

is a basis of $H^0(M, L^m)$. Let $\widetilde{\mathcal{H}}_L$ be its universal cover. By Theorem 10.4 we know that the group $PGL\,(N_m, \mathbb{C})$ acts on $\widetilde{\mathcal{H}}_L$ and the family $\widetilde{\mathcal{X}}_L \to \widetilde{\mathcal{H}}_L$ holomorphically and without fixed points. Furthermore, it was proved in [**18**] and [**25**] that the group $PGL\,(N_m, \mathbb{C})$ also acts properly on $\widetilde{\mathcal{H}}_L$.

We define the Teichmüller space of M with polarization L by

$$\mathcal{T} = \mathcal{T}_L(M) = \widetilde{\mathcal{H}}_L / PGL\,(N_m, \mathbb{C})\,.$$

One of the most important features of the Teichmüller space is the existence of universal family.

THEOREM 10.5. *There exist a family of marked polarized CY manifolds $\pi : \mathcal{U}_L \to \mathcal{T}_L(M)$ such that there is a point $p \in \mathcal{T}_L(M)$ with M_p isomorphic to M as marked polarized CY manifolds and the family has the following properties:*

(1) *$\mathcal{T}_L(M)$ is a smooth complex manifold of dimension* $\dim_{\mathbb{C}} \mathcal{T}_L(M) = h^{n-1,1}(M)$.

(2) *For each point $q \in \mathcal{T}_L(M)$ there is a natural identification*

$$T_q^{1,0}\mathcal{T}_L(M) \cong H^{0,1}\left(M_q, T_{M_q}^{1,0}\right)$$

via the Kodaira-Spencer map.

(3) *Let $\rho : \mathcal{Y} \to \mathcal{C}$ be a family of marked polarized CY manifold such that there is a point $x \in \mathcal{C}$ whose fiber $\rho^{-1}(x)$ is isomorphic to M_p as marked polarized CY manifolds. Then there is a unique holomorphic map $f : (\mathcal{Y} \to \mathcal{C}) \to (\mathcal{U}_L \to \mathcal{T}_L(M))$, defined up to biholomorphic maps on the fibers whose induced maps on $H_n(M, \mathbb{Z})$ are the identity map, such that f maps the fiber $\rho^{-1}(x)$ to the fiber M_p and the family $\mathcal{Y}$ is just the pullback of $\mathcal{U}_L$ via the map f. Furthermore, the map $\widetilde{f} : \mathcal{C} \to \mathcal{T}_L(M)$ induced by f is unique.*

It follows directly that

PROPOSITION 10.1. *The Teichmüller space $\mathcal{T} = \mathcal{T}_L(M)$ is a smooth complex manifold and is simply connected.*

In the rest of this paper by the Teichmüller space $\mathcal{T}$ of $(M, L, (\gamma_1, \cdots, \gamma_{b_n}))$ we always mean the reduced irreducible component of $\widetilde{\mathcal{H}}_L / PGL\,(N_m, \mathbb{C})$ with the fixed polarization L.

It follows from its construction and Theorem 10.4 that the universal family $\mathcal{U}_L$ over the Teichmüller space $\mathcal{T}$ is diffeomorphic to $M_p \times \mathcal{T}$ as a C^∞ family where $p \in \mathcal{T}$ is any point and M_p is the corresponding CY manifold.

10.3. The Classifying Space of Variation of Polarized Hodge Structures. Now we recall the construction of the classifying space of variation of polarized Hodge structures and its basic properties such as the description of its real and complex tangent spaces and the Hodge metric. See [**27**] for details.

In the construction of the Teichmüller space $\mathcal{T}$ we fixed a marking of the background manifold X, namely a basis of $H_n(X,\mathbb{Z})/tor$. This gives us canonical identifications of the middle dimensional de Rahm cohomology of different fibers over $\mathcal{T}$. Namely for any two distinct points $p,q\in\mathcal{T}$ we have the canonical identification

$$H^n(M_p)\cong H^n(M_q)\cong H^n(X)$$

where the coefficient ring is $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$.

Since the polarization $[L]$ is an integeral class, it defines a map

$$L: H^n(X,\mathbb{Q})\to H^{n+2}(X,\mathbb{Q})$$

given by $A\mapsto c_1(L)\wedge A$ for any $A\in H^n(X,\mathbb{Q})$. We denote by $H^n_{pr}(X)=\ker(L)$ the primitive cohomology groups where, again, the coefficient ring is $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$. For any $p\in\mathcal{T}$ we let $H^{k,n-k}_{pr}(M_p)=H^{k,n-k}(M_p)\cap H^n_{pr}(M_p,\mathbb{C})$ and denote its dimension by $h^{k,n-k}$. The Poincaré bilinear form Q on $H^n_{pr}(X,\mathbb{Q})$ is defined by

$$Q(u,v)=(-1)^{\frac{n(n-1)}{2}}\int_X u\wedge v$$

for any d-closed n-forms u,v on X. The bilinear form Q is symmetric if n is even and is skew-symmetric if n is odd. Furthermore, Q is non-degenerate and can be extended to $H^n_{pr}(X,\mathbb{C})$ bilinearly. For any point $q\in\mathcal{T}$ we have the Hodge decomposition

$$H^n_{pr}(M_q,\mathbb{C})=H^{n,0}_{pr}(M_q,\mathbb{C})\oplus\cdots\oplus H^{0,n}_{pr}(M_q,\mathbb{C}) \tag{10.9}$$

which satisfies

$$\dim_{\mathbb{C}} H^{k,n-k}_{pr}(M_q,\mathbb{C})=h^{k,n-k} \tag{10.10}$$

and the Hodge-Riemann relations

$$Q\left(H^{k,n-k}_{pr}(M_q,\mathbb{C}),H^{l,n-l}_{pr}(M_q,\mathbb{C})\right)=0 \ \text{ unless } \ k+l=n \tag{10.11}$$

and

$$\left(\sqrt{-1}\right)^{2k-n}Q\left(v,\overline{v}\right)>0 \ \text{ for } \ v\in H^{k,n-k}_{pr}(M_q,\mathbb{C})\setminus\{0\}. \tag{10.12}$$

The above Hodge decomposition of $H^n_{pr}(M_q,\mathbb{C})$ can also be described via the Hodge filtration. Let $f^k=\sum_{i=k}^n h^{i,n-i}$. We let

$$F^k=F^k(M_q)=H^{n,0}_{pr}(M_q,\mathbb{C})\oplus\cdots\oplus H^{k,n-k}_{pr}(M_q,\mathbb{C})$$

and we have decreasing filtration

$$H^n_{pr}(M_q,\mathbb{C})=F^0(M_q)\supset\cdots\supset F^n(M_q).$$

We know that

$$\dim_{\mathbb{C}} F^k=f^k, \tag{10.13}$$

$$H_{pr}(X,\mathbb{C})=F^k(q)\oplus\overline{F^{n-k+1}(q)} \tag{10.14}$$

and

$$(10.15)\qquad H_{pr}^{k,n-k}(M_q,\mathbb{C})=F^k(M_q)\cap\overline{F^{n-k}(M_q)}.$$

In term of the Hodge filtration $F^n\subset\cdots\subset F^0=H_{pr}^n(M_q,\mathbb{C})$ the Hodge-Riemann relations can be written as

$$(10.16)\qquad Q\left(F^k,F^{n-k+1}\right)=0$$

and

$$(10.17)\qquad Q\left(Cv,\overline{v}\right)>0\ \text{ if }\ v\neq 0$$

where C is the Weil operator given by $Cv=\left(\sqrt{-1}\right)^{2k-n}v$ when $v\in H_{pr}^{k,n-k}(M_q,\mathbb{C})$. The classifying space D of variation of polarized Hodge structures with data (10.13) is the space of all such Hodge filtrations

$$D=\left\{F^n\subset\cdots\subset F^0=H_{pr}^n(X,\mathbb{C})\mid(10.13),(10.16)\text{ and }(10.17)\text{ hold}\right\}.$$

The compact dual $\check{D}$ of D is

$$\check{D}=\left\{F^n\subset\cdots\subset F^0=H_{pr}^n(X,\mathbb{C})\mid(10.13)\text{ and }(10.16)\text{ hold}\right\}.$$

The classifying space $D\subset\check{D}$ is an open set. We note that the conditions (10.13), (10.16) and (10.17) imply the identity (10.14).

An important feature of the variation of polarized Hodge structures is that both D and $\check{D}$ can be written as quotients of semi-simple Lie groups. Let $H_{\mathbb{R}}=H_{pr}^n(X,\mathbb{R})$ and $H_{\mathbb{C}}=H_{pr}^n(X,\mathbb{C})$. We consider the real and complex semi-simple Lie groups

$$G_{\mathbb{R}}=\{\sigma\in GL(H_{\mathbb{R}})\mid Q(\sigma u,\sigma v)=Q(u,v)\}$$

and

$$G_{\mathbb{C}}=\{\sigma\in GL(H_{\mathbb{C}})\mid Q(\sigma u,\sigma v)=Q(u,v)\}.$$

The real group $G_{\mathbb{R}}$ acts on D and the complex group $G_{\mathbb{C}}$ acts on $\check{D}$ where both actions are transitive. This implies that both D and $\check{D}$ are smooth. Furthermore, we can embed the real group into the complex group naturally as real points.

We now fix a reference point $O=\{F_0^k\}\in D\subset\check{D}$ and let B be the isotropy group of O under the action of $G_{\mathbb{C}}$ on $\check{D}$. Let $\left\{H_0^{k,n-k}\right\}$ be the corresponding Hodge decomposition where $H_0^{k,n-k}=F_0^k\cap\overline{F_0^{n-k}}$. Let $V=G_{\mathbb{R}}\cap B$. Then we have

$$(10.18)\qquad D=G_{\mathbb{R}}/V\quad\text{and}\quad\check{D}=G_{\mathbb{C}}/B.$$

Following the argument in [**27**] we let

$$H_0^+=\bigoplus_{i\text{ is even}}H_0^{i,n-i}\qquad H_0^-=\bigoplus_{i\text{ is odd}}H_0^{i,n-i}$$

and let K be the isotropy group of H_0^+ in $G_{\mathbb{R}}$. We note that H_0^+ and H_0^- are defined over $\mathbb{R}$ and are orthogonal with respect to Q when n is even. When n is odd they are conjugate to each other. Thus K is also the isotropy group of H_0^-. In both cases K is the maximal compact subgroup of $G_{\mathbb{R}}$ containing V. This implies that $\widetilde{D} = G_{\mathbb{R}}/K$ is a symmetric space of noncompact type and D is a fibration over $\widetilde{D}$ whose fibers are isomorphic to K/V.

REMARK 10.2. *In the following we will only consider primitive cohomology classes and we will drop the mark "pr". Furthermore, Since we only need to use the component of $G_{\mathbb{R}}$ containing the identity, we will denote again by $G_{\mathbb{R}}$ and K the components of the real group and its corresponding maximal compact subgroup which contain the identity.*

We fix a point $p \in \mathcal{T}$ and let $O = \Phi(p) \in D \subset \check{D}$. For $0 \leq k \leq n$ we let $H_0^{k,n-k} = H^{k,n-k}(M_p)$. Now we let $\mathfrak{g} = \mathfrak{g}_{\mathbb{C}}$ be the Lie algebra of $G_{\mathbb{C}}$ and let $\mathfrak{g}_0 = \mathfrak{g}_{\mathbb{R}}$ be the Lie algebra of $G_{\mathbb{R}}$. The real Lie algebra $\mathfrak{g}_0$ can also be embedded into $\mathfrak{g}$ naturally as real points. The Hodge structure $\left\{H_0^{k,n-k}\right\}$ induces a weight 0 Hodge structure on $\mathfrak{g}$. Namely $\mathfrak{g} = \bigoplus_p \mathfrak{g}^{p,-p}$ where

$$\mathfrak{g}^{p,-p} = \left\{X \in \mathfrak{g} \mid X\left(H_0^{k,n-k}\right) \subset H_0^{k+p,n-k-p}\right\}.$$

Let B be the isotropy group of $O \in \check{D}$ under the action of $G_{\mathbb{C}}$ and let $\mathfrak{b}$ be the Lie algebra of B. Then

$$\mathfrak{b} = \bigoplus_{p \geq 0} \mathfrak{g}^{p,-p}.$$

Let $V = B \cap G_{\mathbb{R}}$ be the isotropy group of $O \in D$ under the action of $G_{\mathbb{R}}$ and let $\mathfrak{v}$ be its Lie algebra. We have

$$\mathfrak{v} = \mathfrak{b} \cap \mathfrak{g}_0 \subset \mathfrak{g}.$$

Now we have

$$\mathfrak{v} = \mathfrak{g}_0 \cap \mathfrak{b} = \mathfrak{g}_0 \cap \mathfrak{b} \cap \overline{\mathfrak{b}} = \mathfrak{g}_0 \cap \mathfrak{g}^{0,0}.$$

Let θ be the Weil operator of the weight 0 Hodge structure on $\mathfrak{g}$. Then for any $v \in \mathfrak{g}^{p,-p}$ we have $\theta(v) = (-1)^p v$. The eigenvalues of θ are ± 1. Let $\mathfrak{g}^+$ be the eigenspace of 1 and let $\mathfrak{g}^-$ be the eigenspace of -1. Then we have

$$\mathfrak{g}^+ = \bigoplus_{p \text{ even}} \mathfrak{g}^{p,-p} \quad \text{and} \quad \mathfrak{g}^- = \bigoplus_{p \text{ odd}} \mathfrak{g}^{p,-p}.$$

We note here that, in the above expression, p can be either positive or negative. Let $\mathfrak{k}$ be the Lie algebra of K, the maximal compact subgroup of $G_{\mathbb{R}}$ containing V. By the work of Schmid [**27**] we know that

LEMMA 10.3. *The Lie algebra $\mathfrak{k}$ is given by $\mathfrak{k} = \mathfrak{g}_0 \cap \mathfrak{g}^+$. Furthermore, if we let $\mathfrak{p}_0 = \mathfrak{g}_0 \cap \mathfrak{g}^-$, then*

$$\mathfrak{g}_0 = \mathfrak{k} \oplus \mathfrak{p}_0$$

is a Cartan decomposition of $\mathfrak{g}_0$. The space $\mathfrak{p}_0$ is Ad_V invariant.

We call such a Cartan decomposition the canonical Cartan decomposition. Here we recall that if $\mathfrak{g}_0 = \mathfrak{k} \oplus \mathfrak{p}_0$ is a Cartan decomposition of the real semisimple Lie algebra $\mathfrak{g}_0$, then we know that $\mathfrak{k}$ is a Lie subalgebra, $[\mathfrak{p}_0, \mathfrak{p}_0] \subset \mathfrak{k}$ and $[\mathfrak{p}_0, \mathfrak{k}] \subset \mathfrak{p}_0$.

By the expression of $\mathfrak{v}$ and $\mathfrak{k}$ we have the identification

$$\mathfrak{k}/\mathfrak{v} \cong \mathfrak{g}_0 \cap \left(\bigoplus_{p \neq 0,\ p \text{ is even}} \mathfrak{g}^{p,-p} \right) \tag{10.19}$$

and the identification

$$T^{\mathbb{R}}_O D \cong \mathfrak{k}/\mathfrak{v} \oplus \mathfrak{p}_0. \tag{10.20}$$

Now we look at the complex structures on D. By the above identification we know that for each element $X \in T^{\mathbb{R}}_O D$ we have the unique decomposition $X = X_+ + X_-$ where $X_+ \in \bigoplus_{p>0} \mathfrak{g}^{-p,p}$ and $X_- \in \bigoplus_{p>0} \mathfrak{g}^{p,-p}$. We define the complex structure J on $T^{\mathbb{R}}_O D$ by

$$JX = iX_+ - iX_-. \tag{10.21}$$

Now we use left translation by elements in $G_{\mathbb{R}}$ to move this complex structure to every point in D. Namely, for any point $\alpha \in D$ we pick $g \in G_{\mathbb{R}}$ such that $g(O) = \alpha$. If $X \in T^{\mathbb{R}}_\alpha D$, then we define $JX = (l_g)_* \circ J \circ \left(l_{g^{-1}}\right)_* (X)$.

LEMMA 10.4. *J is an invariant integrable complex structure on D. Furthermore, it coincides with the complex structure on D induced by the inclusion $D \subset \check{D} = G_{\mathbb{C}}/B$.*

This lemma is well known. See [**6**] and [**19**] for details.

There is a natural metric on D induced by the Killing form. By the Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k} \oplus \mathfrak{p}_0$ we know that the Killing form κ on $\mathfrak{g}_0$ is positive definite on $\mathfrak{p}_0$ and is negative definite on $\mathfrak{k}/\mathfrak{v}$. By the identification (10.20), for real tangent vectors $X, Y \in T^{\mathbb{R}}_O D$, if $X = X_1 + X_2$ and $Y = Y_1 + Y_2$ where $X_1, Y_1 \in \mathfrak{k}/\mathfrak{v}$ and $X_2, Y_2 \in \mathfrak{p}_0$, we let

$$\widetilde{\kappa}(X, Y) = -\kappa(X_1, Y_1) + \kappa(X_2, Y_2). \tag{10.22}$$

Then $\widetilde{\kappa}$ is a positive definite symmetric bilinear form on $T^{\mathbb{R}}_O D$. Now we use left translation of elements in $G_{\mathbb{R}}$ to move this metric to the real tangent space of every point in D and we obtain a Riemannian metric on D. This is the Hodge metric defined by Griffiths and Schmid in [**6**].

10.4. Global Torelli Theorem and Applications. Now we describe the global Torelli theorem.

THEOREM 10.6. *Let (M, L) be a polarized CY manifold of dimension n and let $\mathcal{T}$ be its Teichmüller space. Let D be the classifying space of the variation of Hodge structures according to the middle cohomology of M. Let $\Phi : \mathcal{T} \to D$ be the period map which maps each point $q \in \mathcal{T}$ to the Hodge decomposition of the middle dimensional primitive cohomology of M_q which is a point in D. Then the period map $\Phi : \mathcal{T} \to D$ is injective.*

Let us describe the main idea of proving the global Torelli theorem. See [**11**] for details. In fact we have proved a stronger result. For any distinct points $p, q \in \mathcal{T}$, in [**11**] we showed that the lines $F^n(M_p)$ and $F^n(M_q)$ do not coincide. This means the first Hodge bundle already determines polarized marked Calabi-Yau structures.

The first main component in the proof of Theorem 10.6 is Theorem 10.3, namely the Kähler forms ω of the polarized CY metrics are invariant. From this we know that all the complex structures corresponding to all points in $\mathcal{T}$ are tamed by ω. If we fix a base point $0 \in \mathcal{T}$ then for any point $q \in \mathcal{T}$ such that $q \neq 0$, the complex structure on M_q is obtained by deforming the complex structure on M_0 via a unique Beltrami differential $\varphi(q) \in A^{0,1}\left(M_0, T^{1,0}_{M_0}\right)$. Thus we obtained the assigning map

$$\rho : \mathcal{T} \to A^{0,1}\left(M_0, T^{1,0}_{M_0}\right)$$

by letting $\rho(q) = \varphi(q)$. The assigning map ρ is holomorphic. In fact, in [**10**] we proved that the assigning map ρ is a holomorphic embedding.

According to the work of Todorov [**31**] and our work [**11**] we know that $\Omega_q = \sum_{k=0}^{n} \frac{1}{k!}\left(\wedge^k \varphi(q) \lrcorner \Omega_0\right)$ is a smooth $(n,0)$-form on M_q where Ω_0 is a properly normalized holomorphic $(n,0)$-form on M_0. It follows from Theorem 10.2 and Lemma 10.2 that Ω_q is a holomorphic $(n,0)$-form on M_q. Since the cohomology classes $[\Omega_0]$ and $[\Omega_q]$ are generators of the Hodge lines $F^n(M_p)$ and $F^n(M_q)$, it is enough to show that these two classes are not proportional. Now we look at the Calabi-Yau equation

$$c_n \Omega_p \wedge \overline{\Omega}_p = \omega_p^n$$

where $p \in \mathcal{T}$ is any point, Ω_p is a properly normalized nowhere vanishing holomorphic $(n,0)$-form on M_p, ω_p is the Kähler form of the polarized CY metric on M_p and $c_n = (-1)^{\frac{n(n-1)}{2}}\left(\frac{\sqrt{-1}}{2}\right)^n$. It follows from Theorem 10.3 and the Calabi-Yau equation that if $[\Omega_q] = c[\Omega_0]$, then $c = 1$ and $\varphi(q) = 0$ which means that M_0 and M_q are isomorphic. This contradicts the assumption that $q \neq 0$ as points in $\mathcal{T}$ and the global Torelli theorem follows.

By using the same method we proved the global Torelli theorem of the Teichmüller space of polarized Hyper-Kähler manifolds:

Theorem 10.7. *Let (M, L) be a polarized Hyper-Kähler manifold and let $\mathcal{T}$ be its Teichmüller space. Let D be the classifying space of variation of polarized weight 2 Hodge structures according to the data of (M, L). Then the period map $\Phi : \mathcal{T} \to D$ which maps each point $q \in \mathcal{T}$ to the Hodge decomposition of the second primitive cohomology of M_q is injective.*

In [**10**] we gave another proof of the global Torelli theorem of the Teichmüller space of polarized Hyper-Kähler manifolds by directly showing that the cohomology expansion of the canonical $(2,0)$-forms has no quantum correction. This implies that the Hodge completion of the Teichmüller space is biholomorphic to the classifying space via the Harish-Chandra realization.

Another important property of the Teichmüller space is the existence of holomorphic flat connections. We proved the following theorem in [**10**].

THEOREM 10.8. *There exists affine structures on the Teichmüller space $\mathcal{T}$ of polarized CY manifolds. The affine structures are given by global holomorphic flat connections on $\mathcal{T}$.*

It is not difficult by using elementary Lie algebra arguments to establish the affine structure on the complement of the Schubert cycle which is the nilpotent orbit containing the base point. In our case we need to prove that the image of the Teichmüller space under the period map do not intersect certain component of the Schubert cycle which is codimension one. We call this the partial global transversality. The problem when partial global transversality holds is very important in the study of global behavior of the period map. The Griffiths transversality is too weak to deal with such global problems of the period map. In [**10**] we obtained the partial global transversality by using Yau's solution of the Calabi conjecture.

As a corollary we proved the holomorphic embedding theorem of the Teichmüller space of polarized CY manifolds in [**10**]:

THEOREM 10.9. *The Teichmüller space of polarized CY manifolds can be holomorphically embedded into the Euclidean space of same dimension. Furthermore, its Hodge completion is a domain of holomorphy and there exists a unique Kähler-Einstein metric on the completion.*

The methods that we used in [**11**] can be used to prove the global Torelli theorem for a large class of manifolds of general type. Furthermore, these methods can be used to treat the invariance of the plurigenera for Kähler manifolds where the projective case was proved by Siu.

References

[1] L. Ahlfors and L. Bers. Riemann's mapping theorem for variable metrics. *Ann. of Math. (2)*, 72:385–404, 1960.

[2] J. Bass and A. Todorov. The analogue of the Dedekind eta function for CY manifolds. I. *J. Reine Angew. Math.*, 599:61–96, 2006.

[3] M. Bershadsky, S. Cecotti, H. Ooguri, and C. Vafa. Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes. *Comm. Math. Phys.*, 165(2):311–427, 1994.

[4] S. Y. Cheng and S. T. Yau. On the existence of a complete Kähler metric on non-compact complex manifolds and the regularity of Fefferman's equation. *Comm. Pure Appl. Math.*, 33(4):507–544, 1980.

[5] P. Deligne and D. Mumford. The irreducibility of the space of curves of given genus. *Inst. Hautes Études Sci. Publ. Math.*, (36):75–109, 1969.

[6] F. Griffits and V. Šmid. Locally homogeneous complex manifolds. *Uspehi Mat. Nauk*, 26(5(161)):117–168, 1971. Translated from the English (Acta Math. **1**23 (1969), 253–302) by B. L. Žitnickiĭ.

[7] J. Harer and D. Zagier. The Euler characteristic of the moduli space of curves. *Invent. Math.*, 85(3):457–485, 1986.

[8] L. Ji, K. Liu, X. Sun, and S.-T. Yau. Gauss-bonnet theorem on the moduli space of riemann surfaces. preprint, 2009.

[9] L. Keen. Collars on Riemann surfaces. In *Discontinuous groups and Riemann surfaces (Proc. Conf., Univ. Maryland, College Park, Md., 1973)*, pages 263–268. Ann. of Math. Studies, No. 79. Princeton Univ. Press, Princeton, N.J., 1974.

[10] K. Liu, X. Sun, A. Todorov, and S.-T. Yau. Affine structures on the teichmüller space of polarized calabi-yau manifolds. preprint, 2009.

[11] K. Liu, X. Sun, A. Todorov, and S.-T. Yau. Global torelli theorem of the teichmüller space of polarized calabi-yau manifolds. preprint, 2009.

[12] K. Liu, X. Sun, and S.-T. Yau. Canonical metrics on the moduli space of Riemann surfaces. I. *J. Differential Geom.*, 68(3):571–637, 2004.

[13] K. Liu, X. Sun, and S.-T. Yau. Canonical metrics on the moduli space of Riemann surfaces. II. *J. Differential Geom.*, 69(1):163–216, 2005.

[14] K. Liu, X. Sun, and S.-T. Yau. Good geometry on the curve moduli. *Publ. Res. Inst. Math. Sci.*, 44(2):699–724, 2008.

[15] K. Liu, X. Sun, and S.-T. Yau. New results on the geometry of the moduli space of Riemann surfaces. *Sci. China Ser. A*, 51(4):632–651, 2008.

[16] K. Liu, X. Sun, and S.-T. Yau. Good metrics on the moduli space of Riemann surface I. preprint, 2009.

[17] K. Liu, X. Sun, and S.-T. Yau. Good metrics on the moduli space of Riemann surface II. preprint, 2009.

[18] K. Liu, A. Todorov, S.-T. Yau, and K. Zuo. Shafarevich's conjecture for CY manifolds. I. *Q. J. Pure Appl. Math.*, 1(1):28–67, 2005.

[19] Z. Lu and X. Sun. Weil-Petersson geometry on moduli space of polarized Calabi-Yau manifolds. *J. Inst. Math. Jussieu*, 3(2):185–229, 2004.

[20] V. Markovic and D. Saric. The universal properties of teichmüller spaces. Surveys in Differential Geometry, 2009.

[21] H. Masur. Extension of the Weil-Petersson metric to the boundary of Teichmuller space. *Duke Math. J.*, 43(3):623–635, 1976.

[22] C. T. McMullen. The moduli space of Riemann surfaces is Kähler hyperbolic. *Ann. of Math. (2)*, 151(1):327–357, 2000.

[23] D. Mumford. Hirzebruch's proportionality theorem in the noncompact case. *Invent. Math.*, 42:239–272, 1977.

[24] R. C. Penner. Perturbative series and the moduli space of Riemann surfaces. *J. Differential Geom.*, 27(1):35–53, 1988.

[25] I. I. Pyatetskii-Shapiro and I. R. Shafarevich. A torelli theorem for algebraic surfaces of type $k3$. *Izv. Akad. Nauk SSSR Ser. Mat.*, 35:530–572, 1971.

[26] L. Saper. L^2-cohomology of the Weil-Petersson metric. In *Mapping class groups and moduli spaces of Riemann surfaces (Göttingen, 1991/Seattle, WA, 1991)*, volume 150 of *Contemp. Math.*, pages 345–360. Amer. Math. Soc., Providence, RI, 1993.

[27] W. Schmid. Variation of Hodge structure: the singularities of the period mapping. *Invent. Math.*, 22:211–319, 1973.

[28] W.-X. Shi. Ricci flow and the uniformization on complete noncompact Kähler manifolds. *J. Differential Geom.*, 45(1):94–220, 1997.

[29] Y. Shimizu and K. Ueno. *Advances in moduli theory*, volume 206 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 2002. Translated from the 1999 Japanese original, Iwanami Series in Modern Mathematics.

[30] G. Tian. Smoothness of the universal deformation space of compact Calabi-Yau manifolds and its Petersson-Weil metric. In *Mathematical aspects of string theory (San Diego, Calif., 1986)*, volume 1 of *Adv. Ser. Math. Phys.*, pages 629–646. World Sci. Publishing, Singapore, 1987.

[31] A. N. Todorov. The Weil-Petersson geometry of the moduli space of SU($n \geq 3$) (Calabi-Yau) manifolds. I. *Comm. Math. Phys.*, 126(2):325–346, 1989.

[32] S. Trapani. On the determinant of the bundle of meromorphic quadratic differentials on the Deligne-Mumford compactification of the moduli space of Riemann surfaces. *Math. Ann.*, 293(4):681–705, 1992.

[33] S. A. Wolpert. Chern forms and the Riemann tensor for the moduli space of curves. *Invent. Math.*, 85(1):119–145, 1986.

[34] S. A. Wolpert. Asymptotics of the spectrum and the Selberg zeta function on the space of Riemann surfaces. *Comm. Math. Phys.*, 112(2):283–315, 1987.

[35] S. A. Wolpert. The hyperbolic metric and the geometry of the universal curve. *J. Differential Geom.*, 31(2):417–472, 1990.

[36] S. A. Wolpert. Geometry of the Weil-Petersson completion of Teichmüller space. In *Surveys in differential geometry, Vol. VIII (Boston, MA, 2002)*, Surv. Differ. Geom., VIII, pages 357–393. Int. Press, Somerville, MA, 2003.

[37] S. A. Wolpert. Understanding weil-petersson curvature. arXiv:0809.3699, 2008.

[38] S. A. Wolpert. Extension of the Weil-Petersson connection. *Duke Math. J.*, 146(2):281–303, 2009.

[39] S. A. Wolpert. The Weil-Petersson metric geometry. In *Handbook of Teichmüller theory. Vol. II*, volume 13 of *IRMA Lect. Math. Theor. Phys.*, pages 47–64. Eur. Math. Soc., Zürich, 2009.

[40] S. Yeung. Geometry of domains with the uniform squeezing property. arXiv:0906.4647, 2009.