The global existence of smooth solutions of relativistic string equations in the Schwarzschild space-time

Chun-Lei He* and De-Xing Kong†‡

August 1, 2009

Abstract

This paper concerns the motion of relativistic strings in the Schwarzschild space-time. As a general framework, we first analyze the basic equations for the motion of a p-dimensional extended object in a general enveloping space-time $(\mathscr{N}, \tilde{g})$, which is a given Lorentzian manifold, and then particularly investigate the interesting properties enjoyed by the equations for the motion of relativistic strings in the Schwarzschild space-time. Based on this, under suitable assumptions we prove the global existence of smooth solutions of the Cauchy problem for the equations for the motion of relativistic strings with small arc length in the Schwarzschild space-time.

Key words and phrases: Schwarzschild space-time, relativistic string, nonlinear wave equation, classical solution, global existence, event horizon.

2000 Mathematics Subject Classification: 35Q75, 35L70, 70H40.

*Department of Mathematics, Anhui Normal University, Wuhu 241000, China;
†Department of Mathematics, Zhejiang University, Hangzhou 310027, China;
‡Corresponding author: kong@cms.zju.edu.cn.

1 Introduction

It is well known that, in particle physics the string model is frequently used to study the structure of hardrons. A free string is a one-dimensional physical object whose motion is represented by a time-like surface. In this paper, we study the nonlinear dynamics of relativistic strings moving in the Schwarzschild space-time.

In mathematics, the the extremal surfaces in a physical space-time include the following four types: space-like, time-like, light-like or mixed types. For the case of the space-like minimal (or maximal) surfaces in the Minkowski space-time, we refer to the classical papers by Calabi [7] and by Cheng and Yau [8]. The case of time-like surfaces the Minkowski space-time has been investigated by several authors (e.g. [2] and [28]). Barbashov, Nesterenko and Chervyakov [2] study the nonlinear differential equations describing in differential geometry the minimal surfaces in the Minkowski space-time and provided examples with exact solutions. Milnor [28] generate examples that display considerable variety in the shape of entire time-like minimal surfaces in the 3-dimensional Minkowski space-time $\mathbb{R}^{1+2}$ and show that such surfaces need not be planar. Gu investigates the extremal surfaces of mixed type in the n-dimensional Minkowski space-time (cf. [16]) and constructes many complete extremal surfaces of mixed type in the 3-dimensional Minkowski space-time (cf. [17]). Recently, Kong et al re-study the equation for time-like extremal surfaces in the Minkowski space-time $\mathbb{R}^{1+n}$, which corresponds to the motion of an open string in $\mathbb{R}^{1+n}$ (see [21]-[22]). For the multidimensional versions, Hoppe et al derive the equation for a classical relativistic open membrane moving in the Minkowski space-time $\mathbb{R}^{1+3}$, which is a nonlinear wave equation corresponding to the extremal hypersurface equation in $\mathbb{R}^{1+3}$, and give some special classical solutions (cf. [4], [18]). The Cauchy problem with small initial data for the minimal surface equation in the Minkowski space-time has been studied successfully by Lindblad [27] and, by Chae and Huh [9] in a more general framework. They prove the global existence of smooth solutions for sufficiently small initial data with compact support, using the null forms in Christodoulou and Klainerman's style (cf. [10] and [20]).

In the paper [25], the authors investigate the dynamics of relativistic (in particular, closed) strings moving in the Minkowski space-time $\mathbb{R}^{1+n}$ ($n \geq 2$). They first derive a system with n nonlinear wave equations of Born-Infeld type which governs the motion of the string. This system can also be used to describe the extremal surfaces in $\mathbb{R}^{1+n}$. Then they show that this system enjoys some interesting geometric properties. Based on this, they give a sufficient and necessary condition for the global existence of extremal surfaces without space-like point in $\mathbb{R}^{1+n}$ with given initial data. This result corresponds to the global propagation of nonlinear waves for the system describing the motion of the string in $\mathbb{R}^{1+n}$. Moreover, a great deal of numerical analysis are investigated, and the numerical results show that, in phase space, various topological singularities develop in finite time in the motion of the string. More recently, Kong and Zhang furthermore study the motion of relativistic strings in the Minkowski space $\mathbb{R}^{1+n}$ (see [24]). Surprisingly, they obtain a general solution formula for this complicated system of nonlinear equations. Based on this solution formula, they successfully show that the motion of closed strings is always time-periodic. Moreover, they further extend the solution formula to finite relativistic strings.

Here we would like to mention some important results related to this topic. As we know, the Born-Infeld theory has recently received much attention mainly due to the fact that the Born-Infeld type Lagrangian naturally appear in the string theory and the relativity theory. This triggers the revival of interests in the original Born-Infeld electromagnetism (cf. Born and Infeld [5]) and the exploration of Born-Infeld gauge theory (cf. Gibbons [12]). From the mathematical point of view, this theory is a nonlinear generalization of the Maxwell theory. Gibbons [12] gives a systematic study of the Born-Infeld theory and obtained exact solutions in numerous situations. Recently Brenier [6] even carried out a study of the theory in the connection to the hydrodynamics.

However, in a curved space-time there are only few results to obtain (see [15] and Sections 24 and 32 in [1]). In the present paper, we consider the motion of relativistic strings in the Schwarzschild space-time. We first analyze the basic equations for the motion of a p-dimensional extended object in a general enveloping space-time $(\mathscr{N}, \tilde{g})$, which is a given Lorentzian manifold, and then in particular investigate the interesting properties enjoyed by the equations for the motion of relativistic strings in the Schwarzschild space-time. Based on this, under suitable assumptions we prove a global existence theorem on smooth solutions of the Cauchy problem for the equations for the motion of relativistic strings with small arc length in the Schwarzschild space-time.

The paper is organized as follows. In Section 2, we study the basic equations for the motion of a p-dimensional extended object in a given Lorentzian manifold. In particular, in Section 3 we investigate the equations for the motion of relativistic strings in the Schwarzschild space-time and show some interesting properties enjoyed by these equations. Based on this, in Section 4 we prove a global existence theorem for the motion of a relativistic string with small arc length in the Schwarzschild space-time. A summary and some discussions are given in Section 5.

2 Basic equations within general framework

In this section, we investigate the basic equations of the motion for a p-dimensional extended object in the enveloping space-time $(\mathscr{N}, \tilde{g})$, which is a given Lorentzian manifold.

Since the world sheet of the p-dimensional extended object corresponds to a $(p+1)$-dimensional extremal sub-manifold, denoted by $\mathscr{M}$, we may choose the local coordinates $(u^0, u^1, \cdots, u^p)$ in $\mathscr{M}$. Let the position vector in the space-time $(\mathscr{N}, \tilde{g})$ be

$$X(u^0, u^1, \cdots, u^p) = (x^0(u^0, u^1, \cdots, u^p), x^1(u^0, u^1, \cdots, u^p), \cdots, x^n(u^0, u^1, \cdots, u^p)). \tag{2.1}$$

Denote

$$x^A_\mu = \frac{\partial x^A}{\partial u^\mu} \quad \text{and} \quad x^A_{\mu\nu} = \frac{\partial^2 x^A}{\partial u^\mu \partial u^\nu} \quad (A, B = 0, 1, \cdots, n). \tag{2.2}$$

Then the induced metric of the sub-manifold $\mathscr{M}$ can be written as $g = (g_{\mu\nu})$, where

$$g_{\mu\nu} = \tilde{g}_{AB} x^A_\mu x^B_\nu \quad (\mu, \nu = 0, 1, \cdots, p). \tag{2.3}$$

As a result, the corresponding Euler-Lagrange equations for the p-dimensional extended object moving in the Lorentzian space-time $(\mathscr{N}, \tilde{g})$ read

$$g^{\mu\nu}\left(x^C_{\mu\nu} + \tilde{\Gamma}^C_{AB} x^A_\mu x^B_\nu - \Gamma^\rho_{\mu\nu} x^C_\rho\right) = 0 \quad (C = 0, 1, \cdots, n), \tag{2.4}$$

where $g^{-1} = (g^{\mu\nu})$ is the inverse of g and $\tilde{\Gamma}^C_{AB}$, $\Gamma^\rho_{\mu\nu}$ stand for the connections of the metric $\tilde{g}$ and the induced metric g, respectively.

For the convenience of the following discussion, we introduce notations

$$\begin{cases} Q \triangleq (X_0, X_1, \cdots, X_p), & G = (G_{AB})_{(1+n)\times(1+n)} \triangleq Q g^{-1} Q^T, \\ M \triangleq I - G\tilde{g}, & E \triangleq (E_0, E_1, \cdots, E_n)^T, \end{cases} \tag{2.5}$$

where I is the $(n+1)\times(n+1)$ identity matrix, $X_\mu = (x^0_\mu, x^1_\mu, \cdots, x^n_\mu)^T$ $(\mu = 0, 1, \cdots, p)$ and

$$E_C = g^{\mu\nu}\left(x^C_{\mu\nu} + \tilde{\Gamma}^C_{AB} x^A_\mu x^B_\nu\right) \quad (C = 0, 1, \cdots, n). \tag{2.6}$$

The main result in this section is the following theorem.

Theorem 2.1 *The left hand side of (2.4) can be exactly rewritten in the form ME, i.e.,*

$$g^{\mu\nu}\left(x^C_{\mu\nu} + \tilde{\Gamma}^C_{AB} x^A_\mu x^B_\nu - \Gamma^\rho_{\mu\nu} x^C_\rho\right) = E_C - \sum_{A,B=0}^{n} G_{CA} \tilde{g}_{AB} E_B, \tag{2.7}$$

and then, the equations (2.4) for the motion of extended object are equivalent to

$$ME = 0. \tag{2.8}$$

Proof. It suffices to verify that

$$g^{\mu\nu}\Gamma^{\rho}_{\mu\nu}x^C_\rho = \sum_{A,B=0}^{n} G_{CA}\tilde{g}_{AB}E_B \quad (C=0,1,\cdots,n). \tag{2.9}$$

On one hand, calculating the left hand side of (2.9) yields

$$\begin{aligned}
g^{\mu\nu}\Gamma^{\rho}_{\mu\nu}x^C_\rho &= g^{\mu\nu}x^C_\rho g^{\rho\sigma}\left(\frac{\partial g_{\sigma\mu}}{\partial u^\nu} - \frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial u^\sigma}\right)\\
&= g^{\mu\nu}g^{\rho\sigma}x^C_\rho\left[\frac{\partial}{\partial u^\nu}\left(\tilde{g}_{AD}x^A_\mu x^D_\sigma\right) - \frac{1}{2}\frac{\partial}{\partial u^\sigma}\left(\tilde{g}_{AB}x^A_\mu x^B_\nu\right)\right]\\
&= g^{\mu\nu}g^{\rho\sigma}x^C_\rho\left[\frac{\partial \tilde{g}_{AD}}{\partial x^B}x^B_\nu x^A_\mu x^D_\sigma + \tilde{g}_{AD}x^A_{\mu\nu}x^D_\sigma + \tilde{g}_{AD}x^A_\mu x^D_{\nu\sigma} - \frac{1}{2}\frac{\partial \tilde{g}_{AB}}{\partial x^D}x^D_\sigma x^A_\mu x^B_\nu - \tilde{g}_{AB}x^A_{\mu\sigma}x^B_\nu\right]\\
&= g^{\mu\nu}g^{\rho\sigma}x^C_\rho x^D_\sigma\left[\left(\frac{\partial \tilde{g}_{AD}}{\partial x^B} - \frac{1}{2}\frac{\partial \tilde{g}_{AB}}{\partial x^D}\right)x^A_\mu x^B_\nu + \tilde{g}_{AD}x^A_{\mu\nu}\right].
\end{aligned} \tag{2.10}$$

On the other hand, the right hand side of (2.9) can be reformulated as

$$\begin{aligned}
\sum_{A,B=0}^{n} G_{CA}\tilde{g}_{AB}E_B &= \sum_{A,B=0}^{n} g^{\rho\sigma}x^C_\rho x^A_\sigma \tilde{g}_{AB}g^{\mu\nu}\left(x^B_{\mu\nu} + \tilde{\Gamma}^B_{EF}x^E_\mu x^F_\nu\right)\\
&= \sum_{A,B=0}^{n} g^{\rho\sigma}x^C_\rho x^A_\sigma \tilde{g}_{AB}g^{\mu\nu}\left[x^B_{\mu\nu} + \tilde{g}^{BD}\left(\frac{\partial \tilde{g}_{DE}}{\partial x^F} - \frac{1}{2}\frac{\partial \tilde{g}_{EF}}{\partial x^D}\right)x^E_\mu x^F_\nu\right]\\
&= g^{\mu\nu}g^{\rho\sigma}x^C_\rho x^D_\sigma\left[\left(\frac{\partial \tilde{g}_{AD}}{\partial x^B} - \frac{1}{2}\frac{\partial \tilde{g}_{AB}}{\partial x^D}\right)x^A_\mu x^B_\nu + \tilde{g}_{AD}x^A_{\mu\nu}\right].
\end{aligned} \tag{2.11}$$

Combining (2.10) and (2.11) gives the desired (2.9). This proves Theorem 2.1. ∎

Remark 2.1 *Because the equations for the motion of the extended object are equivalent to the equations (2.8), we can only consider the equations*

$$E = 0. \tag{2.12}$$

It is obvious that the solutions of $E=0$ must be the solutions of the equations (2.4). Furthermore, since the equations (2.4) are independent of the choice of coordinate charts, we may choose some special coordinate charts to simplify these equations. Historically, for the case of Riemannian manifolds, the first special coordinate system is the so-called harmonic coordinate system which obeys the equations $\nabla_\nu\nabla^\nu x^A = 0$; while for the case of Lorentzian manifolds, the corresponding one is the wave coordinate system, which satisfies the wave coordinate condition $g^{\mu\nu}\Gamma^\rho_{\mu\nu} = 0$ for all ρ. Under this coordinate system, the equations (2.4) obviously reduce to the equations (2.12).

Remark 2.2 *If the Lorentzian manifold $(\mathscr{N}, \tilde{g})$ is flat, then the corresponding connection vanishes, i.e., $\tilde{\Gamma}^C_{AB} = 0$ $(A,B,C=0,1,2,3)$. In this case, the equations (2.4) become*

$$g^{\mu\nu}\left(x^C_{\mu\nu} - \Gamma^\rho_{\mu\nu}x^C_\rho\right) = 0 \quad (C=0,1,\cdots,n);$$

while the equations (2.12) reduce to

$$g^{\mu\nu}x^C_{\mu\nu}=0 \quad (C=0,1,\cdots,n),$$

which go back to the equations studied by Kong et al (cf. [25] and [24]).

Remark 2.3 *Under the wave coordinate system, the equations (2.4) is nothing but the equations (2.12), which leads to*

$$g^{\mu\nu}\Gamma^\rho_{\mu\nu}x^C_\rho=0 \quad (C=0,1,\cdots,n). \tag{2.13}$$

Unfortunately, since the connection $\Gamma^\rho_{\mu\nu}$ of the induced metric g contains the second-order derivatives of the unknown functions x^C $(C=0,1,\cdots,n)$, it is not easy to solve the equations (2.13).

The rest of the section is devoted to the study on the rank of the matrix M. Since we are only interested in the physical motion, we may assume that the sub-manifold $\mathscr{M}$ is C^2 and *time-like*, i.e.,

$$\Delta \triangleq \det g<0. \tag{2.14}$$

This implies that the world sheet of the extended object is time-like, and then the motion satisfies the causality.

Theorem 2.2 *Under the assumption (2.14), it holds that*

$$\text{rank } M=n-p. \tag{2.15}$$

Moreover, the nonzero eigenvalues of M are all equal to 1.

Proof. In fact, it is easy to see that

$$|\lambda I-M|=\frac{1}{|-g|}|-g|\left|(\lambda-1)I+Qg^{-1}Q^T\tilde{g}\right|=\frac{1}{|-g|}\begin{vmatrix}(\lambda-1)I & Q\\ Q^T\tilde{g} & -g\end{vmatrix}. \tag{2.16}$$

Noting the fact that $|(\lambda-1)I|$ is equal to zero if and only if $\lambda=1$, we can choose a sequence $\{t_k\}$ such that

$$t_k\to 1 \quad \text{and} \quad |(\lambda-t_k)I|\neq 0.$$

We now consider the matrixes $(\lambda-t_k)I$ instead of $(\lambda-1)I$. Notice that

$$(\lambda-t_k)I\to(\lambda-1)I \quad \text{as} \quad t_k\to 1.$$

A direct calculation gives

$$\begin{aligned}\begin{vmatrix}(\lambda-t_k)I & Q\\ Q^T\tilde{g} & -g\end{vmatrix}&=|(\lambda-t_k)I|\left|-g-Q^T\tilde{g}((\lambda-t_k)I)^{-1}Q\right|=(-1)^{1+p}(\lambda-t_k)^{1+n}|g+(\lambda-t_k)^{-1}g|\\&=(-1)^{1+p}(\lambda-t_k)^{n-p}(\lambda-t_k+1)^{1+p}|g|=|-g|(\lambda-t_k)^{n-p}(\lambda-t_k+1)^{1+p}.\end{aligned}$$

Thus, we obtain

$$|\lambda I - M| = \frac{1}{|-g|} \lim_{t_k \to 1} \begin{vmatrix} (\lambda - t_k)I & Q \\ Q^T \tilde{g} & -g \end{vmatrix} = \lambda^{1+p}(\lambda - 1)^{n-p}. \tag{2.17}$$

This proves Theorem 2.2. $\blacksquare$

Remark 2.4 *Theorems 2.1 and 2.2 show that there only exist $n-p$ independent equations in (2.8). Moreover, it holds that*

$$MQQ^T = 0, \quad Q^T M Q = 0. \tag{2.18}$$

In fact, the first equation in (2.18) comes directly from the following fact

$$MQQ^T = (I - G\tilde{g})QQ^T = QQ^T - Qg^{-1}Q^T\tilde{g}QQ^T = 0. \tag{2.19}$$

The second equation in (2.18) can be proved in a similar way.

3 The equations for the motion of relativistic strings in the Schwarzschild space-time

The Schwarzschild space-time is a fundamental physical space-time, it plays an important role in general relativity, modern cosmology and the physics of black holes. This kind of space-time is stationary, spherically symmetric and asymptotically flat. This section is devoted to the study on the equations for the motion of relativistic strings in the Schwarzschild space-time. In the spherical coordinates (τ, r, α, β), the Schwarzschild metric $\tilde{g}$ reads

$$ds^2 = -\left(1 - \frac{2m}{r}\right) d\tau^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 \left(d\alpha^2 + \sin^2 \alpha d\beta^2\right), \tag{3.1}$$

where m is a positive constant standing for the universe mass. However, if the enveloping space-time is Schwarzschild, we do not think that the spherical coordinates is a good choice for the study on general motion of relativistic strings in the Schwarzschild space-time, since the angle variables will bring some difficulties (see the details in Appendix). Hence, in the present paper we shall adopt the Schwarzschild metric in the Cartesian coordinates, denoted by (x^0, x^1, x^2, x^3).

By a direct calculation, in the Cartesian coordinates (x^0, x^1, x^2, x^3) the Schwarzschild metric reads

$$ds^2 = -\left(1 - \frac{2m}{r}\right) (dx^0)^2 + \sum_{i,j=1}^{3} \frac{2m}{r - 2m} \frac{x^i x^j}{r^2} dx^i dx^j + \sum_{i=1}^{3} (dx^i)^2, \tag{3.2}$$

where

$$r = \left(\sum_{i=1}^{3} (x^i)^2\right)^{\frac{1}{2}}.$$

In this paper, we take the parameters of world sheet of the motion of the relativistic string in the Schwarzschild space-time as the following form

$$(t,\theta) \longrightarrow (x^0(t,\theta), x^1(t,\theta), x^2(t,\theta), x^3(t,\theta)). \tag{3.3}$$

Denote

$$X(t,\theta) = (x^1(t,\theta), x^2(t,\theta), x^3(t,\theta))^T, \quad \bar{X} = (x^0(t,\theta), (X(t,\theta))^T)^T.$$

For two given vectors $X = (x^1, x^2, x^3)$ and $Y = (y^1, y^2, y^3)$, the inner product of them is defined by

$$\langle X, Y\rangle = \sum_{i=1}^{3} x^i y^i.$$

Thus, in the coordinates (t,θ) the induced metric of the sub-manifold $\mathscr{M}$ reads

$$g = (g_{\mu\nu})^1_{\mu,\nu=0},$$

where

$$\begin{cases} g_{00} = -\dfrac{r-2m}{r}(x^0_t)^2 + |X_t|^2 + \dfrac{2m}{r^2(r-2m)}\langle X, X_t\rangle^2, \\ g_{01} = g_{10} = -\dfrac{r-2m}{r}x^0_t x^0_\theta + \langle X_t, X_\theta\rangle + \dfrac{2m}{r^2(r-2m)}\langle X, X_t\rangle\langle X, X_\theta\rangle, \\ g_{11} = -\dfrac{r-2m}{r}(x^0_\theta)^2 + |X_\theta|^2 + \dfrac{2m}{r^2(r-2m)}\langle X, X_\theta\rangle^2. \end{cases} \tag{3.4}$$

As before, let $g^{-1} = (g^{\mu\nu})$ be the inverse matrix of g.

In the present situation, the time-like assumption (2.14) becomes

$$\begin{aligned} \Delta \triangleq\ & \det g = g_{00}g_{11} - g_{01}^2 \\ =\ & -\frac{r-2m}{r}\left[(x^0_t|X_\theta|)^2 + (x^0_\theta|X_t|)^2 - 2x^0_t x^0_\theta\langle X_t, X_\theta\rangle\right] + |X_t|^2|X_\theta|^2 - \langle X_t, X_\theta\rangle^2 \\ & -\frac{2m}{r^3}\left[(x^0_t)^2\langle X, X_\theta\rangle^2 + (x^0_\theta)^2\langle X, X_t\rangle^2 - 2x^0_t x^0_\theta\langle X, X_t\rangle\langle X, X_\theta\rangle\right] \\ & +\frac{2m}{r^2(r-2m)}\left[|X_t|^2\langle X, X_\theta\rangle^2 + |X_\theta|^2\langle X, X_t\rangle^2 - 2\langle X, X_t\rangle\langle X, X_\theta\rangle\langle X_t, X_\theta\rangle\right] \\ <\ & 0, \end{aligned} \tag{3.5}$$

and the system (2.12), i.e.,

$$E_C = g^{\mu\nu}\left(x^C_{\mu\nu} + \tilde{\Gamma}^C_{AB}x^A_\mu x^B_\nu\right) = 0 \quad (C = 0,1,2,3) \tag{3.6}$$

can be rewritten in the following form

$$\begin{aligned} & g_{11}x^0_{tt} - 2g_{01}x^0_{t\theta} + g_{00}x^0_{\theta\theta} + \\ & \quad \frac{2m}{r^2(r-2m)}(g_{11}x^0_t\langle X, X_t\rangle - g_{01}x^0_t\langle X, X_\theta\rangle - g_{01}x^0_\theta\langle X, X_t\rangle + g_{00}x^0_\theta\langle X, X_\theta\rangle) = 0 \end{aligned} \tag{3.7}$$

and

$$\begin{aligned}&g_{11}X_{tt}-2g_{01}X_{t\theta}+g_{00}X_{\theta\theta}+\frac{m(r-2m)}{r^4}(g_{11}(x_t^0)^2-2g_{01}x_t^0x_\theta^0+g_{00}(x_\theta^0)^2)X+\\&\frac{2m}{r^3}(g_{11}|X_t|^2-2g_{01}\langle X_t,X_\theta\rangle+g_{00}|X_\theta|^2)X+\\&\frac{m(3r-4m)}{r^5(2m-r)}(g_{11}\langle X,X_t\rangle^2-2g_{01}\langle X,X_t\rangle\langle X,X_\theta\rangle+g_{00}\langle X,X_\theta\rangle^2)X=0.\end{aligned}\tag{3.8}$$

Remark 3.1 *The mapping (3.3) described by the system (3.6) is essentially a wave map from the Minkowski space-time $\mathbb{R}^{1,1}$ to the Schwarzschild space-time. Gu [14] proved successfully the global existence of smooth solution of the Cauchy problem for harmonic maps defined on the Minkowski space-time $\mathbb{R}^{1,1}$. According to the authors' knowledge, there exist only a few results on the wave map from $\mathbb{R}^{1,1}$ to the Schwarzschild space-time.*

Let

$$\begin{cases}\bar{u}=(u^0,u)^T\triangleq\bar{X}=(x^0,x^1,x^2,x^3)^T, \quad \bar{v}=(v^0,v)^T\triangleq\bar{X}_t=(x_t^0,x_t^1,x_t^2,x_t^3)^T,\\ \bar{w}=(w^0,w)^T\triangleq\bar{X}_\theta=(x_\theta^0,x_\theta^1,x_\theta^2,x_\theta^3)^T\end{cases}\tag{3.9}$$

and

$$U=(U^1,\cdots,U^{12})^T\triangleq(\bar{u},\bar{v},\bar{w})^T.\tag{3.10}$$

Then the equations (3.7)-(3.8) can be equivalently rewritten as

$$U_t+AU_\theta+B=0,\tag{3.11}$$

where

$$A=\begin{pmatrix}0&0&0\\0&-\dfrac{2g_{01}}{g_{11}}I_{4\times4}&\dfrac{g_{00}}{g_{11}}I_{4\times4}\\0&-I_{4\times4}&0\end{pmatrix}_{12\times12}$$

and

$$B=(B^1,\cdots,B^{12})^T,$$

in which

$$\begin{cases}B^i=-v^{i-1}\quad(i=1,2,3,4),\\ B^5=\dfrac{2m}{|u|^2(|u|-2m)}\left[v^0\langle u,v\rangle-\dfrac{g_{01}}{g_{11}}v^0\langle u,w\rangle-\dfrac{g_{01}}{g_{11}}w^0\langle u,v\rangle+\dfrac{g_{00}}{g_{11}}w^0\langle u,w\rangle\right],\\ B^i=u^{i-5}\times\left\{\dfrac{m(|u|-2m)}{|u|^4}\left[(v^0)^2-\dfrac{2g_{01}}{g_{11}}v^0w^0+\dfrac{g_{00}}{g_{11}}(w^0)^2\right]+\dfrac{2m}{|u|^3}\left[|v|^2-\dfrac{2g_{01}}{g_{11}}\langle v,w\rangle+\dfrac{g_{00}}{g_{11}}|w|^2\right]\right.\\ \qquad\left.+\dfrac{m(3|u|-4m)}{|u|^5(2m-|u|)}\left[\langle u,v\rangle^2-\dfrac{2g_{01}}{g_{11}}\langle u,v\rangle\langle u,w\rangle+\dfrac{g_{00}}{g_{11}}\langle u,w\rangle^2\right]\right\}\quad(i=6,7,8),\\ B^i=0\quad(i=9,10,11,12).\end{cases}\tag{3.12}$$

By a direct calculation, the eigenvalues of the matrix A read

$$\begin{cases} \lambda_i = \lambda_0 \triangleq 0 \quad (i = 1, 2, 3, 4), \\ \lambda_i = \lambda_- \triangleq \dfrac{-g_{01} - \sqrt{(g_{01})^2 - g_{00}g_{11}}}{g_{11}} \quad (i = 5, 6, 7, 8), \\ \lambda_i = \lambda_+ \triangleq \dfrac{-g_{01} + \sqrt{(g_{01})^2 - g_{00}g_{11}}}{g_{11}} \quad (i = 9, 10, 11, 12). \end{cases} \tag{3.13}$$

The right eigenvector corresponding to λ_i $(i = 1, 2, \cdots, 12)$ can be chosen as

$$\begin{cases} r_i = (e_i, 0, 0, 0, 0, 0, 0, 0, 0)^T \quad (i = 1, 2, 3, 4), \\ r_i = (0, 0, 0, 0, -\lambda_- e_{i-4}, e_{i-4})^T \quad (i = 5, 6, 7, 8), \\ r_i = (0, 0, 0, 0, -\lambda_+ e_{i-8}, e_{i-8})^T \quad (i = 9, 10, 11, 12), \end{cases} \tag{3.14}$$

where

$$e_1 = (1, 0, 0, 0), \quad e_2 = (0, 1, 0, 0), \quad e_3 = (0, 0, 1, 0), \quad e_4 = (0, 0, 0, 1).$$

While, the left eigenvector corresponding to λ_i $(i = 1, 2, \cdots, 12)$ can be taken as

$$\begin{cases} l_i = (e_i, 0, 0, 0, 0, 0, 0, 0, 0) \quad (i = 1, 2, 3, 4), \\ l_i = (0, 0, 0, 0, e_{i-4}, \lambda_+ e_{i-4}) \quad (i = 5, 6, 7, 8), \\ r_i = (0, 0, 0, 0, e_{i-8}, \lambda_- e_{i-8}) \quad (i = 9, 10, 11, 12). \end{cases} \tag{3.15}$$

Summarizing the above discussion gives

Proposition 3.1 *Under the assumption (3.5), the system (3.11) is a non-strictly hyperbolic system with twelve eigenvalues (see (3.13)), and the corresponding right (resp. left) eigenvectors can be chosen as (3.14) (resp. (3.15)).*

Proposition 3.2 *Under the assumption (3.5), the system (3.11) is linearly degenerate in the sense of Lax (see [26]).*

Proof. Obviously, it is easy to see that

$$\nabla\lambda_0 \cdot r_i = 0 \quad (i = 1, 2, 3, 4).$$

We next calculate the invariants $\nabla\lambda_- \cdot r_i$ $(i = 5, 6, 7, 8)$ and $\nabla\lambda_+ \cdot r_i$ $(i = 9, 10, 11, 12)$.

For every $i \in \{5, 6, 7, 8\}$, by calculations, we obtain

$$\nabla\lambda_- \cdot r_i = -\lambda_- \frac{\partial\lambda_-}{\partial U^i} + \frac{\partial\lambda_-}{\partial U^{i+4}} \equiv 0.$$

Similarly, we have

$$\nabla\lambda_+ \cdot r_i \equiv 0 \qquad (i = 9, 10, 11, 12).$$

Thus, the proof of Proposition 3.2 is completed. ■

On the one hand, we have

$$\left\{\begin{array}{lll}\dfrac{\partial\lambda_-}{\partial t}+\lambda_+\dfrac{\partial\lambda_-}{\partial\theta} & = & \displaystyle\sum_{\mu=0}^{3}\left[\frac{\partial\lambda_-}{\partial u^\mu}u_t^\mu+\frac{\partial\lambda_-}{\partial v^\mu}v_t^\mu+\frac{\partial\lambda_-}{\partial w^\mu}w_t^\mu+\lambda_+\left(\frac{\partial\lambda_-}{\partial u^\mu}u_\theta^\mu+\frac{\partial\lambda_-}{\partial v^\mu}v_\theta^\mu+\frac{\partial\lambda_-}{\partial w^\mu}w_\theta^\mu\right)\right]\\ & = & \displaystyle\sum_{\mu=0}^{3}\left(\frac{\partial\lambda_-}{\partial u^\mu}v^\mu+\frac{\lambda_-}{\partial u^\mu}\lambda_+w^\mu\right)-\sum_{\mu=0}^{3}\frac{\partial\lambda_-}{\partial v^\mu}B^{\mu+5}.\end{array}\right. \tag{3.16}$$

On the other hand, by straightforward computations, we obtain

$$\frac{\partial\lambda_-}{\partial u^0}=0,\qquad \frac{\partial\lambda_-}{\partial v^0}=-\frac{1}{\sqrt{-\Delta}}\frac{|u|-2m}{|u|}(\lambda_-w^0+v^0), \tag{3.17}$$

$$\begin{aligned}\frac{\partial\lambda_-}{\partial u^i} = {} & \frac{\lambda_-}{\sqrt{-\Delta}}\left\{-\frac{2m}{|u|^3}u^iv^0w^0-\frac{2m(3|u|-4m)}{|u|^4(|u|-2m)^2}u^i\langle u,v\rangle\langle u,w\rangle+\frac{2mv^i\langle u,w\rangle}{|u|^2(|u|-2m)}+\frac{2mw^i\langle u,v\rangle}{|u|^2(|u|-2m)}\right\}+\\ & \frac{1}{2\sqrt{-\Delta}}\left\{-\frac{2m}{|u|^3}u^i(v^0)^2-\frac{2m(3|u|-4m)}{|u|^4(|u|-2m)^2}u^i\langle u,v\rangle^2+\frac{4mv^i\langle u,v\rangle}{|u|^2(|u|-2m)}\right\}+\\ & \frac{\lambda_-^2}{2\sqrt{-\Delta}}\left\{-\frac{2m}{|u|^3}u^i(w^0)^2-\frac{2m(3|u|-4m)}{|u|^4(|u|-2m)^2}u^i\langle u,w\rangle^2+\frac{4mw^i\langle u,w\rangle}{|u|^2(|u|-2m)}\right\}\quad (i=1,2,3)\end{aligned} \tag{3.18}$$

and

$$\frac{\partial\lambda_-}{\partial v^i}=\frac{1}{\sqrt{-\Delta}}\left\{\lambda_-w^i+v^i+\frac{2m}{|u|^2(|u|-2m)}(\lambda_-u^i\langle u,w\rangle+u^i\langle u,v\rangle)\right\}\quad (i=1,2,3). \tag{3.19}$$

Substituting (3.17)-(3.19) into (3.16) leads to

$$\frac{\partial\lambda_-}{\partial t}+\lambda_+\frac{\partial\lambda_-}{\partial\theta}=0. \tag{3.20}$$

Similarly, we can show

$$\frac{\partial\lambda_+}{\partial t}+\lambda_-\frac{\partial\lambda_+}{\partial\theta}=0. \tag{3.21}$$

Thus, we have proved the following theorem.

Theorem 3.1 *Under the assumption (3.5), λ_- (resp. λ_+) is a Riemann invariant corresponding to λ_+ (resp. λ_-). Moreover, these two Riemann invariants satisfy*

$$\frac{\partial\lambda_-}{\partial t}+\lambda_+\frac{\partial\lambda_-}{\partial\theta}=0,\qquad \frac{\partial\lambda_+}{\partial t}+\lambda_-\frac{\partial\lambda_+}{\partial\theta}=0. \tag{3.22}$$

The system (3.22) plays an important role in our argument.

Let

$$P^\mu=v^\mu+\lambda_-w^\mu,\quad Q^\mu=v^\mu+\lambda_+w^\mu\quad (\mu=0,1,2,3) \tag{3.23}$$

and introduce

$$\begin{cases} \bar{S} = (S^0, S)^T = (S^0, S^1, S^2, S^3) \triangleq (u^0, u^1, u^2, u^3), \\ \bar{P} = (P^0, P)^T = (P^0, P^1, P^2, P^3), \\ \bar{Q} = (Q^0, Q)^T = (Q^0, Q^1, Q^2, Q^3). \end{cases} \tag{3.24}$$

Define

$$R = (R^1, \cdots, R^{12})^T \triangleq (\bar{S}, \bar{P}, \bar{Q})^T. \tag{3.25}$$

By direct calculations, it is easy to verify that R_i $(i = 1, \cdots, 12)$ satisfy

$$\begin{cases} \dfrac{\partial S^\mu}{\partial t} + \lambda_0 \dfrac{\partial S^\mu}{\partial \theta} = \dfrac{\lambda_+ P^\mu - \lambda_- Q^\mu}{\lambda_+ - \lambda_-} \qquad (\mu = 0, 1, 2, 3), \\ \dfrac{\partial P^0}{\partial t} + \lambda_+ \dfrac{\partial P^0}{\partial \theta} = \dfrac{-m}{|S|^2(|S| - 2m)} (P^0 \langle S, Q \rangle + Q^0 \langle S, P \rangle), \\ \dfrac{\partial P^i}{\partial t} + \lambda_+ \dfrac{\partial P^i}{\partial \theta} = S^i \left[\dfrac{m(2m - |S|)}{|S|^4} P^0 Q^0 - \dfrac{2m}{|S|^3} \langle P, Q \rangle + \dfrac{m(3|S| - 4m)}{|S|^5(|S| - 2m)} \langle S, P \rangle \langle S, Q \rangle \right] \quad (i = 1, 2, 3), \\ \dfrac{\partial Q^0}{\partial t} + \lambda_- \dfrac{\partial Q^0}{\partial \theta} = \dfrac{-m}{|S|^2(|S| - 2m)} (P^0 \langle S, Q \rangle + Q^0 \langle S, P \rangle), \\ \dfrac{\partial Q^i}{\partial t} + \lambda_- \dfrac{\partial Q^i}{\partial \theta} = S^i \left[\dfrac{m(2m - |S|)}{|S|^4} P^0 Q^0 - \dfrac{2m}{|S|^3} \langle P, Q \rangle + \dfrac{m(3|S| - 4m)}{|S|^5(|S| - 2m)} \langle S, P \rangle \langle S, Q \rangle \right] \quad (i = 1, 2, 3). \end{cases} \tag{3.26}$$

Remark 3.2 *Noting (3.22) and (3.25), we observe that, once we can solve $\lambda_\pm$ from the system (3.22), the system (3.26) becomes a semilinear hyperbolic system of first order.*

4 Global existence

This section is devoted to the study on the global existence of smooth solutions of the Cauchy problem for the equations for the motion of relativistic strings in the Schwarzschild space-time.

Consider the Cauchy problem for the equations (3.6) (or (3.7)-(3.8)) with the initial data

$$x^C(0, \theta) = p^C(\theta), \quad x_t^C(0, \theta) = q^C(\theta) \quad (C = 0, 1, 2, 3), \tag{4.1}$$

where $p^C(\theta)$ are C^2-smooth functions with bounded C^2-norm, while $q^C(\theta)$ are C^1-smooth functions with bounded C^1-norm. $p = (p^0, p^1, p^2, p^3), q = (q^0, q^1, q^2, q^3)$ stand for the initial position and initial velocity of the string under consideration, respectively. In order to state our main result in this section, we need some preliminaries.

4.1 Preliminaries

Stimulated by the discussion in Section 3, we now consider the Cauchy problem for the system (3.22) with the initial data

$$t = 0 : \ \lambda_\pm = \Lambda_\pm(\theta), \tag{4.2}$$

where $\Lambda_\pm(\theta)$ are two C^1-smooth functions with bounded C^1-norm. The following lemma comes from Kong and Tsuji [23] (or see [25]).

Lemma 4.1 *Suppose that the initial data $\Lambda_\pm(\theta)$ satisfy*

$$\Lambda_-(\theta) < \Lambda_+(\theta), \quad \forall\, \theta \in \mathbb{R}. \tag{4.3}$$

Then the Cauchy problem (3.22), (4.2) admits a unique global C^1 solution $\lambda_\pm = \lambda_\pm(t,\theta)$ on $\mathbb{R}^+ \times \mathbb{R}$, if and only if, for every fixed $\theta_2 \in \mathbb{R}$, it holds that

$$\Lambda_-(\theta_1) < \Lambda_+(\theta_2), \quad \forall\, \theta_1 < \theta_2. \tag{4.4}$$

Moreover, if the assumption (4.4) is satisfied, then the global smooth solution $\lambda_\pm = \lambda_\pm(t,\theta)$ satisfies

$$\lambda_-(t,\theta) < \lambda_+(t,\theta), \quad \forall\, (t,\theta) \in \mathbb{R}^+ \times \mathbb{R}. \tag{4.5}$$

Remark 4.1 *The assumption (4.3) guarantees that the system (3.22) is strictly hyperbolic near the initial time, while (4.4) is a necessary and sufficient condition guaranteeing that the system (3.22) is strictly hyperbolic on the domain where the smooth solution exists, i.e., $\mathbb{R}^+ \times \mathbb{R}$ (in fact, the whole (t,x)-plane).*

According to (3.4), we introduce

$$g_{00}[p,q](\theta) = -\frac{r_0(\theta) - 2m}{r_0(\theta)} \left(q^0(\theta)\right)^2 + \sum_{i=1}^{3} \left(q^i(\theta)\right)^2 + \frac{2m}{(r_0(\theta)^2)(r_0(\theta) - 2m)} \left(\sum_{i=1}^{3} p^i(\theta) q^i(\theta)\right)^2, \tag{4.6}$$

etc., and denote

$$\lambda^0_\pm(\theta) = \frac{-g_{01}[p,q](\theta) \pm \sqrt{\left(g_{01}[p,q](\theta)\right)^2 - g_{00}[p,q](\theta) g_{11}[p,q](\theta)}}{g_{11}[p,q](\theta)}, \tag{4.7}$$

where

$$r_0(\theta) = \left(\sum_{i=1}^{3} (p^i(\theta))^2\right)^{\frac{1}{2}}. \tag{4.8}$$

In order to apply Lemma 4.1, we assume that the initial data p, q satisfies

Assumption (H$_1$): $\lambda^0_\pm(\theta)$ are C^1-smooth functions with bounded C^1-norm;

Assumption (H$_2$): $\lambda^0_+(\theta) > \lambda^0_-(\theta), \quad \forall\, \theta \in \mathbb{R}$;

Assumption (H$_3$): For every fixed $\theta_2 \in \mathbb{R}$, it holds that

$$\lambda^0_+(\theta_2) > \lambda^0_-(\theta_1), \quad \forall\, \theta_1 < \theta_2.$$

Remark 4.2 *In the following argument the assumption (H_2) can be replaced by the following stronger hypothesis*

Assumption ($\mathbf{H}_2'$): *There exists a positive constant κ such that*

$$\lambda_+^0(\theta) \geq \lambda_-^0(\theta) + \kappa, \quad \forall\, \theta \in \mathbb{R}.$$

In fact, if we only suppose the assumption (H_2) is true, then in the following argument, it suffices to prove the existence of the solution on the domain of determinacy of any given interval $[-M, M]$, where M is an arbitrary positive number. Therefore, for simplicity, in what follows we always suppose that the assumption (H_2') is satisfied. However, all results are true for the case of the assumption (H_2).

In order to show that there indeed exist some initial data (p, q) satisfying the assumptions (H_1)-(H_3), as an example, we consider the following initial data

$$\begin{cases} p = (p^0, p^1, p^2, p^3) = \left(\overline{p^0}, \overline{p^1}, \overline{p^2}, \overline{p^3}\right) + \varepsilon\left(0, \hat{p^1}(\theta), \hat{p^2}(\theta), \hat{p^3}(\theta)\right), \\ q = (q^0, q^1, q^2, q^3) = \left(1, \varepsilon\hat{q^1}(\theta), \varepsilon\hat{q^2}(\theta), \varepsilon\hat{q^3}(\theta)\right), \end{cases} \tag{4.9}$$

where $\left(\overline{p^0}, \overline{p^1}, \overline{p^2}, \overline{p^3}\right)$ is a constant vector with the property

$$\bar{r}_0 \triangleq \sqrt{\sum_{i=1}^{3}\left(\overline{p^i}\right)^2} > 2m, \tag{4.10}$$

$\hat{p^i}(\theta)$ $(i = 1, 2, 3)$ are C^2-smooth functions with bounded C^2-norm and satisfy

$$\sum_{i=1}^{3}\left(\hat{p^i}_\theta(\theta)\right)^2 = \text{constant} \triangleq \mathscr{L} \tag{4.11}$$

(without loss of generality, we assume $\mathscr{L} = 1$), $\hat{q^i}(\theta)$ $(i = 1, 2, 3)$ are C^1-smooth functions with bounded C^1-norm, while ε is a positive small parameter. Thus, in the present situation we have

$$\begin{cases} g_{00}[p, q](\theta) \sim -\dfrac{\bar{r}_0 - 2m}{\bar{r}_0}, \quad g_{01}[p, q](\theta) \sim O(\varepsilon^2), \\ g_{11}[p, q](\theta) \sim \varepsilon^2 + \dfrac{2m}{(\bar{r}_0)^2(\bar{r}_0 - 2m)}\left(\displaystyle\sum_{i=1}^{3}\overline{p^i}\hat{p^i}_\theta(\theta)\right)^2\varepsilon^2, \quad \forall\, \theta \in \mathbb{R}, \\ g_{00}[p, q](\theta)g_{11}[p, q](\theta) < 0, \end{cases} \tag{4.12}$$

provided that $\varepsilon > 0$ is suitably small. The third inequality in (4.12) gives

$$\sqrt{(g_{01}[p, q](\theta))^2 - g_{00}[p, q](\theta)g_{11}[p, q](\theta)} > |g_{01}[p, q](\theta)|, \quad \forall\, \theta \in \mathbb{R}. \tag{4.13}$$

Noting (4.13), we observe from the second and third equations in (3.13) that

$$\lambda_+^0(\theta) > 0 > \lambda_-^0(\theta), \quad \forall\, \theta \in \mathbb{R}, \tag{4.14}$$

provided that $\varepsilon > 0$ is suitably small. Obviously, the initial data given by (4.9) satisfies the assumptions (H_1)-(H_3) as long as the parameter $\varepsilon > 0$ is suitably small.

We now turn to the Cauchy problem for the system (3.7)-(3.8) with the initial data (4.1). Throughout of this paper, we always assume that the initial data (4.1) satisfies the assumptions (H_1)-(H_3).

First, we consider the Cauchy problem for the system (3.22) with the initial data

$$t = 0: \quad \lambda_\pm = \lambda_\pm^0(\theta). \tag{4.15}$$

Obviously, under the assumptions (H_1)-(H_3) it follows from Lemma 4.1 that the Cauchy problem (3.22), (4.15) admits a unique global C^1 solution $\lambda_\pm = \lambda_\pm(t,\theta)$ on $\mathbb{R}^+ \times \mathbb{R}$, moreover the solution $\lambda_\pm = \lambda_\pm(t,\theta)$ satisfies (4.5). It is easy to show that, for the solution $\lambda_\pm = \lambda_\pm(t,\theta)$, the following identity

$$\partial_t \left(\frac{2}{\lambda_+ - \lambda_-} \right) + \partial_\theta \left(\frac{\lambda_+ + \lambda_-}{\lambda_+ - \lambda_-} \right) = 0 \tag{4.16}$$

holds (see [31]). This allows us to introduce the following transformation of the variables

$$(t,\theta) \longrightarrow (\tau,\vartheta), \tag{4.17}$$

which is defined by

$$\tau = t, \quad \vartheta = \vartheta(t,\theta), \tag{4.18}$$

where $\vartheta = \vartheta(t,\theta)$ is given by

$$\begin{cases} d\vartheta = \dfrac{2}{\lambda_+(t,\theta) - \lambda_-(t,\theta)} d\theta - \dfrac{\lambda_+(t,\theta) + \lambda_-(t,\theta)}{\lambda_+(t,\theta) - \lambda_-(t,\theta)} dt, \quad \forall\, (t,\theta) \in \mathbb{R}^+ \times \mathbb{R} \\ \vartheta(0,\theta) = \Theta_0(\theta) \triangleq \displaystyle\int_0^\theta \frac{2}{\lambda_+^0(\zeta) - \lambda_-^0(\zeta)} d\zeta, \quad \forall\, \theta \in \mathbb{R}. \end{cases} \tag{4.19}$$

Lemma 4.2 *Under the assumptions (H_1)-(H_3), the mapping defined by (4.17)-(4.19) is globally diffeomorphic; moreover, it holds that*

$$\frac{\partial}{\partial t} + \lambda_+ \frac{\partial}{\partial \theta} = \frac{\partial}{\partial \tau} + \frac{\partial}{\partial \vartheta}, \quad \frac{\partial}{\partial t} + \lambda_- \frac{\partial}{\partial \theta} = \frac{\partial}{\partial \tau} - \frac{\partial}{\partial \vartheta}. \tag{4.20}$$

Proof. It is obvious that the mapping defined by (4.17)-(4.19) is well-defined on $\mathbb{R}^+ \times \mathbb{R}$.

We now calculate

$$\mathscr{J} \triangleq \frac{\partial(\tau,\vartheta)}{\partial(t,\theta)} = \begin{vmatrix} 1 & 0 \\ -\dfrac{\lambda_+(t,\theta) + \lambda_-(t,\theta)}{\lambda_+(t,\theta) - \lambda_-(t,\theta)} & \dfrac{2}{\lambda_+(t,\theta) - \lambda_-(t,\theta)} \end{vmatrix} = \frac{2}{\lambda_+(t,\theta) - \lambda_-(t,\theta)} \neq 0 \tag{4.21}$$

for every $(t,\theta) \in \mathbb{R}^+ \times \mathbb{R}$.

On the one hand, we introduce

$$\tilde{\lambda}_{\pm}(\tau,\vartheta)=\lambda_{\pm}(t,\theta). \tag{4.22}$$

It follows from (4.19) that

$$\tilde{\lambda}_{\pm}(0,\vartheta)=\lambda_{\pm}^{0}(\Phi_0(\vartheta))\triangleq\tilde{\Lambda}_{\pm}(\vartheta), \tag{4.23}$$

where $\Phi_0=\Phi_0(\vartheta)$ is the inverse function of $\vartheta=\Theta_0(\theta)$ which is defined by the second equation in (4.19).

On the other hand, noting the first equation in (4.19), we have

$$\begin{aligned}\frac{\partial\tilde{\lambda}_+}{\partial\tau}-\frac{\partial\tilde{\lambda}_+}{\partial\vartheta} &= \frac{\partial\lambda_+}{\partial t}\frac{\partial t}{\partial\tau}+\frac{\partial\lambda_+}{\partial\theta}\frac{\partial\theta}{\partial\tau}-\frac{\partial\lambda_+}{\partial t}\frac{\partial t}{\partial\vartheta}-\frac{\partial\lambda_+}{\partial\theta}\frac{\partial\theta}{\partial\vartheta}\\ &= \frac{\partial\lambda_+}{\partial t}+\frac{\lambda_++\lambda_-}{2}\frac{\partial\lambda_+}{\partial\theta}-\frac{\lambda_+-\lambda_-}{2}\frac{\partial\lambda_+}{\partial\theta}\\ &= \frac{\partial\lambda_+}{\partial t}+\lambda_-\frac{\partial\lambda_+}{\partial\theta}\\ &= 0.\end{aligned} \tag{4.24}$$

Similarly, it holds that

$$\frac{\partial\tilde{\lambda}_-}{\partial\tau}+\frac{\partial\tilde{\lambda}_-}{\partial\vartheta}=\frac{\partial\lambda_-}{\partial t}+\lambda_+\frac{\partial\lambda_-}{\partial\theta}=0. \tag{4.25}$$

Noting (4.24)-(4.25) and using (4.23) leads to

$$\tilde{\lambda}_{\pm}(\tau,\vartheta)=\tilde{\Lambda}_{\pm}(\vartheta\pm\tau)=\lambda_{\pm}^{0}(\Phi_0(\vartheta\pm\tau)). \tag{4.26}$$

This allows us to solve out $\vartheta=\vartheta(t,\theta)$ from (4.19).

In fact, it follows from the first equation in (4.19) that

$$\begin{aligned}d\theta &= \frac{\tilde{\lambda}_+-\tilde{\lambda}_-}{2}d\vartheta+\frac{\tilde{\lambda}_++\tilde{\lambda}_-}{2}d\tau\\ &= \frac{\lambda_+^0(\Phi_0(\vartheta+\tau))-\lambda_-^0(\Phi_0(\vartheta-\tau))}{2}d\vartheta+\frac{\lambda_+^0(\Phi_0(\vartheta+\tau))+\lambda_-^0(\Phi_0(\vartheta-\tau))}{2}d\tau.\end{aligned} \tag{4.27}$$

(4.27) defines a unique function $\theta=\Phi(\tau,\vartheta)$ with the initial data

$$\Phi(0,\vartheta)=\Phi_0(\vartheta). \tag{4.28}$$

Introduce

$$\Theta(t,\theta)=\frac{1}{2}\int_0^{\theta+t}\lambda_+^0(\Phi_0(\zeta))d\zeta-\frac{1}{2}\int_0^{\theta-t}\lambda_-^0(\Phi_0(\zeta))d\zeta. \tag{4.29}$$

It is easy to check that

$$\vartheta=\Theta(t,\theta) \tag{4.30}$$

is the inverse function of $\theta=\Phi(\tau,\vartheta)$. (4.30) is the desired $\vartheta=\vartheta(t,\theta)$.

We now prove that, under the assumptions (H$_1$)-(H$_3$), the mapping defined by (4.17)-(4.19) is globally diffeomorphic.

On the one hand, noting (4.29) and (4.30), we observe that, for any fixed $\tau \in \mathbb{R}^+$ (i.e., $t \in \mathbb{R}^+$),

$$\begin{aligned}\vartheta_\theta = \Theta_\theta &= \frac{1}{2}\left[\lambda_+^0(\Phi_0(\vartheta+\tau)) - \lambda_-^0(\Phi_0(\vartheta-\tau))\right] \\ &= \frac{1}{2}\left[\lambda_+^0(\vartheta_+) - \lambda_-^0(\vartheta_-)\right] \neq 0,\end{aligned} \tag{4.31}$$

where

$$\vartheta_\pm = \Phi_0(\theta \pm t). \tag{4.32}$$

See Figure 1 for the geometric meaning of $\vartheta_\pm$ defined by (4.32).

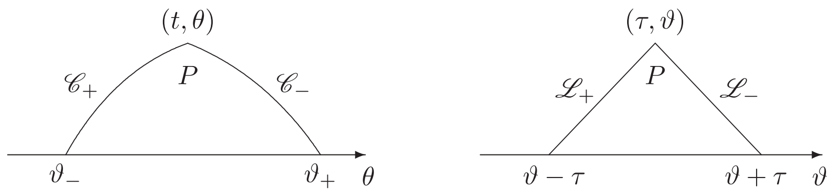

Figure 1: The same point P in the (t,θ)-plane and (τ,ϑ)-plane and the corresponding characteristics, where $\mathscr{L}_\pm$ are straight lines.

On the other hand, we can prove that the mapping defined by (4.17)-(4.19) is proper.

In fact, by (4.29) and (4.30), for any fixed $\tau \in \mathbb{R}^+$ (i.e., $t \in \mathbb{R}^+$), it holds that

$$\begin{aligned}\vartheta = \Theta(t,\theta) &= \frac{1}{2}\int_0^{\theta+t} \lambda_+^0(\Phi_0(\zeta))d\zeta - \frac{1}{2}\int_0^{\theta-t} \lambda_-^0(\Phi_0(\zeta))d\zeta. \\ &= \int_0^{\vartheta_+} \frac{\lambda_+^0(\eta)}{\lambda_+^0(\eta)-\lambda_-^0(\eta)}d\eta - \int_0^{\vartheta_-} \frac{\lambda_-^0(\eta)}{\lambda_+^0(\eta)-\lambda_-^0(\eta)}d\eta \\ &= \vartheta_+ + \int_{\vartheta_-}^{\vartheta_+} \frac{\lambda_-^0(\eta)}{\lambda_+^0(\eta)-\lambda_-^0(\eta)}d\eta.\end{aligned} \tag{4.33}$$

Noting (4.32) and the second equation in (4.19) yields

$$|\vartheta_+(t,\theta) - \vartheta_-(t,\theta)| = |\Phi_0(\theta+t) - \Phi_0(\theta-t)| \leq 2t \times \max_{\xi\in\mathbb{R}} |\Phi_0'(\xi)| \leq 2t \times \max_{\xi\in\mathbb{R}} \left\{\frac{|\lambda_+^0(\xi)-\lambda_-^0(\xi)|}{2}\right\}. \tag{4.34}$$

Thus, we have

$$\left|\int_{\vartheta_-}^{\vartheta_+} \frac{\lambda_-^0(\eta)}{\lambda_+^0(\eta)-\lambda_-^0(\eta)}d\eta\right| \leq t \times \max_{\xi\in\mathbb{R}}\left\{|\lambda_+^0(\xi)-\lambda_-^0(\xi)|\right\} \times \max_{\eta\in\mathbb{R}}|\lambda_-^0(\eta)| \times \frac{1}{\min\limits_{\eta\in\mathbb{R}}\left\{|\lambda_+^0(\eta)-\lambda_-^0(\eta)|\right\}}. \tag{4.35}$$

Here we have made use of the assumptions (H$_1$) and (H$_2'$). Therefore, it follows from (4.33) that

$$\vartheta \longrightarrow \pm\infty \quad \text{if and only if} \quad \vartheta_+ \longrightarrow \pm\infty. \tag{4.36}$$

Noting (4.32) and the second equation in (4.19) again gives

$$\vartheta_+ \longrightarrow \pm\infty \quad \text{if and only if} \quad \theta \longrightarrow \pm\infty. \tag{4.37}$$

By (4.36) and (4.37), the mapping defined by (4.17)-(4.19) is proper. Combining this fact and (4.21) and using the Hadamard's Lemma, we prove that the mapping defined by (4.17)-(4.19) is globally diffeomorphic.

(4.20) comes from (4.24) and (4.25) directly. This proves Lemma 4.2. ■

Remark 4.3 *In fact, (4.21) holds if and only if the assumption (H_3) is satisfied (here we assume that the assumption (H_2) is always true).*

Lemma 4.3 *Under the assumptions (H_1)-(H_3), for any given function $h = h(t, \theta)$ defined on $\mathbb{R}^+ \times \mathbb{R}$ it holds that*

$$BV(h(t, \cdot)) = BV(\tilde{h}(t, \cdot)), \tag{4.38}$$

provided that the function $h(t, \cdot)$ is in the BV class for every fixed $t \in \mathbb{R}^+$, where $\tilde{h}$ is defined by

$$\tilde{h}(t, \vartheta) = h(t, \theta(t, \vartheta)). \tag{4.39}$$

Lemma 4.3 is obvious, here we omit its proof.

Remark 4.4 *The mapping defined by (4.17)-(4.19) is somewhat similar to the transformation between the Euler version and Lagrange version for one-dimensional gas dynamics.*

4.2 Main result and its proof

We now state our main result and give its proof.

Consider the Cauchy problem for the equation (3.6) (or (3.7)-(3.8)) with the initial data (4.1), i.e.,

$$\begin{cases} g^{\mu\nu}\left(x^C_{\mu\nu} + \tilde{\Gamma}^C_{AB} x^A_\mu x^B_\nu\right) = 0, \\ t = 0: \quad x^C = p^C(\theta), \quad x^C_t = q^C(\theta) \end{cases} \quad (C = 0, 1, 2, 3). \tag{4.40}$$

We have

Theorem 4.1 *Suppose that $g_{\mu\nu}$ is the Schwarzschild metric, $p^C(\theta)$ $(C = 0, 1, 2, 3)$ are C^2-smooth functions with bounded C^2-norm and satisfy*

$$\left|(p^1(\theta), p^2(\theta), p^3(\theta))\right| \triangleq \sqrt{\sum_{i=1}^{3}(p^i(\theta))^2} \geq 2m + \hat{\delta} \quad (\textit{where } \hat{\delta} \textit{ is a positive constant}), \tag{4.41}$$

$q^C(\theta)$ $(C = 0, 1, 2, 3)$ are C^1-smooth functions with bounded C^1-norm. Suppose furthermore that the assumptions (H_1)-(H_3) are satisfied. Then there exists a positive constant ε such that the Cauchy problem (4.40) admits a unique global C^2-smooth solution $x^C = x^C(t, \theta)$ for all $t \in \mathbb{R}$, provided that

$$\int_{-\infty}^{\infty} \left| \frac{dp^C(\theta)}{d\theta} \right| d\theta \leq \varepsilon \quad and \quad \int_{-\infty}^{\infty} \left| q^C(\theta) \right| d\theta \leq \varepsilon. \tag{4.42}$$

Remark 4.5 *The inequality (4.41) implies that the initial string lies in in the Schwarzschild exterior space-time. The first inequality in (4.42) implies that the BV-norm of $p^C(\theta)$ is small, that is, the arc length of the initial string is small; while the second inequality in (4.42) implies that the L^1-norm of the initial velocity is small. The physical meaning of Theorem 4.1 is as follows: for a string with small arc length, the smooth motion exists globally (or say, no singularity appears in the whole motion process), provided that the L^1-norm of the initial velocity is small. In geometry, Theorem 4.1 gives a global existence result on smooth solutions of a wave map from the Minkowski space-time $\mathbb{R}^{1+1}$ to the Schwarzschild space-time.*

Noting the second and third equalities in (3.9), (3.23) and the assumption (H_1), we observe that there exists a positive constant k_0 independent of ε, such that

$$\int_{-\infty}^{\infty} |P_0^\mu(\theta)| \, d\theta \leq k_0 \varepsilon \quad \text{and} \quad \int_{-\infty}^{\infty} |Q_0^\mu(\theta)| \, d\theta \leq k_0 \varepsilon \quad (\mu = 0, 1, 2, 3), \tag{4.43}$$

where

$$P_0^\mu(\theta) = P^\mu(0, \theta), \quad Q_0^\mu(\theta) = Q^\mu(0, \theta),$$

in which $P^\mu(t, \theta)$ and $Q^\mu(t, \theta)$ are defined by (3.23).

Obviously, in order to prove Theorem 4.1, it suffices to show the following theorem.

Theorem 4.2 *Under the assumptions of Theorem 4.1, there exists a positive constant ε such that the Cauchy problem*

$$\begin{cases} (3.26), \\ t = 0: \quad S^\mu = p^\mu(\theta), \quad P^\mu = P_0^\mu(\theta), \quad Q^\mu = Q_0^\mu(\theta) \qquad (\mu = 0, 1, 2, 3) \end{cases} \tag{4.44}$$

admits a unique global C^1-smooth solution for all $t \in \mathbb{R}$, provided that (4.43) is satisfied.

Remark 4.6 *In fact, we can show that the C^2-smooth solution of the Cauchy problem (4.40) is equivalent to the C^1-smooth solution of the Cauchy problem (4.44).*

By Lemma 4.2, the Cauchy problem (4.44) can be equivalently rewritten as

$$
\begin{cases}
\dfrac{\partial S^\mu}{\partial \tau} + \lambda_0 \dfrac{\partial S^\mu}{\partial \vartheta} = \dfrac{\lambda_+ P^\mu - \lambda_- Q^\mu}{\lambda_+ - \lambda_-} \quad (\mu = 0,1,2,3),\\
\dfrac{\partial P^0}{\partial \tau} + \dfrac{\partial P^0}{\partial \vartheta} = \dfrac{-m}{|S|^2(|S|-2m)}(P^0\langle S, Q\rangle + Q^0 \langle S, P\rangle),\\
\dfrac{\partial P^i}{\partial \tau} + \dfrac{\partial P^i}{\partial \vartheta} = S^i \left[\dfrac{m(2m-|S|)}{|S|^4}P^0 Q^0 - \dfrac{2m}{|S|^3}\langle P, Q\rangle + \dfrac{m(3|S|-4m)}{|S|^5(|S|-2m)}\langle S, P\rangle \langle S, Q\rangle\right] \quad (i = 1,2,3),\\
\dfrac{\partial Q^0}{\partial \tau} - \dfrac{\partial Q^0}{\partial \vartheta} = \dfrac{-m}{|S|^2(|S|-2m)}(P^0\langle S, Q\rangle + Q^0 \langle S, P\rangle),\\
\dfrac{\partial Q^i}{\partial \tau} - \dfrac{\partial Q^i}{\partial \vartheta} = S^i \left[\dfrac{m(2m-|S|)}{|S|^4}P^0 Q^0 - \dfrac{2m}{|S|^3}\langle P, Q\rangle + \dfrac{m(3|S|-4m)}{|S|^5(|S|-2m)}\langle S, P\rangle \langle S, Q\rangle\right] \quad (i = 1,2,3),\\
\tau = 0: \ \ S^\mu = \tilde{p}^\mu(\vartheta), \quad P^\mu = \tilde{P}_0^\mu(\vartheta), \quad Q^\mu = \tilde{Q}_0^\mu(\vartheta) \qquad (\mu = 0,1,2,3).
\end{cases}
\tag{4.45}
$$

In (4.45), all the unknown functions should be $\tilde{S}^\mu(\tau,\vartheta)$, $\tilde{P}^\mu(\tau,\vartheta)$, $\tilde{Q}^\mu(\tau,\vartheta)$, however, for simplicity, in (4.45) and in what follows, we still use the symbols S^μ, P^μ and Q^μ to stand for $\tilde{S}^\mu(\tau,\vartheta)$, $\tilde{P}^\mu(\tau,\vartheta)$ and $\tilde{Q}^\mu(\tau,\vartheta)$ $(\mu = 0,1,2,3)$, respectively. Thus, by Lemma 4.3, in order to prove Theorem 4.2, it suffices to show

Theorem 4.3 *Under the assumptions in Theorem 4.1, there exists a positive constant ε such that the Cauchy problem (4.45) admits a unique C^1-smooth solution for all $t \in \mathbb{R}$, provided that*

$$
\int_{-\infty}^{\infty} \left|\tilde{P}_0^\mu(\vartheta)\right| d\vartheta \le k_0 \varepsilon \quad \text{and} \quad \int_{-\infty}^{\infty} \left|\tilde{Q}_0^\mu(\vartheta)\right| d\vartheta \le k_0 \varepsilon. \tag{4.46}
$$

Remark 4.7 *In fact, by Lemma 4.3, the inequality (4.43) is equivalent to the inequality (4.46).*

In order to prove Theorem 4.3, we need the following two lemmas which come from [32] and are essentially due to Schartzman [29]-[30].

Lemma 4.4 *Let $\phi = \phi(t,x)$ be a C^1 function satisfying*

$$
\begin{cases}
\phi_t + c\phi_x = F, \quad \forall\, (t,x) \in (0,T] \times \mathbb{R},\\
t = 0: \ \ \phi = g(x),
\end{cases}
\tag{4.47}
$$

where $c \neq 0$ is a constant and $T > 0$ is a fixed real number. Then

$$
\int_{-\infty}^{\infty} |\phi(t,x)| dx \le \int_{-\infty}^{\infty} |g(x)| dx + \int_0^T \int_{-\infty}^{\infty} |F(t,x)| dx dt, \quad \forall\, t \in [0,T], \tag{4.48}
$$

provided that the right-hand side of the inequality is bounded.

Lemma 4.5 *Let $\phi = \phi(t,x)$ and $\psi = \psi(t,x)$ be two C^1 functions satisfying*

$$
\begin{cases}
\phi_t + \phi_x = F, \quad \forall\, (t,x) \in (0,T] \times \mathbb{R},\\
t = 0: \ \ \phi = g_1(x),
\end{cases}
\tag{4.49}
$$

$$\begin{cases} \psi_t - \psi_x = G, \quad \forall\, (t,x) \in (0,T] \times \mathbb{R}, \\ t = 0: \quad \psi = g_2(x), \end{cases} \tag{4.50}$$

respectively, where $T > 0$ is a fixed real number. Then

$$\begin{aligned} \int_0^T \int_{-\infty}^{\infty} |\phi(t,x)||\psi(t,x)| dx dt \quad \leq \quad & \Big(\int_{-\infty}^{\infty} |g_1(x)| dx + \int_0^T \int_{-\infty}^{\infty} |F(t,x)| dx dt \Big) \times \\ & \Big(\int_{-\infty}^{\infty} |g_2(x)| dx + \int_0^T \int_{-\infty}^{\infty} |G(t,x)| dx dt \Big), \end{aligned} \tag{4.51}$$

provided that the two factors on the right-hand side of the inequality are bounded.

The proof of Lemmas 4.4-4.5 can be found in [32] or [29]-[30].

We next prove Theorem 4.3.

Proof of Theorem 4.3. By the existence and uniqueness of local C^1 solution of the Cauchy problem for quasilinear hyperbolic systems, in order to prove Theorem 4.3, it suffices to establish a uniform *a priori* estimate on the C^0 norm of U and $\frac{\partial U}{\partial \vartheta}$ on the existence domain of the C^1 solution $U = U(\tau, \vartheta)$ of the Cauchy problem (4.45), where

$$U = \left(S^0, S^1, S^2, S^3, P^0, P^1, P^2, P^3, Q^0, Q^1, Q^2, Q^3\right). \tag{4.52}$$

On the other hand, notice that the PDEs in (4.45) constitute a diagonal semi-linear hyperbolic system. By the theory of diagonal semi-linear hyperbolic systems, in order to establish a uniform *a priori* estimate on the C^1 norm of the C^1 solution $U = U(\tau, \vartheta)$, it suffices to establish a uniform *a priori* estimate on the C^0 norm of the solution $U = U(\tau, \vartheta)$.

Noting (4.41) gives

$$\left|(p^1(0), p^2(0), p^3(0))\right| \geq 2m + \hat{\delta}. \tag{4.53}$$

For the time being, it is supposed that, on the existence domain of the C^1 solution $U = U(\tau, \vartheta)$, we have

$$\left|S(\tau, \vartheta) - (p^1(0), p^2(0), p^3(0))\right| \leq \delta, \tag{4.54}$$

where $\delta > 0$ is a small constant independent of ε and $\hat{\delta}$. At the end of the proof of Theorem 4.3, we shall explain that the hypothesis (4.54) is reasonable.

By (4.54), on the existence domain of the C^1 solution $U = U(\tau, \vartheta)$ it holds that

$$M - \delta \leq |S(\tau, \vartheta)| \leq M + \delta, \quad M_i - \delta \leq \left|S^i(\tau, \vartheta)\right| \leq M_i + \delta \quad (i = 1, 2, 3), \tag{4.55}$$

where

$$M = \left|(p^1(0), p^2(0), p^3(0))\right|, \quad M_i = \left|p^i(0)\right|. \tag{4.56}$$

Combining (4.53)-(4.56) yields

$$|S| - 2m \geq M - \delta - 2m \geq 2m + \hat{\delta} - \delta - 2m = \hat{\delta} - \delta > 0, \tag{4.57}$$

where we choose δ so small that

$$\delta < \hat{\delta}. \tag{4.58}$$

We first establish a uniform *a prior* estimate on the supreme of

$$V = \left(P^0, P^1, P^2, P^3, Q^0, Q^1, Q^2, Q^3\right) \tag{4.59}$$

on any given time interval $[0, T]$.

To do so, let

$$\mathscr{V}_\infty(T) = \sup_{0 \leq \tau \leq T} \max_{\mu=0,1,2,3} \left\{ \sup_{\vartheta \in \mathbb{R}} |P^\mu(\tau, \vartheta)|, \ \sup_{\vartheta \in \mathbb{R}} |Q^\mu(\tau, \vartheta)| \right\}, \tag{4.60}$$

$$\mathscr{V}_1(T) = \max_{\mu=0,1,2,3} \sup_{0 \leq \tau \leq T} \left\{ \int_{-\infty}^{\infty} |P^\mu(\tau, \vartheta)| d\vartheta, \ \int_{-\infty}^{\infty} |Q^\mu(\tau, \vartheta)| d\vartheta \right\}, \tag{4.61}$$

$$\widetilde{\mathscr{V}_1}(T) = \max_{\mu=0,1,2,3} \left\{ \sup_{L_-} \int_{L_-} |P^\mu| d\tau, \ \sup_{L_+} \int_{L_+} |Q^\mu| d\tau \right\}, \tag{4.62}$$

where $L_\pm$ stand for given characteristics (corresponding to the eigenvalues ± 1, respectively) on the domain $0 \leq \tau \leq T$, i.e.,

$$L_\pm : \quad \vartheta = \alpha \pm \tau \quad (\tau \in [0, T]),$$

in which $\alpha \in \mathbb{R}$ stands for the intersection point of $L_\pm$ with the ϑ-axis.

Introduce

$$\mathscr{Q}_V(T) = \sum_{\mu,\nu=0,1,2,3} \int_0^T \int_{\mathbb{R}} |P^\mu(\tau, \vartheta)||Q^\nu(\tau, \vartheta)| d\vartheta d\tau. \tag{4.63}$$

Noting (4.55), (4.57) and using the equations for P^μ and Q^μ in (4.45), by Lemma 4.5 we have

$$\mathscr{Q}_V(T) \leq c_1 \left(\mathscr{V}_1(0) + \mathscr{Q}_V(T)\right)^2, \tag{4.64}$$

here and hereafter c_i $(i = 1, 2, \cdots)$ stand for some positive constants independent of ε, but depending on δ, $\hat{\delta}$ and M. Noting the assumption (4.46) and the definition of $\mathscr{V}_1(T)$, we obtain

$$\mathscr{V}_1(0) \leq k_0 \varepsilon. \tag{4.65}$$

Thus, it follows from (4.64) that

$$\mathscr{Q}_V(T) \leq c_1 \left(k_0 \varepsilon + \mathscr{Q}_V(T)\right)^2. \tag{4.66}$$

By the method of *continuous induction* (see Hörmander [19]), we can obtain from (4.66) that

$$\mathscr{Q}_V(T) \leq k_1 \varepsilon^2, \tag{4.67}$$

provided that $\varepsilon > 0$ is suitably small, here and hereafter k_i $(i = 1, 2, \cdots)$ stand for some positive constants independent of ε but depending on δ, $\hat{\delta}$ and M.

On the other hand, by Lemma 4.4, it follows from the equations for P^μ and Q^μ in (4.45) that

$$\begin{aligned} \mathscr{V}_1(T) &\leq \mathscr{V}_1(0) + c_2 \mathscr{Q}_V(T) \\ &\leq k_0\varepsilon + c_2 k_1 \varepsilon^2 \\ &\leq 2k_0\varepsilon, \end{aligned} \tag{4.68}$$

provided that $\varepsilon > 0$ is suitably small. In (4.68) we have made use of (4.65) and (4.67).

We now estimate $\widetilde{\mathscr{V}_1}(T)$.

To do so, we first estimate

$$\int_{L_-} |P^0| d\tau,$$

where L^- stands for any given characteristic (corresponding to the eigenvalue -1) on the domain $0 \leq \tau \leq T$, i.e.,

$$L_- : \quad \vartheta = \alpha - \tau \quad (\tau \in [0, T]), \tag{4.69}$$

where α is the ϑ-coordinate of the intersection point, denoted by D_1, of L_- with the ϑ-axis. Let D_2 be the intersection point of L_- with the line $\tau = T$. We draw the characteristic corresponding to the eigenvalue $+1$ from the point D_2 downward which intersects $\tau = 0$ with a point denoted by D_3. We rewrite the second equation in (4.45) as

$$d\left\{\left|P^0(\tau,\vartheta)\right|(d\vartheta - d\tau)\right\} = \operatorname{sgn}\left(P^0\right) \frac{-m}{|S|^2(|S|-2m)}(P^0\langle S, Q\rangle + Q^0\langle S, P\rangle) d\tau \wedge d\vartheta, \tag{4.70}$$

and integrate (4.70) in the triangle domain $\widehat{D_1 D_2 D_3}$ to get

$$\int_{L_-} \left|P^0(\tau,\vartheta)\right| |-d\tau - d\tau| \leq \int_{D_3}^{D_1} \left|P^0(0,\vartheta)\right| d\vartheta + \iint_{\widehat{D_1 D_2 D_3}} \frac{m\left(\left|P^0\right| |\langle S, Q\rangle| + \left|Q^0\right| |\langle S, P\rangle|\right)}{|S|^2(|S|-2m)} d\vartheta dt. \tag{4.71}$$

This leads to

$$\int_{L_-} \left|P^0(\tau,\vartheta)\right| d\tau \leq \frac{1}{2}\{\mathscr{V}_1(0) + c_3\mathscr{Q}_V(T)\} \leq \frac{1}{2}\{k_0\varepsilon + c_3 k_1\varepsilon^2\} \leq c_4\varepsilon. \tag{4.72}$$

Here we have made use of (4.65) and (4.67). Similarly, we can prove

$$\int_{L_-} \left|P^i(\tau,\vartheta)\right| d\vartheta \leq c_5\varepsilon \quad \text{and} \quad \int_{L_+} |Q^\mu(\tau,\vartheta)| d\vartheta \leq c_6\varepsilon \quad (i = 1,2,3;\ \ \mu = 0,1,2,3). \tag{4.73}$$

Combining (4.72) and (4.73) gives

$$\widetilde{\mathscr{V}_1}(T) \leq k_2\varepsilon. \tag{4.74}$$

We turn to estimate $\mathscr{V}_\infty(T)$.

We first estimate $\left|P^0(\tau,\vartheta)\right|$.

Integrating the second equation in (4.45) along the characteristic $L_+ : \vartheta = \alpha + \tau$ leads to

$$P^0(\tau,\vartheta) = P^0(0,\alpha) + \int_0^\tau \left\{ \frac{-m}{|S|^2(|S|-2m)} (P^0\langle S, Q\rangle + Q^0\langle S, P\rangle) \right\} (\eta, \alpha+\eta) d\eta. \tag{4.75}$$

It follows from (4.75) that

$$\left| P^0(\tau,\vartheta) \right| \leq \mathscr{V}_\infty(0) + c_7 \mathscr{V}_\infty(T) \widetilde{\mathscr{V}}_1(T). \tag{4.76}$$

Similarly, we have

$$\left| P^i(\tau,\vartheta) \right| \leq \mathscr{V}_\infty(0) + c_8 \mathscr{V}_\infty(T) \widetilde{\mathscr{V}}_1(T) \quad (i = 1,2,3) \tag{4.77}$$

and

$$|Q^\mu(\tau,\vartheta)| \leq \mathscr{V}_\infty(0) + c_9 \mathscr{V}_\infty(T) \widetilde{\mathscr{V}}_1(T) \quad (\mu = 0,1,2,3). \tag{4.78}$$

Thus, combining (4.76)-(4.78) gives

$$\mathscr{V}_\infty(T) \leq \mathscr{V}_\infty(0) + c_{10} \mathscr{V}_\infty(T) \widetilde{\mathscr{V}}_1(T) \leq \mathscr{V}_\infty(0) + c_{10} k_2 \varepsilon \mathscr{V}_\infty(T). \tag{4.79}$$

In (4.79), we have made use of (4.74). It follows from (4.79) that

$$\mathscr{V}_\infty(T) \leq 2\mathscr{V}_\infty(0), \tag{4.80}$$

provided that $\varepsilon > 0$ is suitably small.

We finally estimate $|S^\mu(\tau,\vartheta)|$ $(\mu = 0,1,2,3)$.

It follows from the first equation in (4.45)

$$S^\mu(\tau,\vartheta) = S^\mu(0,\vartheta) + \int_0^\tau \frac{\lambda_+ P^\mu - \lambda_- Q^\mu}{\lambda_+ - \lambda_-} (\eta,\vartheta) d\eta. \tag{4.81}$$

Noting the assumptions (H_1), (H_2) (for simplicity, (H_2')), we obtain from (4.81) that

$$|S^\mu(\tau,\vartheta) - S^\mu(0,\vartheta)| \leq c_{11} \Big(\int_0^\tau |P^\mu(\eta,\vartheta)|\, d\eta + \int_0^\tau |Q^\mu(\eta,\vartheta)|\, d\eta \Big) \leq c_{12} \bar{\mathscr{V}}_1(T). \tag{4.82}$$

where

$$\bar{\mathscr{V}}_1(T) = \max_{\mu=0,1,2,3} \sup_{\vartheta \in \mathbb{R}} \left\{ \int_0^T |P^\mu(\eta,\vartheta)|\, d\eta, \int_0^T |Q^\mu(\eta,\vartheta)|\, d\eta \right\}.$$

Similar to (4.74), we can prove

$$\bar{\mathscr{V}}_1(T) \leq k_3 \varepsilon. \tag{4.83}$$

Thus, it follows from (4.82) that

$$|S^\mu(\tau,\vartheta) - S^\mu(0,\vartheta)| \leq c_{12} k_3 \varepsilon. \tag{4.84}$$

On the other hand, by Leibniz integral rule, it holds that

$$|S^\mu(0,\vartheta) - S^\mu(0,0)| = \left| \int_0^\vartheta \frac{dS^\mu}{d\vartheta}(0,\vartheta) d\vartheta \right| \leq \int_{-\infty}^\infty \left| \frac{dp^\mu(\vartheta)}{d\vartheta} \right| d\vartheta \leq \varepsilon. \tag{4.85}$$

Here we have made use of the first inequality in (4.42). Thus, combining (4.84) and (4.85) gives

$$\begin{aligned} |S^\mu(\tau,\vartheta)-S^\mu(0,0)| &\le |S^\mu(\tau,\vartheta)-S^\mu(0,\vartheta)+S^\mu(0,\vartheta)-S^\mu(0,0)| \\ &\le |S^\mu(\tau,\vartheta)-S^\mu(0,\vartheta)|+|S^\mu(0,\vartheta)-S^\mu(0,0)| \\ &\le c_{12}k_3\varepsilon+\varepsilon \\ &= (1+c_{12}k_3)\varepsilon. \end{aligned} \tag{4.86}$$

This leads to

$$|S^\mu(\tau,\vartheta)| \le |S^\mu(0,0)|+(1+c_{12}k_3)\varepsilon, \quad \forall\, (\tau,\vartheta)\in[0,T]\times\mathbb{R}. \tag{4.87}$$

Obviously, (4.80) and (4.87) gives a uniform *a priori* estimate on the C^0 norm of the Cauchy problem (4.45).

At the end of the proof, we explain the hypothesis (4.54) is reasonable.

It follows from (4.86) that

$$\left|S(\tau,\vartheta)-(p^1(0),p^2(0),p^3(0))\right| = |S(\tau,\vartheta)-S(0,0)| \le \sqrt{3}(1+c_{12}k_3)\varepsilon. \tag{4.88}$$

Taking ε suitably small gives

$$\left|S(\tau,\vartheta)-(p^1(0),p^2(0),p^3(0))\right| \le \frac{1}{2}\delta. \tag{4.89}$$

(4.89) implies the reasonablity of the hypothesis (4.54).

Thus, the proof of Theorem 4.3 is completed. ∎

5 Appendix

This appendix concerns the motion of relativistic string ($p=1$) in a special enveloping space-time $(\mathscr{N},\tilde{g})$ — the Schwarzschild space-time in which the metric $\tilde{g}$ in the spherical coordinates (τ,r,α,β) reads

$$ds^2 = -\left(1-\frac{2m}{r}\right)dt^2+\left(1-\frac{2m}{r}\right)^{-1}dr^2+r^2\left(d\alpha^2+\sin^2\alpha d\beta^2\right). \tag{5.1}$$

In the spherical coordinates (τ,r,α,β), the parameter form of the motion of the relativistic string under consideration in the Schwarzschild space-time may take the following form

$$(\tau,\theta)\longrightarrow(t(\tau,\theta),r(\tau,\theta),\alpha(\tau,\theta),\beta(\tau,\theta)). \tag{5.2}$$

In the coordinates (τ,θ), the induced metric of the sub-manifold $\mathscr{M}$ reads $g=(g_{\mu\nu})^1_{\mu,\nu=0}$, where

$$\begin{cases} g_{00} = -\left(1-\dfrac{2m}{r}\right)t_\tau^2+\left(1-\dfrac{2m}{r}\right)^{-1}r_\tau^2+r^2\alpha_\tau^2+r^2\sin^2\alpha\beta_\tau^2, \\ g_{01}=g_{10} = -\left(1-\dfrac{2m}{r}\right)t_\tau t_\theta+\left(1-\dfrac{2m}{r}\right)^{-1}r_\tau r_\theta+r^2\alpha_\tau\alpha_\theta+r^2\sin^2\alpha\beta_\tau\beta_\theta, \\ g_{11} = -\left(1-\dfrac{2m}{r}\right)t_\theta^2+\left(1-\dfrac{2m}{r}\right)^{-1}r_\theta^2+r^2\alpha_\theta^2+r^2\sin^2\alpha\beta_\theta^2. \end{cases} \tag{5.3}$$

Moreover, we denote the inverse of g by $g^{-1} = (g^{\mu\nu})$.

Throughout this appendix, we assume that the sub-manifold $\mathscr{M}$ is C^2 and *time-like*, i.e.,

$$\begin{aligned}\Delta \triangleq \det g = &-\left(1-\frac{2m}{r}\right) r^2 \sin^2\alpha (t_\tau \beta_\theta - \beta_\tau t_\theta)^2 - \left(1-\frac{2m}{r}\right) r^2 (t_\tau \alpha_\theta - \alpha_\tau t_\theta)^2 \\ &- (t_\tau r_\theta - r_\tau t_\theta)^2 + \left(1-\frac{2m}{r}\right)^{-1} r^2 \sin^2\alpha (r_\tau \beta_\theta - \beta_\tau r_\theta)^2 \\ &+ \left(1-\frac{2m}{r}\right)^{-1} r^2 (r_\tau \alpha_\theta - \alpha_\tau r_\theta)^2 + r^4 \sin^2\alpha (\alpha_\tau \beta_\theta - \beta_\tau \alpha_\theta)^2 < 0.\end{aligned} \tag{5.4}$$

In the present situation, the system

$$E_C = g^{\mu\nu}\left(x^C_{\mu\nu} + \tilde{\Gamma}^C_{AB} x^A_\mu x^B_\nu\right) \quad (C = 0, 1, 2, 3) \tag{5.5}$$

can be rewritten in the following form

$$g^{00} t_{\tau\tau} + 2g^{01} t_{\tau\theta} + g^{11} t_{\theta\theta} + \frac{2m}{r^2}\left(1-\frac{2m}{r}\right)^{-1}\left(g^{00} t_\tau r_\tau + g^{01} t_\tau r_\theta + g^{01} t_\theta r_\tau + g^{11} t_\theta r_\theta\right) = 0, \tag{5.6}$$

$$\begin{aligned}&g^{00} r_{\tau\tau} + 2g^{01} r_{\tau\theta} + g^{11} r_{\theta\theta} + \frac{m}{r^2}\left(1-\frac{2m}{r}\right)(g^{00} t_\tau^2 + 2g^{01} t_\tau t_\theta + g^{11} t_\theta^2) \\ &\quad - \frac{m}{r^2}\left(1-\frac{2m}{r}\right)^{-1}(g^{00} r_\tau^2 + 2g^{01} r_\tau r_\theta + g^{11} r_\theta^2) - r\left(1-\frac{2m}{r}\right)(g^{00}\alpha_\tau^2 + 2g^{01}\alpha_\tau\alpha_\theta + g^{11}\alpha_\theta^2) \\ &\quad - r\sin^2\alpha\left(1-\frac{2m}{r}\right)(g^{00}\beta_\tau^2 + 2g^{01}\beta_\tau\beta_\theta + g^{11}\beta_\theta^2) = 0,\end{aligned} \tag{5.7}$$

$$\begin{aligned}&g^{00}\alpha_{\tau\tau} + 2g^{01}\alpha_{\tau\theta} + g^{11}\alpha_{\theta\theta} + \frac{2}{r}\left(g^{00}\alpha_\tau r_\tau + g^{01}\alpha_\tau r_\theta + g^{01}\alpha_\theta r_\tau + g^{11}\alpha_\theta r_\theta\right) \\ &\quad - \sin\alpha\cos\alpha(g^{00}\beta_\tau^2 + 2g^{01}\beta_\tau\beta_\theta + g^{11}\beta_\theta^2) = 0,\end{aligned} \tag{5.8}$$

$$\begin{aligned}&g^{00}\beta_{\tau\tau} + 2g^{01}\beta_{\tau\theta} + g^{11}\beta_{\theta\theta} + \frac{2}{r}\left(g^{00}\beta_\tau r_\tau + g^{01}\beta_\tau r_\theta + g^{01}\beta_\theta r_\tau + g^{11}\beta_\theta r_\theta\right) \\ &\quad + \frac{2\cos\alpha}{\sin\alpha}(g^{00}\alpha_\tau\beta_\tau + g^{01}\alpha_\tau\beta_\theta + g^{01}\beta_\tau\alpha_\theta + g^{11}\alpha_\theta\beta_\theta) = 0.\end{aligned} \tag{5.9}$$

Let

$$\begin{cases} U_1 = r, \quad U_2 = \alpha, \quad U_3 = t_\tau, \quad U_4 = r_\tau, \quad U_5 = \alpha_\tau, \\ U_6 = \beta_\tau, \quad U_7 = t_\theta, \quad U_8 = r_\theta, \quad U_9 = \alpha_\theta, \quad U_{10} = \beta_\theta \end{cases} \tag{5.10}$$

and

$$U = (U_1, U_2, \cdots, U_{10})^T. \tag{5.11}$$

Then the equations (5.6)-(5.9) can be equivalently rewritten as

$$U_\tau + AU_\theta + B = 0, \tag{5.12}$$

where

$$A=\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & \frac{2g^{01}}{g^{00}} & 0 & 0 & 0 & \frac{g_{11}}{g^{00}} & 0 & 0 & 0\\
0 & 0 & 0 & \frac{2g^{01}}{g^{00}} & 0 & 0 & 0 & \frac{g_{11}}{g^{00}} & 0 & 0\\
0 & 0 & 0 & 0 & \frac{2g^{01}}{g^{00}} & 0 & 0 & 0 & \frac{g_{11}}{g^{00}} & 0\\
0 & 0 & 0 & 0 & 0 & \frac{2g^{01}}{g^{00}} & 0 & 0 & 0 & \frac{g_{11}}{g^{00}}\\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0
\end{pmatrix}$$

and

$$B=(B_1,B_2,B_3,B_4,B_5,B_6,0,0,0,0)^T,$$

in which

$$\begin{cases}
B_1=-r_\tau, \quad B_2=-\alpha_\tau, \quad B_3=\dfrac{2m}{r^2}\left(1-\dfrac{2m}{r}\right)^{-1}\left(t_\tau r_\tau+\dfrac{g^{01}}{g^{00}}t_\tau r_\theta+\dfrac{g^{01}}{g^{00}}t_\theta r_\tau+\dfrac{g^{11}}{g^{00}}t_\theta r_\theta\right),\\
B_4=\dfrac{m}{r^2}\left(1-\dfrac{2m}{r}\right)\left(t_\tau^2+2\dfrac{g^{01}}{g^{00}}t_\tau t_\theta+\dfrac{g^{11}}{g^{00}}t_\theta^2\right)-\dfrac{m}{r^2}\left(1-\dfrac{2m}{r}\right)^{-1}\left(r_\tau^2+2\dfrac{g^{01}}{g^{00}}r_\tau r_\theta+\dfrac{g^{11}}{g^{00}}r_\theta^2\right)\\
\qquad -r\left(1-\dfrac{2m}{r}\right)\left(\alpha_\tau^2+2\dfrac{g^{01}}{g^{00}}\alpha_\tau\alpha_\theta+\dfrac{g^{11}}{g^{00}}\alpha_\theta^2\right)-r\sin^2\alpha\left(1-\dfrac{2m}{r}\right)\left(\beta_\tau^2+2\dfrac{g^{01}}{g^{00}}\beta_\tau\beta_\theta+\dfrac{g^{11}}{g^{00}}\beta_\theta^2\right),\\
B_5=\dfrac{2}{r}\left(\alpha_\tau r_\tau+\dfrac{g^{01}}{g^{00}}\alpha_\tau r_\theta+\dfrac{g^{01}}{g^{00}}\alpha_\theta r_\tau+\dfrac{g^{11}}{g^{00}}\alpha_\theta r_\theta\right)-\sin\alpha\cos\alpha\left(\beta_\tau^2+2\dfrac{g^{01}}{g^{00}}\beta_\tau\beta_\theta+\dfrac{g^{11}}{g^{00}}\beta_\theta^2\right),\\
B_6=\dfrac{2}{r}\left(\beta_\tau r_\tau+\dfrac{g^{01}}{g^{00}}\beta_\tau r_\theta+\dfrac{g^{01}}{g^{00}}\beta_\theta r_\tau+\dfrac{g^{11}}{g^{00}}\beta_\theta r_\theta\right)+\dfrac{2\cos\alpha}{\sin\alpha}\left(\alpha_\tau\beta_\tau+\dfrac{g^{01}}{g^{00}}\alpha_\tau\beta_\theta+\dfrac{g^{01}}{g^{00}}\beta_\tau\alpha_\theta+\dfrac{g^{11}}{g^{00}}\alpha_\theta\beta_\theta\right).
\end{cases} \tag{5.13}$$

By a direct calculation, the eigenvalues of $A(U)$ read

$$\begin{cases}
\lambda_i=\lambda_-\triangleq\dfrac{g^{01}+\sqrt{(g^{01})^2-g^{00}g^{11}}}{g^{00}}=\dfrac{-g_{01}-\sqrt{(g_{01})^2-g_{00}g_{11}}}{g_{11}} & (i=1,2,3,4),\\
\lambda_i=\lambda_+\triangleq\dfrac{g^{01}-\sqrt{(g^{01})^2-g^{00}g^{11}}}{g^{00}}=\dfrac{-g_{01}+\sqrt{(g_{01})^2-g_{00}g_{11}}}{g_{11}} & (i=5,6,7,8),\\
\lambda_i=\lambda_0\triangleq 0 \quad (i=9,10).
\end{cases} \tag{5.14}$$

The right eigenvector corresponding to λ_i $(i=1,2,\cdots,10)$ can be chosen as

$$\begin{cases}
r_i=(0,0,-\lambda_- e_i,e_i)^T \quad (i=1,2,3,4), \qquad r_i=(0,0,-\lambda_+ e_{i-4},e_{i-4})^T \quad (i=5,6,7,8),\\
r_9=(1,0,0,0,0,0,0,0,0,0)^T, \quad r_{10}=(0,1,0,0,0,0,0,0,0,0)^T,
\end{cases} \tag{5.15}$$

where

$$e_1=(1,0,0,0), \quad e_2=(0,1,0,0), \quad e_3=(0,0,1,0), \quad e_4=(0,0,0,1);$$

while, the left eigenvector corresponding to λ_i $(i=1,2,\cdots,10)$ can be taken as

$$\begin{cases} l_i = (0,0,e_i,\lambda_+ e_i) \quad (i=1,2,3,4), \qquad l_i=(0,0,e_{i-4},\lambda_- e_{i-4}) \quad (i=5,6,7,8), \\ l_9 = (1,0,0,0,0,0,0,0,0,0), \quad l_{10}=(0,1,0,0,0,0,0,0,0,0). \end{cases} \tag{5.16}$$

Summarizing the above discussion yields

Proposition 5.1 *Under the assumption (5.4), the system (5.12) is a non-strictly hyperbolic system with ten eigenvalues (see (5.14)), and the right (resp. left) eigenvectors can be chosen as (5.15) (resp. (5.16)).*

Proposition 5.2 *Under the assumption (5.4), the system (5.12) is linearly degenerate in the sense of Lax (see [26]), i.e.,*

$$\nabla\lambda_- \cdot r_i \equiv 0 \quad (i=1,2,3,4), \qquad \nabla\lambda_+ \cdot r_i \equiv 0 \quad (i=5,6,7,8).$$

Theorem 5.1 *Under the assumption (5.4), λ_- (resp. λ_+) is a Riemann invariant corresponding to λ_+ (resp. λ_-). Moreover, these two Riemann invariants satisfy*

$$\frac{\partial\lambda_-}{\partial\tau}+\lambda_+\frac{\partial\lambda_-}{\partial\theta}=0, \qquad \frac{\partial\lambda_+}{\partial\tau}+\lambda_-\frac{\partial\lambda_+}{\partial\theta}=0. \tag{5.17}$$

The equations (5.17) play an important role in the study of the motion of relativistic string in the Schwarzschild space-time.

Similarly, we may introduce the following Riemann invariants

$$R_i=U_i \ \ (i=1,2), \quad R_i=U_i+\lambda_- U_{i+4} \quad (i=3,4,5,6), \quad R_i=U_{i-4}+\lambda_+ U_i \quad (i=7,8,9,10). \tag{5.18}$$

It is easy to verify that R_i $(i=1,2,\cdots,10)$ satisfy

$$\begin{cases} \dfrac{\partial R_i}{\partial\tau}+\lambda_0\dfrac{\partial R_i}{\partial\theta}+B_i=0 \qquad (i=1,2), \\ \dfrac{\partial R_i}{\partial\tau}+\lambda_+\dfrac{\partial R_i}{\partial\theta}+B_i=0 \qquad (i=3,4,5,6), \\ \dfrac{\partial R_i}{\partial\tau}+\lambda_-\dfrac{\partial R_i}{\partial\theta}+B_{i-4}=0 \qquad (i=7,8,9,10). \end{cases} \tag{5.19}$$

Obviously, from (5.18) we can solve out U_i $(i=1,2,\cdots,10)$ by utilizing R_i $(i=1,2,\cdots,10)$ and $\lambda_\pm$, and the resulting expressions of B_i in (5.13) (equivalently, (5.19)) can be represented by the unknowns R_i $(i=1,\cdots,10)$ and $\lambda_\pm$ which can be obtained by (5.17), more exactly,

$$\begin{cases} B_1=\dfrac{\lambda_+R_4-\lambda_-R_8}{\lambda_--\lambda_+}, \qquad B_2=\dfrac{\lambda_+R_5-\lambda_-R_9}{\lambda_--\lambda_+}, \qquad B_3=\dfrac{m(R_3R_8+R_4R_7)}{R_1(R_1-2m)}, \\ B_4=(R_1-2m)\left(\dfrac{mR_3R_7}{R_1^3}-\dfrac{mR_4R_8}{R_1(R_1-2m)^2}-R_5R_9-\sin^2R_2R_6R_{10}\right), \\ B_5=\dfrac{1}{R_1}(R_4R_9+R_5R_8)-\sin R_2\cos R_2R_6R_{10}, \\ B_6=\dfrac{1}{R_1}(R_4R_{10}+R_6R_8)+\dfrac{\cos R_2}{\sin R_2}(R_5R_{10}+R_6R_9). \end{cases} \tag{5.20}$$

Notice that the denominator of the second term in the right-hand side of the sixth equality in (5.20) is $\sin R_2$, it is a singularity when R_2 takes the values $k\pi$ ($k \in \mathbb{Z}$), even we only focus our study on the motion of relativistic strings in the exterior Schwarzschild space-time. This makes the estimate on this term very difficult in the study of global existence or blow-up phenomena of smooth solutions for the system (5.20). Therefore, we do not think that the spherical coordinates is a good choice for the study on general motion of relativistic strings in the Schwarzschild space-time, this is the reason why we adopt the Schwarzschild metric in the Cartesian coordinates in the present paper.

Acknowledgements. This work was supported in part by the NNSF of China (Grant No. 10971190), the Qiu-Shi Chair Professor Fellowship from Zhejiang University and the Foundation for University's Excellent Youth Scholars from the Anhui Educational Committee (Grant No. 2009SQRZ025ZD).

References

[1] Barbashov B.M., Nesterenko V.V.: *Introduction to the relativistic string theory.* Translated from the Russian by T. Yu. Dumbrajs. World Scientific Publishing Co., Inc., Teaneck, NJ, 1990.

[2] Barbashov B.M., Nesterenko V.V., Chervyakov A.M.: General solutions of nonlinear equations in the geometric theory of the relativistic string. Comm. Math. Phys. **84**, 471-481 (1982)

[3] Boillat G.: Chocs caractéristiques. C. R. Acad. Sci., Paris, Sér. A **274**, 1018-1021 (1972)

[4] Bordemann M., Hoppe J.: The dynamics of relativistic membranes II: Nonlimear waves and covariantly reduced membrane equations. Physics Letters B **325**, 359-365 (1994)

[5] Born M., Infeld L.: Foundation of the new field theory. Proc. Roy. Soc. London **A144**, 425-451 (1934)

[6] Brenier Y.: Some Geometric PDEs Related to Hydrodynamics and Electrodynamics. Proceedings of ICM 2002 **3**, 761-772 (2002)

[7] Calabi E.: Examples of Bernstein problems for some nonlinear equations. in 1970 Global Analysis (Proc. Sympos. Pure Math., Vol. XV, Berkeley, Calif., 1968), Amer. Math. Soc., Providence, R.I., 223-230.

[8] Cheng S.Y., Yau S.T.: Maximal space-like hypersurfaces in the Lorentz-Minkowski spaces. Ann. of Math. **104**, 407-419 (1976)

[9] Dongho Chae, Hyungjin Huh: Global existence for small initial data in the Born-Infeld equations. J. Math. Phys. **44**, 6132-6139 (2003)

[10] Christodoulou D.: Global solutions of nonlinear hyperbolic equations for small initial data. Comm. Pure Appl. Math. **39**, 267-282 (1986)

[11] Freistühler H.: Linear degeneracy and shock waves. Math. Zeit. **207**, 583-596 (1991)

[12] Gibbons G.W.: Born-Infeld particles and Dirichlet p-branes. Nucl. Phys. B **514**, 603-639 (1998)

[13] Green M.B., Schwarz J.H., Witten E.: *Superstring theory: 1. Introduction.* Second edition, Cambridge Monographs on Mathematical Physics, Cambridge University Press, Cambridge, 1988.

[14] Gu C.H.: On the Cauchy problem for harmonic maps defined on two-dimensional Minkowski space. Comm. Pure Appl. Math. **33**, 727-737 (1980).

[15] Gu C.H.: On the motion of a string in a curved space-time, Proc. of 1982 Grossmann Symposium, 139-142.

[16] Gu C.H.: Extremal surfaces of mixed type in Minkowski space $\mathbb{R}^{n+1}$. in Variational methods (Paris, 1988), 283-296, Progr. Nonlinear Differential Equations Appl. **4**, Birkh$\ddot{a}$user Boston, Boston, MA, 1990.

[17] Gu C.H.: Complete extremal surfaces of mixed type in 3-dimensional Minkowski space. Chin. Ann. Math. **15B**, 385-400 (1994)

[18] Hoppe J.: Some classical solutions of relativistic membrane equations in 4-space-time dimensions. Phys. Lett. B **329**, 10-14 (1994)

[19] Hörmander L.: Lectures on Nonlinear Hyperbolic Differential Equations. Mathématiques & Applications **26**, Springer, 1997.

[20] Klainerman S.: The null condition and global existence to nonlinear wave equations. in Lectures in Appl. Math. **23**, Amer. Math. Soc., Providence, RI, 1986, 293-326.

[21] Kong D.X.: A nonlinear geometric equation related to electrodynamics. Europhys. Lett. **66**, 617-623 (2004)

[22] Kong D.X., Sun Q.Y., Zhou Y.: The equation for time-like extremal surfaces in Minkowski space $\mathbb{R}^{1+n}$. Journal of Mathematical Physics **47**, 013503 (2006)

[23] Kong D.X., Tsuji M.: Global solutions for 2×2 hyperbolic systems with linearly degenerate characteristics. Funkcialaj Ekvacioj **42**, 129-155 (1999)

[24] Kong D.X., Zhang Q.: Solution formula and time periodicity for the motion of relativistic strings in the Minkowski space $\mathbb{R}^{1+n}$. Physica D: Nonlinear Phenomena **238**, 902-922 (2009)

[25] Kong D.X., Zhang Q., Zhou Q.: The dynamics of relativistic strings moving in the Minkowski space $\mathbb{R}^{1+n}$. Communications in Mathematical Physics **269** 153-174 (2007)

[26] Lax P.D.: Hyperbolic systems of conservation laws II. Commun. Pure Appl. Math. **10**, 537-556 (1957)

[27] Lindblad H.: A remark on global existence for small initial data of the minimal surface equation in Minkowskian space time. Proc. Amer. Math. Soc. **132**, 1095-1102 (2004)

[28] Milnor T.: Entire timelike minimal surfaces in $E^{3,1}$. Michigan Math. J. **37**, 163-177 (1990)

[29] Schatzman M.: Continuous Glimm functionals and uniqueness of solutions of the Riemann problem. Indiana University Mathematics Journal **34**, 533-589 (1985).

[30] Schatzman M.: The Geometry of Continuous Glimm Functionals. Lectures in Applied Mathematics **23**, 417-439 (1986).

[31] Serre D.: *Systems of Conservation Laws 2: Geometric Structures, Oscillations, and Initial-Boundary Value Problems*, Canmbridge University Press, Cambridge, 2000.

[32] Zhou Y.: Global classical solutions to quasilinear hyperbolic systems with weak linear degeneracy. Chin. Ann. Math. **25B**, 37-56 (2004).