Time-Periodic Solutions of the Einstein’s Field Equations II

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In this paper, we construct several kinds of new time-periodic solutions of the vacuum Einstein’s field equations whose Riemann curvature tensors vanish, keep finite or take the infinity at some points in these space-times, respectively. The singularities of these new time-periodic solutions are investigated and some new physical phenomena are found. The applications of these solutions in modern cosmology and general relativity can be expected.

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1. Introduction. The Einstein’s field equations are the fundamental equations in general relativity and play an essential role in cosmology. This paper concerns the time-periodic solutions of the following vacuum Einstein’s field equations

\[
G_{\mu\nu} \triangleq R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0,
\]

or equivalently,

\[
R_{\mu\nu} = 0,
\]

where \(g_{\mu\nu} (\mu, \nu = 0, 1, 2, 3)\) is the unknown Lorentzian metric, \(R_{\mu\nu}\) is the Ricci curvature tensor, \(R\) is the scalar curvature and \(G_{\mu\nu}\) is the Einstein tensor.

It is well known that the exact solutions of the Einstein’s field equations play a crucial role in general relativity and cosmology. Typical examples are the Schwarzschild solution and Kerr solution. Although many interesting and important solutions have been obtained (see, e.g., [1] and [5]), there are still many fundamental open problems. One such problem is if there exists a “time-periodic” solution, which contains physical singularities such as black hole to the vacuum Einstein’s field equations (1). We shall construct three kinds of new time-periodic solutions of the vacuum Einstein’s field equations (1) whose Riemann curvature tensors vanish, keep finite or go to the infinity at some points in these space-times respectively. The singularities of these new time-periodic solutions are investigated and new physical phenomena are found. Moreover, the applications of these solutions in modern cosmology and general relativity may be expected. In the forthcoming paper [4], we shall construct a time-periodic solution of the Einstein’s field equations with black hole, which describes the time-periodic cosmology with many new and interesting physical phenomena.

2. Procedure of finding new solutions.

We consider the metric of the following form

\[
(g_{\mu\nu}) = \begin{pmatrix}
  u & v & p & 0 \\
  v & 0 & 0 & 0 \\
  p & 0 & f & 0 \\
  0 & 0 & h & 0
\end{pmatrix},
\]

vanishing Riemann curvature tensor but is inhomogeneous, anisotropic and not asymptotically flat. In particular, this space-time does not contain any essential singularity, but contains some non-essential singularities which correspond to steady event horizons, time-periodic event horizon and has some interesting new physical phenomena.

In this paper, we focus on finding the time-periodic solutions, which contain physical singularities such as black hole to the vacuum Einstein’s field equations (1). We shall construct three kinds of new time-periodic solutions of the vacuum Einstein’s field equations (1) whose Riemann curvature tensors vanish, keep finite or go to the infinity at some points in these space-times respectively. The singularities of these new time-periodic solutions are investigated and new physical phenomena are found. Moreover, the applications of these solutions in modern cosmology and general relativity may be expected. In the forthcoming paper [4], we shall construct a time-periodic solution of the Einstein’s field equations with black hole, which describes the time-periodic cosmology with many new and interesting physical phenomena.
where $u, v, p, f$ and $h$ are smooth functions of the coordinates $(t, x, y, z)$. It is easy to verify that the determinant of $(g_{\mu\nu})$ is given by

$$g \triangleq \det(g_{\mu\nu}) = -v^2 fh.$$  

(4)

Throughout this paper, we assume that $g < 0$. \((H)\)

Without loss of generality, we may suppose that $f$ and $g$ keep the same sign, for example,

$$f < 0 \text{ (resp. } f > 0) \text{ and } h < 0 \text{ (resp. } g > 0).$$  

(5)

In what follows, we solve the Einstein’s field equations (2) under the framework of the Lorentzian metric of the form (3).

By a direct calculation, we have the Ricci tensor

$$R_{11} = \frac{1}{2} \left\{ \frac{v_x}{v} \left( f_x + \frac{h_x}{h} \right) + \frac{1}{2} \left[ \left( f_x \right)^2 + \left( \frac{h_x}{h} \right)^2 \right] - \left( f_{xx} + \frac{h_{xx}}{h} \right) \right\}.$$  

(6)

It follows from (2) that

$$\frac{v_x}{v} \left( f_x + \frac{h_x}{h} \right) + \frac{1}{2} \left[ \left( f_x \right)^2 + \left( \frac{h_x}{h} \right)^2 \right] - \left( f_{xx} + \frac{h_{xx}}{h} \right) = 0.$$  

(7)

This is an ordinary differential equation of first order on the unknown function $v$. Solving (7) gives

$$v = V(t, y, z) \exp \left\{ \int \Theta(t, x, y, z) dx \right\},$$  

(8)

where

$$\Theta = \left[ \frac{f_{xx}}{f} + \frac{h_{xx}}{h} - \frac{1}{2} \left( \frac{f_x}{f} \right)^2 - \frac{1}{2} \left( \frac{h_x}{h} \right)^2 \right] \frac{fh}{(fh)_x},$$

and $V = V(t, y, z)$ is an integral function depending on $t, y$ and $z$. Here we assume that

$$(fh)_x \neq 0.$$  

(9)

In particular, taking the ansatz

$$f = -K(t, x)^2, \quad h = N(t, y, z)K(t, x)^2$$  

(10)

and substituting it into (8) yields

$$v = VK_x.$$  

(11)

By the assumptions (H) and (9), we have

$$V \neq 0, \quad K \neq 0, \quad K_x \neq 0.$$  

(12)

Noting (10) and (11), by a direct calculation we obtain

$$R_{13} = -\frac{V_x K_x}{KV}.$$  

(13)

It follows from (2) that

$$R_{13} = 0.$$  

Combining (12) and (13) gives

$$V_z = 0.$$  

(14)

This implies that the function $V$ depends only on $t, y$ but is independent of $x$ and $z$. Noting (10)-(11) and using (14), we calculate

$$R_{12} = -\frac{1}{2V} \left( \frac{p_{xx}}{K_x} - \frac{K_x p_x}{K^2} - \frac{2pK_x}{K^2} + \frac{2K_x V_y}{K} \right).$$  

(15)

Solving $p$ from the equation $R_{12} = 0$ yields

$$p = AK^2 + V_y K + \frac{B}{K},$$  

(16)

where $A$ and $B$ are integral functions depending on $t, y$ and $z$. Noting (10)-(11) and using (14) and (16), we observe that the equation $R_{23} = 0$ is equivalent to

$$B_z - 2K^3 A_z = 0.$$  

(17)

Since $K$ is a function depending only on $t, x$, and $A, B$ are functions depending on $t, y$ and $z$, we can obtain that

$$B = 2K^3 A + C(t, x, y),$$  

(18)

where $C$ is an integral function depending on $t, x$ and $y$. For simplicity, we take

$$A = B = C = 0.$$  

(19)
Thus, (16) simplifies to
\[ p = V_y K. \]  
(20)

From now on, we assume that the function \( N \) only depends on \( y \), that is to say,
\[ N = N(y). \]  
(21)

Substituting (10)-(11), (14) and (20)-(21) into the equation \( R_{02} = 0 \) yields
\[ u_y V_y + V(u_{yz} - 4V_y K_{xt}) = 0. \]  
(22)

Solving \( u \) from the equation (22) leads to
\[ u = 2K_y V. \]  
(23)

Noting (10)-(11), (14), (20)-(21) and (23), by a direct calculation we obtain
\[ R_{03} = 0, \]  
(24)

\[ \begin{cases} R_{22} = (4N^2V^2)^{-1} \left[ 2NV^2 N_{yy} - 4N^2V V_{yy} \\ + 4N^2V_y^2 - 2NV N_y V_y - V^2 N_y^2 \right], \\ R_{33} = -(4N^2V^2)^{-1} \left[ 2NV^2 N_{yy} - 4N^2V V_{yy} \\ + 4N^2V_y^2 - 2NV N_y V_y - V^2 N_y^2 \right] \end{cases} \]  
(25)

and
\[ R_{00} = (2KNV^2)^{-1} \left[ 4NV_yV_y + 2N^2V_y^2 V_{yy} - 2NV_y V_{yy} \\ - 4NV^2 V_{yy} V_y - V N_y V_y + V^2 N_y V_{yy} \right]. \]  
(26)

Therefore, under the assumptions mentioned above, the Einstein’s field equations (2) are reduced to
\[ \begin{align*} - \frac{N_{yy}}{N} + \frac{1}{2} \left( \frac{N_y}{N} \right)^2 + 2 \frac{V_{yy}}{N V} + \frac{N_y V_y}{N V} - 2 \left( \frac{V_y}{V} \right)^2 &= 0 \quad (27) \\
\end{align*} \n
and
\[ 4V_y^2 V_y + 2V^2 V_{yty} - 2V V_{yy} V_y - 4V Y V_{yt} \\
- V V_y V_y N_y + V^2 V_{yt} N_y = 0. \]  
(28)

On the other hand, (27) can be rewritten as
\[ 2 \left( \frac{V_y}{V} \right)_y + \frac{V_y N_y}{V N} - \left( \frac{N_y}{N} \right)_y - \frac{1}{2} \left( \frac{N_y}{N} \right)^2 = 0 \]  
(29)

and (28) is equivalent to
\[ 2 \left( \frac{V_y}{V} \right)_y + \frac{V_y N_y}{V N} + \frac{N_y}{N} = 0. \]  
(30)

Noting (21) and differentiating (29) with respect to \( t \) gives (30) directly. This shows that (29) implies (30). Hence in the present situation, the Einstein’s field equations (2) are essentially (29). Solving \( V \) from the equation (29) yields
\[ V = w(t)|N(y)|^{-1/2} \exp \left\{ q(t) \int |N(y)|^{-1/2} dy \right\}, \]  
(31)

where \( w = w(t) \) and \( q = q(t) \) are two integral functions only depending on \( t \). Thus, we can obtain the following solution of the vacuum Einstein’s field equations in the coordinates \((t, x, y, z)\)
\[ ds^2 = (dt, dx, dy, dz)(g_{\mu\nu})(dt, dx, dy, dz)^T, \]  
(32)

where
\[ (g_{\mu\nu}) = \begin{pmatrix} 2K_y V & K_y V & K V_y & 0 \\
K_y V & 0 & 0 & 0 \\
K V_y & 0 & -K^2 & 0 \\
0 & 0 & 0 & N K^2 \end{pmatrix}, \]  
(33)

in which \( N = N(y) \) is an arbitrary function of \( y \), \( K = K(t, x) \) is an arbitrary function of \( t, x \), and \( V \) is given by (31).

By calculations, the Riemann curvature tensor reads
\[ R_{\alpha\beta\mu\nu} = 0, \quad \forall \alpha\beta\mu\nu \neq 0202 \text{ or } 0303, \]  
(34)

while
\[ R_{0202} = K \omega q |N|^{-1/2} \exp \left\{ q \int |N|^{-1/2} dy \right\} \]  
(35)

and
\[ R_{0303} = K \omega q |N|^{-1/2} \exp \left\{ q \int |N|^{-1/2} dy \right\}. \]  
(36)

3. Time-periodic solutions. This section is devoted to constructing some new time-periodic solutions of the vacuum Einstein’s field equations.

3.1 Regular time-periodic space-times with vanishing Riemann curvature tensor. Take \( q = \text{constant} \) and let \( V = \rho(t) \kappa(y) \), where \( \kappa \) is defined by
\[ \kappa(y) = c_1 \sqrt{|N|} \exp \left\{ c_2 \int |N|^{-1/2} dy \right\}, \]  
(37)
in which \(c_1\) and \(c_2\) are two integrable constants. In this case, the solution to the vacuum Einstein’s field equations in the coordinates \((t, x, y, z)\) reads

\[ds^2 = (dt, dx, dy, dz)(g_{\mu\nu})(dt, dx, dy, dz)^T,\]  

(38)

where

\[
(g_{\mu\nu}) = \begin{pmatrix}
2\rho K \partial_t K & \rho K \partial_x K & 0 \\
\rho K \partial_y K & 0 & 0 \\
\rho K \partial_\nu K & 0 & -K^2 & 0 \\
0 & 0 & 0 & NK^2
\end{pmatrix}.
\]  

(39)

**Theorem 1** The vacuum Einstein’s field equations (2) have a solution described by (38) and (39), and the Riemann curvature tensor of this solution vanishes. ■

As an example, let

\[
\begin{align*}
w(t) &= \cos t, \\
q(t) &= 0, \\
K(t, x) &= e^t \sin t, \\
N(y) &= -(2 + \sin y)^2.
\end{align*}
\]  

(40)

In the present situation, we obtain the following solution of the vacuum Einstein’s field equations (2)

\[
(\eta_{\mu\nu}) = \begin{pmatrix}
\eta_{00} & \eta_{01} & \eta_{02} & 0 \\
\eta_{01} & 0 & 0 & 0 \\
\eta_{02} & 0 & 0 & 0 \\
0 & 0 & 0 & \eta_{33}
\end{pmatrix},
\]  

(41)

where

\[
\begin{align*}
\eta_{00} &= 2e^2(2 + \sin y) \cos^2 t, \\
\eta_{01} &= \frac{1}{2}e^2(2 + \sin y) \sin(2t), \\
\eta_{02} &= \frac{1}{2}e^2 \cos y \sin(2t), \\
\eta_{22} &= -[e^2 \sin t]^2, \\
\eta_{33} &= -[e^2(2 + \sin y) \sin t]^2.
\end{align*}
\]  

(42)

By (4),

\[
\eta \triangleq \det(\eta_{\mu\nu}) = -\frac{1}{4}e^{6x}(2 + \sin y)^4 \sin^4 t \sin^2(2t).
\]  

(43)

**Property 1** The solution (41) of the vacuum Einstein’s field equations (2) is time-periodic. ■

**Proof.** In fact, the first equality in (42) implies that

\[\eta_{00} > 0 \quad \text{for} \quad t \neq k\pi + \pi/2 \quad (k \in \mathbb{N}) \quad \text{and} \quad x \neq -\infty.
\]

On the other hand, by direct calculations,

\[
\begin{pmatrix}
\eta_{00} & \eta_{01} \\
\eta_{01} & 0
\end{pmatrix} = -\frac{1}{4}e^{2x}(2 + \sin y)^2 \sin^2(2t) < 0,
\]

and

\[
\begin{pmatrix}
\eta_{00} & \eta_{01} & \eta_{02} & 0 \\
\eta_{01} & 0 & 0 & 0 \\
\eta_{02} & 0 & 0 & 0 \\
0 & 0 & 0 & \eta_{33}
\end{pmatrix} = -\frac{1}{4}e^{2x}(2 + \sin y)^2 \sin^2(2t) < 0.
\]

for \(t \neq k\pi, \quad k\pi + \pi/2 \quad (k \in \mathbb{N})\) and \(x \neq -\infty\).

In Property 3 below, we will show that \(t = k\pi, \quad k\pi + \pi/2 \quad (k \in \mathbb{N})\) are the singularities of the space-time described by (41), but they are not essential (or say, physical) singularities, these non-essential singularities correspond to the event horizons of the space-time described by (41) with (42); while, when \(x = -\infty\), the space-time (41) degenerates to a point.

The above discussion implies that the variable \(t\) is a time coordinate. Therefore, it follows from (42) that the Lorentzian metric

\[ds^2 = (dt, dx, dy, dz)(\eta_{\mu\nu})(dt, dx, dy, dz)^T
\]  

(44)

is indeed a time-periodic solution of the vacuum Einstein’s field equations (2), where \((\eta_{\mu\nu})\) is given by (41). This proves Property 1. ■

Noting (34)-(36) and the second equality in (40) gives

**Property 2** The Lorentzian metric (44) (in which \((\eta_{\mu\nu})\) is given by (41) and (42)) describes a regular space-time, this space-time is Riemannian flat, that is to say, its Riemann curvature tensor vanishes. ■

**Remark 1** The first time-periodic solution to the Einstein’s field equations was constructed by Kong and Liu [3]. The time-periodic solution presented in [3] also has the vanishing Riemann curvature tensor.

It follows from (43) that the hypersurfaces \(t = k\pi, \quad k\pi + \pi/2 \quad (k \in \mathbb{N})\) and \(x = \pm\infty\) are singularities of the
space-time (44) (in which \((\eta_{\mu\nu})\) is given by (41) and (42)), however, by Property 2, these singularities are not physical (or say, not essential). According to the definition of event horizon (see e.g., Wald [6]), it is easy to show that the hypersurfaces \(t = k\pi, k\pi + \pi/2\) \((k \in \mathbb{N})\) and \(x = +\infty\) are the event horizons of the space-time (44) (in which \((\eta_{\mu\nu})\) is given by (41) and (42)). Therefore, we have

**Property 3** The Lorentzian metric (44) (in which \((\eta_{\mu\nu})\) is given by (41) and (42)) does not contain any essential singularity. These non-essential singularities consist of the hypersurfaces \(t = k\pi, k\pi + \pi/2\) \((k \in \mathbb{N})\) and \(x = \pm\infty\). The singularities \(t = k\pi, k\pi + \pi/2\) \((k \in \mathbb{N})\) and \(x = +\infty\) correspond to the event horizons, while, when \(x = -\infty\), the space-time (44) degenerates to a point.

We now investigate the physical behavior of the space-time (44).

Fixing \(y\) and \(z\), we get the induced metric

\[
(ds^2)_{\text{ind}} = \eta_{00} dt^2 + 2\eta_{01} dt dx. \tag{45}
\]

Consider the null curves in the \((t, x)\)-plan, which are defined by

\[
\eta_{00} dt^2 + 2\eta_{01} dt dx = 0. \tag{46}
\]

Noting (42) gives

\[
dt = 0 \quad \text{and} \quad \frac{dt}{dx} = -\tan t. \tag{47}
\]

Thus, the null curves and light-cones are shown in Figure 1.

We next study the geometric behavior of the \(t\)-slices.

For any fixed \(t \in \mathbb{R}\), it follows from (44) that the induced metric of the \(t\)-slice reads

\[
(ds^2)_{\text{ind}} = \eta_{22} dy^2 + \eta_{33} dz^2 = -e^{2x} \sin^2 t [dy^2 + (2 + \sin y)^2 dz^2]. \tag{48}
\]

When \(t = k\pi\) \((k \in \mathbb{N})\), the metric (48) becomes

\[
(ds^2)_{\text{ind}} = 0.
\]

This implies that the \(t\)-slice reduces to a point. On the other hand, in the present situation, the metric (44) becomes

\[
(ds^2)_{\text{ind}} = 2e^x (2 + \sin y) dt^2.
\]

When \(t \neq k\pi\) \((k \in \mathbb{N})\), (48) shows that the \(t\)-slice is a three-dimensional cone-like manifold centered at \(x = -\infty\).

**3.2 Regular time-periodic space-times with non-vanishing Riemann curvature tensor.** We next construct the regular time-periodic space-times with non-vanishing Riemann curvature tensor.
To do so, let

\[
\begin{align*}
w(t) &= \cos t, \\
g(t) &= \sin t, \\
K(x, t) &= e^x \sin t, \\
N &= -\frac{1}{(2 + \sin y)^2}.
\end{align*}
\]

Then, by (31),

\[
V = \frac{\cos t \exp \{(2y - \cos y) \sin t\}}{2 + \sin y}.
\]

Thus, in the present situation, we have the following solution of the vacuum Einstein’s field equations (2)

\[
\tilde{\eta}_{\mu\nu} = \begin{pmatrix}
\tilde{\eta}_{00} & \tilde{\eta}_{01} & \tilde{\eta}_{02} & 0 \\
\tilde{\eta}_{01} & \tilde{\eta}_{11} & 0 & 0 \\
\tilde{\eta}_{02} & 0 & \tilde{\eta}_{22} & 0 \\
0 & 0 & 0 & \tilde{\eta}_{33}
\end{pmatrix},
\]

(50)

where

\[
\begin{align*}
\tilde{\eta}_{00} &= \frac{2e^x \cos^2 t \exp \{(2y - \cos y) \sin t\}}{2 + \sin y}, \\
\tilde{\eta}_{01} &= \frac{e^x \sin(2t) \exp \{(2y - \cos y) \sin t\}}{2(2 + \sin y)}, \\
\tilde{\eta}_{02} &= e^x \left\{\sin t \cos t - \frac{\cos t \cos y}{(2 + \sin y)^2}\right\} \sin t \\
&\quad \times \exp \{(2y - \cos y) \sin t\}, \\
\tilde{\eta}_{22} &= -e^{2x} \sin^2 t, \\
\tilde{\eta}_{33} &= -\frac{e^{2x} \sin^2 t}{(2 + \sin y)^2}.
\end{align*}
\]

By (4),

\[
\tilde{\eta} \triangleq \det(\tilde{\eta}_{\mu\nu}) = -\tilde{\eta}_{01}^2 \tilde{\eta}_{22} \tilde{\eta}_{33}
\]

\[
= -\frac{e^{6x + 2(2y - \cos y) \sin t} \sin^2(2t) \sin^4 t}{4(2 + \sin y)^4}.
\]

(52)

Introduce

\[
\Delta(t, x, y) = 6x + 2(2y - \cos y) \sin t.
\]

Thus, it follows from (52) that

\[
\tilde{\eta} < 0
\]

(53)

for \( t \neq k\pi, k\pi + \pi/2 \) \((k \in \mathbb{N})\) and \( \Delta \neq -\infty \). It is obvious that the hypersurfaces \( t = k\pi, k\pi + \pi/2 \) \((k \in \mathbb{N})\) and \( \Delta = \pm \infty \) are the singularities of the space-time described by (50) with (51). As in Subsection 3.1, we can prove that the hypersurfaces \( t = k\pi, k\pi + \pi/2 \) \((k \in \mathbb{N})\) are not essential (or say, physical) singularities, these non-essential singularities correspond to the event horizons of the space-time described by (50) with (51).

Similar to Property 1, we have

**Property 4** The solution (50) (in which \((\tilde{\eta}_{\mu\nu})\) is given by (51)) of the vacuum Einstein’s filed equations (2) is time-periodic. ■

Similar to Property 2, we have

**Property 5** The Lorentzian metric (50) (in which \((\tilde{\eta}_{\mu\nu})\) is given by (51)) describes a regular space-time, this space-time has a non-vanishing Riemann curvature tensor. ■

**Proof.** In the present situation, by (34)

\[
R_{\alpha\beta\mu\nu} = 0, \quad \forall \alpha\beta\mu\nu \neq 0202 \text{ or } 0303,
\]

(54)

while

\[
R_{0202} = e^x(2 + \sin y) \cos^2 t \sin^2 t \\
\quad \times \exp \{(2y - \cos y) \sin t\},
\]

and

\[
R_{0303} = e^x \cos^2 t \sin^2 t \exp \{(2y - \cos y) \sin t\}.
\]

(56)

Property 5 follows from (54)-(56) directly. Thus the proof is completed. 

In particular, when \( t \neq k\pi, k\pi + \pi/2 \) \((k \in \mathbb{N})\), it follows from (55) and (56) that

\[
R_{0202}, R_{0303} \longrightarrow \infty \quad \text{as} \quad x + (2y - \cos y) \sin t \to \infty.
\]

(57)

However, a direct calculation gives

\[
R \triangleq R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} \equiv 0.
\]

(58)

Thus, we obtain

**Property 6** The Lorentzian metric (50) (in which \((\tilde{\eta}_{\mu\nu})\) is given by (51)) does not contain any essential singularity. These non-essential singularities consist of the hypersurfaces \( t = k\pi, k\pi + \pi/2 \) \((k \in \mathbb{N})\) and \( \Delta = \pm \infty \), in
which the hypersurfaces $t = k\pi, k\pi + \pi/2 \ (k \in \mathbb{N})$ are the event horizons. Moreover, the Riemann curvature tensor satisfies the properties (57) and (58).

We next analyze the singularity behavior of $\Delta = \pm \infty$.

**Case 1:** Fixing $y \in \mathbb{R}$, we observe that

$$\Delta \rightarrow \pm \infty \iff x \rightarrow \pm \infty.$$ 

This situation is similar to the case $x \rightarrow \pm \infty$ discussed in Subsection 3.1. That is to say, $x = +\infty$ corresponds to the event horizon, while, when $x \rightarrow -\infty$, the space-time (50) with (51) degenerates to a point.

**Case 2:** Fixing $x \in \mathbb{R}$, we observe that

$$\Delta \rightarrow \pm \infty \iff y \rightarrow \pm \infty.$$ 

In the present situation, it holds that

$$t \neq k\pi \ (k \in \mathbb{N}).$$

Without loss of generality, we may assume that $\sin t > 0$.

For the case that $\sin t < 0$, we have a similar discussion.

Thus, noting (57), we have

$$R_{0202}, R_{0303} \rightarrow \infty \text{ as } y \rightarrow \infty.$$ 

Moreover, by the definition of the event horizon we can show that $y = +\infty$ is not a event horizon. On the other hand, when $y \rightarrow -\infty$, the space-time (50) with (51) degenerates to a point.

**Case 3:** For the situation that $x \rightarrow \pm \infty$ and $y \rightarrow \pm \infty$ simultaneously, we have a similar discussion, here we omit the details.

For the space-time (50) with (51), the null curves and light-cones are shown just as in Figure 1. On the other hand, for any fixed $t \in \mathbb{R}$, the induced metric of the $t$-slice reads

$$ds^2 = \tilde{\eta}_{22}dy^2 + \tilde{\eta}_{33}dz^2 = -e^{2x} \sin^2 t (dy^2 + (2 + \sin y)^{-2}dz^2). \quad (59)$$ 

Obviously, in the present situation, the $t$-slice possesses similar properties shown in the last paragraph in Subsection 3.1.

In particular, if we take $(t, x, y, z)$ as the spherical coordinates $(t, r, \theta, \varphi)$ with $t \in \mathbb{R}$, $r \in [0, \infty)$, $\theta \in [0, 2\pi)$, $\varphi \in [-\pi/2, \pi/2]$, then the metric (50) with (51) describes a regular time-periodic space-time with non-vanishing Riemann curvature tensor. This space-time does not contain any essential singularity, these non-essential singularities consist of the hypersurfaces $t = k\pi, k\pi + \pi/2 \ (k \in \mathbb{N})$ which are the event horizons. The Riemann curvature tensor satisfies (58) and

$$R_{0202}, R_{0303} \rightarrow \infty \text{ as } r \rightarrow \infty.$$ 

Moreover, when $t \neq k\pi \ (k \in \mathbb{N})$, the $t$-slice is a three dimensional bugle-like manifold with the base at $x = 0$; while, when $t = k\pi \ (k \in \mathbb{N})$, the $t$-slice reduces to a point.

### 3.3 Time-periodic space-times with physical singularities.

This subsection is devoted to constructing the time-periodic space-times with physical singularities.

To do so, let

$$\begin{cases}
  w(t) = \cos t, \\
  q(t) = \sin t, \\
  K(x, t) = \sin t \frac{x}{x^2}, \\
  N = -\frac{1}{(2 + \cos y)^2}.
\end{cases} \quad (60)$$

Then, by (31) we have

$$V = \frac{\cos t \exp \{(2y + \sin y)\sin t\}}{2 + \cos y}.$$ 

Thus, in the present situation, the solution of the vacuum Einstein’s field equations (2) in the coordinates $(t, x, y, z)$ reads

$$ds^2 = (dt, dx, dy, dz)(\tilde{\eta}_{\mu\nu})(dt, dx, dy, dz)^T, \quad (61)$$ 

where

$$(\tilde{\eta}_{\mu\nu}) = \begin{pmatrix}
  \tilde{\eta}_{00} & \tilde{\eta}_{01} & \tilde{\eta}_{02} & 0 \\
  \tilde{\eta}_{01} & 0 & 0 & 0 \\
  \tilde{\eta}_{02} & 0 & \tilde{\eta}_{22} & 0 \\
  0 & 0 & 0 & \tilde{\eta}_{33}
\end{pmatrix} \quad (62)$$.
in which
\[
\begin{align*}
\dot{\eta}_{00} &= \frac{2 \cos^2 t \exp \left\{ (\sin y + 2y) \sin t \right\}}{(2 + \cos y)x^2}, \\
\dot{\eta}_{01} &= -\frac{\sin(2t) \exp \left\{ (\sin y + 2y) \sin t \right\}}{(2 + \cos y)x^3}, \\
\dot{\eta}_{02} &= \frac{\sin t}{x^2} \left\{ \frac{\cos t \sin y}{(2 + \cos y)^2} + \frac{\sin(2t)}{2} \right\} \times \exp \left\{ (\sin y + 2y) \sin t \right\}, \\
\dot{\eta}_{22} &= -\frac{\sin^2 t}{x^4}, \\
\dot{\eta}_{33} &= -\frac{\sin^2 t}{(2 + \cos y)^2 x^4}.
\end{align*}
\]

By (4), we have
\[
\dot{\eta} \triangleq \det(\dot{\eta}_{\mu\nu}) = -{(\dot{\eta}_{01})}^2 \dot{\eta}_{22} \dot{\eta}_{33} = -e^{2(2y+\sin y) \sin t} \sin^2(2t) \sin^4 t \frac{1}{x^4 (2 + \cos y)^4}.
\]

It follows from (63) that
\[
\dot{\eta} < 0
\]
for \( t \neq k\pi, k\pi + \pi/2 \ (k \in \mathbb{N}) \) and \( x \neq 0 \). Obviously, the hypersurfaces \( t = k\pi, k\pi + \pi/2 \ (k \in \mathbb{N}) \) and \( x = 0 \) are the singularities of the space-time described by (61) with (62)-(63). As before, we can prove that the hypersurfaces \( t = k\pi, k\pi + \pi/2 \ (k \in \mathbb{N}) \) are not essential (or, say, physical) singularities, and these non-essential singularities correspond to the event horizons of the space-time described by (61) with (62)-(63), however \( x = 0 \) is an essential (or, say, physical) singularity (see Property 8 below).

Similar to Property 1, we have

**Property 7** The solution (61) (in which \((\dot{\eta}_{\mu\nu})\) is given by (62) and (63)) of the vacuum Einstein’s field equations (2) is time-periodic. ■

**Proof.** In fact, the first equality in (63) implies that
\[
\dot{\eta}_{00} > 0 \quad \text{for} \quad t \neq k\pi + \pi/2 \ (k \in \mathbb{N}) \quad \text{and} \quad x \neq 0.
\]

On the other hand, by direct calculations we have
\[
\left| \begin{array}{cc}
\dot{\eta}_{00} & \dot{\eta}_{01} \\
\dot{\eta}_{01} & 0
\end{array} \right| = -\dot{\eta}_{01}^2 < 0,
\]
and
\[
\left| \begin{array}{ccc}
\dot{\eta}_{00} & \dot{\eta}_{01} & \dot{\eta}_{02} \\
\dot{\eta}_{01} & 0 & 0 \\
\dot{\eta}_{02} & 0 & \dot{\eta}_{22}
\end{array} \right| = -\dot{\eta}_{01}^2 \dot{\eta}_{22} > 0
\]

for \( t \neq k\pi, k\pi + \pi/2 \ (k \in \mathbb{N}) \) and \( x \neq 0 \).

The above discussion implies that the variable \( t \) is a time coordinate. Therefore, it follows from (63) that the Lorentzian metric (61) is indeed a time-periodic solution of the vacuum Einstein’s field equations (2), where \((\dot{\eta}_{\mu\nu})\) is given by (63). This proves Property 7. □

**Property 8** When \( t \neq k\pi, k\pi + \pi/2 \ (k \in \mathbb{N}) \), for any fixed \( y \in \mathbb{R} \) it holds that
\[
R_{0202} \rightarrow +\infty \quad \text{and} \quad R_{0303} \rightarrow +\infty, \quad \text{as} \quad x \rightarrow 0.
\]

**Proof.** By direct calculations, we obtain from (35) and (36) that
\[
R_{0202} = \frac{(2 + \cos y) \sin^2(2t) \exp \left\{ (\sin y + 2y) \sin t \right\}}{4x^2},
\]
and
\[
R_{0303} = \frac{\sin^2(2t) \exp \left\{ (\sin y + 2y) \sin t \right\}}{4x^2 (2 + \cos y)}.
\]

(70) follows from (71) and (72) directly. The proof is finished. □

On the other hand, a direct calculation yields
\[
R \triangleq R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} \equiv 0.
\]

Therefore, we have

**Property 9** The Lorentzian metric (61) describes a time-periodic space-time, this space-time contains two kinds of singularities: the hypersurfaces \( t = k\pi, k\pi + \pi/2 \ (k \in \mathbb{N}) \), which are non-essential singularities and correspond to the event horizons, and \( x = 0 \), which is an essential (or, say, physical) singularity. ■
We now analyze the behavior of the singularities of the space-time characterized by (61) with (63).

By (64), we shall investigate the following cases: (a) $t = k\pi, k\pi + \pi/2$ ($k \in \mathbb{N}$); (b) $y \to \pm \infty$; (c) $x \to \pm \infty$; (d) $x \to 0$.

**Case a:** $t = k\pi, k\pi + \pi/2$ ($k \in \mathbb{N}$). According to the definition of the event horizon, the hypersurfaces $t = k\pi, k\pi + \pi/2$ ($k \in \mathbb{N}$) are the event horizons of the space-time described by (61) with (63).

**Case b:** $y \to \pm \infty$. Noting (64), in this case we may assume that $t \neq k\pi$ ($k \in \mathbb{N}$) (if $t = k\pi$, then the situation becomes trivial). Without loss of generality, we may assume that $\sin t > 0$. Therefore, it follows from (71) and (72) that, for any fixed $x \neq 0$ it holds that

$$R_{0202}, R_{0303} \to \infty \quad \text{as} \quad y \to +\infty$$

and

$$R_{0202}, R_{0303} \to 0 \quad \text{as} \quad y \to -\infty.$$  

(74) implies that $y = +\infty$ is also an essential singularity, while $y = -\infty$ is not because of (75).

**Case c:** $x \to \pm \infty$. By (63), in this case the space-time characterized by (61) reduces to a point.

**Case d:** $x \to 0$. Property 8 shows that $x = 0$ is a physical singularity. This is the biggest difference between the space-times presented in Subsections 3.1-3.2 and the one given this subsection. In order to illustrate its physical meaning, we take $(t, x, y, z)$ as the spherical coordinates $(t, r, \theta, \varphi)$ with $t \in \mathbb{R}$, $r \in [0, \infty)$, $\theta \in [0, 2\pi)$, $\varphi \in [-\pi/2, \pi/2)$. In the coordinates $(t, r, \theta, \varphi)$, the metric (61) with (63) describe a time-periodic space-time which possesses three kind of singularities:

(i) $t \neq k\pi$ ($k \in \mathbb{N}$): they are the event horizons;
(ii) $r \to +\infty$: the space-time degenerates to a point;
(iii) $r \to 0$: it is a physical singularity.

For the case (iii), in fact Property 8 shows that every point in the set

$$\mathcal{S}_B \triangleq \{(t, r, \theta, \varphi) \mid r = 0, t \neq k\pi, k\pi + \pi/2 \ (k \in \mathbb{N})\}$$

is a singular point. Noting (34) and (70), we name the set of singular points $\mathcal{S}_B$ as a quasi-black-hole. Property 8 also shows that the space-time (61) is not homogenous and not asymptotically flat. This space-time perhaps has some new applications in cosmology due to the recent WMAP data, since the recent WMAP data show that our Universe exists anisotropy (see [2]). This inhomogenous property of the new space-time (61) may provide a way to give an explanation of this phenomena.

We next investigate the physical behavior of the space-time (61).

Fixing $y$ and $z$, we get the induced metric

$$ds^2 = \hat{\eta}_{00}dt^2 + 2\hat{\eta}_{01}dtdx.$$  

(76)

Consider the null curves in the $(t, x)$-plan defined by

$$\hat{\eta}_{00}dt^2 + 2\hat{\eta}_{01}dtdx = 0.$$  

(77)

Noting (63) leads to

$$dt = 0 \quad \text{and} \quad \frac{dt}{dx} = -\frac{2\tan t}{x}.$$  

(78)

Let

$$\rho = 2\ln |x|.$$  

(79)

Then the second equation in (78) becomes

$$\frac{dt}{d\rho} = -\tan t.$$  

(80)

Thus, in the $(t, \rho)$-plan the null curves and light-cones are shown in Figure 1 in which $x$ should be replaced by $\rho$.

We now study the geometric behavior of the $t$-slices.

For any fixed $t \in \mathbb{R}$, the induced metric of the $t$-slice reads

$$ds^2 = -\frac{\sin^2 t}{x^4}[dy^2 + (2 + \cos y)^{-2}dz^2].$$  

(81)

When $t = k\pi$ ($k \in \mathbb{N}$), the metric (81) becomes

$$ds^2 = 0.$$
This implies that the $t$-slice reduces to a point. On the other hand, in this case the metric (61) becomes
\[ ds^2 = \frac{2}{(2 + \cos y)x^2} dt^2. \]
When $t \neq k\pi$ ($k \in \mathbb{N}$), (81) shows that the $t$-slice is a three-dimensional manifold with cone-like singularities at $x = \infty$ and $x = -\infty$, respectively. In particular, if we take $(t,x,y,z)$ as the spherical coordinates $(t,r,\theta,\varphi)$, then the induced metric (81) becomes
\[ ds^2 = -\frac{\sin^2 t}{r^4} [d\theta^2 + (2 + \cos \theta)^{-2} d\varphi^2]. \] (82)
In this case the $t$-slice is a three-dimensional cone-like manifold centered at $r = \infty$.

At the end of this subsection, we would like to emphasize that the space-time (61) possesses a physical singularity, i.e., $x = 0$ which is named as a quasi-black-hole in this paper.

4. Summary and discussion. In this paper we describe a new method to find exact solutions of the Einstein’s field equations (1). Using our method, we can construct many interesting exact solutions, in particular, the time-periodic solutions of the vacuum Einstein’s field equations. More precisely, we have constructed three kinds of new time-periodic solutions of the vacuum Einstein’s field equations: the regular time-periodic solution with vanishing Riemann curvature tensor, the regular time-periodic solution with finite Riemann curvature tensor and the time-periodic solution with physical singularities. We have also analyzed the singularities of these new time-periodic solutions and investigate some new physical phenomena enjoyed by these new space-times.

In particular, in the spherical coordinates $(t,r,\theta,\varphi)$ we construct a time-periodic space-time with essential singularities. This space-time possesses an interesting and important singularity which is named as a quasi-black-hole. This space-time is inhomogenous and not asymptotically flat and can perhaps be used to explain the phenomenon that our Universe exists anisotropy from the recent WMAP data (see [2]). We believe some applications of these new space-times in modern cosmology and general relativity can be expected.

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