6. Bochner formulas and basic vanishing theorems III [1]

1. Bochner formulas on Kähler manifolds. Let (M,ω) be a compact Käher manifold. If ∇ is the complexifiied Levi-Civita connection and $\nabla=\nabla'+\nabla''$. The formal adjoint operators of ∇', ∇'' and ∇ with respect to the Kähler metric are denoted by δ', $\delta''$and δ. For any $\varphi\in\Gamma(M,\Lambda^{p,q}T^*M)$ and $u\in\Gamma(M,\Lambda^{p-1,q}T^*M)$,

$$(\nabla' u,\varphi)=\int_M \nabla' u\wedge *\overline{\varphi}=-\int_M(-1)^{p+q-1}u\wedge\nabla' *\overline{\varphi}=-\int_M u\wedge *\overline{(*\nabla'' *\varphi)}=(u,-*\nabla'' *\varphi)$$

that is

$$\delta'=-*\nabla'' *$$

Similarly, we have

$$\delta''=-*\nabla' * \quad \text{and} \quad \delta=-*\nabla *$$

Now we consider the $(1,1)$ operator L defined by

$$Lu=\omega\wedge u$$

and its adjoint operator is Λ:

$$(u,\Lambda v)=(Lu,v)$$

Lemma 0.1. *We have*

$$[\overline{\partial}^*,L]=\sqrt{-1}\partial,\qquad [\partial^*,L]=-\sqrt{-1}\overline{\partial}$$
$$[\Lambda,\overline{\partial}]=-\sqrt{-1}\partial^*,\qquad [\Lambda,\partial]=\sqrt{-1}\overline{\partial}^*$$

and

$$[\delta'',L]=\sqrt{-1}\nabla',\qquad [\delta',L]=-\sqrt{-1}\nabla''$$
$$[\Lambda,\nabla'']=-\sqrt{-1}\delta',\qquad [\Lambda,\nabla']=\sqrt{-1}\delta''$$

Proof. We just prove the identities for ∇. We have $\omega=g_{i\overline{j}}dz^i\wedge d\overline{z}^j$. Then

$$L\varphi=\omega\wedge\varphi=g_{i\overline{j}}dz^i\wedge d\overline{z}^j\wedge dz^I\wedge d\overline{z}^J$$

Now we set

$$L\varphi=T_{i_0\cdots i_p\overline{j}_0\cdots\overline{j}_q}dz^{i_0}\wedge\cdots\wedge d\overline{z}^{i_p}\wedge d\overline{z}^{j_0}\wedge\cdots\wedge d\overline{z}^{j_q}$$

where

$$T_{i_0\cdots i_p\overline{j}_0\cdots\overline{j}_q}=(-1)^{p+s+t}\sum_{s=0}^{p}\sum_{t=0}^{q}g_{i_s\overline{j}_t}\varphi_{i_0\cdots\hat{i}_s\cdots i_p\overline{j}_0\cdots\hat{\overline{j}}_t\cdots\overline{j}_q}$$

On the other hand

$$\delta''(L\varphi)=-g^{k\overline{\ell}}i_{\overline{\partial}_\ell}\nabla_k(L\varphi)=(-1)^p g^{k\overline{\ell}}\nabla_k T_{i_0\cdots i_p\overline{\ell}\overline{j}_1\cdots\overline{j}_q}dz^{i_0}\wedge\cdots\wedge dz^{i_p}\wedge d\overline{z}^{j_1}\wedge\cdots\wedge d\overline{z}^{i_q}$$

that is

$$\delta''(L\varphi)=(-1)^{s+t}\sum_{s=0}^{p}\sum_{t=1}^{q}g^{k\overline{\ell}}g_{i_s\overline{j}_t}\varphi_{i_0\cdots\hat{i}_s i_p\overline{\ell}\overline{j}_1\cdots\hat{\overline{j}}_t\overline{j}_q}dz^{i_0}\wedge\cdots\wedge dz^{i_p}\wedge d\overline{z}^{j_1}\wedge\cdots\wedge d\overline{z}^{i_q}$$

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$$+(-1)^s \sum_{s=0}^{p} g^{k\bar{\ell}} g_{i_s\bar{\ell}} \nabla_k \varphi_{i_0\cdots\hat{i}_s\cdots i_p\bar{j}_1\cdots\bar{j}_q} dz^{i_0}\wedge\cdots\wedge dz^{i_p}\wedge d\bar{z}^{j_1}\wedge\cdots\wedge d\bar{z}^{i_q}$$

The second part is

$$(-1)^s \nabla_{i_s}\varphi_{i_0\cdots\hat{i}_s\cdots i_p} dz^{i_0}\wedge\cdots\wedge dz^{i_p}\wedge d\bar{z}^{j_1}\wedge\cdots\wedge d\bar{z}^{i_q} = \nabla'\varphi$$

Similarly

$$L(\delta''\varphi) = L((-1)^{p+1} g^{k\bar{\ell}} \nabla_k \varphi_{i_1\cdots i_p\bar{\ell}\bar{j}_1\cdots\bar{j}_{q-1}} dz^{i_1}\wedge\cdots dz^{i_p}\wedge d\bar{z}^{j_1}\cdots d\bar{z}^{j_{p-1}})$$

$$= (-1)^{s+t}\sum_{s=0}^{p}\sum_{t=1}^{q} g^{k\bar{\ell}} g_{i_s\bar{j}_t} \varphi_{i_0\cdots\hat{i}_s i_p\bar{\ell}\bar{j}_1\cdots\hat{\bar{j}}_t\bar{j}_q} dz^{i_0}\wedge\cdots\wedge dz^{i_p}\wedge d\bar{z}^{j_1}\wedge\cdots\wedge d\bar{z}^{i_q}$$

Then we have

$$[\delta'', L] = \sqrt{-1}\nabla'$$

Here we add the $\sqrt{-1}$ in the operator L. The other identities are the conjugate and adjoint of this. □

Corollary 0.2. *If (X,ω) is a Kähler manifold, then*

$$\bar{\partial}^*\partial + \partial\bar{\partial}^* = \partial^*\bar{\partial} + \bar{\partial}\partial^* = 0$$

and

$$\Delta_\partial = \Delta_{\bar{\partial}} = \frac{1}{2}\Delta_d$$

Theorem 0.3. *$\Delta_{\bar{\partial}}$, Δ_∂ and Δ_d commute with all operators $*$, ∂, $\bar{\partial}$, ∂^*, $\bar{\partial}^*$, L and Λ.*

Proof. By the Kähler identities above. □

Theorem 0.4. *We have*

$$\Delta'' = \Delta'$$

Proof. We have

$$\begin{aligned}
\Delta'' &= \delta''\nabla'' + \nabla''\delta'' \\
&= -\sqrt{-1}(\Lambda\nabla' - \nabla'\Lambda)\nabla'' - \sqrt{-1}\nabla''(\Lambda\nabla' - \nabla'\Lambda) \\
&= -\sqrt{-1}\Lambda\nabla'\nabla'' + \sqrt{-1}\nabla'(-\sqrt{-1}\delta'\nabla''\Lambda) - \sqrt{-1}(\sqrt{-1}\delta' + \Lambda\nabla'')\nabla' + \sqrt{-1}\nabla''\nabla'\Lambda \\
&= \Delta' + \sqrt{-1}[\nabla''\nabla' + \nabla''\nabla', \Lambda] \\
&= \Delta'
\end{aligned}$$

for

$$[\nabla''\nabla' + \nabla''\nabla', \Lambda] = 0(\mathit{Exercise!})$$

We should notice that ∇' and ∇'' are not the (Chern) connections on $T^{*1,0}M$ and $T^{*0,1}M$. □

2. Bochner formulas on holomorphic vector bundles over compact Kähler manifolds. If (E, h, ∇^E) is a holomorphic vector bundle over a compact Kähler manifold (M,ω) with Chern connection ∇^E, then we have the decomposition

$$\nabla^E = \nabla' + \nabla''$$

As above, the formal adjoint operators δ', δ'' could be defined similarly.

Lemma 0.5. *We have*

$$[\delta'', L] = \sqrt{-1}\nabla', \qquad [\delta', L] = -\sqrt{-1}\nabla''$$
$$[\Lambda, \nabla''] = -\sqrt{-1}\delta', \qquad [\Lambda, \nabla'] = \sqrt{-1}\delta''$$

Proof. The proof is the same as Lemma 0.1. □

Theorem 0.6. *We have*

$$\Delta'' = \Delta' + \sqrt{-1}[\Theta^E, \Lambda]$$

Proof. We have

$$\begin{aligned} \Delta'' &= \Delta' + \sqrt{-1}[\nabla''\nabla' + \nabla''\nabla', \Lambda] \\ &= \Delta' + \sqrt{-1}[\Theta^E, \Lambda] \end{aligned}$$

for $\Theta^E = \nabla''\nabla' + \nabla'\nabla''$. □

3. Basic vanishing Theorems.

Lemma 0.7. *Let (M, ω) be a compact Kähler manifold. Then*

$$[L, \Lambda]u = (p + q - n)u$$

if $u \in \Gamma(M, \Lambda^{p,q}T^*M)$.

Theorem 0.8 (Dolbeault).

$$H^{p,q}_{\overline{\partial}}(M, E) = H^q(M, \Omega^p(E))$$

Theorem 0.9 (Serre duality). *The bilinear pairing*

$$H^{p,q}(X, E) \times H^{n-p,n-q}(X, E^*) \longrightarrow \mathbb{C}, \qquad (s, t) \longrightarrow \int_M s \wedge t$$

is a non-degenerate, i.e.

$$H^{p,q}(X, E) \cong H^{n-p,n-q}(X, E^*)^*$$

Now we can give the statement the following theorem

Theorem 0.10 (Kodaira-Nakano). *Let L be a holomorphic line bundle on the compact complex manifold M.*
(a) If L is negative, then

$$H^{p,q}(M, L) = 0 \qquad\qquad for \quad p + q < n$$

(b) If E is positive, then

$$H^{p,q}(M, L) = 0 \qquad\qquad for \quad p + q > n$$

Proof. The part (b) is the Serre duality of (a), so we need to prove (a). Let L be a negative line bundle and choose a Kähler metric on M, such that the Kähler form satisfies

$$\omega = -\sqrt{-1}\Theta^L$$

Then for any $u \in \Gamma(M, \Lambda^{p,q}T^*M \otimes L)$ we have

$$\langle[\sqrt{-1}\Theta^L, \Lambda]u, u\rangle = -(p + q - n)|u|^2$$

That is

$$([\sqrt{-1}\Theta^L, \Lambda]u, u) = -(p+q-n)\|u\|^2$$

So if $u \in \mathcal{H}^{p,q}(X, L)$, then by the Bochner formula

$$(\Delta'' u, u) = (\Delta' u, u) + (\sqrt{-1}[\Theta^L, \Lambda]u, u)$$

we have

$$0 \geq ([\sqrt{-1}\Theta^E, \Lambda]u, u) = -(p+q-n)\|u\|^2$$

That is $u = 0$ if $p + q < n$. □

Corollary 0.11. *If L is negative, then*

$$H^q(M, \Omega^0(L)) = 0, \qquad for\ q \neq n$$

Corollary 0.12. *If $K_M^* \otimes L$ is positive, then*

$$H^q(M, \Omega^0(L)) = 0, \qquad for\ q \neq 0$$

Proof. For we have

$$H^q(M, \Omega^0(L)) = H^{0,q}(M, E) = H^{n,n-q}(M, L^*) = H^{n-q}(M, \Omega^n \otimes L^*) = H^{n-q}(M, K_M \otimes L^*)$$

Since $K_M^* \otimes L$ is positive, we get $K_M \otimes L^*$ is negative. □

5. Generalizations of the Nakano Vanishing Theorems.

Definition 0.13. Let M ba a compact complex manifold and L a hermitian holomorphic line bundle on M. E is said to be **k-positive**(resp. **k-negative**) at the point $x \in M$ if the Chern curvature $\sqrt{-1}\Theta^L$ is semi-positive(resp. semi-negative) and has at least $n-k$ positive (resp. negative) eigenvalues at x. A holomorphic line bundle on M is said to be k-positive (resp. negative) if it carries a hermitian metric that is k-positive (resp. negative) at all points of M.

Theorem 0.14 (Gibrau). *Let (M, ω) be a connected compact Kähler manifold and L a k-positive hermitian holomorphic line bundle on X. Then we have*

$$H^{p,q}(M, L) = 0 \quad for\ p+q > n+k$$

Proof. Let's consider a new Kähler metric on M given by

$$\omega_\epsilon = \epsilon\omega + \sqrt{-1}\Theta^E$$

Then we have

$$\sqrt{-1}\Theta^L = \sqrt{-1}\sum \gamma_j w_j \wedge \overline{w}_j$$

and

$$\omega_\epsilon = \sqrt{-1}\sum(\epsilon + \gamma_j) w_j \wedge \overline{w}_j$$

for some orthonormal basis $(w_1, \cdots, w_n)$(with respect to ω) of T^*M . Then we have

$$\omega_\epsilon = \sqrt{-1}\sum_j \eta_j \wedge \overline{\eta}_j$$

where

$$\eta_j = \sqrt{\epsilon + \gamma_j}\, w_j$$

then $(\eta_1, \cdots, \eta_n)$ is an orthonormal basis with respect to Kähler metric ω_ϵ. Then we have

$$\sqrt{-1}\Theta^L = \sqrt{-1}\sum_j \gamma_{j,\epsilon}\eta_j \wedge \overline{\eta}_j$$

with respect to the Kähler metric ω_ϵ where

$$\gamma_{j,\epsilon} = \frac{\gamma_j}{\epsilon + \gamma_j} \in [0, 1)$$

Now we denote the operators and inner product with respect to the new metric ω_ϵ by the index ϵ. Then we have

$$\langle[\sqrt{-1}\Theta^L, \Lambda_\epsilon]u, u\rangle_\epsilon \geq (\gamma_{1,\epsilon} + \cdots + \gamma_{q,\epsilon} - \gamma_{p+1,\epsilon} - \cdots - \gamma_{n,\epsilon})|u|^2$$

On the other hand we have

$$0 < \gamma_{k+1} \leq \gamma_j \quad \text{if } j \geq k+1$$

then

$$\gamma_{j,\epsilon} = \frac{1}{1+\epsilon/\gamma_j} \leq \frac{1}{1+1/\gamma_{k+1}} \geq 1 - \frac{\epsilon}{\gamma_{k+1}}$$

Then we have

$$\begin{aligned}\langle[\sqrt{-1}\Theta^L, \Lambda_\epsilon]u, u\rangle_\epsilon &\geq ((q-k)(1-\frac{\epsilon}{\gamma_{k+1}}) - (n-p))|u|^2 \\ &= (p+q-n-k-(q-k)\frac{\epsilon}{\gamma_{k+1}})|u|^2\end{aligned}$$

So if we choose ϵ such that

$$p+q-n-k-(q-k)\frac{\epsilon}{\gamma_{k+1}} > 0$$

that is

$$0 < \epsilon < \frac{p+q-n-k}{q-k}\min_{x\in M}\gamma_{k+1}(x)$$

then we have $u \in \mathcal{H}^{p,q}(M, L)$ then $u = 0$ if $p + q > n + k$. □

Corollary 0.15. *Let (M, ω) be a connected compact Kähler manifold and L a k-negative hermitian holomorphic line bundle on X. Then we have*

$$H^{p,q}(M, L) = 0 \quad \textit{for } p+q < n-k$$

5. Applications-Vanishing theorems on $\mathbb{P}^n$. On $\mathbb{P}^n$, we have

$$K_{\mathbb{P}^n} = \det T^{*0,1}\mathbb{P}^n = \Lambda^n T^{*0,1}\mathbb{P}^n = \mathcal{O}_{\mathbb{P}^n}(-n-1)$$

Theorem 0.16. *We have the following vanishing results for $H^{p,q}(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(r)) = 0$:*

(1) $p+q < n, r < 0$;

(2) $p+q > n, r > 0$;

(3) $p = 0, q \neq 0, r > -n-1$;

(4) $p = n, q \neq n$, $r < n+1$.

Proof. (1) and (2) is by vanishing theorems directly. (4) is the Serre dual of (3). For (3), we have

$$H^{0,q}(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(r)) = H^{n,n-q}(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-r)) = H^{n-q}(\mathbb{P}^n, \Omega^0(-r-n-1)) = H^{0,n-q}(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-n-1-r))$$

□

Corollary 0.17. *$H^{0,i}(\mathbb{P}^n, T^{1,0}\mathbb{P}^n) = 0$ for $i \neq 0, 1$.*

Proof. By Kodaira-Serre duality we have

$$H^{0,i}(\mathbb{P}^n, T^{1,0}\mathbb{P}^n) = H^{n-i}(\mathbb{P}^n, K_{\mathbb{P}^n} \otimes T^{*1,0}\mathbb{P}^n) = H^{n-i}(\mathbb{P}^n, \Omega^1(-n-1)) = 0$$

□

We know that $H^0(\mathbb{P}^1, T^{1,0}\mathbb{P}^1) \cong \mathbb{C}^3$.

Corollary 0.18. *$H^{0,1}(\mathbb{P}^1, T^{1,0}\mathbb{P}^1) = 0$.*

Proof. We need

$$H^{0,1}(\mathbb{P}^1, T^{1,0}\mathbb{P}^1) = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^n}(-4)) = 0$$

□

On $\mathbb{P}^1$, we get

$$H^0(\mathbb{P}^1, \mathbb{C}) = H^{1,1}(\mathbb{P}^1, \mathbb{C}) = \mathbb{C}, \quad H^0(\mathbb{P}^1, T^{1,0}\mathbb{P}^1) = \mathbb{C}^3, \quad H^{1,0}(\mathbb{P}^1, T^{1,0}\mathbb{P}^1) = \mathbb{C}$$

$$H^{0,1}(\mathbb{P}^1, \mathbb{C}) = H^{1,0}(\mathbb{P}^1, \mathbb{C}) = 0, \quad H^{0,1}(\mathbb{P}^1, T^{1,0}\mathbb{P}^1) = H^{1,1}(\mathbb{P}^1, T^{1,0}M) = 0$$

6. Exercise: If (M, ω) is a Kähler manifold, then

$$[\nabla''\nabla' + \nabla''\nabla', \Lambda] = 0$$

Hint: Use the second method in Lemma 0.1. The Key point is the Kähler condition. We will see that if M is non-Kähler, $\Delta'' \neq \Delta'$.