References:
1. Jean-Pierre Demailly: Complex analytic and algebraic geometry;
2. K.Kodaira, J.Morrow: Complex manifolds;
3. James D. Lewis: A survey of Hodge conjecture;

1. Holomorphic vector bundles

Definition 0.1. A complex manifold $M$ of complex dimension $n$ is a Hausdorff and second countable topological space, together with coordinate charts $\{(U_j, \phi_j)\}_{j \in J}$ where

1. $\{U_j\}_{j \in J}$ is an open cover of $M$;
2. $\phi_j : U_j \rightarrow V_j$ is a homeomorphism onto an open set $V_j \subset \mathbb{C}^n$;
3. the transition functions
   $$\phi_i \circ \phi_j^{-1} : h_j(U_j \cap U_i) \rightarrow h_i(U_i \cap U_j)$$
   are holomorphic (as functions between complex spaces) for all $i$ and $j$ whenever defined.

Example 0.2 (Complex projective spaces).

$$\mathbb{P}^n = \{ \text{lines in } \mathbb{C}^{n+1} \text{ through } 0 \} = \mathbb{C}^{n+1} \setminus \{0\}/\sim$$

where

$$[Z_0, \cdots, Z_n] \sim [Z_0', \cdots, Z_n']$$

if $Z_i = \lambda Z_i'$ for some $\lambda \in \mathbb{C}^n \setminus \{0\}$, $\forall i$. Define the coordinate charts $(U_i, \phi_i)$

$$U_i = \{ [Z_0, \cdots, Z_n] \mid Z_i \neq 0 \}$$

$$\phi_i : U_i \rightarrow \mathbb{C}^n, \quad \phi_i([Z_0, \cdots, Z_n]) = \left( \frac{Z_0}{Z_i}, \cdots, \frac{Z_{i-1}}{Z_i}, \frac{Z_{i+1}}{Z_i}, \cdots, \frac{Z_n}{Z_i} \right)$$

It is obvious that the transition functions ($i < j$)

$$\phi_i \circ \phi_j^{-1} : \phi_j(U_j \cap U_i) \rightarrow \phi_i(U_i \cap U_j)$$

$$\phi_i \circ \phi_j^{-1}(z_1, \cdots, z_n) = \phi_i([z_1, \cdots, z_j, 1, z_{j+1}, \cdots, z_n])$$

are holomorphic for $z_i \neq 0$.

Definition 0.3. Let $M$ be a complex manifold. A complex vector bundle of rank $r$ over $M$ is a smooth manifold $E$ together with:

1. there exists a family of $\{V_j\}$ of open sets covering $M$ indexed by $J$, and the $V_j$'s are called the coordinate neighborhoods;
(2) for each \( j \in J \), a diffeomorphism
\[
\varphi_j : \pi^{-1}(V_j) \longrightarrow V_j \times \mathbb{C}^r
\]
called the coordinate function (or trivialization) which is required to satisfy the following condition:
\[
\varphi_j : \pi^{-1}(x) \longrightarrow \{ x \} \times \mathbb{C}^r
\]
is a linear isomorphism.

(3) For each \( i, j \in J \), the map
\[
\varphi_{ij} = \varphi_i \circ \varphi_j^{-1} : (V_i \cap V_j) \times \mathbb{C}^r \longrightarrow (V_i \cap V_j) \times \mathbb{C}^r
\]
acts as a linear automorphisms on each fiber \( \{ x \} \times \mathbb{C}^r \). It can be written as
\[
\varphi_{ij}(x, v) = (x, g_{ij}(x) \cdot v), \quad (x, v) \in (V_i \cap V_j) \times \mathbb{C}^r
\]
where \( g_{ij} \in C^\infty(V_i \cap V_j, GL(r, \mathbb{C})) \) satisfies
\[
g_{ij} \circ g_{jk} = g_{ik} \text{ on } v_i \cap V_j \cap V_k \quad (0.1)
\]
and \( g_{ij} \)'s are called transition functions. \( E \) is called the bundle space or the total space, \( M \) is called the base space.

\( E \) is called a holomorphic vector bundle over \( M \) if \( \varphi_i \) are biholomorphic.

**Remark 0.4.** (1) If we replace the vector space \( \mathbb{C} \) by \( \mathbb{R} \), we get the definition of real vector bundles.

(2) It is obvious that given any \( \{ g_{ij} \} \) satisfy the relation 0.1 and 0.2, we can construct a complex vector bundle by
\[
E = \coprod_j (V_j \times \mathbb{C}^r) / \sim
\]
\((x_i, v_i) \sim (x_j, v_j) \) if and only if
\[
x_i = x_j = x, \quad v_j = g_{ij}(x) \cdot v_i
\]
where \((x_i, v_i) \in V_i \times \mathbb{C}^r\) and \((x_j, v_j) \in V_j \times \mathbb{C}^r\)

**Example 0.5 (Holomorphic tangent bundle and cotangent bundle of \( \mathbb{P}^n \)).**
\[
\pi : T\mathbb{P}^n \longrightarrow \mathbb{P}^n
\]
\[
\varphi_i : \pi^{-1}(U_i) \longrightarrow U_i \times \mathbb{C}^n
\]
If \( p \in U_i \) and \( v \in T_p\mathbb{P}^n \subset \pi^{-1}(U_i) \)
\[
v = \sum a_k \frac{\partial}{\partial z_k} \bigg|_p \in \pi^{-1}(U_i)
\]
where (Pay attention to the indices)
\[
z_k^{(i)} = \frac{Z_{k-1}}{Z_i}, k < i; \ z_{k-1}^{(i)} = \frac{Z_k}{Z_i}, k > i
\]
Then
\[ \varphi_i(v) = (p, a_1, \cdots, a_n) \in U_i \times \mathbb{C}^n \]
and
\[ \varphi_{ij} : (U_i \cap U_j) \times \mathbb{C}^n \rightarrow (U_i \cap U_j) \times \mathbb{C}^n \]
\[ \varphi_{ij}(p, a_1, \cdots, a_n) = \varphi_i \left( \sum a_k \frac{\partial}{\partial z_k^{(j)}}|_p \right) = (p, g_{ij} \cdot a) \]
where \( g_{ij} \) is the matrix:
\[ g_{ij} = \left( \frac{\partial z_i^{(i)}}{\partial z_k^{(j)}} \right) \in \mathcal{C}^\infty(U_i \cap U_j, GL(n, \mathbb{C})) \]

It is very easy to check the other conditions.

Similarly, the cotangent bundle \( T^*\mathbb{P}^n \) is given by the transition functions
\[ h_{ij} = t^{-1}g_{ij} \]

**Definition 0.6.** A section \( s \) of the vector bundle \( E \rightarrow M \) over \( U \subset M \) is a \( \mathcal{C}^\infty \)-map
\[ s : U \rightarrow E \]
such that \( s(x) \in E_x \) for every \( x \in U \), and the set of sections over \( U \) is denoted by \( \Gamma(U, E) \). A frame \( e \) for \( E \) over \( U \subset M \) is a collection \( e_1, \cdots, e_n \) of sections over \( U \) such that \( \{e_1(x), \cdots, e_n(x)\} \) is a basis for \( E_x \) for all \( x \in U \). A section \( s \in \Gamma(U, E) \) is said to be holomorphic if \( s : U \rightarrow E \) is a holomorphic map, a frame \( e = (e_1, \cdots, e_r) \) is called holomorphic if each \( e_i \) is holomorphic. In terms of a holomorphic frame \( e = (e_1, \cdots, e_r) \) any section
\[ s(x) = \sum_{i=1}^r s_i(x) \cdot e_i(x) \]
is holomorphic if and only if the functions \( s_i \) are holomorphic.

**Proposition 0.7.** If \( \varphi : E_U \rightarrow U \times \mathbb{C}^r \) is a trivialization of \( E \), then we can get a frame for \( E \) over \( U \) by
\[ e_i(x) = \varphi^{-1}(x, e_i) \]
where \( \{e_i\} \) is the canonical basis of \( \mathbb{C}^r \). Conversely, given any frame \( \{e_i\} \) over \( U \), we can get a trivialization
\[ \varphi : E_U \rightarrow U \times \mathbb{C}^r, \quad \varphi(\lambda) = (x, (\lambda_1, \cdots, \lambda_r)) \]
where \( \lambda = \sum_{i=1}^r \lambda_i e_i \in E_x \).

**Remark 0.8.** If \( s \in \Gamma(M, E) \), we will denote
\[ \varphi_i(s) := s^i = (s_1^i, \cdots, s_r^i)^t, \quad s = \sum_{k=1}^r s_k^i e_k^i \]
under the trivialization \( \varphi_i : E|_{V_i} \rightarrow V_i \times \mathbb{C}^r \) and basis \( e_k^i(x) = \varphi_i(x, e_k) \).
Proposition 0.9. If \( s \in \Gamma(M, E) \), then we have
\[
s_i = g_{ij} \cdot s_j \quad \text{on} \quad V_i \cap V_j
\]

Proof. If \( x \in V_i \cap V_j \),
\[
s(x) = \sum_{k=1}^{r} s_i^k(x) e_k(x) = \sum_{l=1}^{r} s_l^i(x) e_l(x)
\]
By the action of \( \varphi_i \) on both side,
\[
\varphi_i(s(x)) = (x, (s_i^1(x), \cdots, s_i^r(x))) = \sum_{l=1}^{r} s_l^i(x) \varphi_i(\varphi_j^{-1}(x, \epsilon_l)) = (x, g_{ij}(x) \cdot s_j)
\]
That is
\[
s_i = g_{ij} \cdot s_j \quad \text{on} \quad V_i \cap V_j \quad \square
\]

Example 0.10 (Holomorphic sections of \( TP^1 \)). If \( V \) is a holomorphic section of \( TP^1 \), then on the chart \( (U_0, z^{(0)}) \) it has the form
\[
V = V^{(0)} \frac{\partial}{\partial z^{(0)}}
\]
on the chart \( (U_1, z^{(1)}) \) it has the form
\[
V = V^{(1)} \frac{\partial}{\partial z^{(1)}}
\]
and on \( U_0 \cap U_1 \)
\[
V^{(0)} = g_{01} V^{(1)} = \frac{\partial z^{(0)}}{\partial z^{(1)}} V^{(1)}
\]
On the other hand on \( U_0 \cap U_1 \)
\[
z^{(0)} = \frac{Z_1}{Z_0} = \frac{1}{z^{(1)}}
\]
Now we set, on \( U_0 \cap U_1 \)
\[
V^{(1)} = \sum_{k=0}^{\infty} a_k (z^{(1)})^k
\]
than
\[
V^{(0)} = - \sum_{k=0}^{\infty} a_k (z^{(1)})^{k-2} = - \sum_{k=0}^{\infty} a_k (z^{(0)})^{2-k} = - a_0 (z^{(0)})^2 - a_1 z^{(0)} - a_2
\]
Hence there are three independent holomorphic vector fields, i.e.
\[
\dim_{\mathbb{C}} H^0(\mathbb{P}^1, TP^1) = 3
\]
In general
\[
\dim_{\mathbb{C}} H^0(\mathbb{P}^n, TP^n) = n^2 + 2n, \quad \dim_{\mathbb{C}} H^i(\mathbb{P}^n, TP^n) = 0, \ i > 0
\]
We will proof this later.

We shall consider differential forms on \( M \) with values in \( E \), in fact such forms are the sections of tensor product bundle \( \wedge^p T^* M \otimes E \). We shall write
\[
A^p(M, E) = \Gamma(M, \wedge^p T^* M \otimes E)
\]
Definition 0.11. A connection $\nabla$ on the vector bundle $E$ is a map $\nabla : A^p(E) \to A^{p+1}(M, E)$ such that:

$$\nabla(fs_1 + s_2) = df \otimes s_1 + (-1)^p f \wedge \nabla s_1 + \nabla s_2$$

where $f \in A^p(M, E)$ and $s_1, s_2 \in \Gamma(E)$.

Assume that $\varphi : E|_\Omega \to \Omega \times \mathbb{C}^r$ is a trivialization function of $E|_\Omega$, and $e = (e_1, \cdots, e_r)$ is the corresponding frame. If $s \in A^p(\Omega, E)$, then

$$s = \sum_{i=1}^r \sigma^i \otimes e_i, \quad \sigma^i \in A^p(\Omega, \mathbb{C})$$

By definition we have

$$\nabla s = \sum_{i=1}^r (d\sigma^i \otimes e_i + (-1)^p \sigma^i \wedge \nabla e_i)$$

For $\nabla e_j \in A^1(\Omega, E)$, we can write

$$\nabla e_j = \sum_{j=1}^r \theta^k_j \otimes e_k, \quad \theta^k_j \in A^1(\Omega, \mathbb{C})$$

thus

$$\nabla s = \sum_i (d\sigma^i + \sum_j \theta^k_j \wedge \sigma^k) \otimes e_i$$

Under the frame we write

$$\nabla s = d\sigma + \omega \wedge \sigma$$

where $\sigma = (\sigma^1, \cdots, \sigma^r)^t$ is the coordinate of $s$ under the trivialization $\varphi$, $\omega = (\theta^k_j) \in A^1(\Omega, \text{End}(\mathbb{C}^r))$ which is called the connection 1-form of $\nabla$ associated to the trivialization $\varphi$.

Proposition 0.12. If $\tilde{\varphi} : E|_\Omega \to \Omega \times \mathbb{C}^r$ is another trivialization and if we set

$$g = \tilde{\varphi} \circ \varphi^{-1} \in \Gamma(\Omega, \text{End}(\mathbb{C}^r))$$

then the coordinate $\tilde{\sigma} = (\tilde{\sigma}_1, \cdots, \tilde{\sigma}_r)^t$ and $\sigma = (\sigma_1, \cdots, \sigma_r)^t$ is related by

$$\tilde{\sigma} = g \cdot \sigma, \quad \tilde{e} = g^{-1} \cdot e$$

and the corresponding connection 1-forms are related by

$$\tilde{\omega} = dg^{-1} \cdot g + g^{-1} \cdot \omega \cdot g$$

Proof. We have

$$s = \tilde{\sigma} \cdot \tilde{e} = \sigma \cdot e$$

then

$$\tilde{\varphi}(s) = \tilde{\sigma} = \tilde{\varphi}(\sigma \cdot e) = g \cdot \sigma$$

For

$$\nabla \tilde{e} = \tilde{\omega} \otimes \tilde{e}$$

On the other hand

$$\nabla \tilde{e} = \nabla(g^{-1} \cdot e) = dg^{-1} \otimes e + g^{-1} \cdot \omega \otimes e = (dg^{-1} \cdot g + g^{-1} \cdot \omega g) \otimes \tilde{e}$$

that is

$$\tilde{\omega} = dg^{-1} \cdot g + g^{-1} \cdot \omega \cdot g$$

\[\square\]
Caution: We should take care about the product of the matrices in the formulas! The result of last proposition is very different if the matrices product is defined differently.

**Definition 0.13.** The curvature of the connection $\nabla$ is defined by

$$\nabla^2 : \Gamma(E) \longrightarrow \Omega^2(M, E)$$

**Proposition 0.14.** For any $f \in C^\infty(M)$ and $s \in \mathcal{A}^p(M, E)$, we have

$$\nabla^2(f \cdot s) = f \nabla^2 s$$

that is there exists a global 2-form $\Theta \in \mathcal{A}^2(M, \text{Hom}(E, E))$ such that

$$\nabla^2 s = \Theta \wedge s$$

where $\Theta$ is called the curvature tensor of $\nabla$ which is given by

$$\Theta = d\omega - \omega \wedge \omega, \quad \text{i.e.} \quad \Theta^i_j = d\theta^i_j - \theta^k_i \wedge \theta^j_k$$

with respect to some trivialization $\varphi$ and connection 1-form $\omega$.

**Proof.** Under the trivialization we have

$$\nabla^2 s = \nabla((d\sigma^i + \theta^i_j \wedge \sigma^j) \otimes e_i)$$

$$= (d\theta^i_j \wedge \sigma^j - \theta^k_i \wedge d\sigma^j) \otimes e_i + (-1)^{p+1}(d\sigma^i + \theta^i_j \wedge \sigma^j)\theta^k_i \otimes e_k$$

$$= (d\theta^i_j - \theta^k_i \wedge \theta^j_k) \wedge \sigma^j \otimes e_i$$

We have the curvature transformation formula

$$\Theta' = g \cdot \Theta \cdot g^{-1}$$

under the base changing $s' = g \cdot s$.

$$\Theta' s' = \nabla^2 s' = \nabla^2(gs) = g \nabla^2 s = g \Omega s = g \cdot \Theta \cdot g^{-1}s'$$

where we use the $C^\infty$-linear property of $\nabla^2$.

**Definition 0.15.** If $\xi \in \Gamma(M, TM)$, the covariant derivative of a section $s \in \Gamma(M, E)$ in the direction $\xi$ is defined to be the section

$$\nabla_\xi s := \nabla s \cdot \xi \in \Gamma(M, E)$$

**Proposition 0.16.** For all sections $s \in \Gamma(M, E)$ and all vector fields $\xi, \eta \in \Gamma(M, TM)$, we have

$$\nabla_\xi \nabla_\eta s - \nabla_\eta \nabla_\xi s - \nabla_{[\xi,\eta]} s = \Theta(\xi, \eta)s$$

**Proof.** Exercise.