Hyperbolic mean curvature flow: Evolution of plane curves

De-Xing Kong[*],   Kefeng Liu[†]   and   Zeng-Gui Wang[‡§]

Abstract

In this paper we investigate the one-dimensional hyperbolic mean curvature flow for closed plane curves. More precisely, we consider a family of closed curves $F : S^1 \times [0, T) \to \mathbb{R}^2$ which satisfies the following evolution equation

$$\frac{\partial^2 F}{\partial t^2}(u, t) = k(u, t)\vec{N}(u, t) - \nabla\rho(u, t), \quad \forall\, (u, t) \in S^1 \times [0, T)$$

with the initial data

$$F(u, 0) = F_0(u) \quad \text{and} \quad \frac{\partial F}{\partial t}(u, 0) = f(u)\vec{N_0},$$

where k is the mean curvature and $\vec{N}$ is the unit inner normal vector of the plane curve $F(u, t)$, $f(u)$ and $\vec{N_0}$ are the initial velocity and the unit inner normal vector of the initial convex closed curve F_0 respectively, and $\nabla\rho$ is given by

$$\nabla\rho \triangleq \left\langle \frac{\partial^2 F}{\partial s\partial t}, \frac{\partial F}{\partial t} \right\rangle \vec{T},$$

in which $\vec{T}$ stands for the unit tangent vector. The above problem is an initial value problem for a system of partial differential equations for F, it can be completely reduced to an initial value problem for a single partial differential equation for its support function. The latter equation is a hyperbolic Monge-Ampère equation. Based on this, we show that there exists a class of initial velocities such that the solution of the above initial value problem exists only at a finite time interval $[0, T_{\max})$ and when t goes to $T_{\max}$, the solution converges to a point. In the end, we discuss the close relationship between the hyperbolic mean curvature flow and the equations for the evolving relativistic string in the Minkowski space-time $\mathbb{R}^{1,1}$.

Key words and phrases: hyperbolic mean curvature flow, hyperbolic Monge-Ampère equation, closed plane curve, short-time existence.

2000 Mathematics Subject Classification: 58J45, 58J47.

---

[*]Center of Mathematical Sciences, Zhejiang University, Hangzhou 310027, China;

[†]Department of Mathematics, University of California at Los Angeles, CA 90095, USA;

[‡]Department of Mathematics, Shanghai Jiao Tong University, Shanghai 200240, China.

[§]Corresponding author.

1 Introduction

In this paper we study the closed convex evolving plane curves. More precisely, we consider the following initial value problem

$$\begin{cases} \dfrac{\partial^2 F}{\partial t^2}(u,t) = k(u,t)\vec{N}(u,t) - \triangledown\rho(u,t), & \forall\,(u,t) \in S^1 \times [0,T), \\[2mm] F(u,0) = F_0(u), \\[2mm] \dfrac{\partial F}{\partial t}(u,0) = f(u)\vec{N_0}, \end{cases} \tag{1.1}$$

where k is the mean curvature, $\vec{N}$ is the unit inner normal at $F(u,t)$, F_0 stands for a smooth strictly convex closed curve, $f(u)(\geq 0)$ and $\vec{N_0}$ are the initial velocity and inner normal vector of F_0, respectively, and with $\vec{T}$ denoting the unit tangent vector and s the arclength parameter, $\triangledown\rho$ is defined by

$$\triangledown\rho \triangleq \left\langle \frac{\partial^2 F}{\partial s\partial t}, \frac{\partial F}{\partial t} \right\rangle \vec{T}. \tag{1.2}$$

This system is an initial value problem for a system of partial differential equations for F, which can be completely reduced to an initial value problem for a single partial differential equation for its support function. The latter equation is a hyperbolic Monge-Ampère equation. Our first result is the following local existence theorem for the initial value problem (1.1).

Theorem 1.1 (Local existences and uniqueness) *Suppose that F_0 is a smooth strictly convex closed curve. Then there exist a positive T and a family of strictly convex closed curves $F(\cdot,t)$ with $t \in [0,T)$ such that $F(\cdot,t)$ satisfies (1.1), provided that $f(u)$ is a smooth function on S^1.*

Our second main result is the following theorem.

Theorem 1.2 *Suppose that F_0 is a smooth strictly convex closed curve. Then there exists a class of the initial velocities such that the solution of (1.1) with F_0 and f as initial curve and initial velocity of the initial curve, respectively, exists only at a finite time interval $[0, T_{\max})$. Moreover, when $t \to T_{\max}$, the solution $F(\cdot,t)$ converges to a point.*

After rescaling as Gage and Hamilton did, we can see that the limiting solution will be a circle. We will also introduce hyperbolic mean curvature flow with dissipative terms. A close relation between our hyperbolic mean curvature flow and the string evolving in the Minkowski space-time will be derived in the last section of the paper.

For reader's convenience, we briefly discuss some history of parabolic and hyperbolic mean curvature flows.

The parabolic theory for the evolving of plane curves, which in its simplest form is based on the curve shortening equation

$$v = k \qquad (1.3)$$

relating the normal velocity v and the curvature k, has been extremely successful in providing geometers with great insight. For example, Gage and Halmilton [6] proved that, when the curve is strictly convex, the deformation decreases the isoperimetric ratio, and furthermore if it shrinks to a point p, the normalized curves, obtained by "blowing up" the curves at p so that its enclosed areas is π, must tend to the unit circle in a certain sense. Grayson [7] generalized this result and showed that a smooth embedded plane curve first becomes convex and then shrinks to a point in a finite time. These results can be applied to many physical problems such as crystal growth, computer vision and image processing. One of the important applications of mean curvature flow is that Huisken and Ilmanen developed a theory of weak solutions of the inverse mean curvature flow and used it to prove successfully the Riemannian Penrose inequality which plays an important role in general relativity (see [13]).

However, to our knowledge, there is very few hyperbolic versions of mean curvature flow. Melting crystals of helium exhibits a phenomenon generally not found in other materials: oscillations of the solid-liquid interface in which atoms of the solid move only when they melt and enter the liquid (see [7] and references therein). Gurtin and Podio-Guidugli [7] developed a hyperbolic theory for the evolution of plane curves. Rostein, Brandon and Novick-Cohen [23] studied a hyperbolic theory by the mean curvature flow equation

$$v_t + \psi v = k, \qquad (1.4)$$

where v_t is the normal acceleration of the interface, ψ is a constant. A crystalline algorithm was developed for the motion of closed polygonal curves.

The hyperbolic version of mean curvature flow is important in both mathematics and applications, and has attracted many mathematicians to study it. He, Kong and Liu [10] introduced hyperbolic mean curvature flow from geometric point of view. Let $\mathcal{M}$ be a Riemannian manifold and $X(\cdot, t) : \mathcal{M} \to \mathbb{R}^{n+1}$ be a smooth map. When X is an isometric immersion, the Laplacian of F is given by $\triangle X = H\vec{N}$, where H is the mean curvature (i.e., the trace of the second fundamental form) and $\vec{N}$ is the unit inner normal vector. The hyperbolic mean curvature flow is the following partial differential equation of second order

$$\frac{\partial^2}{\partial t^2}X(u,t) = \triangle X \quad \text{or} \quad \frac{\partial^2}{\partial t^2}X(u,t) = H(u,t)\vec{N}(u,t), \quad \forall\, u \in \mathcal{M}, \quad \forall\, t > 0. \qquad (1.5)$$

$X = X(u, t)$ is called a solution of the hyperbolic mean curvature flow if it satisfies the equation (1.5). He, Kong and Liu in [10] proved that the corresponding system of partial differential equations are strictly hyperbolic, and based on this, they also showed that this flow admits a unique short-time smooth solution and possesses the nonlinear stability defined on the Euclidean space with dimension larger than 4. Moreover, the nonlinear wave equations satisfied by curvatures are also derived in [10], these equations will play an important role in future study. The hyperbolic mean curvature flow was considered as one of the general hyperbolic geometric flows introduced by Kong and Liu, see [20] for more discussions for related hyperbolic flows and their applications to geometry and Einstein equations.

Recently, Lefloch and Smoczyk [18] studied the following geometric evolution equation of hyperbolic type which governs the evolution of a hypersurface moving in the direction of its mean curvature vector

$$\begin{cases} \dfrac{\partial^2}{\partial t^2} X = eH(u,t)\vec{N} - \bigtriangledown e, \\[2mm] X(u,0) = X_0, \\[2mm] \left(\dfrac{\partial X}{\partial t} \right)^{\vec{T}_0}_{t=0} = 0, \end{cases} \tag{1.6}$$

where $\vec{T}_0$ stands for the unit tangential vector of the initial hypersurface X_0, $e \triangleq \dfrac{1}{2} \left(\left| \dfrac{d}{dt} X \right|^2 + n \right)$ is the local energy density and $\bigtriangledown e \triangleq \bigtriangledown^i e_i$, in which $e_i = \dfrac{\partial e}{\partial x^i}$. This flow stems from a geometrically natural action containing kinetic and internal energy terms. They have shown that the normal hyperbolic mean curvature flow will blow up in finite time. In the case of graphs, they introduce a concept of weak solution suitably restricted by an entropy inequality and proved that the classical solution is unique in the larger class of entropy solutions. In the special case of one-dimensional graphs, a global-in-time existence result is established. Moreover, an existence theorem has been established under the assumption that the BV norm of initial data is small.

The paper is organized as follows. In Section 2, a hyperbolic Monge-Ampère equation will be derived and a theorem on local existence and uniqueness of the solution, i.e., Theorem 1.1 will be proved. In Section 3, an example is given and then some properties of the evolving curve have been established. The main result — Theorem 1.2 will be proved in Section 4. In section 5, we consider the normal hyperbolic mean curvature flow with the dissipative term and get the hyperbolic equation of S and k respectively. Section 6 is devoted to illustrating the relations between the hyperbolic mean curvature flow and the evolution equations for the relativistic string in the Minkowski space $\mathbb{R}^{1,1}$.

2 Hyperbolic Monge-Ampère equation

Roughly speaking, an evolving curve is a smooth family of curves $u \mapsto F(u, t)$, where $u \in S^1$ and $t \in [0, T)$, in which T is called the duration of F. For a given curve $F(\cdot, t)$, the underlying physics must be independent of the choice of the parameter u, and hence can involve F only through intrinsic quantities such as curvature, normal acceleration and normal velocity, which are independent of parametrization. On the other hand, this invariance allows us to use any convenient parametrization. The following notion is needed in our study.

Definition 2.1 *A curve $F : S^1 \times [0, T) \to \mathbb{R}^2$ evolves normally if*

$$\left\langle \frac{\partial F}{\partial t}, \frac{\partial F}{\partial u} \right\rangle = 0 \tag{2.1}$$

for all $(u, t) \in S^1 \times [0, T)$.

Definition 2.1 can be found in [3] and [18]. In this paper, we shall restrict our attention to the parametrization (2.1). Such a parametrization significantly simplifies the analysis. The normally evolving curve was first investigated by Angenent and Gurtin [3] and then further studied by Lefloch and Smoczyk [18]. The following result is important, it shows that within a large class of time-dependent curves there is no essential loss of generality in limiting attention to curves that evolves normally.

Lemma A *If the evolving curve $\mathscr{C}$ is closed, then there is a parameter change ϕ for $\mathscr{C}$ such that $\mathscr{C} \circ \phi$ is a normally evolving curve.*

This important lemma was proved in [3]. In fact, for the initial value problem (1.1), the initial velocity field is normal to the curve, it can be proved that this property is preserved during the evolution, that is to say, the flow (1.1) is automatically a normal flow[1]

$$\begin{cases} \dfrac{\partial F}{\partial t} = \sigma(u, t)\vec{N}, \\ F(u, 0) = F_0(u), \end{cases} \tag{2.2}$$

where $\sigma(u, t) = f(u) + \int_0^t k(u, \xi)d\xi$, hence we have

$$\frac{\partial \sigma}{\partial t} = k(u, t), \quad \sigma \frac{\partial \sigma}{\partial s} = \left\langle \frac{\partial^2 F}{\partial s \partial t}, \frac{\partial F}{\partial t} \right\rangle. \tag{2.3}$$

Here we denote by $s = s(\cdot, t)$ the arclength parameter of the curve $F(\cdot, t) : S^1 \to \mathbb{R}^2$. The operator $\partial/\partial s$ is given in terms of u by

$$\frac{\partial}{\partial s} = \frac{1}{v} \frac{\partial}{\partial u},$$

---

[1] Although the partial differential equation in (2.2) only contains the first order derivative of F with respect to t, i.e., F_t, it is non-local partial differential equation, and it is not easier to handle than the second order partial differential equation in (1.1).

where

$$v = \sqrt{(\partial x/\partial u)^2 + (\partial y/\partial u)^2} = |\partial F/\partial u|.$$

By Frenet formula,

$$\frac{\partial \vec{T}}{\partial s} = k\vec{N}, \quad \frac{\partial \vec{N}}{\partial s} = -k\vec{T}.$$

Then $\{\vec{T}, \vec{N}\}$ is an orthogonal basis of $\mathbb{R}^2$. Let us denote θ to be the unit outer normal angle for a convex closed curve $F : S^1 \to \mathbb{R}^2$. Hence,

$$\vec{N} = (-\cos\theta, -\sin\theta), \quad \vec{T} = (-\sin\theta, \cos\theta),$$

and by Frenet formula, we have

$$\frac{\partial \theta}{\partial s} = k.$$

Furthermore,

$$\frac{\partial N}{\partial t} = -\frac{\partial \theta}{\partial t}\vec{T}, \quad \frac{\partial T}{\partial t} = \frac{\partial \theta}{\partial t}\vec{N}.$$

Using the previous definition, we have

$$\frac{\partial^2}{\partial t \partial s} = k\sigma \frac{\partial}{\partial s} + \frac{\partial^2}{\partial s \partial t},$$

and observing

$$\vec{T}(u, t) = \frac{\partial F}{\partial s}(u, t),$$

we deduce

$$\frac{\partial \vec{T}}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\partial F}{\partial s}\right) = \frac{\partial \sigma}{\partial s}\vec{N}$$

and

$$\frac{\partial \vec{N}}{\partial t} = -\frac{\partial \sigma}{\partial s}\vec{T},$$

hence,

$$\frac{\partial \theta}{\partial t} = \frac{\partial \sigma}{\partial s}.$$

Suppose that $F(u, t) : S^1 \times [0, T) \to \mathbb{R}^2$ is a family of convex curves satisfying the curve shortening flow (1.1). Let us use the normal angle to parameterize each convex curve $F(\cdot, t)$, i.e., set

$$\widetilde{F}(\theta, \tau) = F(u(\theta, \tau), t(\theta, \tau)),$$

where $t(\theta, \tau) = \tau$. Here, $\vec{N}$ and $\vec{T}$ are independent of the parameter τ, which can be proved as follows:

$$0 = \frac{\partial \theta}{\partial \tau} = \frac{\partial \theta}{\partial u}\frac{\partial u}{\partial \tau} + \frac{\partial \theta}{\partial t},$$

then

$$\frac{\partial \theta}{\partial t} = -\frac{\partial \theta}{\partial u}\frac{\partial u}{\partial \tau} = -\frac{\partial \theta}{\partial s}\frac{\partial s}{\partial u}\frac{\partial u}{\partial \tau} = -kv\frac{\partial u}{\partial \tau}.$$

Hence,

$$\frac{\partial \vec{T}}{\partial \tau} = \frac{\partial \vec{T}}{\partial t} + \frac{\partial \vec{T}}{\partial u}\frac{\partial u}{\partial \tau} = \frac{\partial \theta}{\partial t}\vec{N} + \frac{\partial \vec{T}}{\partial s}\frac{\partial s}{\partial u}\frac{\partial u}{\partial \tau} = \left(\frac{\partial \theta}{\partial t} + kv\frac{\partial u}{\partial \tau}\right)\vec{N} = 0$$

and

$$\frac{\partial \vec{N}}{\partial \tau} = \frac{\partial \vec{N}}{\partial t} + \frac{\partial \vec{N}}{\partial u}\frac{\partial u}{\partial \tau} = -\frac{\partial \theta}{\partial t}\vec{T} + \frac{\partial \vec{N}}{\partial s}\frac{\partial s}{\partial u}\frac{\partial u}{\partial \tau} = -\left(\frac{\partial \theta}{\partial t} + kv\frac{\partial u}{\partial \tau}\right)\vec{T} = 0.$$

By the chain rule,

$$\frac{\partial \widetilde{F}}{\partial \tau} = \frac{\partial F}{\partial u}\frac{\partial u}{\partial \tau} + \frac{\partial F}{\partial t}$$

and

$$\frac{\partial^2 \widetilde{F}}{\partial \tau^2} = \frac{\partial F}{\partial u}\frac{\partial^2 u}{\partial \tau^2} + \frac{\partial^2 F}{\partial u^2}\left(\frac{\partial u}{\partial \tau}\right)^2 + 2\frac{\partial^2 F}{\partial u\partial t}\frac{\partial u}{\partial \tau} + \frac{\partial^2 F}{\partial t^2}.$$

The support function of F is given by

$$S(\theta, \tau) = \langle \widetilde{F}(\theta, \tau), -\vec{N}\rangle,$$

$$= \langle \widetilde{F}(\theta, \tau), (\cos\theta, \sin\theta)\rangle$$

$$= x(\theta, \tau)\cos\theta + y(\theta, \tau)\sin\theta.$$

Its derivative satisfies

$$S_\theta(\theta, \tau) = -x(\theta, \tau)\sin\theta + y(\theta, \tau)\cos\theta + \langle \widetilde{F}_\theta(\theta, \tau), (\cos\theta, \sin\theta)\rangle$$

$$= -x(\theta, \tau)\sin\theta + y(\theta, \tau)\cos\theta$$

$$= \langle \tilde{F}(\theta, \tau), \vec{T}\rangle,$$

where

$$\langle \widetilde{F}_\theta(\theta, \tau), (\cos\theta, \sin\theta)\rangle = 0,$$

namely, the tangent vector is orthogonal to the unit normal vector. And then the curve can be represented by the support function

$$\begin{cases} x = S\cos\theta - S_\theta \sin\theta, \\ y = S\sin\theta + S_\theta \cos\theta. \end{cases} \tag{2.4}$$

Thus all geometric quantities of the curve can be represented by the support function. In particular, the curvature can be written as

$$k = \frac{1}{S_{\theta\theta} + S}.$$

In fact, by the definition of the support function,

$$S_{\theta\theta} + S = -x_\theta \sin\theta + y_\theta \cos\theta - x\cos\theta - y\sin\theta + x\cos\theta + y\sin\theta$$

$$= \left\langle \frac{\partial \widetilde{F}}{\partial \theta}, \vec{T}\right\rangle = \left\langle \frac{\partial \widetilde{F}}{\partial s}\frac{\partial s}{\partial \theta}, \vec{T}\right\rangle = \frac{1}{k}.$$

We know that the support function

$$S(\theta, \tau) = \left\langle \widetilde{F}(\theta, \tau), -\vec{N} \right\rangle$$

satisfies

$$S_\tau = \left\langle \frac{\partial \widetilde{F}}{\partial \tau}, -\vec{N} \right\rangle + \left\langle \widetilde{F}(\theta, \tau), \frac{\partial \vec{N}}{\partial \tau} \right\rangle$$

$$= \left\langle \frac{\partial F}{\partial u} \frac{\partial u}{\partial \tau} + \frac{\partial F}{\partial t}, -\vec{N} \right\rangle$$

$$= \langle \frac{\partial F}{\partial t}, -\vec{N} \rangle = -\tilde{\sigma}(\theta, \tau),$$

where

$$\left\langle \widetilde{F}(\theta, \tau), \frac{\partial \vec{N}}{\partial \tau} \right\rangle = 0$$

is obtained by $\dfrac{\partial \vec{N}}{\partial \tau} = 0$. Moreover,

$$S_{\tau\tau} = \left\langle \frac{\partial^2 \widetilde{F}}{\partial \tau^2}, -\vec{N} \right\rangle + \left\langle \frac{\partial \widetilde{F}}{\partial \tau}, -\frac{\partial \vec{N}}{\partial \tau} \right\rangle$$

$$= \left\langle \frac{\partial F}{\partial u} \frac{\partial^2 u}{\partial \tau^2} + \frac{\partial^2 F}{\partial u^2} \left(\frac{\partial u}{\partial \tau} \right)^2 + 2 \frac{\partial^2 F}{\partial u \partial t} \frac{\partial u}{\partial \tau} + \frac{\partial^2 F}{\partial t^2}, -\vec{N} \right\rangle$$

$$= \left\langle \frac{\partial^2 F}{\partial u^2} \left(\frac{\partial u}{\partial \tau} \right)^2 + 2 \frac{\partial^2 F}{\partial u \partial t} \frac{\partial u}{\partial \tau} + \frac{\partial^2 F}{\partial t^2}, -\vec{N} \right\rangle$$

$$= \left\langle \frac{\partial^2 F}{\partial u^2} \left(\frac{\partial u}{\partial \tau} \right)^2 + \frac{\partial^2 F}{\partial u \partial t} \frac{\partial u}{\partial \tau}, -\vec{N} \right\rangle + \left\langle \frac{\partial^2 F}{\partial u \partial t} \frac{\partial u}{\partial \tau} + \frac{\partial^2 F}{\partial t^2}, -\vec{N} \right\rangle$$

$$= \left\langle \left(\frac{\partial F}{\partial u} \right)_\tau, -\vec{N} \right\rangle \frac{\partial u}{\partial \tau} + \left\langle \frac{\partial^2 F}{\partial u \partial t} \frac{\partial u}{\partial \tau} + \frac{\partial^2 F}{\partial t^2}, -\vec{N} \right\rangle$$

$$= \left\langle \frac{\partial^2 F}{\partial u \partial t} \frac{\partial u}{\partial \tau}, -\vec{N} \right\rangle - k.$$

In terms of the normal evolving curve, we have

$$\left\langle \frac{\partial F}{\partial t}, \vec{T} \right\rangle \equiv 0 \quad \text{for all } t \in [0, T).$$

By the formula

$$S_\tau = \left\langle \frac{\partial F}{\partial t}, -\vec{N} \right\rangle,$$

we get

$$S_{\theta\tau} = \left\langle \frac{\partial^2 F}{\partial u \partial t} \frac{\partial u}{\partial \theta}, -\vec{N} \right\rangle + \left\langle \frac{\partial F}{\partial t}, -\frac{\partial \vec{N}}{\partial \theta} \right\rangle$$

$$= \left\langle \frac{\partial^2 F}{\partial u \partial t} \frac{\partial u}{\partial \theta}, -\vec{N} \right\rangle + \left\langle \frac{\partial F}{\partial t}, \vec{T} \right\rangle$$

$$= \left\langle \frac{\partial^2 F}{\partial u \partial t} \frac{\partial u}{\partial \theta}, -\vec{N} \right\rangle = \frac{1}{\partial \theta / \partial u} \left\langle \frac{\partial^2 F}{\partial u \partial t}, -\vec{N} \right\rangle$$

$$= \frac{1}{(\partial \theta / \partial s)(\partial s / \partial u)} \left\langle \frac{\partial^2 F}{\partial u \partial t}, -\vec{N} \right\rangle = \frac{1}{kv} \left\langle \frac{\partial^2 F}{\partial u \partial t}, -\vec{N} \right\rangle.$$

Noting that
$$S_\theta = \langle \widetilde{F}, \vec{T} \rangle,$$

we have
$$S_{\tau\theta} = \left\langle \frac{\partial \widetilde{F}}{\partial \tau}, \vec{T} \right\rangle + \left\langle \widetilde{F}, \frac{\partial \vec{T}}{\partial \tau} \right\rangle = \left\langle \frac{\partial \widetilde{F}}{\partial \tau}, \vec{T} \right\rangle$$
$$= \left\langle \frac{\partial F}{\partial u} \frac{\partial u}{\partial \tau} + \frac{\partial F}{\partial t}, \vec{T} \right\rangle$$
$$= \left\langle \frac{\partial F}{\partial u} \frac{\partial u}{\partial \tau}, \vec{T} \right\rangle = v \frac{\partial u}{\partial \tau}.$$

Hence, the support function S satisfies
$$S_{\tau\tau} = \left\langle \frac{\partial^2 F}{\partial u \partial t} \frac{\partial u}{\partial \tau}, -\vec{N} \right\rangle - k$$
$$= kv \frac{\partial u}{\partial \tau} S_{\theta\tau} - k \tag{2.5}$$
$$= kS_{\theta\tau}^2 - k = (S_{\theta\tau}^2 - 1)k,$$

namely,
$$S_{\tau\tau} = \frac{S_{\theta\tau}^2 - 1}{S_{\theta\theta} + S}, \quad \forall\, (\theta, \tau) \in S^1 \times [0, T). \tag{2.6}$$

Then, it follows from (1.1) that
$$
\begin{cases}
SS_{\tau\tau} + S_{\tau\tau} S_{\theta\theta} - S_{\theta\tau}^2 + 1 = 0, \\
S(\theta, 0) = h(\theta), \\
S_\tau(\theta, 0) = -\widetilde{f}(\theta),
\end{cases} \tag{2.7}
$$

where h is the support function of F_0, and $\widetilde{f}$ is the initial velocity of the initial curve F_0.

For an unknown function $z = z(\theta, \tau)$ defined for $(\theta, \tau) \in \mathbb{R}^2$, the corresponding *Monge-Ampère equation* reads
$$A + Bz_{\tau\tau} + Cz_{\tau\theta} + Dz_{\theta\theta} + E(z_{\tau\tau} z_{\theta\theta} - z_{\theta\tau}^2) = 0, \tag{2.8}$$

the coefficients A, B, C, D and E depends on τ, θ, S, S_τ, S_θ. We say that the equation (2.8) is *τ-hyperbolic* for S, if
$$\triangle^2(\tau, \theta, z, z_\tau, z_\theta) \triangleq C^2 - 4BD + 4AE > 0$$

and
$$z_{\theta\theta} + B(\tau, \theta, z, z_\tau, z_\theta) \neq 0.$$

We state the initial values $z(0, \theta) = z_0(\theta)$, $z_\tau(0, \theta) = z_1(\theta)$ for the unknown function on the $\theta \in [0, 2\pi]$. Moreover, we require the following *τ-hyperbolicity* condition:
$$\triangle^2\,(0, \theta, z_0, z_1, z_0') = (C^2 - 4BD + 4A)|_{t=0} > 0,$$
$$z_0'' + B(0, \theta, z_0, z_1, z_0') \neq 0,$$

9

in which $z_0' = \dfrac{dz_0}{d\theta}$ and $z_0'' = \dfrac{d^2 z_0}{d\theta^2}$.

It is easy to see that the equation (2.7) is a hyperbolic Monge-Amère equation, in which

$$A = 1, \ B = S, \ C = D = 0, \ E = 1.$$

In fact,

$$\triangle^2(\tau, \theta, S, S_\tau, S_\theta) = C^2 - 4BD + 4A$$

$$= 0^2 - 4S \times 0 + 4 \times 1 = 4 > 0$$

and

$$S_{\theta\theta} + B(\tau, \theta, S, S_\tau, S_\theta) = S_{\theta\theta} + S = \frac{1}{k} \neq 0.$$

Furthermore, if we assume that $h(\theta)$ is third and $\widetilde{f}(\theta)$ is twice continuous by differentiable on the real axis, then the initial conditions satisfies

$$\triangle^2(0, \theta, h, -\widetilde{f}, h_\theta) = 4 > 0$$

and

$$h_{\theta\theta} + B(0, \theta, h, -\widetilde{f}, h_\theta) = h_{\theta\theta} + h = \frac{1}{k_0} \neq 0.$$

This implies that the equation (2.7) is a *hyperbolic Monge-Ampère equation* on S. By the standard theory of hyperbolic equations (e.g., [9], [11], [16], [19] or [26]), we have

Theorem 2.1 (Local existences and uniqueness) *Suppose that F_0 is a smooth strictly convex closed curve. Then there exist a positive T and a family of strictly convex closed curves $F(\cdot, t)$ (in which $t \in [0, T)$) such that $F(\cdot, t)$ satisfies (1.1) (or 2.7), provided that $f(u)$ is a smooth function on S^1.*

Theorem 2.1 is nothing but Theorem 1.1, which is one of main results in this paper.

3   An example and some propositions

In this section, we will give an example to understand further the normal hyperbolic mean curvature flow. For simplicity we replace τ by t.

Example 3.1 *Consider $F(\cdot, t)$ to be a family of round circles with the radius $R(t)$ centered at the origin. The support function and the curvature are given by $S(\theta, t) = R(t)$ and $k(\theta, t) = 1/R(t)$, respectively. Substituting these into (1.1) gives*

$$\begin{cases} R_{tt} = -\dfrac{1}{R} \ , \\ R(0) = r_0 > 0, \quad R_t(0) = r_1. \end{cases} \tag{3.1}$$

For this initial value problem, we have the following lemma which is given in [10].

Lemma 3.1 *For arbitrary initial data $r_0 > 0$, if the initial velocity $r_1 \leqslant 0$, the solution $R = R(t)$ decreases and shrinks to a point at time T^* (where $T^* \leq \sqrt{\frac{\pi}{2}} r_0$, and the equality holds if and only if $r_1 = 0$); if the initial velocity is positive, the solution R increases first and then decreases and shrinks to a point in a finite time.*

Remark 3.1 *In fact, this phenomena can also be interpreted by physical principles. From (2.7), we can see that the direction of acceleration is always the same as the inner normal vector. Thus, if $R_t(0) \leqslant 0$, i.e., the initial velocity is in accordance with the unit inner normal vector, then evolving circle will shrink to a point at a finite time; if $r_t(0) > 0$, i.e., the initial velocity is in accordance with the outer unit normal vector, then the evolving sphere will expand first and then shrink to a point at a finite time. In the equation (2.7), $S_t(\theta, 0) = -\widetilde{f} \leq 0$, i.e., we assume the initial velocity always accords with the initial unit inner normal, hence only the first phenomena will happen.*

In what follows, we shall establish some properties enjoyed by the hyperbolic mean curvature flow.

Consider the following general second-order operator

$$L[w] \triangleq a w_{\theta\theta} + 2b w_{\theta t} + c w_{tt} + d w_\theta + e w_t, \tag{3.2}$$

where a, b, and c are twice continuously differentiable and d and e are continuously differentiable functions of θ and t. The operator L is said to be *hyperbolic* at a point (θ, t), if

$$b^2 - ac > 0.$$

It is hyperbolic in a domain D if it is hyperbolic at each point of D, and *uniformly hyperbolic* in D if there is a constant μ such that

$$b^2 - ac \geq \mu > 0$$

in D.

We suppose that w and the conormal derivative

$$\frac{\partial w}{\partial \nu} \triangleq -b \frac{\partial w}{\partial \theta} - c \frac{\partial w}{\partial t}$$

are given at $t = 0$.

We associate with L the **adjoint operator**

$$L^*[\omega] \triangleq (a\omega)_{\theta\theta} + 2(b\omega)_{\theta t} + (c\omega)_{tt} - (d\omega)_\theta - (e\omega)_t$$

$$= a\omega_{\theta\theta} + 2b\omega_{\theta t} + c\omega_{tt} + (2a_\theta + 2b_t - d)\omega_x + (2b_\theta + 2c_t - 3)\omega_t$$

$$+ (a_{\theta\theta} + 2b_{\theta t} + c_{tt} - d_\theta - e_t)\omega.$$

11

Now we shall show that for any hyperbolic operator L there is a function l which satisfies the following condition

$$\begin{cases} 2\sqrt{b^2 - ac}\left[l_t - \dfrac{1}{c}(\sqrt{b^2 - ac} - b)l_\theta\right] + lK_+ \geq 0, \\[2mm] 2\sqrt{b^2 - ac}\left[l_t + \dfrac{1}{c}(\sqrt{b^2 - ac} - b)l_\theta\right] + lK_- \geq 0, \\[2mm] (L^* + g)[\omega] \geq 0, \end{cases}$$

in a sufficiently small strip $0 \leq t \leq t_0$, where

$$K_+ \triangleq K_+(\theta, t) \triangleq (\sqrt{b^2 - ac})_\theta + \frac{b}{c}(\sqrt{b^2 - ac})_\theta + \frac{1}{c}(b_\theta + c_t - e))\sqrt{b^2 - ac}$$
$$+ \left[-\frac{1}{2c}(b^2 - ac)_\theta + a_\theta + b_t - d - \frac{b}{c}(b_\theta + c_t - e)\right],$$

and

$$K_- \triangleq K_-(\theta, t) \triangleq (\sqrt{b^2 - ac})_\theta + \frac{b}{c}(\sqrt{b^2 - ac})_\theta + \frac{1}{c}(b_\theta + c_t - e))\sqrt{b^2 - ac}$$
$$- \left[-\frac{1}{2c}(b^2 - ac)_\theta + a_\theta + b_t - d - \frac{b}{c}(b_\theta + c_t - e)\right].$$

We let

$$l(\theta, t) \triangleq 1 + \alpha t - \beta t^2. \tag{3.3}$$

A computation shows that the above condition are

$$\begin{cases} 2\sqrt{b^2 - ac}(\alpha - 2\beta t) + (1 + \alpha t - \beta t^2)K_+ \geq 0, \\[2mm] 2\sqrt{b^2 - ac}(\alpha - 2\beta t) + (1 + \alpha t - \beta t^2)K_- \geq 0, \\[2mm] -2c\beta + (2b_{\theta + 2c_t - e})(\alpha - 2\beta t) \\[2mm] \qquad + (a_{\theta\theta} + 2b_{\theta t} + c_{tt} - d_\theta - e_t + g)(1 + \alpha t - \beta t^2) \geq 0. \end{cases} \tag{3.4}$$

Since all the coefficients and their derivatives which appear in the above expressions are supposed bounded and since $-c$ and $\sqrt{b^2 - ac}$ have positive lower bounds, the first two expressions above are positive at $t = 0$ if α is chosen sufficiently large. The third expression are positive at $t = 0$ if β is chosen sufficiently large. With these values of α and β there is a number $t_0 > 0$ such $l(\theta, t) > 0$ and all the inequalities hold for $0 \leq t \leq t_0$.

With l given by (3.3), the condition on the conormal derivative becomes

$$\frac{\partial \omega}{\partial \nu} + (b_\theta + c_t - e + c\alpha)\omega \leq 0 \quad \text{at} \quad t = 0.$$

If we select a constant M so large that

$$M \geq -[b_\theta + c_t - e + c\alpha] \quad \text{on} \ \ \Gamma_0. \tag{3.5}$$

Then we obtain the following maximum principle for a strip adjacent to the θ-axis(see [22]).

Lemma 3.2 *Suppose that the coefficients of the operator L given by (3.2) are bounded and have bounded first and second derivatives. Let D be an admissible domain. If t_0 and M are selected in accordance with (3.4) and (3.5), then any function w which satisfies*

$$\begin{cases} (L+g)[w] \geq 0 & \text{in } D, \\[2mm] \dfrac{\partial w}{\partial \nu} - Mw \leq 0 & \text{on } \Gamma_0, \\[2mm] w \leq 0 & \text{on } \Gamma_0, \end{cases}$$

also satisfies $w \leq 0$ in the part of D which lies in the strip $0 \leq t \leq t_0$. The constants t_0 and M depend only on lower bounds for $-c$ and $\sqrt{b^2 - ac}$ and on bounds for the coefficients of L and their derivatives.

Proposition 3.1 (Containment principle) *Let F_1 and $F_2 : S^1 \times [0,T) \to \mathbb{R}^2$ be two convex solutions of (1.1) (or (2.7)). Suppose that $F_2(\cdot,0)$ lies in the domain enclosed by $F_1(\cdot,0)$, and $f_2(u) \geq f_1(u)$. Then $F_2(\cdot,t)$ is contained in the domain enclosed by $F_1(\cdot,t)$ for all $t \in [0,T)$.*

Proof. Let $S_1(\theta,t)$ and $S_2(\theta,t)$ be the support functions of $F_1(\cdot,t)$ and $F_2(\cdot,t)$ respectively. Then S_1 and S_2 satisfies the same equation (2.7) with $S_2(\theta,0) \leq S_1(\theta,0)$ and $S_{2t}(\theta,0) \leq S_{1t}(\theta,0)$ for $\theta \in S^1$.

Let

$$w(\theta,t) \triangleq S_2(\theta,t) - S_1(\theta,t),$$

then w satisfies the following equation

$$\begin{cases} w_{tt} = [1 - S_{1\theta t} S_{2\theta t}]\, k_1 k_2 w_{\theta\theta} + (k_1 S_{1\theta t} + k_2 S_{2\theta t})w_{\theta t} + [1 - S_1(\theta,t) S_2(\theta,t)]\, k_1 k_2 w, \\[2mm] w_t(\theta,0) = f_1(\theta) - f_2(\theta) = w_1(\theta), \\[2mm] w(\theta,0) = h_2(\theta) - h_1(\theta) = w_0(\theta). \end{cases} \qquad (3.6)$$

Define the operator L by

$$L[w] \triangleq [1 - S_{1\theta t} S_{2\theta t}]\, k_1 k_2 w_{\theta\theta} + (k_1 S_{1\theta t} + k_2 S_{2\theta t})w_{\theta t} - w_{tt}. \qquad (3.7)$$

In view of (3.7), we know that

$$a = [1 - S_1(\theta,t) S_2(\theta,t)]\, k_1 k_2, \quad b = \frac{1}{2}(k_1 S_{1\theta t} + k_2 S_{2\theta t}) \quad \text{and} \quad c = -1$$

are twice continuously differentiable and $d = e = 0$ are continuously differentiable functions of θ and t. By a direct computation, we get

$$\begin{aligned} b^2 - ac &= \frac{1}{4}(k_1 S_{1\theta t} + k_2 S_{2\theta t})^2 - [1 - S_1(\theta,t) S_2(\theta,t)]\, k_1 k_2 \cdot (-1) \\[2mm] &= \frac{1}{4}(k_1 S_{1\theta t} - k_2 S_{2\theta t})^2 + k_1 k_2 \geq \min_{\theta \in [0,2\pi]} \{k_{10}(\theta) k_{20}(\theta)\} > 0. \end{aligned}$$

Hence the operator L is defined by (3.7) is hyperbolic in $S^1 \times [0, T)$ and it is *uniformly hyperbolic* in $S^1 \times [0, T)$, since there is a constant $\mu = \min_{\theta \in [0, 2\pi]}\{k_{10}(\theta)k_{20}(\theta)\}$ such that $b^2 - ac \geq \mu = \min_{\theta \in [0, 2\pi]}\{k_{10}(\theta)k_{20}(\theta)\} > 0$ in $S^1 \times [0, T)$.

By Lemma 3.2, we deduce that

$$S_2(\theta, t) \leq S_1(\theta, t)$$

for all $t \in [0, T)$. Thus, the proof is completed. ■

Proposition 3.2 (Preserving convexity) *Let k_0 be the mean curvature of F_0 and let $\delta = \min_{\theta \in [0, 2\pi]}\{k_0(\theta)\} > 0$. Then for a C^4-solution S of (2.7), one has*

$$k(\theta, t) \geq \delta$$

for $t \in [0, T_{\max})$, where $[0, T_{\max})$ is the maximal time interval for the solution $F(\cdot, t)$ of (1.1).

Proof. Since the initial curve is strictly convex, by Theorem 2.1, we know that the solution of (2.7) remains strictly convex on some short time interval $[0, T)$ with some $T \leq T_{\max}$ and its support function satisfies

$$S_{tt} = (S_{\theta t}^2 - 1)k = \frac{S_{\theta t}^2 - 1}{S_{\theta\theta} + S}, \quad \forall (\theta, t) \in S^1 \times [0, T).$$

By taking derivative in time t, we have

$$\begin{aligned}
k_t &= \left(\frac{1}{S_{\theta\theta} + S}\right)_t \\
&= -\frac{1}{(S_{\theta\theta} + S)^2}[S_{\theta\theta t} + S_t] \\
&= -k^2[S_{\theta\theta t} + S_t] \\
&= k^2[\tilde{\sigma}_{\theta\theta} + \tilde{\sigma}],
\end{aligned}$$

hence

$$S_{\theta\theta t} + S_t = -(S_{\theta\theta} + S)^2 k_t = -\frac{1}{k^2}k_t,$$

$$S_{\theta\theta\theta t} + S_{\theta t} = \left(-\frac{1}{k^2}k_t\right)_\theta = \frac{2}{k^3}k_t k_\theta - \frac{1}{k^2}k_{\theta t}$$

14

and

$$k_{tt} = \frac{2}{(S_{\theta\theta}+S)^3}[S_{\theta\theta t}+S_t]^2 - \frac{1}{(S_{\theta\theta}+S)^2}[S_{\theta\theta tt}+S_{tt}]$$

$$=2k^3\left(-\frac{1}{k^2}k_t\right)^2 - k^2\left\{[(S_{\theta t}^2-1)k]_{\theta\theta}+(S_{\theta t}^2-1)k\right\}$$

$$=\frac{2}{k}k_t^2 - k^2\left\{(S_{\theta t}^2-1)_{\theta\theta}k + 2(S_{\theta t}^2-1)_\theta k_\theta + (S_{\theta t}^2-1)k_{\theta\theta} + (S_{\theta t}^2-1)k\right\}$$

$$=\frac{2}{k}k_t^2 - k^2(S_{\theta t}^2-1)(k_{\theta\theta}+k) - k^2\left\{2(S_{\theta t}S_{\theta\theta t})_\theta k + 4S_{\theta t}S_{\theta\theta t}k_\theta\right\}$$

$$=\frac{2}{k}k_t^2 - k^2(S_{\theta t}^2-1)(k_{\theta\theta}+k) - k^2\left\{2(S_{\theta\theta t}^2+S_{\theta t}S_{\theta\theta\theta t})k + 4S_{\theta t}(S_{\theta\theta}+S-S)_t k_\theta\right\}$$

$$=\frac{2}{k}k_t^2 - k^2(S_{\theta t}^2-1)(k_{\theta\theta}+k) - k^2\{2[(S_{\theta\theta t}+S_t)^2 - 2S_{\theta\theta t}S_t - S_t^2 + S_{\theta t}(S_{\theta\theta}+S)_{\theta t} - S_{\theta t}^2]k$$

$$- 4S_{\theta t}\frac{1}{k^2}k_t k_\theta - 4S_{\theta t}S_t k_\theta\}$$

$$=\frac{2}{k}k_t^2 - k^2(S_{\theta t}^2-1)(k_{\theta\theta}+k) - k^2\left\{2\left[(S_{\theta\theta t}+S_t)^2 - 2(S_{\theta\theta t}+S_t)S_t + S_t^2\right.\right.$$

$$\left.\left. -S_{\theta t}^2 + S_{\theta t}\left(\frac{1}{k}\right)_{\theta t}\right]k - 4S_{\theta t}\frac{1}{k^2}k_t k_\theta - 4S_{\theta t}S_t k_\theta\right\}$$

$$=\frac{2}{k}k_t^2 - k^2(S_{\theta t}^2-1)(k_{\theta\theta}+k) - k^2\left\{2\left[\left(-\frac{1}{k^2}k_t\right)^2 - 2\left(-\frac{1}{k^2}k_t\right)S_t + S_t^2 - S_{\theta t}^2\right.\right.$$

$$\left.\left. +S_{\theta t}\left(\frac{2}{k^3}k_t k_\theta - \frac{1}{k^2}k_{\theta t}\right)\right]k - 4S_{\theta t}\frac{1}{k^2}k_t k_\theta - 4S_{\theta t}S_t k_\theta\right\}$$

$$=\frac{2}{k}k_t^2 - k^2(S_{\theta t}^2-1)(k_{\theta\theta}+k) - k^2\left\{\frac{2}{k^3}k_t^2 + \frac{4}{k}S_t k_t + 2S_t^2 - 2S_{\theta t}^2 k\right.$$

$$\left. +\frac{4}{k^2}S_{\theta t}k_t k_\theta - \frac{2}{k}S_{\theta t}k_{\theta t} - 4S_{\theta t}\frac{1}{k^2}k_t k_\theta - 4S_{\theta t}S_t k_\theta\right\}$$

$$=k^2(1-S_{\theta t}^2)k_{\theta\theta} + 2kS_{\theta t}k_{\theta t} + 4k^2 S_{\theta t}S_t k_\theta - 4kS_t k_t + (S_{\theta t}^2+1-2S_t^2)k^3.$$

Thus, the curvature k satisfies the following equation

$$k_{tt} = k^2(1-S_{\theta t}^2)k_{\theta\theta} + 2kS_{\theta t}k_{\theta t} + 4k^2 S_{\theta t}S_t k_\theta - 4kS_t k_t + (S_{\theta t}^2+1-2S_t^2)k^3. \qquad (3.8)$$

Define the operator L as follows

$$L[k] \triangleq k^2(1-S_{\theta t}^2)k_{\theta\theta} + 2kS_{\theta t}k_{\theta t} - k_{tt} + 4k^2 S_{\theta t}S_t k_\theta - 4kS_t k_t. \qquad (3.9)$$

In terms of (3.6),

$$a = k^2(1-S_{\theta t}^2), \quad b = kS_{\theta t} \quad \text{and} \quad c = -1$$

are twice continuously differentiable and

$$d = 4k^2 S_{\theta t}S_t \quad \text{and} \quad e = -4kS_t$$

are continuously differentiable functions of θ and t. By the direct computation,

$$b^2 - ac = (kS_{\theta t})^2 - k^2(1-S_{\theta t}^2)\cdot(-1) = k^2 > 0,$$

hence the operator L is defined by (3.9) is hyperbolic in the domain $S^1 \times [0, T)$.

We consider the problem of determining a function $k(\theta, t)$ which satisfies

$$
\begin{cases}
(L + \widetilde{h})[k] \triangleq k^2(1 - S_{\theta t}^2)k_{\theta\theta} + 2kS_{\theta t}k_{\theta t} + 4k^2 S_{\theta t}S_\theta k_\theta \\
\qquad\qquad - 4kS_t k_t + k^2(S_{\theta t}^2 + 1 - 2S_t^2)k = 0 \quad \text{in } S^1 \times [0, \widetilde{T}), \\
k(\theta, 0) = k_0(\theta) \quad \text{on } \Gamma_0, \\
0 \leq \dfrac{\partial k}{\partial \nu} \triangleq -bk_\theta - ck_t = \gamma(\theta) \quad \text{on } \Gamma_0.
\end{cases}
\tag{3.10}
$$

We can find that a function $\widetilde{k}(\theta, t) = \min_{\theta \in [0, 2\pi]} \{k_0(\theta)\} = \delta$ which satisfies

$$
\begin{cases}
(L + \widetilde{h})[\widetilde{k}] = 0 \quad \text{in } S^1 \times [0, \widetilde{T}), \\
\widetilde{k}(\theta, 0) \leq k_0(\theta) \quad \text{on } \Gamma_0, \\
\dfrac{\partial \widetilde{k}}{\partial \nu} - M\widetilde{k} \leq \gamma(\theta) - Mk_0(\theta) \quad \text{on } \Gamma_0,
\end{cases}
\tag{3.11}
$$

where Γ_0 is the initial domain, and M is the constant given by (3.5). If k and if $\widetilde{k}$ satisfies (3.11), we may apply Lemma 3.2 to $\widetilde{k} - k$ and conclude that

$$
\widetilde{k} \leq k(\theta, t) \quad \text{in } S^1 \times [0, t_0)
$$

with $t_0 \leq T$. This implies that the solution $F(\cdot, t)$ is convex on $[0, T_{\max})$. Moreover, the curvature of $F(\cdot, t)$ has a uniform positive lower bound $\min_{\theta \in S^1}\{k_0(\theta)\}$ on $S^1 \times [0, T_{\max})$. Thus, the proof is completed. ∎

The following lemmas will be useful later.

Lemma 3.3 *The arclength $\mathscr{L}(t)$ of the closed curve $F(\cdot, t)$ satisfies*

$$
\frac{d\mathscr{L}(t)}{dt} = -\int_0^{2\pi} \widetilde{\sigma}(\theta, t)d\theta
$$

and

$$
\frac{d^2\mathscr{L}(t)}{dt^2} = \int_0^{2\pi} \left[\left(\frac{\partial\widetilde{\sigma}}{\partial\theta}\right)^2 k - k\right] d\theta.
$$

Proof. By the definitino of arclength,

$$
\mathscr{L}(t) = \int_0^{2\pi} v(\theta, t)d\theta.
$$

By a direct calculation,

$$
\begin{aligned}
\frac{d\mathscr{L}(t)}{dt} &= \int_0^{2\pi} \frac{\partial v}{\partial t}d\theta \\
&= \int_0^{2\pi} -\widetilde{\sigma}(\theta, t)k(\theta, t)v(\theta, t)d\theta \\
&= -\int_0^{2\pi} \widetilde{\sigma}(\theta, t)d\theta,
\end{aligned}
$$

16

and then

$$\frac{d^2\mathscr{L}(t)}{dt^2} = -\int_0^{2\pi} \frac{\partial}{\partial t}\left(\widetilde{\sigma}(\theta,t)\right)d\theta$$

$$= \int_0^{2\pi} \left(S_{\theta t}^2 - 1\right)kd\theta$$

$$= \int_0^{2\pi} \left[\left(\frac{\partial\widetilde{\sigma}}{\partial\theta}\right)^2 k - k\right]d\theta.$$

Thus, the proof is completed.   ∎

Lemma 3.4 *The area $\mathscr{A}(t)$ enclosed by the closed curve $F(\cdot,t)$ satisfies*

$$\frac{d\mathscr{A}(t)}{dt} = \int_0^{2\pi} \frac{S_t}{k}d\theta,$$

$$\frac{d^2\mathscr{A}(t)}{dt^2} = -2\pi + \int_0^{2\pi} S_t^2 d\theta,$$

$$\frac{d^3\mathscr{A}(t)}{dt^3} = \int_0^{2\pi} (S_{\theta t}^2 - 1)kS_t d\theta.$$

Proof. The area $\mathscr{A}(t)$ enclosed by the convex curve is defined by

$$\mathscr{A}(t) = -\frac{1}{2}\int_0^{2\pi} \left\langle \widetilde{F}(\theta,t), v(\theta,t)N(\theta,t)\right\rangle d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \frac{S}{k}d\theta.$$

Then,

$$\frac{d\mathscr{A}(t)}{dt} = \frac{1}{2}\int_0^{2\pi} \left[\frac{S_t}{k} - \frac{S}{k^2}k_t\right]d\theta,$$

$$= \frac{1}{2}\int_0^{2\pi} \left[\frac{S_t}{k} + S(S_{\theta\theta t} + S_t)\right]d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \left[\frac{S_t}{k} + (S_{\theta\theta} + S)S_t\right]d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \left[\frac{S_t}{k} + \frac{S_t}{k}\right]d\theta$$

$$= \int_0^{2\pi} \frac{S_t}{k}d\theta,$$

and then,

$$\frac{d^2\mathscr{A}(t)}{dt^2} = \int_0^{2\pi} \frac{\partial}{\partial t}\left(\frac{S_t}{k}d\theta\right)$$

$$= \int_0^{2\pi} \left[\frac{S_{tt}}{k} - \frac{S_t}{k^2}k_t\right]d\theta$$

$$= \int_0^{2\pi} \left[S_{\theta t}^2 - 1 + S_t(S_{\theta\theta t} + S_t)\right]d\theta$$

$$= \int_0^{2\pi} (S_{\theta t}^2 - 1 + S_t^2 - S_{\theta t}^2)d\theta$$

$$= -2\pi + \int_0^{2\pi} S_t^2 d\theta.$$

17

Finally,

$$\frac{d^3\mathscr{A}(t)}{dt^3} = 2\int_0^{2\pi} S_t S_{tt} d\theta$$
$$= 2\int_0^{2\pi} S_t(S_{\theta t}^2 - 1)k d\theta.$$

Thus, the proof is completed. ■

Lemma 3.5 *Under the Proposition 3.2, the following inequality holds*

$$\left(\frac{\partial\widetilde{\sigma}}{\partial\theta}\right)^2 - 1 < 0 \quad \text{for all} \quad t \in [0, T_{\max}).$$

Proof. Since

$$\frac{\partial\sigma}{\partial t} = k > 0 \quad \text{for all} \quad t \in [0, T_{\max}),$$

then

$$\sigma(u, t) > \sigma(u, 0) \quad \text{for all} \quad t \in (0, T_{\max}),$$

i.e.,

$$\widetilde{\sigma}(\theta, t) = \sigma(u, t) > \sigma(u, 0) = \widetilde{\sigma}(\theta, 0) \quad \text{for all} \quad t \in (0, T_{\max}),$$

hence,

$$\frac{\partial\widetilde{\sigma}}{\partial t} > 0 \quad \text{for all} \quad t \in [0, T_{\max}).$$

On the other hand, by the chain rule,

$$\frac{\partial\sigma}{\partial t} = \frac{\partial\widetilde{\sigma}}{\partial\theta}\frac{\partial\theta}{\partial t} + \frac{\partial\widetilde{\sigma}}{\partial t}$$
$$= \frac{\partial\widetilde{\sigma}}{\partial\theta}\frac{\partial\sigma}{\partial s} + \frac{\partial\widetilde{\sigma}}{\partial t}$$
$$= \frac{\partial\widetilde{\sigma}}{\partial\theta}\frac{\partial\widetilde{\sigma}}{\partial\theta}\frac{\partial\theta}{\partial s} + \frac{\partial\widetilde{\sigma}}{\partial t},$$

hence,

$$\frac{\partial\widetilde{\sigma}}{\partial t} = \left[1 - \left(\frac{\partial\widetilde{\sigma}}{\partial\theta}\right)^2\right]k > 0,$$

therefore,

$$\left(\frac{\partial\widetilde{\sigma}}{\partial\theta}\right)^2 - 1 < 0 \quad \text{for all} \quad t \in [0, T_{\max}).$$

Thus, the proof is completed. ■

4   Shrinking to a point — Proof of Theorem 1.2

From Example 3.1, we know that, when $t \to T^*$, the solution $F(\cdot, t)$ converges to a point. In this section, we will show that such phenomenon actually holds for all the evolutions of strictly convex closed curves with suitably initial velocities. In other words, we will prove the following theorem which implies Theorem 1.2.

Theorem 4.1 *Suppose that F_0 is a smooth strictly convex closed curve and the initial velocity f satisfies all assumptions mentioned in Section 3. Then the solution of (1.1) with F_0 and f as initial curve and initial velocity of the initial curve, respectively, exists only at a finite time interval $[0, T_{\max})$. Moreover, when $t \to T_{\max}$, the solution $F(\cdot, t)$ converges to a point.*

Proof. Let $[0, T_{\max})$ be the maximal time interval for the solution $F(\cdot, t)$ of (1.1) with F_0 and f as initial curve and initial velocity of the initial curve, respectively. We divide the proof into four steps.

Step 1. Preserving convexity

By Proposition 3.2, we know that the solution $F(\cdot, t)$ remains strictly convex on $[0, T_{\max})$ and the curvature of $F(\cdot, t)$ has a uniform positive lower bound $\min_{\theta \in S^1} \{k_0(\theta)\}$ on $S^1 \times [0, T_{\max})$.

Step 2. Finite time existence

Enclose the initial curve F_0 by a large circle γ_0 with the normal initial velocity equals to the normal initial velocity of the initial curve F_0. Then evolve γ_0 by the flow (1.1) to get a solution $\gamma(\cdot, t)$. By Example 3.1, we know that the solution $\gamma(\cdot, t)$ exists only at a finite time interval $[0, T^*)$, and $\gamma(\cdot, t)$ converges to a point when $t \to T^*(< +\infty)$. Applying Proposition 3.1 (containment principle), we deduce that $F(\cdot, t)$ is always enclosed by $\gamma(\cdot, t)$ for all $t \in [0, T^*)$. Thus we conclude that the solution $F(\cdot, t)$ must become singular at some time $T_{\max} \le T^*$.

Step 3. Hausdorff convergence

Note that $F(\cdot, t_2)$ is enclosed by $F(\cdot, t)$ whenever $t_2 > t_1$ by the evolution equation (1.1) (or (2.7)). In other words, $F(\cdot, t)$ is shrinking. Let us recall the following classical result in convex geometry (see [24]).

Blaschke Selection Theorem *Let K_j be a sequence of convex sets which are contained in a bounded set. Then there exists a subsequence K_{j_k} and a convex set K such that K_{j_k} converges to K in the Hausdorff metric.*

Thus, by using this result, we can directly deduce that $F(\cdot, t)$ converges to a (maybe degenerate and nonsmooth) weakly convex curve $F(\cdot, T_{\max})$ in the Haufdorff metric.

Step 4. Shrinking to a point

Noting that
$$\left(\frac{\partial \tilde{\sigma}}{\partial \theta} \right)^2 - 1 < 0 \quad \text{for all} \quad t \in [0, T_{\max}),$$

we obtain from Lemma 3.2 that

$$\frac{d^2 \mathscr{L}(t)}{dt^2} < 0, \qquad \frac{d\mathscr{L}(t)}{dt} < 0 \quad \text{for all} \quad t \in [0, T_{\max}).$$

Hence, there exists a finite time T_0 such that $\mathscr{L}(T_0) = 0$, provided that $T_0 \leq T_{\max}$. There will be two cases:

Case I: $T_0 \leq T_{\max}$. On the one hand, there exists a unique classical solution of the Cauchy problem (1.1) on the interval $[0, T_0)$; on the other hand, when t goes to T_0, $\mathscr{L}(t)$ tends to zero, i.e.,

$$\mathscr{L}(t) \longrightarrow 0 \quad as \quad t \nearrow T_0.$$

This implies that the curvature k goes to infinity when t tends to T_0, and then the solution will blow up at the time T_0. Therefore, by the definition of $T_{\max}$, we have

$$T_0 = T_{\max}.$$

That is, when $t \nearrow T_{\max}$, the solution $F(\cdot, t)$ converges to a point.

Case II: $T_0 > T_{\max}$. In the present situation,

$$\mathscr{L}(T_{\max}) > 0.$$

Then $F(\cdot, T_{\max})$ must be a line segment. It is clear that $\min_{\theta \in S^1}\{k(\theta, t)\}$ tends to zero, when t goes to T_{max}. But in Step 1, we have shown that the curvature of $F(\cdot, t)$ has a uniform positive lower bound. Hence, Case II is not possible. Thus, the proof of Theorem 4.1 is completed. ■

5 Normal hyperbolic mean curvature flow with dissipation

In this section, we consider the normal hyperbolic mean curvature flow with dissipative term

$$\begin{cases} \dfrac{\partial^2 F}{\partial t^2}(u, t) = k(u, t)\vec{N}(u, t) - \nabla\rho(u, t) + d\dfrac{\partial F}{\partial t}, \quad \forall \, (u, t) \in S^1 \times [0, T), \\[2mm] F(u, 0) = F_0(u), \\[2mm] \dfrac{\partial F}{\partial t}(u, 0) = f(u)\vec{N_0}, \end{cases} \qquad (5.1)$$

where k is the mean curvature, $\vec{N}$ is the inner unit normal at $F(u, t)$, F_0 stands for the initial strictly convex smooth closed curve, $f(u)$ and $\vec{N_0}$ are the initial velocity and inner normal vector of F_0, respectively, d is a negative constant and $\nabla\rho$ is denoted by

$$\nabla\rho \triangleq \left\langle \frac{\partial^2 F}{\partial s \partial t}, \frac{\partial F}{\partial t} \right\rangle \vec{T}.$$

For (5.1), if we assume
$$\left\langle \frac{\partial F}{\partial t}, \vec{N} \right\rangle = v,$$
then we obtain from (5.1) that
$$v_t = k + dv, \tag{5.2}$$
which is the same as the equation (1.4).

Similar to Section 2, we can also derive a hyperbolic Monge-Ampère equation.

In fact, let us use the normal angle to parameterize each convex curve $F(\cdot, t)$, that is, set
$$\widetilde{F}(\theta, \tau) = F(u(\theta, \tau), t(\theta, \tau)),$$
where $t(\theta, \tau) = \tau$. Here $\vec{N}$, $\vec{T}$ and θ are independent of the parameter τ. Let
$$\frac{\partial F}{\partial t} = a(u, t)\vec{N} + b(u, t)\vec{T} \quad \text{for all} \quad t \in [0, T),$$
then we have
$$\begin{aligned}
\frac{\partial \theta}{\partial t} &= \frac{\partial a(u, t)}{\partial s} + k(u, t)b(u, t) \\
&= \frac{\partial \widetilde{a}(\theta, \tau)}{\partial \theta}\frac{\partial \theta}{\partial s} + k(\theta, \tau)\widetilde{b}(\theta, \tau) \\
&= k(\theta, \tau)\left(\frac{\partial \widetilde{a}(\theta, \tau)}{\partial \theta} + \widetilde{b}(\theta, \tau) \right).
\end{aligned}$$
On the other hand, the support function satisfies
$$S_\tau = -\widetilde{a},$$
hence,
$$\frac{\partial \theta}{\partial t} = k\left(\frac{\partial}{\partial \theta}(-S_\tau) + \widetilde{b} \right) = k(-S_{\theta\tau} + \widetilde{b}).$$
Then the support function $S(\theta, \tau)$ satisfies the following equation
$$S_{\tau\tau} = [(S_{\theta\tau} - \widetilde{b})^2 - 1]k + dS_\tau,$$
namely,
$$S_{\tau\tau} = \frac{(S_{\theta\tau} - \widetilde{b})^2 - 1}{S_{\theta\theta} + S} + dS_\tau. \tag{5.3}$$
The equation (5.3) is equivalent to the following equation
$$SS_{\tau\tau} + 2\widetilde{b}S_{\theta\tau} - dS_\tau S_{\theta\theta} + S_{\tau\tau}S_{\theta\theta} - S_{\theta\tau}^2 + 1 - \widetilde{b}^2 - dSS_\tau = 0. \tag{5.4}$$

Denote
$$A = 1 - \widetilde{b}^2 - dSS_\tau, \quad B = S, \quad C = 2\widetilde{b}, \quad D = -dS_\tau \quad \text{and} \quad E = 1,$$

then

$$\triangle^2(\tau, \theta, S, S_\tau, S_\theta) = C^2 - 4BD + 4AE$$

$$= (2\widetilde{b})^2 - 4S(-dS_\tau) + 4(1 - \widetilde{b}^2 - dSS_\tau)$$

$$= 4 > 0,$$

$$S_{\theta\theta} + B(\tau, \theta, S, S_\tau, S_\theta) = S_{\theta\theta} + S = \frac{1}{k} \neq 0.$$

Furthermore, we state the initial values $S(0, \theta) = h_0(\theta)$, $S_\tau(0, \theta) = -\widetilde{f}(\theta)$ for the unknown function on the $\theta \in [0, 2\pi]$, $h(\theta)$ being third and $\widetilde{f}(\theta)$ twice continuous by differentiable on the real axis. Moreover, we require the τ-hyperbolicity condition:

$$\triangle^2(0, \theta, \epsilon h, -\epsilon \widetilde{f}, h_\theta) = (C^2 - 4BD + 4A)|_{t=0} = 4 > 0,$$

$$S_{0\theta\theta} + B(0, \theta, \epsilon h, -\epsilon \widetilde{f}, h_\theta) = h_{\theta\theta} + h = \frac{1}{k_0} \neq 0.$$

This implies the equation (5.4) is a hyperbolic Monge-Ampère equation on S. Then the support function S satisfies the following initial value problem

$$\begin{cases} SS_{\tau\tau} + 2\widetilde{b}S_{\theta\tau} - dS_{\theta\tau} + S_{\tau\tau}S_{\theta\theta} - S_{\theta\tau}^2 + 1 - \widetilde{b}^2 - dS_\tau = 0, \\ S(\theta, 0) = h(\theta), \\ S_\tau(\theta, 0) = -\widetilde{f}(\theta), \end{cases} \tag{5.5}$$

where h is the support function of F_0, and $\widetilde{f}$ is the initial velocity of the initial curve F_0.

Similarly, we can get the curvature the curvature k satisfies the following equation,

$$k_{\tau\tau} = k^2[1 - (S_{\theta\tau} - \tilde{b})^2])k_{\theta\theta} + 2k(S_{\theta\tau} - \tilde{b})k_{\theta\tau} + 4k^2(S_{\theta\tau} - \tilde{b})(S_\tau + \tilde{b}_\theta)k_\theta$$

$$+ (d - 4kS_\tau - 4k\tilde{b}_\theta)k_\tau + [S_{\theta\tau}^2 + 1 - 2S_\tau^2 - 4s_\tau\tilde{b}_\theta - \tilde{b}^2 + 2(S_{\theta\tau} - \tilde{b})\tilde{b}_{\theta\theta}]k^3. \tag{5.6}$$

It is easy to verify that the equation (5.6) is also a hyperbolic equation about τ.

Motivated by the theory of dissipative hyperbolic equations (see [15], [21]), we will study the initial value problem (5.5) and the initial value problem for (5.6) in the forthcoming paper.

6 Relation between the hyperbolic mean curvature flow and the string evolution equation in the Minkowski space $\mathbb{R}^{1,1}$

In this section, we study the relation between the hyperbolic mean curve flow and the evolution equation for the string in the Minkowski space $\mathbb{R}^{1,1}$.

Let $z = (z_0, z_1)$ be a position vector of a point in the two-dimensional Minkowski space $\mathbb{R}^{1,1}$. The scalar product of two vectors z and $w = (w_0, w_1)$ is

$$\langle z, w \rangle = -z_0 w_0 + z_1 w_1.$$

The Lorentz metric of $\mathbb{R}^{1,1}$ reads

$$ds^2 = -dt^2 + du^2.$$

A massless closed curve moving in two-dimensional Minkowski space can be defined by making its action proportional to the two-dimensional area swept out in the Minkowski space. We are interested in the following motion of one-dimensional Riemannian manifold in $\mathbb{R}^{1,2}$ with the following parameter

$$(t, u) \to \widetilde{X} = (t, X(t, u)), \tag{6.1}$$

where $u \in \mathscr{M}$ and $\widetilde{X}(\cdot, t)$ be a positive vector of a point in the Minkowski space $\mathbb{R}^{1,2}$. The induced Lorentz metric reads

$$\begin{cases} \widetilde{g}_{00} = -1 + \left(\dfrac{\partial X}{\partial t}, \dfrac{\partial X}{\partial t} \right), \\ \widetilde{g}_{01} = \widetilde{g}_{10} = \left(\dfrac{\partial X}{\partial t}, \dfrac{\partial X}{\partial u} \right), \\ \widetilde{g}_{11} = g_{11} = \left(\dfrac{\partial X}{\partial u}, \dfrac{\partial X}{\partial u} \right), \end{cases} \tag{6.2}$$

i.e., the Lorentz metric becomes

$$ds^2 = (dt, du) A (dt, du)^T,$$

where

$$A = \begin{pmatrix} |X_t|^2 - 1 & \langle X_t, X_u \rangle \\ \langle X_t, X_u \rangle & |X_u|^2 \end{pmatrix}, \tag{6.3}$$

in which

$$|X_t|^2 = \langle X_t, X_t \rangle, \quad |X_u|^2 = \langle X_u, X_u \rangle.$$

Kong, Zhang and Zhou in [17] investigated the dynamics of relativistic (in particular, closed) strings moving in the Minkowski space $\mathbb{R}^{1,n}(n \geq 2)$. By the variational method, they get the following equation

$$|X_u|^2 X_{tt} - 2\langle X_t, X_u \rangle X_{tu} + (|X_t|^2 - 1) X_{uu} = 0. \tag{6.4}$$

Except the variational method, by vanishing mean curvature of the sub-manifold $\mathscr{M}$, we can obtain the following equation for the motion of $\mathscr{M}$ in the Minkowski space $\mathbb{R}^{1,2}$

$$\widetilde{g}^{\alpha\beta} \nabla_\alpha \nabla_\beta \widetilde{X} = \widetilde{g}^{\alpha\beta} \left(\frac{\partial^2 \widetilde{X}}{\partial x^\alpha \partial x^\beta} - \widetilde{\Gamma}^\gamma_{\alpha\beta} \frac{\partial \widetilde{X}}{\partial x^\gamma} \right) = 0, \tag{6.5}$$

where $\alpha, \beta = 0, 1$. It is convenient to fix the parametrization partially (see Albrecht and Turok [1], Turok and Bhattacharjee [27]) by requiring

$$\widetilde{g}_{01} = \widetilde{g}_{10} = \left(\frac{\partial X}{\partial t}, \frac{\partial X}{\partial u} \right) = 0, \tag{6.6}$$

that is, we require the additional gauge condition that the string velocity be orthogonal to the string tangent direction. We assume that the surface is C^2 and *time-like*, i.e.,

$$(|X_t|^2 - 1)|X_u|^2 - \langle X_t, X_u \rangle^2 < 0,$$

equivalently,

$$(1 - |X_t|^2) > 0.$$

Obviously, the equation (6.5) is equivalent to

$$\left(\frac{\partial^2 X}{\partial t^2}, \frac{\partial X}{\partial t} \right) - g^{11} \left(|X_t|^2 - 1 \right) \left(\frac{\partial^2 X}{\partial t \partial u}, \frac{\partial X}{\partial u} \right) = 0, \tag{6.7}$$

$$\frac{\partial^2 X}{\partial t^2} + g^{11} \left(\frac{\partial^2 X}{\partial u^2} - \Gamma_{11}^1 \frac{\partial X}{\partial u} \right) \left(|X_t|^2 - 1 \right) - \frac{1}{|X_t|^2 - 1} \left(\frac{\partial^2 X}{\partial t^2}, \frac{\partial X}{\partial t} \right) \frac{\partial X}{\partial t}$$
$$+ g^{11} \left(\frac{\partial^2 X}{\partial t \partial u}, \frac{\partial X}{\partial t} \right) \frac{\partial X}{\partial u} + g^{11} \left(\frac{\partial^2 X}{\partial t \partial u}, \frac{\partial X}{\partial u} \right) \frac{\partial X}{\partial t} = 0. \tag{6.8}$$

It is easy to verify that the system $(6.7) - (6.8)$ is equivalent to (6.7) and the following equation

$$\frac{\partial^2 X}{\partial t^2} + g^{11} \left(\frac{\partial^2 X}{\partial u^2} - \Gamma_{11}^1 \frac{\partial X}{\partial u} \right) \left(|X_t|^2 - 1 \right) + g^{11} \left(\frac{\partial^2 X}{\partial t \partial u}, \frac{\partial X}{\partial t} \right) \frac{\partial X}{\partial u} = 0,$$

namely,

$$\frac{\partial^2 X}{\partial t^2} = (1 - |X_t|^2) k \vec{N} - \frac{1}{|X_u|^2} \left(\frac{\partial^2 X}{\partial t \partial u}, \frac{\partial X}{\partial t} \right) \frac{\partial X}{\partial u}. \tag{6.9}$$

Remark 6.1 *The equation* (6.9) *is similar to the equation in* (1.1)*, both of them evolve normally. The difference between the equation in* (1.1) *and the equation* (6.9) *is only the normal acceleration of the evolving curve. Because*

$$1 - |X_t|^2 > 0,$$

that is, the velocity of the string is always less than the velocity of light which is meaning in the classical physics, the motion of the string in the Minkowski space $\mathbb{R}^{1,1}$ *can be regarded as one of applications of general normal hyperbolic mean curvature flow.*

References

[1] A. Albrecht and N. Tuok, *Evolution of cosmic string*, Physics Review Letters 54 (1985), 1868-1871.

[2] L. Alvarez, F. Guichard, P. L. Lions and J. M. Morel, *Axioms and fundamental equations of image processing*, Arch. Rational. Mech. Anal. 123 (1993), 199-257.

[3] S. Angenent and M. E. Gurtin, *Multiplhase thermomechanics with an interfacial structure 2. evolution of an isothermal interface*, Arch. Rational. Mech. Anal. 108 (1989), 323-391.

[4] F. Cao, *Geometric curve evolution and image processing*, Lecture Notes in Mathematics 1805, Springer, Berlin, 2003.

[5] D. Christodoulou, *Global solution of nonlinear hyperbolic equations for small initial data*, Comm. Pure Appl. Math. 39 (1986), 367-282.

[6] M. Gage and R. Hamilton, *The heat equation shrinking convex plane curves*, J. Diff. Geom. 23 (1986), 417-491.

[7] M. Grayson, *Shortening embedded curves*, Ann. of Math. 101 (1989), 71-111.

[8] M. E. Gurtin and P. Podio-Guidugli, *A hyperbolic theory for the evolution of plane curves*, SIAM. J. Math. Anal. 22 (1991), 575-586.

[9] J. Hadamard, Le probléme de Cauchy et les èquations aux dérivées partielles linéaries hyperboliques, Hermann, Paris, 1932.

[10] C.-L. He, D.-X. Kong and K.-F. Liu, *Hyperbolic mean curvature flow*, submitted.

[11] L. Hömander, Lectures on Nonlinear Hyperbolic Differential Equations, Mathématiques And Applications 26, Springer-Verlag, Berlin, 1997.

[12] G. Huisken, *Asymptotic behavior for singularities of the mean curvature flow*, J. Diff. Geom. 31 (1990), 285-299.

[13] G. Huisken and T. Ilmanen, *The inverse mean curvature flow and the Riemannian Penrose inequality*, J. Diff. Geom. 59 (2001), 353-437.

[14] S. Klainerman, *Global existence for nonlinear wave equations*, Comm. Pure Appl. Math. 33 (1980), 43-101.

[15] D.-X. Kong, *Maximum principle in nonlinear hyperbolic systems and its applications*, Nonlinear Analysis, Theory, Method & Applications 32 (1998), 871-880.

[16] D.-X. Kong and H.-R. Hu, *Geometric approach for finding exact solutions to nonlinear partial differential equations*, Physics Letters A 246 (1998) 105-112.

[17] D.-X. Kong, Q. Zhang and Q. Zhou, *The dynamics of relativistic string moving in the Minkowski space $\mathbb{R}^{1+n}$*, Commun. Math. Phys. 269 (2007), 153-174.

[18] P. G. Lefloch and K. Smoczyk, *The hyperbolic mean curvature flow*, arXiv:0712.0091v1, 2007.

[19] H. Lewy, *Ueber das Anfangswertproblem einer hyperbolischen nichtlinearen partiellen Differentialgleichung zweiter Ordnung mit zwei unabhangigen Veranderlichen*, Math. Annal. 98 (1928), 179-191.

[20] K. Liu, *Hyperbolic geometric flow*, Lecture at International Conference of Elliptic and Parabolic Differential Equations, Hangzhou, August 20, 2007. Available at preprint webpage of Center of Mathematical Science, Zhejiang University.

[21] T. Nishida, *Nonlinear hyperbolic equations and related topics in fluid dynamics*, Publications Mathématiqées D'osay 78-02, Paris-Sud, 1978.

[22] M. H. Protter and H. F. Weinberger, Maximum Principles in Differential Equations, Springer-Verlag, New York Inc, 1984.

[23] H. G. Rotstein, S. Brandon and A. Novick-Cohen, *Hyperbolic flow by mean curvature*, Journal of Crystal Growth 198-199 (1999), 1256-1261.

[24] R. Schneider, Convex Bodies: The Brum-Minkowski Theory, Cambridge University Press, 1993.

[25] K. Tso, *Deforming a hypersurface by its Gauss-Kronecker curvature*, Commu. Pure. Appl. Math. 38 (1985), 867-882 .

[26] M. Tsuji, *Formation of singularties for Monge-Ampère equations*, Bull. Sci. Math. 119 (1995), 433-457.

[27] N. Turok and P. Bhattacharjee, *Stretching cosmic strings*, Physical Review D 29 (1983), 1557-1562.

[28] D. V. Tynitskii, *The Cauchy problem for a hyperbolic Monge-Ampère equation*, Mathmatical Notes 51 (1992), 582-589.

[29] S.-T. Yau, *Review of geometry and analysis*, Asian J. Math. 4 (2000), 235-278.

[30] X.-P. Zhu, Lectures on Mean Curvature Flows, Studies in Advanced Mathematics 32, AMS/IP, 2002.