Solution formula and time periodicity for the motion of relativistic strings in the Minkowski space $\mathbb{R}^{1+n}$

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Abstract

In this paper we study the motion of relativistic strings in the Minkowski space $\mathbb{R}^{1+n}$. We show a surprising result: this system of equations containing $n$ nonlinear PDEs of second order possesses a general solution formula. By introducing a new concept called generalized time-periodic function, we prove that, if the initial data is periodic, then the smooth solution of the Cauchy problem for this kind of equations is generalized time-periodic. Thus the space-periodicity implies the time-periodicity for the classical solution of the Cauchy problem for this kind of equations. This fact yields an interesting physical phenomenon: the motion of closed strings possesses the generalized time-periodicity, which is consistent with the results from recent numerical studies. We further extend our results to finite strings with homogenous Dirichlet boundary conditions. In particular, we prove that the solution to the equations for the motion of relativistic strings satisfies a characteristic-quadrilateral identity, which plays an important role in solving the mixed initial-boundary value problem for finite strings with the inhomogenous Dirichlet boundary conditions.

Key words and phrases: Equations for the motion of relativistic string, general solution formula, generalized periodic function, characteristic-quadrilateral identity.

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1 Introduction

In particle physics, the string model is used to study the structure of hadrons. A free string is a one-dimensional physical object whose motion is represented by a time-like surface. In our recent work [13], we investigate the nonlinear dynamics of relativistic (in particular, closed) strings moving in the Minkowski space $\mathbb{R}^{1+n}$ $(n \geq 2)$. The main results obtained in [13] are as follows.

Let $X = (t, x_1, \cdots, x_n)$ be a position vector of a point in the $(1+n)$-dimensional Minkowski space $\mathbb{R}^{1+n}$. Consider the motion of a relativistic string and let $x_i = x_i(t, \theta)$ ($i = 1, \cdots, n$) be the local equation of its world surface, where $(t, \theta)$ are the surface parameters. The equations governing the motion of the string read

$$|x_\theta|^2 x_{tt} - 2\langle x_t, x_\theta \rangle x_{t\theta} + (|x_t|^2 - 1)x_{\theta\theta} = 0.$$  \hfill (1.1)

The system (1.1) contains $n$ nonlinear partial differential equations of second order. These equations also describe the extremal surfaces in the Minkowski space $\mathbb{R}^{1+n}$. Consider the Cauchy problem for the equations (1.1) with the following initial data

$$t = 0: \quad x = p(\theta), \quad x_t = q(\theta),$$  \hfill (1.2)

where $p$ is a given $C^2$ vector-valued function and $q$ is a given $C^1$ vector-valued function. The Cauchy problem (1.1)-(1.2) describes the motion of a free relativistic string in the Minkowski space $\mathbb{R}^{1+n}$ with the initial position $p(\theta)$ and initial velocity (in the classical sense) $q(\theta)$. In particular, when $p(\theta)$ and $q(\theta)$ are periodic, the string under consideration is a closed one. Let

$$\Lambda_\pm(\theta) = \frac{1}{|p'(\theta)|^2} \left( -\langle q(\theta), p'(\theta) \rangle \pm \sqrt{\langle q(\theta), p'(\theta) \rangle^2 - (|q(\theta)|^2 - 1)|p'(\theta)|^2} \right)$$  \hfill (1.3)

and

$$\mathcal{L}(\theta) = \langle q(\theta), p'(\theta) \rangle^2 - (|q(\theta)|^2 - 1)|p'(\theta)|^2.$$  \hfill (1.4)

In physics, $\Lambda_\pm(\theta)$ are the initial characteristic propagation speeds of the point parameterized by $\theta$, and $\mathcal{L}(\theta)$ is the Lagrangian energy density. The following theorem has been proved in [13]:

**Theorem A** Suppose that there exist two constants $\Lambda_*$ and $\Lambda^*$ such that

$$\Lambda_* \leq \Lambda_\pm(\theta) \leq \Lambda^* \quad \text{and} \quad \Lambda_-(\theta) < \Lambda_+(\theta), \quad \forall \theta \in \mathbb{R}.$$  \hfill (1.5)

Then the Cauchy problem (1.1)-(1.2) admits a unique global $C^2$ solution $x = x(t, \theta)$ on $\mathbb{R}^+ \times \mathbb{R}$, if and only if, for every fixed $\theta_2 \in \mathbb{R}$, it holds that

$$\Lambda_-(\theta_1) < \Lambda_+(\theta_2), \quad \forall \theta_1 \in (-\infty, \theta_2).$$  \hfill (1.6)
Moreover, under the assumptions (1.5)-(1.6), the global $C^2$ solution $x = x(t, \theta)$ satisfies that, for any arbitrary fixed $(t, \theta) \in \mathbb{R}^+ \times \mathbb{R}$, either $x_\theta(t, \theta) = 0$ or

$$\langle x_t(t, \theta), x_\theta(t, \theta) \rangle^2 - (|x_t(t, \theta)|^2 - 1)|x_\theta(t, \theta)|^2 > 0. \quad (1.7)$$

As a consequence, when the initial data (1.2) is periodic, Theorem A can be rewritten as the following theorem.

**Theorem B** Suppose that the initial data $p(\theta)$ and $q(\theta)$ are periodic functions with the period $\mathcal{P}$, and

$$\mathcal{L}(\theta) > 0, \quad \forall \, \theta \in [0, \mathcal{P}]. \quad (1.8)$$

Then the Cauchy problem (1.1)-(1.2) admits a unique global $C^2$ solution $x = x(t, \theta)$ on $\mathbb{R}^+ \times \mathbb{R}$, if and only if,

$$\max_{\theta \in [0, \mathcal{P}]} \Lambda_-(\theta) < \min_{\theta \in [0, \mathcal{P}]} \Lambda_+(\theta). \quad (1.9)$$

Moreover, under the assumptions (1.8)-(1.9), the global $C^2$ solution $x = x(t, \theta)$ satisfies that, for any arbitrary fixed $(t, \theta) \in \mathbb{R}^+ \times \mathbb{R}$, either $x_\theta(t, \theta) = 0$ or

$$\langle x_t(t, \theta), x_\theta(t, \theta) \rangle^2 - (|x_t(t, \theta)|^2 - 1)|x_\theta(t, \theta)|^2 > 0. \quad (1.10)$$

The first inequality in (1.5) implies that the initial characteristic speeds are bounded, and the second one implies that the initial moving speed (in the sense of relativity) of the string is less than the speed of light. The inequality (1.6) is a necessary and sufficient condition that guarantees the global existence of the extremal surface which contains no space-like and light-like point, but perhaps contains singular points. All results presented in [13] have been obtained under the assumptions (1.5)-(1.6).

Representations of solutions in closed form for partial differential equations are very important and interesting in both mathematics and applied fields. Unfortunately, most nonlinear partial differential equations in general do not possess representations of solutions in closed form. However, in this paper we show a surprising fact that under the assumptions (1.5)-(1.6), the set of $n$ nonlinear PDEs of second order given by (1.1) possesses a general solution formula. By introducing a new concept of generalized time-periodic function, we prove that, if the initial data is periodic, then the smooth solution of the Cauchy problem (1.1)-(1.2) is generalized time-periodic, namely, the space-periodicity also implies the time-periodicity for the classical solution of the Cauchy problem.
(1.1)-(1.2). This fact yields an interesting physical phenomenon: the motion of closed strings is always generalized time-periodic. We further extend our result to the mixed initial-boundary value problems for strings with homogenous Dirichlet boundary conditions at their end points. We also present a new method to solve the mixed initial-boundary value problems for strings with the inhomogenous Dirichlet boundary conditions. This kind of problem plays an important role in the string theory and particle physics. The key point of this method is to establish a characteristic-quadrilateral identity (see Theorem 6.2). Several numerical examples are presented.

On this research topic, a well known result has been obtained by Kibble and Turok [12]: they show that any initially static string will collapse after half an oscillation period, and also show that there exists solutions in which the strings never self-intersect. Here we would like to point out that, our formula presented in this paper can be used to investigate the self-intersection of general closed strings including both the initially static strings and initially non-static strings. Our numerical examples show that the initially non-static strings may also collapse in finite time.

The paper is organized as follows. For the sake of completeness, in Section 2 we briefly recall some basic properties enjoyed by the equations (1.1). Section 3 is devoted to deriving the solution formula for the equations (1.1). In Section 4 we study the generalized time-periodicity of the smooth solution of the Cauchy problem for (1.1) with periodic initial data (1.2), and show that the space-periodicity also implies the time-periodicity for the classical solution of the Cauchy problem (1.1)-(1.2). In Section 5, we derive the solution formula for the mixed initial-boundary value problems for the equations (1.1) with homogenous Dirichlet boundary conditions. Since the inhomogenous Dirichlet boundary conditions are important in both mathematics and physics, and the method employed in Section 5 does not work for this case, in Section 6 we investigate the mixed initial-boundary value problem for the equations (1.1) with the inhomogenous Dirichlet boundary conditions. Section 7 is devoted to the numerical examples to illustrate the main results. A summary and discussions are given in Section 8.

2 Preliminaries

Let

\[ u = x_t, \quad v = x_\theta, \]

(2.1)
where \( u = (u_1, \ldots, u_n) \) and \( v = (v_1, \ldots, v_n) \). Then (1.1) can be equivalently rewritten as

\[
\begin{align*}
\begin{cases}
  u_t - \frac{2\langle u, v \rangle}{|v|^2} u_\theta + \frac{|u|^2 - 1}{|v|^2} v_\theta = 0, \\
v_t - u_\theta = 0
\end{cases}
\end{align*}
\]

(2.2)

for classical solutions, provided that \(|v|^2 > 0\). Setting \( U = (u, v)^T \), we can rewrite (2.2) as

\[
U_t + A(U)U_\theta = 0,
\]

(2.3)

where

\[
A(U) = \begin{bmatrix}
-\frac{2\langle u, v \rangle}{|v|^2} I_{n \times n} & \frac{|u|^2 - 1}{|v|^2} I_{n \times n} \\
-I_{n \times n} & 0
\end{bmatrix}.
\]

(2.4)

By a direct calculation, the eigenvalues of \( A(U) \) are

\[
\lambda_1 \equiv \cdots \equiv \lambda_n = \lambda_-, \quad \lambda_{n+1} \equiv \cdots \equiv \lambda_{2n} = \lambda_+,
\]

(2.5)

where

\[
\lambda_{\pm} = \frac{1}{|v|^2} \left( -\langle u, v \rangle \pm \sqrt{\langle u, v \rangle^2 - (|u|^2 - 1)|v|^2} \right).
\]

(2.6)

By [13], the characteristic fields \( \lambda_{\pm} \) are linearly degenerate in the sense of Lax (cf. [15]), and the system (2.3) is rich (cf. [19]). The Riemann invariants for the system (2.3) constructed in [13] are

\[
R_i = u_i + \lambda_- v_i, \quad R_{i+n} = u_i + \lambda_+ v_i \quad (i = 1, \ldots, n).
\]

(2.7)

They satisfy

\[
\frac{\partial R_i}{\partial t} + \lambda_+ \frac{\partial R_i}{\partial \theta} = 0, \quad \frac{\partial R_{i+n}}{\partial t} + \lambda_- \frac{\partial R_{i+n}}{\partial \theta} = 0 \quad (i = 1, \ldots, n).
\]

(2.8)

Another important fact shown in [13] is that \( \lambda_{\pm} \) satisfy the following equations

\[
\frac{\partial \lambda_-}{\partial t} + \lambda_+ \frac{\partial \lambda_-}{\partial \theta} = 0, \quad \frac{\partial \lambda_+}{\partial t} + \lambda_- \frac{\partial \lambda_+}{\partial \theta} = 0.
\]

(2.9)

Consider the Cauchy problem for the system (2.9) with the following initial data

\[
t = 0 : \quad \lambda_{\pm} = \Lambda_{\pm}(\theta),
\]

(2.10)

where \( \Lambda_{\pm}(\theta) \) are defined by (1.3). Define

\[
\rho(\theta) = \int_0^{\theta} \frac{2}{\Lambda_+(\xi) - \Lambda_-(\xi)} d\xi
\]

(2.11)

and

\[
\Theta(t, \sigma) = \frac{1}{2} \int_0^{\sigma+t} \Lambda_+(\rho(\xi)) d\xi - \frac{1}{2} \int_0^{\sigma-t} \Lambda_-(\rho(\xi)) d\xi.
\]

(2.12)
Let $\theta = \varrho(\sigma)$ be the inverse function of $\sigma = \rho(\theta)$ and $\eta = \Phi(t, \theta)$ be the inverse function of $\theta = \Theta(t, \eta)$. The solution of the Cauchy problem (2.9)-(2.10) can be expressed as (cf. [13])

$$
\lambda_\pm(t, \theta) = \Lambda_\pm(\varrho(\Phi(t, \theta) \pm t)).
$$

(2.13)

On the other hand, consider the Cauchy problem for the system (2.8) with the initial data

$$
t = 0: \begin{cases}
R_i(t, \theta) = q_i(\theta) + \Lambda_-^\prime(\theta)p_i(t, \theta), \\
R_{i+n}(t, \theta) = q_i(\theta) + \Lambda_+(\theta)p_i(t, \theta) \overset{\Delta}{=} R_{i+n}^0(\theta), \quad (i = 1, \cdots, n),
\end{cases}
$$

(2.14)

where $p_i$ and $q_i$ are the $i$-th components of $p$ and $q$ respectively. The solution of the Cauchy problem (2.8), (2.14) reads

$$
R_i(t, \theta) = R_i^0(\varrho(\Phi(t, \theta) - t)), \quad R_{i+n}(t, \theta) = R_{i+n}^0(\varrho(\Phi(t, \theta) + t)) \quad (i = 1, \cdots, n).
$$

(2.15)

In terms of $\lambda_\pm(t, \theta)$ and $R_j(t, \theta)$ ($j = 1, \cdots, 2n$), the solution of the Cauchy problem (1.1)-(1.2) is given by (cf. [13])

$$
x(t, \theta) = p(\theta) + \int_0^t u(s, \theta) ds,
$$

(2.16)

where $u = (u_1, \cdots, u_n)$ and

$$
u_i = \frac{\lambda_+(t, \theta)R_i(t, \theta) - \lambda_-(t, \theta)R_{i+n}(t, \theta)}{\lambda_+(t, \theta) - \lambda_-(t, \theta)} \quad (i = 1, \cdots, n).
$$

(2.17)

The equality (2.16) gives an exact representation, involving two independent and arbitrary vector-valued functions, $p(x)$ and $q(x)$, of general solution of the equations (1.1), where $p(x)$ and $q(x)$ should be chosen such that the conditions (1.5)-(1.6) are satisfied. However, it is difficult to analyze the properties of the solution of the Cauchy problem (1.1)-(1.2) from the representation (2.16). Therefore, in next section we develop a parametric form of the solution formula (2.16) which allows us to analyze the properties and time-periodicity of the solution of the Cauchy problem (1.1)-(1.2).

### 3 Solution formula in parametric form

In this section, we develop a parametric version of the solution formula (2.16) of the Cauchy problem (1.1)-(1.2). Throughout this section, we assume that (1.5)-(1.6) hold.

Introduce two new variables $\theta_\pm(t, \theta)$, which are determined by

$$
\sigma \pm t = \int_0^{\theta_{\mp}} 2 \frac{d\zeta}{\Lambda_+(\zeta) - \Lambda_-(\zeta)},
$$

(3.1)

where, as before, $\sigma = \Phi(t, \theta)$ is the inverse function of $\theta = \Theta(t, \sigma)$. It follows from (3.1) that

$$
t = \int_{\theta_-}^{\theta_+} \frac{1}{\Lambda_+(\zeta) - \Lambda_-(\zeta)} d\zeta.
$$

(3.2)
In terms of $\theta_{\pm}$, the equation (2.12) can be written as
\[
\theta = \int_{0}^{\theta_+} \frac{\Lambda_+(\zeta)}{\Lambda_+(\zeta) - \Lambda_-(\zeta)} d\zeta - \int_{0}^{\theta_-} \frac{\Lambda_-(\zeta)}{\Lambda_+(\zeta) - \Lambda_-(\zeta)} d\zeta.
\] (3.3)

This is equivalent to
\[
\int_{\theta}^{\theta_+} \frac{\Lambda_+(\zeta)}{\Lambda_+(\zeta) - \Lambda_-(\zeta)} d\zeta = \int_{\theta}^{\theta_-} \frac{\Lambda_-(\zeta)}{\Lambda_+(\zeta) - \Lambda_-(\zeta)} d\zeta.
\] (3.4)

Define
\[
\Lambda_{\pm}^+(t, \theta) = \Lambda_+(\theta_{\pm}(t, \theta)), \quad \Lambda_{\pm}^-(t, \theta) = \Lambda_-(\theta_{\pm}(t, \theta)).
\] (3.5)

Then, (2.17) becomes
\[
u_i(t, \theta) = \frac{\Lambda_+^+(q_i^- + \Lambda_-^- p_i'^-) - \Lambda_-^-(q_i^+ + \Lambda_+^+ p_i'^+)}{\Lambda_+^- - \Lambda_-^-} (i = 1, \cdots, n),
\] (3.6)

where
\[
q_i^\pm = q_i(\theta_{\pm}), \quad p_i'^\pm = p_i'(\theta_{\pm}),
\] (3.7)
in which $p_i$ and $q_i$ are the $i$-th components of the initial data $p$ and $q$, respectively. The component form of (2.16) is
\[
x_i(t, \theta) = p_i(\theta) + \int_{0}^{t} u_i(\tau, \theta) d\tau \quad (i = 1, \cdots, n).
\] (3.8)

In what follow, by means of $\theta_-$ and $\theta_+$, we derive the representation of the integral $\int_{0}^{t} u_i(\tau, \theta) d\tau$.

Differentiating both sides of (3.2) gives
\[
dt = \frac{1}{\Lambda_+^- - \Lambda_-^-} d\theta_+ - \frac{1}{\Lambda_+^- - \Lambda_-^-} d\theta_-.
\] (3.9)

Hence it follows from (3.6) that
\[
\int_{0}^{t} u_i(\tau, \theta) d\tau = \int_{\theta}^{\theta_+} G_i(\tau, \theta) \left\{ \frac{1}{\Lambda_+^+(\tau, \theta) - \Lambda_+^-(\tau, \theta)} - \frac{1}{\Lambda_-^-(\tau, \theta) - \Lambda_-^+(\tau, \theta)} \right\} \frac{d\tilde{\theta}_+}{\Lambda_-^- - \Lambda_-^-} d\tilde{\theta}_+,
\] (3.10)

where
\[
G_i(\tau, \theta) = \frac{\Lambda_+^+(q_i^- + \Lambda_-^- p_i'^-) - \Lambda_-^-(q_i^+ + \Lambda_+^+ p_i'^+)}{\Lambda_+^- - \Lambda_-^-} (\tau, \theta)
\] (3.11)

and
\[
\tau = \int_{\theta_-}^{\theta_+} \frac{1}{\Lambda_+(\zeta) - \Lambda_-(\zeta)} d\zeta.
\] (3.12)

Notice that, for any fixed $\theta$, the relation between $\theta_+$ and $\theta_-$ are determined by (3.4). Then it follows from (3.4) that
\[
\frac{\Lambda_+^+}{\Lambda_+^- - \Lambda_-^-} d\theta_+ = \frac{\Lambda_-^-}{\Lambda_+^- - \Lambda_-^-} d\theta_-,
\]

namely,
\[
\frac{d\theta_-}{d\theta_+} = \frac{\Lambda_+^+(\Lambda_-^- - \Lambda_-^-)}{\Lambda_-^- (\Lambda_+^- - \Lambda_+^-)}.
\] (3.13)
Substituting (3.13) into (3.10) leads to

\[
\int_0^t u_i(\tau, \theta) d\tau = \int_0^\theta \frac{G_i(\tau, \theta)}{A_i^+(\tau, \theta) - A_i^-(\tau, \theta)} \left( 1 - \frac{\Lambda_i^+(\tau, \theta)}{\Lambda_i^-(\tau, \theta)} \right) d\theta^+ \\
= - \int_0^\theta \frac{\Lambda_i^+(q_i^+ + \Lambda_i^- p_i^-) - \Lambda_i^-(q_i^+ + \Lambda_i^+ p_i^+)}{A_i^+(\Lambda_i^+ - \Lambda_i^-)} (\tau, \theta) d\theta^+ \\
= \int_0^{\theta^+} \frac{q_i^+ + \Lambda_i^+ p_i^+}{A_i^+ - \Lambda_i^-} (\tau, \theta) d\theta^+ - \int_0^{\theta^-} \frac{q_i^- + \Lambda_i^- p_i^-}{A_i^- - \Lambda_i^+} (\tau, \theta) d\theta^- \\
= \int_0^{\theta^+} q_i + \Lambda_i^+ p_i^+ (\zeta) d\zeta - \int_0^{\theta^-} q_i + \Lambda_i^- p_i^- (\zeta) d\zeta, 
\]

where, from the third equality to the fourth equality in (3.14), we have applied (3.13) to the second integral. Therefore, noting (1.3), we obtain from (3.14) that

\[
\int_0^t u_i(\tau, \theta) d\tau = \frac{1}{2} \int_0^{\theta^+} \frac{|p_i'(\zeta)|^2 q_i(\zeta) - (q(\zeta), p_i'(\zeta)) p_i'(\zeta)}{\sqrt{(q(\zeta), p_i'(\zeta))}^2 - (|q(\zeta)|^2 - 1)|p_i'(\zeta)|^2} d\zeta + \\
\frac{1}{2} \left\{ \int_0^{\theta^+} p_i'(\zeta) d\zeta + \int_0^{\theta^-} p_i'(\zeta) d\zeta \right\}. 
\]

Substituting (3.15) into (3.8) yields

\[
x_i(t, \theta) = \frac{p_i(\theta^+) + p_i(\theta^-)}{2} + \frac{1}{2} \int_{\theta^-}^{\theta^+} \frac{|p_i'(\zeta)|^2 q_i(\zeta) - (q(\zeta), p_i'(\zeta)) p_i'(\zeta)}{\sqrt{(q(\zeta), p_i'(\zeta))}^2 - (|q(\zeta)|^2 - 1)|p_i'(\zeta)|^2} d\zeta. 
\]

This gives a parametric version of the solution formula (2.16). The formula (3.16) has the following advantages:

(a) For arbitrary fixed \( \theta \), by (3.2) and (3.4), we observe that \( \theta_{\pm} \) are the functions of \( t \in \mathbb{R}^+ \). Therefore, (3.16) gives an exact description on the trajectory of the point \( \theta \) on the string.

(b) The parametric form also has advantages in numerical evaluation. To vary \( t \) and \( \theta \), we can simply vary \( \theta^+ \) and \( \theta^- \). Then we can determine the value of \( x(t, \theta) \) in the whole \( (t, \theta) \)-plane by (3.2), (3.4) and (3.16). The periodicity of \( x(t, \theta) \), which we will establish in Section 4, implies that we only need to evaluate the expressions (3.2), (3.4) and (3.16) for one period only. This will make the evaluation even much fast and more accurate.

Summarizing the above argument gives

**Theorem 3.1** Under the assumptions (1.5)-(1.6), the Cauchy problem (1.1)-(1.2) admits a unique global \( C^2 \) solution \( x = x(t, \theta) \) on \( \mathbb{R}^+ \times \mathbb{R} \), and this solution can be expressed by (3.16), where \( \theta_{\pm}(t, \theta) \) are determined by (3.2) and (3.4).
**Remark 3.1** The extremal surfaces in the Minkowski space include the following four types: space-like, time-like, light-like or mixed types. The surface defined by the solution (3.16) of the Cauchy problem (1.1)-(1.2) is globally time-like. The space-like extremal surfaces in the Minkowski space are essentially regarded as minimal surfaces in the Euclidean space. The Bjorling problem in classical differential geometry is a geometrical problem of constructing an extremal surface which passes through a given curve and is tangent to a family of planes given along that curve. Coordinates of the minimal surface in the three-dimensional Euclidean space are defined by the Schwarz formula (see [9]); and for general n-dimensional Euclidean space, by the Beckenbach formula (see [3]):

\[
x_i(z, w) = \frac{x_i(z) + x_i(w)}{2} + \frac{\sqrt{-1}}{2} \sum_{j=1}^{n} \int_{w}^{z} Z_{ij} dx_j.
\]

(3.17)

where \( z \) and \( w \) are complex variables. This surface passes through a given curve whose coordinates \( x_i(\cdot) \) are analytic functions. The planes tangent to the minimal surface have, along this curve, direction cosine given by analytic functions \( Z_{ij}(\cdot) \):

\[
Z_{ij}Z_{ij}' = \delta_{jj}', \quad Z_{ij} = -Z_{ji}.
\]

The direction cosine \( Z_{ij} \) can be expressed by the metric tensor and vectors tangent to the surface:

\[
Z_{ij} = \frac{1}{\sqrt{g}} \frac{\partial(x_i, x_j)}{\partial(z, w)}.
\]

(3.18)

Inserting (3.18) into (3.17) gives the Euclidean analog of the formula (3.16).

**Remark 3.2** The solution formula (3.16) of the Cauchy problem (1.1)-(1.2) in the relativistic string theory represents reformulation for the hyperbolic (time-like) case of the Bjorling problem in the Minkowski space. However, the formula (3.17) gives reformulation for the elliptic (space-like) case of the Bjorling problem. Equations (3.16) and (3.17) are the counterparts of time-like and space-like extremal surfaces, respectively, in the framework of Bjorling problem in the Minkowski space.

Now we would like to provide a geometric interpretation of the new variables \( \theta_{\pm}(t, \theta) \). To do so, we consider the Cauchy problem (2.9)-(2.10). For every fixed point \((t, \theta)\) in the domain where the smooth solution exists, we can draw two characteristics passing through the point \((t, \theta)\). It follows from the first inequality in (1.5) that these two characteristics intersect the \( \theta \)-axis at two points, denoted by \((0, \alpha(t, \theta))\) and \((0, \beta(t, \theta))\) respectively, where \((0, \alpha(t, \theta))\) (resp. \((0, \beta(t, \theta))\)) is the intersection point of the \( \theta \)-axis with the fast (resp. slow) characteristic passing through the point \((t, \theta)\). Obviously,

\[
\alpha(t, \theta) \leq \beta(t, \theta).
\]
Such \((\alpha, \beta)\) are called the characteristic coordinates of the point \((t, \theta)\). On the other hand, for any given \((\alpha, \beta)\) with \(\alpha \leq \beta\), we may draw the fast (resp. slow) characteristic passing through the point \((0, \alpha)\) (resp. \((0, \beta)\)). Under the assumptions of (1.5) and (1.6), these two characteristics must intersect and we denote the intersection point by \((t, \theta)\). By the standard method of characteristics, it is easy to prove

\[
t = \int_{\alpha}^{\beta} \frac{d\zeta}{\Lambda_+ (\zeta) - \Lambda_- (\zeta)} < \infty
\]  

(3.19)

and

\[
\theta = \frac{1}{2} \left\{ \alpha + \beta + \int_{\alpha}^{\beta} \frac{\Lambda_+ (\zeta) + \Lambda_- (\zeta)}{\Lambda_+ (\zeta) - \Lambda_- (\zeta)} d\zeta \right\} = \int_{0}^{\beta} \frac{\Lambda_+ (\zeta)}{\Lambda_+ (\zeta) - \Lambda_- (\zeta)} d\zeta - \int_{0}^{\alpha} \frac{\Lambda_- (\zeta)}{\Lambda_+ (\zeta) - \Lambda_- (\zeta)} d\zeta.
\]  

(3.20)

Comparing (3.19) and (3.20) with (3.2) and (3.3), we observe that \((\theta_-, \theta_+)\) are nothing but the characteristic coordinates of the point \((t, \theta)\). Sometimes \((t, \theta)\) is called the image of \((\theta_-, \theta_+)\), while \((\theta_-, \theta_+)\) is called the preimage of \((t, \theta)\). The geometric meaning of \(\theta_{\pm}\) is illustrated in Figure 1 where \(C_{\pm}\) stand for the \(\lambda_{\pm}\)-characteristics passing through the point \((t, \theta)\). \(\theta_-\) remains constant on \(C_+\) and \(\theta_+\) remains constant on \(C_-\).

![Figure 1: Geometric meaning of \(\theta_-\) and \(\theta_+\)](image)

We comment that the Cauchy problem for the Born-Infeld equation

\[
(1 + \psi^2) \psi_{tt} - 2\psi_t \psi_\theta \psi_{t\theta} + (\psi^2_t - 1) \psi_{\theta\theta} = 0
\]  

(3.21)

with the initial data

\[
\psi(0, \theta) = \psi_0(\theta), \quad \psi_t(0, \theta) = \psi_1(\theta),
\]  

(3.22)

is a special case of motion of relativistic strings in the \((1+2)\)-dimensional Minkowski space \(\mathbb{R}^{1+2}\), where \(\psi_0(\theta)\) and \(\psi_1(\theta)\) are smooth functions satisfying \(\psi^2_t - \psi^2_0 < 1\). To show this, we consider \(x = (x_1, x_2)\) with the initial data

\[
p(\theta) = (\theta, \psi_0(\theta)), \quad q(\theta) = (0, \psi_1(\theta)).
\]
Then the \( x_2 \)-component of (1.1) is identical to (3.21). It follows from (3.2), (3.4) and (3.16) that the solution of the Cauchy problem (3.21)-(3.22) is given by

\[
t = \frac{1}{2} \int_{\theta_-}^{\theta_+} \frac{1 + \psi_0'(\zeta)}{\sqrt{1 + \psi_0'^2(\zeta) - \psi_1^2(\zeta)}} d\zeta,
\]

(3.23)

\[
\theta = \frac{\theta_- + \theta_+}{2} - \frac{1}{2} \int_{\theta_-}^{\theta_+} \frac{\psi_0'(\zeta)\psi_1(\zeta)}{\sqrt{1 + \psi_0'^2(\zeta) - \psi_1^2(\zeta)}} d\zeta,
\]

(3.24)

\[
x_2 = \psi = \frac{\psi_0(\theta_-) + \psi_0(\theta_+)}{2} + \frac{1}{2} \int_{\theta_-}^{\theta_+} \frac{\psi_1(\zeta)}{\sqrt{1 + \psi_0'^2(\zeta) - \psi_1^2(\zeta)}} d\zeta.
\]

(3.25)

In this case, \( x_1 = \theta \). The expressions (3.23)-(3.25) are identical to the ones obtained previously by Lochak [17]. Therefore, the Cauchy problem (3.21)-(3.22) for the Born-Infeld equation is indeed a special case of motion of relativistic strings in the Minkowski space.

### 4 Generalized time-periodicity of the motion of closed strings

In this section, we consider the motion of a closed string in the Minkowski space \( \mathbb{R}^{1+n} \). The motion of the closed string corresponds to the case that the initial data \( p(\theta), q(\theta) \) are periodic functions. By introducing a new concept named as **generalized time-periodic function**, we prove that the smooth solution of the Cauchy problem (1.1)-(1.2) with periodic initial data is **generalized time-periodic**. In other words, the motion of closed strings possesses the generalized time-periodicity. Throughout this section, we assume that (1.8) and (1.9) are satisfied.

#### 4.1 Generalized time-periodic functions

In this subsection, we introduce the following concept named as **generalized time-periodic functions**. Consider a vector-valued function \( f(t, \theta) \) defined on the domain \( \mathbb{R}^+ \times \mathbb{R} \).

**Definition 4.1** The function \( f(t, \theta) \) is called to be generalized time-periodic, if there exists a positive constant \( T \) independent of \( t \) and \( \theta \), a constant \( \varphi(T) \) and a constant vector \( \mathcal{D}(T) \) such that

\[
f(t + T, \theta + \varphi(T)) = \mathcal{D}(T) + f(t, \theta), \quad \forall (t, \theta) \in \mathbb{R}^+ \times \mathbb{R},
\]

(4.1)

where \( \varphi(T) \) and \( \mathcal{D}(T) \) depend only on \( T \) but are independent of \( t \) and \( \theta \). We call \( T \), \( \varphi(T) \) and \( \mathcal{D}(T) \) the generalized period, phase shift and translation displacement of \( f \), respectively.

**Remark 4.1** Obviously, if \( f(t, \theta) \) is a classical periodic function with respect to the variable \( t \), i.e., for every fixed \( \theta \in \mathbb{R} \)

\[
f(t + T, \theta) = f(t, \theta),
\]

for
then \( f(t, \theta) \) must be generalized time-periodic. In general, the converse is not true. For example, the function \( f(t, \theta) = t + \sin(t + \theta) \) is generalized time-periodic, but not classical time-periodic.

**Remark 4.2** If \( f = f(t, \theta) \) is a \( k \)-dimensional vector function, say, \( f = (f_1(t, \theta), \cdots, f_k(t, \theta)) \), then the function \( f(t, \theta) \) defines a surface \( S \) in the Minkowski space \( \mathbb{R}^{1+k} \) with the coordinates \( (x_0, x_1, \cdots, x_n) \), which can be expressed by:

\[
x_0 = t, \quad x_i = f_i(t, \theta) \quad (i = 1, \cdots, k).
\]

Thus, \( x_i = f_i(t, \cdot) \) \( (i = 1, \cdots, k) \) defines a plane curve, denoted by \( C_t \), which is the intersector curve between the surface \( S \) and \( t \)-hyperplane. If \( f(t, \theta) \) is a generalized time-periodic function in the sense of Definition 4.1, then the intersector curve \( C_{t+T} \) is a translation displacement of \( C_t \). Therefore, they have identical shapes. But we point out that the corresponding points on these curves may not have the same value of parameters due to the phase shift. More precisely, if \( f(t, \theta) \) is a generalized time-periodic function, then there exists a translation mapping \( T \) depending on \( T \) but independent of \( t \) and \( \theta \) such that

\[
C_{t+T} = T C_t.
\]

(4.2)

In particular, if the translation displacement vanishes, that is, \( D(T) = 0 \), then the curves \( C_t \) and \( C_{t+T} \) coincide in the projective space \( \mathbb{R}^n \) in the \( t \)-direction of the Minkowski space \( \mathbb{R}^{1+n} \). This shows that the motion of the curve \( C_0 \) (or more generally, \( C_t \) for every fixed \( t \)) possesses the time periodicity.

**Remark 4.3** Consider a generalized time-periodic function \( f(t, \theta) \), and assume that the spatial parameter \( \theta \) has a classical period \( \mathcal{P} \). In general, the phase shift \( \varphi(T) \) may not be multiple of the spatial period \( \mathcal{P} \). Only when \( \varphi(T) \) is a multiple of \( \mathcal{P} \), the function \( f \) is time-periodic in the classical sense; otherwise, it is generalized time-periodic.

As in Remark 4.2, let \( C_t \) be the plane curve induced by the function \( f(t, \theta) \). We introduce

**Definition 4.2** The motion of the plane curve \( C_0 \) is said to possess the generalized time-periodicity, if there exist a positive constant \( T \) and a translation mapping \( T \) depending on \( T \) but independent of \( t \) and \( \theta \) such that (4.2) holds for every \( t \in \mathbb{R}^+ \).

**Remark 4.4** The definition of generalized periodicity can be easily extended to a (vector-valued) function \( f(t, \theta) \) defined on the domain \( \mathbb{R} \times \mathbb{R} \) and a (vector-valued) function \( f(t, \theta_1, \cdots, \theta_n) \) defined on the domain \( \mathbb{R}^+ \times \mathbb{R}^n \) or \( \mathbb{R} \times \mathbb{R}^n \).
The concept “generalized time-periodic function” defined by Definition 4.1 plays an important role in the present paper. In what follows, we give two simple examples to illustrate this concept.

**Example 4.1** Consider the following function

\[ f(t, \theta) = (f_1(t, \theta), f_2(t, \theta)) = (a_1 t + c_1 \cos t \cos(\theta + bt), a_2 t + c_2 \cos t \sin(\theta + bt)), \quad (4.3) \]

where \(a_1, a_2, b, c_1, c_2\) are constants. It is easy to see

\[ f(t + 2\pi, \theta - 2\pi b) = (2\pi a_1, 2\pi a_2) + f(t, \theta). \quad (4.4) \]

Obviously, \(f(t, \theta)\), defined by (4.3), is a generalized time-periodic function, and the corresponding generalized period \(T\), phase shift \(\varphi(T)\) and translation displacement \(D(T)\) are \(2\pi\), \(-2\pi b\) and \((2\pi a_1, 2\pi a_2)\), respectively.

In Example 4.1, if \((a_1, a_2, b) = (0, 0, 0)\), then \(f(t, \theta)\) becomes a classical periodic function with respect to the variable \(t\). If \(b\) is a rational number, say, \(k/m\), where \(k\) is an integer, and \(m\) is a positive integer. After \(m\) periods, the aggregated phase shift is \(-2\pi mb\), namely, \(-2k\pi\). This implies that, for any fixed \(\theta\) at the time \(t\), after \(m\) periods, it goes back to the original position, that is, \(f(t + 2m\pi, \theta - 2k\pi) = f(t, \theta)\). On the other hand, if \(b\) is an irrational number, even when the shape of string at a late time coincides with that at the initial time, no point on the string will return to its original location.

**Remark 4.5** If we replace the terms \(a_1 t\) and \(a_2 t\) in (4.3) by \(a_1 t^2\) and \(a_2 t^2\), respectively, then the resulting function \(f(t, \theta)\) is no longer generalized time-periodic function, since we can not find a positive constant \(T\) such that (4.1) holds. In fact, there are some intersectional terms including the factor \(tT\) in the formula of \(f(t + T, \theta + \varphi(T))\), and these terms make (4.3) no longer generalized time-periodic.

**Example 4.2** Consider the following Cauchy problem for linear wave equation

\[
\begin{align*}
\begin{cases}
u_{tt} - \nu_{xx} = 0 & \text{in } (0, \infty) \times \mathbb{R}, \\
u = g, \quad \nu_t = h & \text{on } \{t = 0\} \times \mathbb{R},
\end{cases}
\end{align*}
\]

where \(g, h\) are periodic functions with period \(T\). The d’Alembert’s formula gives its solution

\[ u(t, x) = \frac{g(x + t) + g(x - t)}{2} + \frac{1}{2} \int_{x-t}^{x+t} h(\zeta) d\zeta, \quad (4.6) \]
We calculate $u(t + T, x)$:
\[
 u(t + T, x) = \frac{g(x + t + T) + g(x - t - T)}{2} + \frac{1}{2} \int_{x-t-T}^{x+t+T} h(\zeta) d\zeta
\]
\[
 = \frac{g(x + t) + g(x - t)}{2} + \frac{1}{2} \left\{ \int_{x-t-T}^{x-t} + \int_{x-t}^{x+t} + \int_{x+t}^{x+t+T} \right\} h(\zeta) d\zeta
\]
\[
 = \frac{g(x + t) + g(x - t)}{2} + \frac{1}{2} \int_{x-t}^{x+t} h(\zeta) d\zeta + \int_{0}^{T} h(\zeta) d\zeta
\]
\[
 = u(t, x) + \int_{0}^{T} h(\zeta) d\zeta.
\]
(4.7)

Clearly, (4.7) shows that the solution $u(t, x)$ is a generalized time-periodic function in the sense of Definition 4.1, and the corresponding generalized period, phase shift and translation displacement are $T$, $0$ and $\int_{0}^{T} h(\zeta) d\zeta$, respectively.

4.2 Generalized time-periodicity of the motion of closed strings

In this subsection, we prove that the solution formula (3.16) is a generalized time-periodic function in the sense of Definition 4.1, if the initial data $p(\theta)$, $q(\theta)$ are periodic functions.

Let $\mathcal{P}$ be the period of $p(\theta)$ and $q(\theta)$, and $(\theta_{-}, \theta_{+})$ be the characteristic coordinates of $(t, \theta)$. We next calculate the image, denoted by $(\tilde{t}, \tilde{\theta})$, of the characteristic coordinates $(\theta_{-}, \theta_{+} + \mathcal{P})$. It follows from (3.2) that
\[
 \tilde{t} = \int_{\theta_{-}}^{\theta_{+} + \mathcal{P}} \frac{1}{\Lambda_{+}(\zeta) - \Lambda_{-}(\zeta)} d\zeta
\]
\[
 = \left\{ \int_{\theta_{-}}^{\theta_{+}} + \int_{\theta_{+}}^{\theta_{+} + \mathcal{P}} \right\} \frac{1}{\Lambda_{+}(\zeta) - \Lambda_{-}(\zeta)} d\zeta
\]
\[
 = t + \int_{0}^{\mathcal{P}} \frac{1}{\Lambda_{+}(\zeta) - \Lambda_{-}(\zeta)} d\zeta \triangleq t + T.
\]
(4.8)

Furthermore, by (3.3) we have
\[
 \tilde{\theta} = \int_{0}^{\theta_{+} + \mathcal{P}} \frac{\Lambda_{+}(\zeta)}{\Lambda_{+}(\zeta) - \Lambda_{-}(\zeta)} d\zeta - \int_{0}^{\theta_{-}} \frac{\Lambda_{-}(\zeta)}{\Lambda_{+}(\zeta) - \Lambda_{-}(\zeta)} d\zeta
\]
\[
 = \left\{ \int_{0}^{\theta_{+}} + \int_{\theta_{+}}^{\theta_{+} + \mathcal{P}} \right\} \frac{\Lambda_{+}(\zeta)}{\Lambda_{+}(\zeta) - \Lambda_{-}(\zeta)} d\zeta - \int_{0}^{\theta_{-}} \frac{\Lambda_{-}(\zeta)}{\Lambda_{+}(\zeta) - \Lambda_{-}(\zeta)} d\zeta
\]
\[
 = \theta + \int_{0}^{\mathcal{P}} \frac{\Lambda_{+}(\zeta)}{\Lambda_{+}(\zeta) - \Lambda_{-}(\zeta)} d\zeta \triangleq \theta + \varphi_{+}(T).
\]
(4.9)
We now calculate \( x(t + T, \theta + \varphi(T)) \).

It follows from (3.16) that

\[
x_i(t + T, \theta + \varphi(T)) = x_i(\tilde{t}, \tilde{\theta})
\]

\[
= \frac{p_i(\theta_+ + \mathcal{P}) + p_i(\theta_-)}{2} + \int_{\theta_-}^{\theta_+} \mathcal{Q}_i(\zeta)d\zeta
\]

\[
= \frac{p_i(\theta_+) + p_i(\theta_-)}{2} + \left\{ \int_{\theta_-}^{\theta_+} + \int_{\theta_+}^{\theta_-} \right\} \mathcal{Q}_i(\zeta)d\zeta
\]

\[
= x_i(t, \theta) + \int_0^\mathcal{P} \mathcal{Q}_i(\zeta)d\zeta,
\]

where

\[
\mathcal{Q}_i(\zeta) = \frac{|p_i'(\zeta)|^2 q_i(\zeta) - \langle q(\zeta), p_i'(\zeta) \rangle p_i(\zeta)}{2 \sqrt{|q(\zeta), p_i'(\zeta)|^2 - (|q(\zeta)|^2 - 1)|p_i'(\zeta)|^2}}.
\]

In (4.10), we have made use of the periodicity of \( p \) and \( q \).

Equation (4.10) shows that the solution \( x = x(t, \theta) \), given by (3.16), is a generalized time-periodic function in the sense of Definition 4.1, and the corresponding period, phase shift and translation displacement are \( T, \varphi(T) \) and \( \mathcal{D}(T) = (\mathcal{D}_1(T), \cdots, \mathcal{D}_n(T)) \), respectively, where

\[
T = \int_0^\mathcal{P} \frac{1}{\Lambda_+(\zeta) - \Lambda_-(\zeta)} d\zeta, \quad \varphi(T) = \int_0^\mathcal{P} \frac{\Lambda_+(\zeta)}{\Lambda_+(\zeta) - \Lambda_-(\zeta)} d\zeta, \quad \mathcal{D}(T) = \int_0^\mathcal{P} \mathcal{Q}_i(\zeta)d\zeta. \tag{4.12}
\]

**Remark 4.6** Consider the motion of a point \( \theta \) in the string and let \( \theta^t \) be the position of \( \theta \) at the time \( t \). The equation (4.9) implies that the transition length of \( \theta \) is \( \varphi(T) \) when \( \theta \) moves one period \( \mathcal{P} \), say, from some \( \theta_+ \) to \( \theta_+ + \mathcal{P} \). In other words, in this case we have \( \theta^t + T - \theta^t = \varphi(T) \), where \( \varphi(T) \) is given by the second equality in (4.12).

In the above argument, we fix \( \theta_- \) and let \( \theta_+ \) vary over one period \( \mathcal{P} \). Similarly, if we fix \( \theta_+ \) and let \( \theta_- \) vary over one period, then we will have a phase shift \( \varphi_-(T) \) given by

\[
\varphi_-(T) = \int_0^\mathcal{P} \frac{\Lambda_-(\zeta)}{\Lambda_+(\zeta) - \Lambda_-(\zeta)} d\zeta. \tag{4.13}
\]

This implies that the phase shift may not be unique (see Figure 2).
We now consider a point of the string at the time \( t \), denoted by \( \theta^t \). After one-generalized-period motion, the point \( \theta^t \) moves to the point \( \theta^{t+T}_+ \) at the time \( t+T \), if we fix \( \theta_- \). On the other hand, \( \theta^t \) moves to the point \( \theta^{t+T}_- \) at the time \( t+T \), if we fix \( \theta_+ \). The corresponding phase shifts are \( \varphi_+ (T) \) and \( \varphi_- (T) \), respectively. Therefore we have

\[
\theta^{t+T}_+ = \theta^t + \varphi_+ (T), \quad \theta^{t+T}_- = \theta^t + \varphi_- (T).
\tag{4.14}
\]

By (4.14), it follows from the second equation in (4.12) and (4.13) that

\[
\theta^{t+T}_+ - \theta^{t+T}_- = \varphi_+ (T) - \varphi_- (T) = \mathcal{P}.
\tag{4.15}
\]

This shows that \( \theta^{t+T}_+ \) and \( \theta^{t+T}_- \) stand for the same point in the string at the time \( t+T \) since the string is a closed one with the period \( \mathcal{P} \) (see Figure 3).

Figure 3: The solution has different phase shifts \( \varphi_\pm (T) \) due to different moving ways, but the resulting positions \( \theta^{t+T}_\pm \) stand for the same point in the string at the time \( t+T \). It is easy to generalize this discussion to different ways of increasing both \( \theta_+ \) and \( \theta_- \). As long as the combined increments of \( \theta_+ \) and \( \theta_- \) is one period, they all correspond to the same point after taking into account the periodicity of the string.
Summarizing the above discussions, we have the following theorem.

**Theorem 4.1** Under the assumptions (1.8) and (1.9), the Cauchy problem (1.1)-(1.2) admits a unique global $C^2$ solution $x = x(t, \theta)$ on $\mathbb{R}^+ \times \mathbb{R}$, and this solution can be expressed by (3.16). Moreover, the solution $x = x(t, \theta)$ defined by (3.16) is a generalized time-periodic function in the sense of Definition 4.1, and the corresponding generalized period $T$, phase shift $\varphi_+(T)$ and translation displacement $D(T)$ are given by (4.12).

As the end of this section, we would like to point out that, Theorem 4.1 shows an interesting physical phenomenon: the motion of closed strings possesses the generalized time-periodicity in the sense of Definition 4.2. On the other hand, it is well-known that, a closed string stands for an elementary particle in particle physics, the period of the motion of closed string represents the frequency of the corresponding elementary particle. Theorem 4.1 provides a rigorous mathematical proof on this important physical fact.

5 Initial-boundary value problems with homogenous Dirichlet boundary conditions

In this section, we study the motion of a string with open ends. Thus we must include the boundary conditions at the end points. We will consider homogenous Dirichlet boundary conditions in this section and inhomogenous Dirichlet boundary conditions, which has considerable complication, in the next section.

We study the classical $C^2$ solution, $x = x(t, \theta)$, satisfying (1.1) on a time-space region of the form $[0, \infty) \times [0, L]$ and obeying the homogenous Dirichlet boundary conditions

$$x(t, \theta)|_{\theta=0,L} = 0 \quad \text{for} \quad t > 0$$

(5.1)

at the end points of the string together with the initial conditions

$$x(0, \theta) = f(\theta), \quad x_t(0, \theta) = g(\theta),$$

(5.2)

where $(f, g) \in C^2([0, L]) \times C^1([0, L])$. In order for $x$ belonging to $C^2([0, \delta] \times [0, L])$ for arbitrarily small $\delta > 0$, the initial data $f$ and $g$ must necessarily satisfy the compatibility conditions

$$f(\theta)|_{\theta=0,L} = f''(\theta)|_{\theta=0,L} = 0, \quad g(\theta)|_{\theta=0,L} = 0.$$

(5.3)

We have
Suppose that the compatibility conditions in (5.3) are satisfied and suppose further-
that

\[ \min \{ \inf_{\theta \in [0, L]} \Gamma_+ (\pm f(\theta), g(\theta)) \} > \max \{ \sup_{\theta \in [0, L]} \Gamma_- (\pm f(\theta), g(\theta)) \}, \]

(5.4)

where \( \Gamma_\pm \) are defined by

\[ \Gamma_\pm (\mu(\theta), \nu(\theta)) = \frac{1}{|\mu'(\theta)|^2} \left[ -\langle \mu'(\theta), \nu(\theta) \rangle \pm \sqrt{\mathcal{L}(\mu'(\theta), \nu(\theta))} \right], \]

(5.5)
in which

\[ \mathcal{L}(\mu'(\theta), \nu(\theta)) = \langle \nu(\theta), \mu'(\theta) \rangle - (|\nu(\theta)|^2 - 1)|\mu'(\theta)|^2. \]

(5.6)

Then the initial-boundary value problem (1.1), (5.2), (5.1) admits a unique global \( C^2 \) solution
\( x = x(t, \theta) \) on \([0, \infty) \times [0, L]\); Moreover, the global \( C^2 \) solution \( x = x(t, \theta) \) satisfies that, for any
\( (t, \theta) \in [0, \infty) \times [0, L] \), either \( x_\theta(t, \theta) = 0 \) or

\[ \langle x_t(t, \theta), x_\theta(t, \theta) \rangle^2 - (|x_t(t, \theta)|^2 - 1)|x_\theta(t, \theta)|^2 > 0. \]

(5.7)

**Remark 5.1.** As a special case, if

\[ \Gamma_+(f(\theta), g(\theta)) > 0 > \Gamma_-(f(\tilde{\theta}), g(\tilde{\theta})), \quad \forall \ \theta, \tilde{\theta} \in [0, L], \]

then the hypothesis (5.4) is satisfied, and then the conclusions of Theorem 5.1 hold.

**Proof of Theorem 5.1.** Due to the homogenous Dirichlet boundary conditions (5.3), we can
extend any \( C^2 \) solution \( x = x(t, \theta) \) from the interval \([0, L]\) to the interval \([-L, L]\) by

\[ \tilde{x}(t, \theta) = \begin{cases} x(t, \theta) & \text{for } \theta \in [0, L], \\ -x(t, -\theta) & \text{for } \theta \in [-L, 0]. \end{cases} \]

(5.8)

and then the extended \( \tilde{x}(t, \theta) \) is \( 2L \)-periodic. One easily checks that if the given initial data has
the property (5.3), the extended initial data given by

\[ \tilde{f}(0, \theta) = \begin{cases} -f(-\theta) & \text{for } \theta \in [-L, 0], \\ f(\theta) & \text{for } \theta \in [0, L], \end{cases} \]

\[ \tilde{g}(0, \theta) = \begin{cases} -g(-\theta) & \text{for } \theta \in [-L, 0], \\ g(\theta) & \text{for } \theta \in [0, L]. \end{cases} \]

(5.9)
is \( 2L \)-periodic. When the compatibility conditions in (5.3) are satisfied, this extended solution \( \tilde{x} \) is
a \( C^2 \) solution of the equations (1.1) with the initial data \((\tilde{f}, \tilde{g})\). Therefore, under the assumptions
of Theorem 5.1, we make use of Theorem B and obtain the following lemma.

**Lemma 5.1.** Under the assumptions of Theorem 5.1, the Cauchy problem for (1.1) with the fol-
lowing \( 2L \)-periodic initial data

\[ x(0, \theta) = \tilde{f}(\theta), \quad x_t(0, \theta) = \tilde{g}(\theta) \]

(5.10)
admits a unique global $C^2$ solution $x = \tilde{x}(t, \theta)$ on the domain $\mathbb{R}^+ \times \mathbb{R}$; moreover, for any $(t, \theta) \in [0, \infty) \times [0, L]$, either $\tilde{x}_\theta(t, \theta) = 0$ or

$$(\tilde{x}_t(t, \theta), \tilde{x}_\theta(t, \theta))^2 - (|\tilde{x}_t(t, \theta)|^2 - 1)|\tilde{x}_\theta(t, \theta)|^2 > 0, \quad \forall (t, x) \in \mathbb{R}^+ \times \mathbb{R}. \quad (5.11)$$

Let $x(t, \theta)$ be the restriction of $\tilde{x}(t, \theta)$ on the region $[0, \infty) \times [0, L]$. It is obvious that $x(t, \theta)$ is the unique global $C^2$ solution of the mixed initial-boundary value problem (1.1), (5.1) and (5.2), and (5.7) directly follows (5.11). This proves Theorem 5.1.

If we replace the initial data $(p, q)$ by $(\tilde{f}, \tilde{g})$ in (3.2), (3.3) and (3.16), then the resulting formula is the solution formula to the Cauchy problem for system (1.1) with initial data (5.10). Let $x(t, \theta)$ be the restriction of this formula in the region $[0, \infty) \times [0, L]$. Clearly, $x(t, \theta)$ is the solution formula to the initial-boundary value problem (1.1), (5.1) and (5.2).

Notice that the period of the initial data (5.10) is $2L$. We denote the initial characteristic speeds corresponding to $(\tilde{f}, \tilde{g})$ by $\Lambda_{\pm}(\tilde{f}(\theta), \tilde{g}(\theta))$. In the present situation, the generalized period for the solution of the Cauchy problem (1.1) and (5.10) is

$$T = \int_0^{2L} \frac{1}{\Lambda_+(\tilde{f}(\zeta), \tilde{g}(\zeta)) - \Lambda_-(\tilde{f}(\zeta), \tilde{g}(\zeta))} d\zeta, \quad (5.12)$$

the translation displacement $\mathcal{D}(T) = (\mathcal{D}_1(T), \cdots, \mathcal{D}_n(T))$ is given by

$$\mathcal{D}_i(T) = \int_0^{2L} \frac{|\tilde{f}(\zeta)|^2 \tilde{a}_i(\zeta) - (\tilde{g}(\zeta), \tilde{f}(\zeta)) \tilde{f}'(\zeta)}{2 \sqrt{(\tilde{g}(\zeta), \tilde{f}(\zeta))^2 - (|\tilde{g}(\zeta)|^2 - 1)|\tilde{f}'(\zeta)|^2}} d\zeta \quad (i = 1, \cdots, n), \quad (5.13)$$

and the phase shifts read

$$\varphi_{\pm}(T) = \int_0^{2L} \frac{\Lambda_{\pm}(\tilde{f}(\zeta), \tilde{g}(\zeta))}{\Lambda_+(\tilde{f}(\zeta), \tilde{g}(\zeta)) - \Lambda_-(\tilde{f}(\zeta), \tilde{g}(\zeta))} d\zeta, \quad (5.14)$$

where $\varphi_{\pm}(T)$ are the phase shifts induced by $\theta_{\pm}$.

Noting that the functions $\tilde{f}$ and $\tilde{g}$ are odd and $2L$-periodic, we obtain from (5.13) and (5.14) that

$$\mathcal{D}_i(T) = \int_{-L}^{L} \frac{|\tilde{f}(\zeta)|^2 \tilde{g}(\zeta) - (\tilde{g}(\zeta), \tilde{f}(\zeta)) \tilde{f}'(\zeta)}{2 \sqrt{(\tilde{g}(\zeta), \tilde{f}(\zeta))^2 - (|\tilde{g}(\zeta)|^2 - 1)|\tilde{f}'(\zeta)|^2}} d\zeta = 0 \quad (i = 1, \cdots, n) \quad (5.15)$$
\[ \varphi_{\pm}(T) = \int_{-L}^{L} \frac{\Lambda_{\pm}(\tilde{f}(\zeta), \tilde{g}(\zeta))}{\Lambda_{+}(\tilde{f}(\zeta), \tilde{g}(\zeta)) - \Lambda_{-}(\tilde{f}(\zeta), \tilde{g}(\zeta))} d\zeta \]

\[ = \int_{-L}^{L} \frac{-\langle \tilde{g}(\zeta), \tilde{f}'(\zeta) \rangle \pm \sqrt{\langle \tilde{g}(\zeta), \tilde{f}'(\zeta) \rangle^2 - (|\tilde{g}(\zeta)|^2 - 1)|\tilde{f}'(\zeta)|^2}}{2\sqrt{\langle \tilde{g}(\zeta), \tilde{f}'(\zeta) \rangle^2 - (|\tilde{g}(\zeta)|^2 - 1)|\tilde{f}'(\zeta)|^2}} d\zeta \pm L \]

\[ = \pm L, \]

respectively. This implies that the solution to the Cauchy problem (1.1) and (5.10) is time-periodic in the classical sense, and time period is \(2L\). This leads to an interesting physical phenomenon: *when the compatibility conditions in (5.3) hold, the motion of an open finite string possesses the time periodicity* in the classical sense. See Section 7 for detailed numerical illustrations.

### 6 Initial-boundary value problems with inhomogenous Dirichlet boundary conditions

The initial-boundary value problem for the equations (1.1) with the inhomogenous Dirichlet boundary conditions plays an important role in the string theory and particle physics (see [6]). This section is devoted to solving this kind of problem.

To do so, we first establish the following *characteristic-quadrilateral identity* (6.1) (see (A1) in Appendix for more general version). In the domain where the \(C^2\) solution of the Cauchy problem (1.1) and (1.2) exists, we consider the characteristic quadrilateral with vertices \(A, B, C, D\) which is bounded by the \(\lambda_+\)-characteristics \(\overline{AB}\) and \(\overline{DC}\) and the \(\lambda_-\)-characteristics \(\overline{BC}\) and \(\overline{AD}\). See Figure 4. Here we assume that the \(C^2\) solution of the Cauchy problem (1.1)-(1.2) can be expressed by (3.16).
Figure 4: Characteristic quadrilateral with vertices $A, B, C, D$

We have

**Theorem 6.1 (Characteristic-quadrilateral identity)** On the existence domain of the $C^2$ solution of the Cauchy problem (1.1)-(1.2), it holds that

$$x(A) + x(C) = x(B) + x(D),$$

(6.1)

where $x(A)$ stands for the value of the solution $x = x(t, x)$ at the point $A$, etc.

**Proof.** Let $\alpha$ and $\tilde{\alpha}$ be the spatial coordinates of the intersection points of the $\lambda_+$-characteristics $\overrightarrow{AB}$ and $\overrightarrow{DC}$ with the $\theta$-axis, and $\beta$ and $\tilde{\beta}$ be the spatial coordinates of the intersection points of the $\lambda_-$-characteristics $\overrightarrow{AD}$ and $\overrightarrow{BC}$ with the $\theta$-axis, respectively. See Figure 4. It follows from (3.16) that

$$
\begin{align*}
    x_i(A) &= \frac{p_i(\alpha) + p_i(\beta)}{2} + \frac{1}{2} \int_\alpha^{\beta} \frac{|p'(\zeta)|^2 q_i(\zeta) - \langle q(\zeta), p'(\zeta) \rangle p_i'(\zeta)}{\sqrt{(q(\zeta), p'(\zeta))^2 - (|q(\zeta)|^2 - 1)|p'(\zeta)|^2}} d\zeta, \\
    x_i(B) &= \frac{p_i(\alpha) + p_i(\tilde{\alpha})}{2} + \frac{1}{2} \int_\alpha^{\tilde{\alpha}} \frac{|p'(\zeta)|^2 q_i(\zeta) - \langle q(\zeta), p'(\zeta) \rangle p_i'(\zeta)}{\sqrt{(q(\zeta), p'(\zeta))^2 - (|q(\zeta)|^2 - 1)|p'(\zeta)|^2}} d\zeta, \\
    x_i(C) &= \frac{p_i(\tilde{\alpha}) + p_i(\tilde{\beta})}{2} + \frac{1}{2} \int_\tilde{\alpha}^{\tilde{\beta}} \frac{|p'(\zeta)|^2 q_i(\zeta) - \langle q(\zeta), p'(\zeta) \rangle p_i'(\zeta)}{\sqrt{(q(\zeta), p'(\zeta))^2 - (|q(\zeta)|^2 - 1)|p'(\zeta)|^2}} d\zeta, \\
    x_i(D) &= \frac{p_i(\tilde{\beta}) + p_i(\beta)}{2} + \frac{1}{2} \int_\tilde{\beta}^{\beta} \frac{|p'(\zeta)|^2 q_i(\zeta) - \langle q(\zeta), p'(\zeta) \rangle p_i'(\zeta)}{\sqrt{(q(\zeta), p'(\zeta))^2 - (|q(\zeta)|^2 - 1)|p'(\zeta)|^2}} d\zeta.
\end{align*}
$$

(6.2)

(6.1) comes from (6.2) directly. Thus, the proof is complete. □

We now consider the mixed initial-boundary value problem for the equations (1.1) with the inhomogenous Dirichlet boundary conditions. That is to say, we investigate the classical $C^2$ solution, $x = x(t, \theta)$, satisfying (1.1) on a time-space region of the form $[0, \infty) \times [0, L]$ and obeying the initial conditions

$$x(0, \theta) = f(\theta), \quad x_i(0, \theta) = g(\theta)$$

(6.3)
and the inhomogenous Dirichlet boundary conditions

\[ x(t, \theta)|_{\theta = 0} = h_0(t), \quad x(t, \theta)|_{\theta = L} = h_L(t) \quad \text{for} \quad t > 0 \]  

(6.4)
at the end points of the string, where \( h_0, h_L \) are \( C^2 \)-smooth functions, and \( (f, g) \in C^2([0, L]) \times C^1([0, L]) \). Moreover, we assume that

\[ \mathcal{L}(\theta) = (g(\theta), f'(\theta))^2 - (|g(\theta)|^2 - 1)|f'(\theta)|^2 > 0, \quad \forall \theta \in [0, L]. \]

(6.5)

In order for \( x \) belonging to \( C^2([0, \delta] \times [0, L]) \) for arbitrarily small \( \delta > 0 \), the initial data \( f \) and \( g \) must necessarily satisfy the compatibility conditions

\[
\begin{cases}
    f(0) = h_0(0), & g(0) = h'_0(0), \\
    f'(0) = h''_0(0) - 2g(0), & f''(0) = 0,
\end{cases}
\]

(6.6)

**Remark 6.1** The condition (6.6) actually is sufficient for \( x \in C^2 \). If (6.6) is not satisfied, we would find that \( x \) has a jump along the characteristics passing through the points \((0, 0)\) and \((0, L)\). For example, if \( f(0) \neq h_0(0) \), then \( x \) has a jump along the characteristic passing through the origin.

Define two functions on \( \mathbb{R} \), denoted by \( \tilde{f} \) and \( \tilde{g} \), which satisfy

\[
\begin{align*}
\{ & \tilde{f} \in C^2_B(\mathbb{R}), \quad \tilde{g} \in C^1_B(\mathbb{R}), \\
& \tilde{f}(x) = f(x), \quad \tilde{g}(x) = g(x) \quad \text{on} \quad [0, L], \\
& \tilde{\Lambda}_-(\theta_1) < \tilde{\Lambda}_+(\theta_2), \quad \forall \theta_1 \in (-\infty, \theta_2), \\
& \tilde{\mathcal{L}}(\theta) > 0, \quad \forall \theta \in \mathbb{R},
\end{align*}
\]

(6.7)

where \( C^k_B(\mathbb{R}) \) stands for the class of \( C^k \) functions with bounded \( C^k \)-norm, \( \tilde{\Lambda}_\pm(\theta) \) and \( \tilde{\mathcal{L}}(\theta) \) are defined by (1.3) and (1.4), respectively, in which \((p, q)\) is replaced by \((\tilde{f}, \tilde{g})\). Noting (6.7), by Theorem 3.1 we know that the Cauchy problem for the equations (1.1) with the initial data

\[
t = 0: \quad x = \tilde{f}(\theta), \quad x_t = \tilde{g}(\theta)
\]

(6.8)

has a unique global \( C^2 \) solution \( x = \tilde{x}(t, \theta) \), moreover this solution can be explicitly expressed by (3.16).

**Definition 6.1** The boundary conditions \((h_0(t), h_L(t))\) are called to be globally solvable, if there exist two functions \( \tilde{f} \) and \( \tilde{g} \) satisfying (6.7) such that

\[
\tilde{x}(t, 0) = h_0(t), \quad \tilde{x}(t, L) = h_L(t) \quad \text{for} \quad t > 0,
\]

(6.9)

where \( x = \tilde{x}(t, \theta) \) is the \( C^2 \) solution of the Cauchy problem (1.1), (6.8), which can be expressed by (3.16).
Theorem 6.2 Suppose that the compatibility conditions (6.6) are satisfied, and the boundary conditions \((h_0, h_L)\) are globally solvable. Then the mixed initial-boundary value problem (1.1), (6.3) and (6.4) has a unique global \(C^2\) solution \(x = x(t, \theta)\) on the time-space region \([0, \infty) \times [0, L]\).

The proof is obvious, here we omit it.

Remark 6.2 It follows from Section 5 that, under the hypotheses (5.3) and (5.4), the homogeneous boundary conditions \((0, 0)\) are globally solvable.

Theorem 6.2 is a result on the global existence of the \(C^2\) solution of the mixed initial-boundary value problem (1.1), (6.3) and (6.4). However, it is not easy to compute the solution for this kind of problem. In what follows, based on Theorem 6.2 we present a method to compute the solution for the mixed initial-boundary value problem (1.1), (6.3) and (6.4) in the time-space region \([0, \infty) \times [0, L]\). Here we would like to point out that, since our main aim is to present a method to compute the solution and to drive the solution formula, in the following argument we assume that the solution exists on the strip \(\{(t, x)| t > 0, \ 0 < x < L\}\), and for the simplicity, we assume

\[
\lambda_- < 0 < \lambda_+.
\]  

(6.10)

We divide the strip \(\{(t, x)| t > 0, \ 0 < x < L\}\) into a number of regions by the characteristics through the corners and through the points of intersections of the characteristics with the boundaries, etc. as shown in Figure 5.

![Figure 5: The strip divided into a number of regions by the characteristics](image)

In the region \(\Omega_1\) the solution \(x\) is determined by the formula (3.16) from the initial data alone.
In a point $A$ of the region $\Omega_2$ we form the characteristic quadrilateral with vertices $A, B, C, D$ and get $x(A)$ from (6.1) (more precisely, (A1) in the Appendix) as

$$x(A) = -x(C) + x(B) + x(D),$$

with $x(B)$ known from boundary condition (6.4) and $x(C), x(D)$ known since $C, D$ lie in $\Omega_1$. Similarly, we get $x$ successively in all points of the regions $\Omega_3, \Omega_4, \cdots$. Thus, we successfully solve the mixed initial-boundary value problem (1.1), (6.3) and (6.4) in the whole time-space region $[0, \infty) \times [0, L]$.

In the above discussion, we show how to determine the solution over the whole region. In what follows, we drive the solution formula for any fixed point $(t, x)$ in terms of the boundary conditions and the initial data.

For any fixed point $C$ of the strip $[0, \infty) \times [0, L]$, we draw two characteristics backwards in time starting from $C$, denote the points of intersections of the characteristics with the boundaries by $A_1, B_1$. Starting from $A_1$ (resp. $B_1$), we draw backwards in time the $\lambda_-$-characteristic (resp. $\lambda_+$-characteristic), and denote the point of intersection of the characteristic with the boundary $\theta = L$ (resp. $\theta = 0$) by $B_2$ (resp. $A_2$), $\cdots$. In this way, we can obtain three sequences of intersection points $\{A_i\}, \{B_i\}$ and $\{C_i\}$. See Figure 6. Because of the boundedness of $\lambda_k$, each of these sequences only contains finite points. Therefore, we have four and only four cases shown in Figure 6.

Figure 6: Characteristics passing through the point $C$
In what follows, we give the solution formula for the initial-boundary value problem (1.1), (6.3) and (6.4).

Case (a): the case shown in Figure 6(a)

By the characteristic-quadrilateral identity, we have

\[
\begin{align*}
x(C) &= x(A_1) + x(B_1) - x(C_1), \\
x(C_1) &= x(A_2) + x(B_2) - x(C_2), \\
& \cdots, \\
x(C_{k-1}) &= x(A_k) + x(B_k) - x(C_k).
\end{align*}
\]

(6.11)

It follows from (6.11) that

\[
x(C) = (-1)^k x(C_k) - \sum_{i=1}^{k} (-1)^i [x(A_i) + x(B_i)].
\]

(6.12)

In (6.12), \(x(A_i)\) and \(x(B_i)\) \((i = 1, \cdots, k)\) are given by the boundary condition (6.4), and \(x(C_k)\) is given by the solution formula (3.16), which is analytically expressed by the initial data (6.3).

Case (b): the case shown in Figure 6(b)

In this case, (6.12) is still true. Furthermore, in the present situation we have

\[
x(C_k) = x(A_{k+1}) + x(E) - x(A_{k+2})
\]

(6.13)

and

\[
x(E) = x(B_{k+1}) + x(C_{k+1}) - x(B_{k+2}).
\]

(6.14)
Then, we get
\[ x(C_k) = x(A_{k+1}) - x(A_{k+2}) + x(B_{k+1}) - x(B_{k+2}) + x(C_{k+1}). \] (6.15)

Thus, we obtain
\[ x(C) = (-1)^k x(C_{k+1}) - \sum_{i=1}^{k+2} (-1)^i [x(A_i) + x(B_i)] , \] (6.16)
where \( x(A_i) \) and \( x(B_i) \) \((i = 1, \cdots , k + 1)\) are given by the boundary condition (6.4), \( x(A_{k+2}) \) and \( x(B_{k+2}) \) are given by the initial data (6.3), and \( x(C_{k+1}) \) is given by the solution formula (3.16).

**Case (c):** the case shown in Figure 6(c)

Similar to Case (b), in this case (6.12) still holds. Moreover, in the present situation we have
\[ x(C_k) = x(C_{k+1}) + x(B_{k+1}) - x(B_{k+2}). \] (6.17)
Therefore, we obtain
\[ x(C) = (-1)^k x(C_{k+1}) - \sum_{i=1}^{k+2} (-1)^i x(A_i) - \sum_{i=1}^{k} (-1)^i x(B_i), \] (6.18)
where \( x(A_i) \) \((i = 1, \cdots , k)\) and \( x(B_j) \) \((j = 1, \cdots , k + 1)\) are given by the boundary condition (6.4), \( x(B_{k+2}) \) is given by the initial data (6.3), and \( x(C_{k+1}) \) is given by the solution formula (3.16).

**Case (d):** the case shown in Figure 6(d)

This situation is very similar to Case (c), we can easily obtain
\[ x(C) = (-1)^k x(C_{k+1}) - \sum_{i=1}^{k+2} (-1)^i x(A_i) - \sum_{i=1}^{k} (-1)^i x(B_i), \] (6.19)
where \( x(A_i) \) \((i = 1, \cdots , k + 1)\) and \( x(B_j) \) \((j = 1, \cdots , k)\) are given by the boundary condition (6.4), \( x(A_{k+2}) \) is given by the initial data (6.3), and \( x(C_{k+1}) \) is given by the solution formula (3.16).

We can put the solution formulas of the above four cases into one unified form. Let \( m \) and \( n \) be the largest indexes of \( A \) and \( B \), respectively, namely \((A_1, A_2, \cdots, A_m)\) and \((B_1, B_2, \cdots, B_n)\). Here \( A_m \) can be either a point located at the left boundary or a point at the initial line. Similar is true for \( B_n \). Combining (6.12), (6.16), (6.18) and (6.19), we have
\[ x(C) = (-1)^m x(C_{m+1}) - \sum_{i=1}^{m} (-1)^i x(A_i) - \sum_{i=1}^{n} (-1)^i x(B_i). \] (6.20)
In particular, we would like to point out that \(|m - n|\) can only be 0 or 2.

(6.20) gives the solution formula for the initial-boundary value problem (1.1), (6.3) and (6.4).
7 Numerical illustrations

This section is devoted to some numerical examples to illustrate our main results.

In Section 3, we succeed in providing a parametric version (3.16) of general solution formula to a system of nonlinear equations for the motion of relativistic strings in the Minkowski space \( \mathbb{R}^{1+n} \). This solution formula can describe the global time-like extremal surfaces with singular points. The numerical example 1 given below illustrates an extremal surface with singular points in the Minkowski space \( \mathbb{R}^{1+2} \), which is evaluated and described by the solution formula (3.16).

**Numerical example 1** Consider the motion of a unit circle with the initial velocity

\[
t = 0 : x_t = (0, -0.99 \sin(3\theta)).
\]  

Figure 7 is the corresponding extremal surface (over one period) in the Minkowski space \( \mathbb{R}^{1+2} \); Figure 8 is the projection of the above extremal surface in the \((x, y)\)-plane. From Figures 7-8, we clearly observe that this surface contains singular points and the motion of this closed string with an initial shape of a unit circle possesses the time periodicity.

Figure 7: An extremal surface with singular points in the Minkowski space \( \mathbb{R}^{1+2} \)
In Section 5, we extend the main result to the mixed initial-boundary value problems for strings with homogenous Dirichlet boundary conditions at their end points. The numerical example 2 given below describes the dynamics of an open string with homogenous Dirichlet boundary conditions at its end points.

**Numerical example 2** Consider the motion of an open string with the initial shape

\[ x(0, \theta) = (\sin(2\pi \theta), \sin(4\pi \theta)), \quad \theta \in [0, 1], \]

the initial velocity

\[ x_t(0, \theta) = (0, 0.5 \sin(2\pi \theta)), \quad \theta \in [0, 1] \]

and homogenous Dirichlet boundary conditions at their end points.

Obviously, the compatibility conditions in (5.3) are satisfied. Figure 9 bellow gives the corresponding extremal surface in the Minkowski space \(\mathbb{R}^{1+2}\) formed by the motion of this string; Figure 10 is the projection of the extremal surface in the \((x, y)\)-plane; Figure 11 shows the dynamics of the string moving in the Minkowski space \(\mathbb{R}^{1+2}\). Here we particularly point out that, in Figure 9 the blue part stands for the extremal surface formed by the motion of the open string, while the red part stands for the extremal surface formed by the motion of the extended part (image string) on the interval \([-1, 0]\) of the extended string. In order to understand how a point in the string moves in the Minkowski space \(\mathbb{R}^{1+2}\), we mark a point of the string with \(\theta = \frac{1}{2}\) by a marker \(a\), and the corresponding point at \(\theta = -\frac{1}{2}\) in the extended part by \(c\), see the first picture in Figure 11. Since we only consider the homogenous Dirichlet boundary conditions, the far end point of real part of the string and its image, namely, the far end point of extended part of the string always
coincide, then we may denote them by $b$ (see Figure 11). Three curves in Figure 9 stand for the trajectories of the points $a$, $b$ and $c$, where the black curve stands for the trajectory of the point $a$, the green one is the trajectory of $c$, and the blue (or purple) straight line is the trajectory of the point $b$. See Figure 11 for their detailed dynamic processes of these points.

Figure 9: The extremal surface formed by the motion of open string in Numerical example 2

Figure 10: The projection of the above extremal surface in the $(x, y)$-plane
Remark 7.1 From Figure 11, we observe that topological singularities appear in the motion of strings. In order to investigate the development of topological singularities, we have constructed several numerical examples and made some movies which can clearly show how the topological structure changes as the strings move in the Minkowski space $\mathbb{R}^{1+2}$. Please see [14] for these interesting movies.
8 Summary and discussions

The extremal surfaces play an important role in the theoretical apparatus of elementary particle physics. A free string is a one-dimensional physical object whose motion is represented by a
time-like extremal surface in the Minkowski space. In this paper we obtain the following results:

(a) We succeed in providing a parametric version of general solution formula to a system of nonlinear equations for the motion of relativistic strings in the Minkowski space $\mathbb{R}^{1+n}$. We show that this formula is a solution to the time-like (hyperbolic case) extremal surface of the Bjorling-type problem in the Minkowski space, and it is a counterpart of the Beckenbach formula of the Bjorling problem in the $n$-dimensional Euclidean space. In particular, we would like to point out that our solution formula can describe the global time-like extremal surfaces with singular points (see Figure 8).

(b) We prove that the smooth solution of this system is generalized time-periodic in the sense of Definition 4.1, provided that the initial data is periodic with respect to spatial parameter $\theta$. This leads to the following physical phenomenon: the motion of closed strings possesses the generalized time-periodicity. Therefore, the space-periodicity implies the time-periodicity for the relativistic strings moving in the Minkowski space $\mathbb{R}^{1+n}$. This explains the phenomenon of periodicity of solutions observed from the recent numerical studies [13].

(c) We also investigate the general solution formula of a finite segment of a relativistic string with homogenous Dirichlet boundary conditions. Since the inhomogenous Dirichlet boundary conditions play an important role in the string theory and particle physics (see [6]), by introducing and proving an interesting *characteristic-quadrilateral identity* we succeed in solving the mixed initial-boundary value problem for the equations (1.1) with the inhomogenous Dirichlet boundary conditions. It is well known that, Neumann boundary conditions are also important in both mathematics and physics, such a kind of problem is worthy of studying in the future.

(d) We show that the Born-Infeld equation is a special case of the motion of relativistic string in the Minkowski space $\mathbb{R}^{1+2}$. The Born-Infeld equation is the simplest case of nonlinear modification of the Maxwell equations, proposed by Born and Infeld [5] to solve the electrostatic divergence generated by point particles in classical Electrodynamics (see [11], [7]). This equation has a lot of interesting features (see Boillat [4] for some mathematical aspects and Gibbons [10] for its impact in modern high energy physics and string theory), and many important results have been obtained (see [1]-[2], [7]-[8], [16]-[19]). Since the equation (3.21) can be viewed as a special case of the equations (1.1), the following questions arise naturally: whether do the equations (1.1) possess interesting features similar to that of (3.21) (see [1])? From the point of view of mathematics, how do we construct the solutions of the equations (1.1) corresponding to the extremal surfaces of mixed type? From the point of view of physics, what are the Hamiltonian structure, Lax pairs, higher order symmetries, etc.? These problems are fundamental, important and worthy of studying...
Appendix: General characteristic-quadrilateral identity

Let $\Omega$ be a region in $\mathbb{R}^+ \times \mathbb{R}$, and $x = x(t, \theta)$ be a smooth solution of the equations (1.1) in this region, which corresponds to a piece of a time-like surface. Let $A, B, C$ and $D$ be points in $\Omega$, and assume that $A$ and $B$ (resp. $D$ and $C$) are connected by a $\lambda_+$-characteristic and $A$ and $D$ (resp. $B$ and $C$) are connected by a $\lambda_-$-characteristic, see Figure 12. Since the solution $x = x(t, \theta)$ corresponds to a time-like surface, this implies that the system (2.3) is hyperbolic, and that the $\lambda_\pm$-characteristics through the points $A, B, C$ and $D$ always exist in $\Omega$. In what follows, we assume that the characteristic quadrilateral $ABCD$ completely lies in $\Omega$. We have

**Theorem A.1 (Characteristic-quadrilateral identity)** It holds that

$$x(A) + x(C) = x(B) + x(D). \quad (A1)$$

**Proof.** Let $(t_A, \theta_A)$ be coordinates of the point $A$, etc.

**Special case:** $t_B = t_D \triangleq t_0$

Since $x = x(t, \theta)$ is a smooth solution of the equations (1.1), we denote

$$x(t_0, \theta) \triangleq p(\theta), \quad x_t(t_0, \theta) \triangleq q(\theta), \quad \forall \theta \in [\theta_B, \theta_D].$$

By Theorem 3.1, we have

$$\begin{aligned}
    x_i(A) &= \frac{p_i(\theta_B) + p_i(\theta_D)}{2} + \frac{1}{2} \int_{\theta_B}^{\theta_D} \frac{|p'(\zeta)|^2 q_i(\zeta) - \langle q(\zeta), p'(\zeta) \rangle p_i'(\zeta)}{\sqrt{(q(\zeta), p'(\zeta))^2 - (|q(\zeta)|^2 - 1)|p'(\zeta)|^2}} d\zeta,
    \\
    x_i(C) &= \frac{p_i(\theta_B) + p_i(\theta_D)}{2} + \frac{1}{2} \int_{\theta_B}^{\theta_D} \frac{|p'(\zeta)|^2 q_i(\zeta) - \langle q(\zeta), p'(\zeta) \rangle p_i'(\zeta)}{\sqrt{(q(\zeta), p'(\zeta))^2 - (|q(\zeta)|^2 - 1)|p'(\zeta)|^2}} d\zeta.
\end{aligned}$$

Summing up these equations gives

$$x_i(A) + x_i(C) = p_i(\theta_B) + p_i(\theta_D) = x_i(B) + x_i(D).$$

This proves the desired (A1).

**General case:** $t_B \neq t_D$

Without of loss of generality, we may assume $t_B > t_D$. Let $B_1$ be the point of intersection of the line $t = t_B$ with the characteristic $\hat{AD}$, and $B_2$ be the point of intersection of the characteristic $\hat{BC}$ with the $\lambda_+$-characteristic through the point $B_1$, see Figure 12.
Case 1: $t_{B_2} = t_D$

In this case, noting

$$t_B = t_{B_1}, \quad t_{B_2} = t_D,$$

similar to the above special case, we have

\[
\begin{aligned}
&x(A) + x(B_2) = x(B) + x(B_1), \\
x(B_1) + x(C) = x(B_2) + x(D).
\end{aligned}
\]

Summing up them yields the desired (A1).

Case 2: $t_{B_2} \neq t_D$

Without of loss of generality, we assume $t_D > t_{B_2}$. Let $B_3$ be the point of intersection of the line $t = t_{B_2}$ with the characteristic $\widehat{DC}$, and $B_4$ be the point of intersection of the characteristic $\widehat{B_1B_2}$ with the $\lambda_-$-characteristic through the point $B_3$, see Figure 12.

Case 2-1: $t_{B_4} = t_D$

As before, in this case we have

\[
\begin{aligned}
&x(A) + x(B_2) = x(B) + x(B_1), \\
x(B_1) + x(C) = x(B_2) + x(B_3), \\
x(B_1) + x(B_3) = x(B_4) + x(D).
\end{aligned}
\]

Summing up them gives the desired (A1).

Case 2-2: $t_{B_4} \neq t_D$

In this case, we continue to divide the characteristic quadrilateral $B_1B_4B_3D$ in a way used in the above argument. If, by finite steps, the resulting characteristic quadrilateral reduces to the above special case, then summing up these equalities (similar to (A2)) leads to the desired (A1). Otherwise, we continue to divide the resulting characteristic quadrilateral, and denote the resulting
characteristic quadrilateral after \( n \) step by \( \tilde{A}_n\tilde{B}_n\tilde{C}_nD \) (in which \( \tilde{A}_n \) is the highest point, and \( \tilde{C}_n \) is the lowest one). For example, the resulting characteristic quadrilateral \( \tilde{A}_1\tilde{B}_1\tilde{C}_1D \) after the first step is \( B_1B_2CD \), the resulting characteristic quadrilateral \( \tilde{A}_2\tilde{B}_2\tilde{C}_2D \) after the second step is the quadrilateral \( B_1B_4B_3D \), etc. See Figure 12. Noting the boundedness of the characteristics \( \lambda_{\pm} \), we observe that

\[
\tilde{A}_n, \tilde{B}_n, \tilde{C}_n \rightarrow D \quad \text{as} \quad n \rightarrow \infty.
\]

Thus,

\[
\begin{align*}
x(A) + x(B_2) &= x(B) + x(B_1), \\
x(B_4) + x(C) &= x(B_2) + x(B_3), \\
&\ldots \ldots \\
x(\tilde{A}_n) + x(\tilde{C}_n) &= x(\tilde{B}_n) + x(D) + [x(\tilde{A}_n) + x(\tilde{C}_n) - x(\tilde{B}_n) - x(D)].
\end{align*}
\]

Summing up them leads to

\[
x(A) + x(C) = x(B) + x(D) + [x(\tilde{A}_n) + x(\tilde{C}_n) - x(\tilde{B}_n) - x(D)].
\]

Noting (A3) and letting \( n \rightarrow \infty \) yields the desired (A1). The proof is complete. ■

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