A PROOF OF THE FABER INTERSECTION NUMBER CONJECTURE

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Abstract. We prove the Faber intersection number conjecture and other more general results by using a recursion formula of n-point functions for intersection numbers on moduli spaces of curves. We also present several conjectural properties of Gromov-Witten invariants generalizing results on intersection numbers.

1. Introduction

Let $\overline{\mathcal{M}}_{g,n}$ be the moduli space of stable n-pointed genus g complex algebraic curves and ψ_i the first Chern class of the line bundle corresponding to the cotangent space of the universal curve at the ith marked point.

We use Witten's notation

$$\langle\tau_{d_1}\cdots\tau_{d_n}\rangle_g := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1}\cdots\psi_n^{d_n}.$$

Around 1993, Faber [1] proposed remarkable conjectures about the structure of tautological ring $\mathcal{R}^*(\mathcal{M}_g)$. An important part of Faber's conjectures is the intersection number conjecture which asserts that the following relations hold in $\mathcal{R}^{g-2}(\mathcal{M}_g)$,

$$\pi_*(\psi_1^{d_1+1}\ldots\psi_n^{d_n+1}) = \frac{(2g-3+n)!(2g-1)!!}{(2g-1)!\prod_{j=1}^n(2d_j+1)!!}\kappa_{g-2}, \quad \text{for } \sum_{j=1}^n d_j = g-2, \tag{1}$$

where $\pi : \overline{\mathcal{M}}_{g,n} \longrightarrow \overline{\mathcal{M}}_g$ is the forgetful morphism.

The Faber intersection number conjecture computes all top intersections in the tautological ring $\mathcal{R}^*(\mathcal{M}_g)$ and determines its ring structure if we assume Faber's perfect pairing conjecture (which is still open!).

The Faber intersection number conjecture is equivalent to

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1}\ldots\psi_n^{d_n}\lambda_g\lambda_{g-1} = \frac{(2g-3+n)!|B_{2g}|}{2^{2g-1}(2g)!\prod_{j=1}^n(2d_j-1)!!}, \tag{2}$$

where B_{2g} denotes the Bournoulli number. By Mumford's formula [9] for the Chern character of Hodge bundles, the above identity is equivalent to

$$\begin{aligned}\frac{(2g-3+n)!}{2^{2g-1}(2g-1)!\prod_{j=1}^n(2d_j-1)!!} =& \langle\tau_{2g}\prod_{j=1}^n\tau_{d_j}\rangle_g - \sum_{j=1}^n\langle\tau_{d_j+2g-1}\prod_{i\neq j}\tau_{d_i}\rangle_g \\ &+ \frac{1}{2}\sum_{j=0}^{2g-2}(-1)^j\langle\tau_{2g-2-j}\tau_j\prod_{i=1}^n\tau_{d_i}\rangle_{g-1} \\ &+ \frac{1}{2}\sum_{\underline{n}=I\coprod J}\sum_{j=0}^{2g-2}(-1)^j\langle\tau_j\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\tau_{2g-2-j}\prod_{i\in J}\tau_{d_i}\rangle_{g-g'},\end{aligned} \tag{3}$$

where $d_j \geq 1$, $\sum_{j=1}^n d_j = g+n-2$.

The identity (2) can be derived from the degree 0 Virasoro conjecture for $\mathbb{P}^2$ [3]. Givental [4] has announced a proof of Virasoro conjecture for $\mathbb{P}^n$. Y.P. Lee and R. Pandharipande are writing a book supplying the details.

Goulden, Jackson and Vakil [5] recently give a more enlightening proof of identity (1) for up to three points. Their remarkable proof uses relative virtual localization and a combinatorialization of Hodge integrals.

Now we explain our approach to prove identity (3), hence the Faber intersection number conjecture. We will use a recursion formula of n-point functions

$$F_g(x_1,\dots,x_n)=\sum_{\sum d_j=3g-3+n}\langle\tau_{d_1}\cdots\tau_{d_n}\rangle_g\prod_{j=1}^n x_j^{d_j}.$$

Lemma 1.1. [7, 8] *Let $g\geq 0$ and $n\geq 1$. Then*

$$(2g+n-1)F_g(x_1,\dots,x_n)=\frac{\left(\sum_{j=1}^n x_j\right)^3}{12}F_{g-1}(x_1,\dots,x_n)$$
$$+\frac{1}{2\left(\sum_{j=1}^n x_j\right)}\sum_{h=0}^{g}\sum_{\underline{n}=I\coprod J}\left(\sum_{i\in I}x_i\right)^2\left(\sum_{i\in J}x_i\right)^2F_h(x_I)F_{g-h}(x_J).$$

Proof. By induction, we have

$$F_g(x_1,\dots,x_n,0)=\frac{1}{2g+n}\left(\frac{\left(\sum_{j=1}^n x_j\right)^4}{12}F_{g-1}(x_1,\dots,x_n)\right.$$
$$\left.+\frac{1}{2}\sum_{h=0}^{g}\sum_{\underline{n}=I\coprod J}\left(\sum_{i\in I}x_i\right)^2\left(\sum_{i\in J}x_i\right)^2F_h(x_I)F_{g-h}(x_J)+\left(\sum_{j=1}^n x_j\right)F_g(x_1,\dots,x_n)\right)$$
$$=\left(\sum_{j=1}^n x_j\right)F_g(x_1,\dots,x_n).$$

So we need only verify that $F_g(y,x_1,\dots,x_n)$, recursively defined as above, satisfies Witten's KdV coefficient equation (regarding as an ODE in y). By the uniqueness of ODE solutions, this will prove that they are indeed the n-point functions for intersection numbers on moduli spaces of curves. The verification is straightforward but tedious. We have included it in the appendix. ☐

One uses the notation

$$L_g^{a,b}(y,x_1\dots,x_n)=\sum_{h=0}^{g}\sum_{\underline{n}=I\coprod J}(y+\sum_{i\in I}x_i)^a(-y+\sum_{i\in J}x_i)^bF_h(y,x_I)F_{g-h}(-y,x_J).$$

We now prove that the Faber intersection number conjecture can be reduced to statements about the above functions.

Proposition 1.2. *The following three statements together imply identity (3), hence the Faber intersection number conjecture.*

i)
$$\left[L_g^{0,0}(y,x_1\dots,x_n)\right]_{y^{2g-2}}=0;$$

ii) *For* $k > 2g$,

$$\left[L_g^{2,2}(y, x_1 \dots, x_n)\right]_{y^k} = 0;$$

iii) *For* $d_j \geq 1$ *and* $\sum_{j=1}^n d_j = g + n$,

$$\left[L_g^{2,2}(y, x_1 \dots, x_n)\right]_{y^{2g} \prod_{j=1}^n x_j^{d_j}} = \frac{(2g+n+1)!}{4^g (2g+1)! \prod_{j=1}^n (2d_j - 1)!!}.$$

Proof. Since one and two-point functions in genus 0 are

$$F_0(x) = \frac{1}{x^2}, \qquad F_0(x, y) = \frac{1}{x+y} = \frac{1}{y} - \frac{x}{y^2} + \frac{x^2}{y^3} + \cdots,$$

it's consistent to define

$$\langle \tau_{-2} \rangle_0 = 1, \qquad \langle \tau_k \tau_{-1-k} \rangle_0 = (-1)^k, \ k \geq 0.$$

Let A denotes the following term

$$\frac{1}{2} \sum_{\underline{n} = I \coprod J} \sum_{j=0}^{2g-2} (-1)^j \langle \tau_j \prod_{i \in I} \tau_{d_i} \rangle_{g'} \langle \tau_{2g-2-j} \prod_{i \in J} \tau_{d_i} \rangle_{g-g'} + \langle \prod_{j=1}^n \tau_{d_j} \tau_{2g} \rangle_g - \sum_{j=1}^n \langle \tau_{d_j + 2g - 1} \prod_{i \neq j} \tau_{d_i} \rangle_g$$

and

$$B = \frac{1}{2} \sum_{j=0}^{2g-2} (-1)^j \langle \tau_{2g-2-j} \tau_j \prod_{i=1}^n \tau_{d_i} \rangle_{g-1} = \left[\frac{1}{2} F_{g-1}(y, -y, x_1, \dots, x_n)\right]_{y^{2g-2}}.$$

Note that the right-hand side of identity (3) is exactly $A + B$. By allowing the index to run over all integers, we have

$$\begin{aligned} A &= \frac{1}{2} \sum_{\underline{n} = I \coprod J} \sum_{j \in \mathbb{Z}} (-1)^j \langle \tau_j \prod_{i \in I} \tau_{d_i} \rangle_{g'} \langle \tau_{2g-2-j} \prod_{i \in J} \tau_{d_i} \rangle_{g-g'} \\ &= \left[\sum_{h=0}^{g} \sum_{\underline{n} = I \coprod J} F_h(y, x_I) F_{g-h}(-y, x_J)\right]_{y^{2g-2} \prod_{i=1}^n x_i^{d_i}} \\ &= \left[L_g^{0,0}(y, x_1, \dots, x_n)\right]_{y^{2g-2} \prod_{i=1}^n x_i^{d_i}} = 0. \end{aligned}$$

From Lemma 1.1, we have

$$\frac{1}{2} \left(\sum_{j=1}^n x_j\right) F_{g-1}(x_1, \dots, x_n) = \frac{\left(\sum_{j=1}^n x_j\right)^4}{24(2g+n-1)} F_{g-2}(x_1, \dots, x_n)$$

$$+ \frac{1}{2(2g+n-1)} \left(L_{g-1}^{2,2}(y, x_{\underline{n}}) + \sum_{h=0}^{g-1} \sum_{\underline{n} = I \coprod J} \left(\sum_{i \in I} x_i\right)^2 \left(\sum_{i \in J} x_i\right)^2 F_h(y, -y, x_I) F_{g-1-h}(x_J)\right).$$

If we assume statements (ii) and (iii), then we can inductively prove

$$\left[\frac{1}{2} F_g(y, -y, x_1, \dots, x_n)\right]_{y^{2k}} = 0, \quad \text{for } k > g$$

and we have

$$\frac{1}{2} \sum_{j=0}^{2g-2} (-1)^j \langle \tau_{2g-2-j} \tau_j \tau_0 \prod_{i=1}^n \tau_{d_i} \rangle_{g-1} = \frac{(2g+n-2)!}{2^{2g-1} (2g-1)! \prod_{j=1}^n (2d_j - 1)!!},$$

which, by applying string equation and inducting on the maximum index among d_i (see [6]), implies

$$B = \frac{(2g+n-3)!}{2^{2g-1}(2g-1)!\prod_{j=1}^{n}(2d_j-1)!!}.$$

So we see that B is just the constant at the left-hand side of identity (3). $\square$

Proposition 1.2 tells us that in order to prove the Faber intersection number conjecture, we need only prove the three statements (i), (ii) and (iii) about n-point functions. Actually we will prove more general results which are stated as our main theorems, Theorems 2.4, 2.5, 2.6 in the next section. Proposition 1.2, therefore the Faber intersection number conjecture, is a special case of these theorems.

2. Proof of main theorems

The binomial coefficients $\binom{p}{k}$, for $k \geq 0, p \in \mathbb{Z}$ are given by

$$\binom{p}{k} = \begin{cases} 0, & k < 0, \\ 1, & k = 0, \\ \frac{p(p-1)\cdots(p-k+1)}{k!}, & k \geq 1. \end{cases}$$

Lemma 2.1. *Let $a, b \in \mathbb{Z}$ and $n \geq 0$. Then*

$$\sum_{i=0}^{n} \binom{i+a}{i}\binom{n-i+b}{n-i} = \binom{n+a+b+1}{n}.$$

Proof. Since

$$\binom{p}{k} = \binom{p-1}{k} + \binom{p-1}{k-1},$$

denoting the left-hand side of the above equation by $A_n(a,b)$, we have

$$A_n(a,b) = A_n(a-1,b) + A_{n-1}(a,b).$$

First we use induction on n and $|b|$ to prove

$$A_n(0,b) = \binom{n+b+1}{n}.$$

Then we use induction on n and $|a|$ to prove

$$A_n(a,b) = \binom{n+a+b+1}{n}.$$

$\square$

We now prove two lemmas that will serve as base cases for our inductive arguments.

Lemma 2.2. *Let $a, b \in \mathbb{Z}$ and $k \geq 2g-3+a+b$. Then*

i)

$$\left[L_g^{a,b}(y,x)\right]_{y^k} = 0,$$

ii)

$$\left[L_g^{a,b}(y,x)\right]_{y^{2g-4+a+b}x^{g+1}} = \frac{(-1)^b(2g-2+a+b)}{4^g(2g+1)!!}.$$

Proof. We have

$$\sum_{g\geq 0} L_g^{a,b}(y,x_1\dots,x_n)$$
$$=\exp\left(\frac{\sum_{j=1}^n x_j^3}{24}\right)\sum_{\underline{n}=I\coprod J}(y+\sum_{i\in I}x_i)^a(-y+\sum_{i\in J}x_i)^b G(y,x_I)G(-y,x_J),$$

where G is the normalized n-point function defined by

$$G(x_1,\dots,x_n)=\exp\left(\frac{-\sum_{j=1}^n x_j^3}{24}\right)\cdot F(x_1,\dots,x_n).$$

In particular,

$$G(x)=\frac{1}{x^2},\qquad G(x,y)=\frac{1}{x+y}\sum_{k\geq 0}\frac{k!}{(2k+1)!}\left(\frac{1}{2}xy(x+y)\right)^k.$$

The one-point function is due to Witten [10] and the two-point function is due to Dijkgraaf.

For statements (i) and (ii), it's not difficult to see that we need only prove

$$\Big[y^{a-2}(-y+x)^b G_g(-y,x)+(-y)^{b-2}(y+x)^a G_g(y,x)\Big]_{y^k}=0,\quad \text{for } k\geq 2g-3+a+b,$$

and

$$\Big[y^{a-2}(-y+x)^b G_g(-y,x)+(-y)^{b-2}(y+x)^a G_g(y,x)\Big]_{y^{2g-4+a+b}x^{g+1}}=\frac{(-1)^b(2g-2+a+b)}{4^g(2g+1)!!}.$$

Both follow easily from the explicit formula of $G(y,x)$. □

Lemma 2.3. *Let $a,b\in\mathbb{Z}$ and $k\geq a+b-3$. Then*

i)
$$\Big[L_0^{a,b}(y,x_1,\dots,x_n)\Big]_{y^k}=0,$$

ii)
$$\Big[L_0^{a,b}(y,x_1,\dots,x_n)\Big]_{y^{a+b-4}\prod_{j=1}^n x_j}=\frac{(-1)^b(a+b+n-3)!}{(a+b-3)!}.$$

Proof. We have

$$L_0^{a,b}(y,x_1\dots,x_n)=\sum_{\underline{n}=I\coprod J}(y+\sum_{i\in I}x_i)^{|I|-2+a}(-y+\sum_{i\in J}x_i)^{|J|-2+b}.$$

For any monomial $y^k\prod_{j=1}^n x_j^{d_j}$ in $L_0^{a,b}(y,x_1\dots,x_n)$, if $k\geq a+b-3$, then there must be some $d_j=0$. We may assume $d_n=0$, then

$$L_0^{a,b}(y,x_1\dots,x_{n-1},0)$$
$$=\sum_{\underline{n-1}=I\coprod J}\left((y+\sum_{i\in I}x_i)^{|I|-1+a}(-y+\sum_{i\in J}x_i)^{|J|-2+b}+(y+\sum_{i\in I}x_i)^{|I|-2+a}(-y+\sum_{i\in J}x_i)^{|J|-1+b}\right)$$
$$=\left(\sum_{j=1}^{n-1}x_j\right)\sum_{\underline{n-1}=I\coprod J}(x_1+\sum_{i\in I}x_i)^{|I|-2+a}(-x_1+\sum_{i\in J}x_i)^{|J|-2+b}$$

$$= \left(\sum_{j=1}^{n-1} x_j \right) L_0^{a,b}(y, x_1 \ldots, x_{n-1}).$$

So (i) follows by induction on n.

$$\begin{aligned}
&\left[L_0^{a,b}(y, x_1, \ldots, x_n) \right]_{y^{a+b-4} \prod_{j=1}^n x_j} \\
&= (-1)^b \sum_{|I|=0}^{n} \binom{|I|-2+a}{|I|} |I|! \binom{|j|-2+b}{|J|} |J|! \binom{n}{|I|} \\
&= (-1)^b n! \sum_{i=0}^{n} \binom{i-2+a}{i} \binom{n-i-2+b}{n-i} \qquad \text{apply Lemma 2.1} \\
&= (-1)^b n! \binom{a+b+n-3}{n} \\
&= \frac{(-1)^b (a+b+n-3)!}{(a+b-3)!}.
\end{aligned}$$

So we proved (ii). □

Theorem 2.4. *Let $a, b \in \mathbb{Z}$ and $k \geq 2g-3+a+b$. Then*

$$\left[L_g^{a,b}(y, x_1 \ldots, x_n) \right]_{y^k} = 0.$$

Proof. We will argue by induction on g and n, since the theorem holds for $g = 0$ or $n = 1$ proved in the above lemmas. We have

$$\begin{aligned}
&(2g+n-2) L_g^{a,b}(y, x_1 \ldots, x_n) \\
&\quad = \sum_{h=0}^{g} \sum_{\underline{n}=I \coprod J} (y + \sum_{i \in I} x_i)^a (-y + \sum_{i \in J} x_i)^b (2h + |I| - 1) F_h(y, x_I) F_{g-h}(-y, x_J) \\
&\quad + \sum_{h=0}^{g} \sum_{\underline{n}=I \coprod J} (y + \sum_{i \in I} x_i)^a (-y + \sum_{i \in J} x_i)^b F_h(y, x_I) (2(g-h) + |J| - 1) F_{g-h}(-y, x_J).
\end{aligned}$$

By Lemma 1.1, we have

$$\begin{aligned}
&\left[\sum_{h=0}^{g} \sum_{\underline{n}=I \coprod J} (y + \sum_{i \in I} x_i)^a (-y + \sum_{i \in J} x_i)^b (2h + |I| - 1) F_h(y, x_I) F_{g-h}(-y, x_J) \right]_{y^k z} \\
&\qquad = \frac{1}{12} \left[L_{g-1}^{a+3,b}(y, x_1, \ldots, x_n) \right]_{y^k} - \left[L_g^{a,b}(y, x_1 \ldots, x_n) \right]_{y^k} \\
&\qquad\quad + \left[\sum_{h=0}^{g} \sum_{s \geq 0} \binom{a-1}{s} \sum_{\underline{n}=I \coprod J} F_h(x_I) (\sum_{i \in I} x_i)^{s+2} L_{g-h}^{a+1-s,b}(y, x_J) \right]_{y^k}.
\end{aligned}$$

Note that in the last term of the above equation, $|J| < n$. So by induction, for $k \geq 2g-3+a+b$, the sums vanish except for $h = 0$ and $s = 0$, namely the

$$\left[\sum_{\underline{n}=I \coprod J} (\sum_{i \in I} x_i)^{|I|-1} L_g^{a+1,b}(y, x_J) \right]_{y^k}.$$

Let $d_j \geq 1$ for $1 \leq j \leq n$. By induction, it's not difficult to see from the above that

$$(2g+n)\left[L_g^{a,b}(y,x_1\dots,x_n)\right]_{y^k\prod_{j=1}^n x_j^{d_j}} = \frac{1}{12}\left[L_{g-1}^{a+3,b}(y,x_1,\dots,x_n)+L_{g-1}^{a,b+3}(y,x_1,\dots,x_n)\right]_{y^k\prod_{j=1}^n x_j^{d_j}}.$$

By induction, we have

$$\begin{aligned} 0 &= \left(\sum_{i=1}^n x_i\right)\left[L_{g-1}^{a+1,b+1}(y,x_1,\dots,x_n)\right]_{y^k} \quad \text{for } k\geq 2g-3+a+b \\ &= \left[L_{g-1}^{a+2,b+1}(y,x_1,\dots,x_n)+L_{g-1}^{a+1,b+2}(y,x_1,\dots,x_n)\right]_{y^k} \end{aligned}$$

and

$$\begin{aligned} 0 &= \left(\sum_{i=1}^n x_i\right)^3\left[L_{g-1}^{a,b}(y,x_1,\dots,x_n)\right]_{y^k} \quad \text{for } k\geq 2g-5+a+b \\ &= \left[L_{g-1}^{a+3,b}(y,x_1,\dots,x_n)+L_{g-1}^{a,b+3}(y,x_1,\dots,x_n)\right]_{y^k} \\ &\quad + 3\left[L_{g-1}^{a+2,b+1}(y,x_1,\dots,x_n)+L_{g-1}^{a+1,b+2}(y,x_1,\dots,x_n)\right]_{y^k} \\ &= \left[L_{g-1}^{a+3,b}(y,x_1,\dots,x_n)+L_{g-1}^{a,b+3}(y,x_1,\dots,x_n)\right]_{y^k}. \end{aligned}$$

So we have proved that

$$\left[L_g^{a,b}(y,x_1\dots,x_n)\right]_{y^k\prod_{j=1}^n x_j^{d_j}} = 0, \quad \text{for } d_j\geq 1.$$

If some d_j is zero, the above identity still holds by applying the string equation

$$L_g^{a,b}(y,x_1\dots,x_n,0) = \left(\sum_{j=1}^n x_j\right) L_g^{a,b}(y,x_1\dots,x_n).$$

So we conclude the proof of the theorem. □

Theorem 2.5. *Let $a,b\in\mathbb{Z}$, $d_j\geq 1$ and $\sum_j d_j = g+n$. Then*

$$\left[L_g^{a,b}(y,x_1\dots,x_n)\right]_{y^{2g-4+a+b}\prod_{j=1}^n x_j^{d_j}} = \frac{(-1)^b(2g-3+n+a+b)!}{4^g(2g-3+a+b)!\prod_{j=1}^n(2d_j-1)!!}.$$

Proof. As in the proof of the above theorem, we have

$$\begin{aligned} &(2g+n)\left[L_g^{a,b}(y,x_1\dots,x_n)\right]_{y^{2g-4+a+b}\prod_{j=1}^n d_j} \\ &= \frac{1}{12}\left[L_{g-1}^{a+3,b}(y,x_{\underline{n}})+L_{g-1}^{a,b+3}(y,x_{\underline{n}})\right]_{y^{2g-4+a+b}\prod_{j=1}^n d_j} \\ &= -\frac{1}{4}\left[L_{g-1}^{a+2,b+1}(y,x_{\underline{n}})+L_{g-1}^{a+1,b+2}(y,x_{\underline{n}})\right]_{y^{2g-4+a+b}\prod_{j=1}^n d_j} \\ &= -\frac{1}{4}\left[\left(\sum_{i=1}^n x_i\right)L_{g-1}^{a+1,b+1}(y,x_{\underline{n}})\right]_{y^{2g-4+a+b}\prod_{j=1}^n d_j} \end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{4}\sum_{j=1}^{n}\Big[L_{g-1}^{a+1,b+1}(y,x_{\underline{n}})\Big]_{y^{2g-4+a+b}x_j^{d_j-1}\prod_{i\neq j}x_i^{d_i}} \\
&= \frac{(-1)^b(2g-3+n+a+b)!}{4^g(2g-3+a+b)!\prod_{j=1}^n(2d_j-1)!!}\sum_{j=1}^{n}(2d_j-1) \\
&= (2g+n)\frac{(-1)^b(2g-3+n+a+b)!}{4^g(2g-3+a+b)!\prod_{j=1}^n(2d_j-1)!!}.
\end{aligned}
$$

So we have proved the theorem. □

All the three statements in Proposition 1.2 are particular cases of the Theorems 2.4 and 2.5. We thus conclude the proof of the Faber intersection number conjecture.

Let's use the notation

$$
L_g(y,z_{\underline{a}},w_{\underline{b}},x_{\underline{n}})=\sum_{h=0}^{g}\sum_{\underline{n}=I\coprod J}F_h(y,z_1,\dots,z_a,x_I)F_{g-h}(-y,w_1,\dots,w_b,x_J).
$$

Both Theorems 2.4 and 2.5 can be extended without difficulty.

Theorem 2.6. *Let $a\ge 0$, $b\ge 0$, $n\ge 1$. We have*

(1) *For $k\ge 2g-3+a+b$,*

$$
\Big[L_g(y,z_{\underline{a}},w_{\underline{b}},x_{\underline{n}})\Big]_{y^k}=0.
$$

(2) *For $r_j\ge 0$, $s_j\ge 0$, $d_j\ge 1$ and $\sum r_j+\sum s_j+\sum d_j=g+n$,*

$$
\begin{aligned}
&\Big[L_g(y,z_{\underline{a}},w_{\underline{b}},x_{\underline{n}})\Big]_{y^{2g-4+a+b}\prod_{j=1}^a z_j^{r_j}\prod_{j=1}^b w_j^{s_j}\prod_{j=1}^n x_j^{d_j}} \\
&\qquad = \frac{1}{\prod_{j=1}^a(2r_j+1)!!\prod_{j=1}^b(2s_j+1)!!}\cdot\frac{(-1)^b(2g-3+n+a+b)!}{4^g(2g-3+a+b)!\prod_{j=1}^n(2d_j-1)!!}.
\end{aligned}
$$

We may write down the coefficients of $L_g(y,z_{\underline{a}},w_{\underline{b}},x_{\underline{n}})$ explicitly (see [6]). For example, when $a=1,b=0$,

$$
\begin{aligned}
\Big[L_g(y,z,x_{\underline{n}})\Big]_{y^kz^r\prod_{j=1}^n x_j^{d_j}} &= \sum_{\underline{n}=I\coprod J}\sum_{j=0}^{k}(-1)^j\langle\tau_j\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\tau_{k-j}\tau_r\prod_{i\in J}\tau_{d_i}\rangle_{g-g'} \\
&\quad+\langle\tau_{k+2}\tau_r\prod_{j=1}^n\tau_{d_j}\rangle_g-(-1)^k\langle\tau_{k+r+1}\prod_{j=1}^n\tau_{d_j}\rangle_g-\sum_{j=1}^n\langle\tau_{d_j+k+1}\prod_{i\neq j}\tau_{d_i}\rangle_g.
\end{aligned}
$$

When $a=b=1$,

$$
\begin{aligned}
\Big[L_g(y,z,w,x_{\underline{n}})\Big]_{y^kz^rw^s\prod_{j=1}^n x_j^{d_j}} &= \sum_{\underline{n}=I\coprod J}\sum_{j=0}^{k}(-1)^j\langle\tau_j\tau_s\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\tau_{k-j}\tau_r\prod_{i\in J}\tau_{d_i}\rangle_{g-g'} \\
&\quad-\langle\tau_{k+s+1}\tau_r\prod_{j=1}^n\tau_{d_j}\rangle_g-(-1)^k\langle\tau_{k+r+1}\tau_s\prod_{j=1}^n\tau_{d_j}\rangle_g.
\end{aligned}
$$

3. Gromov-Witten invariants

We adopt Gathmann's convention [2] in this section which will simplify the notation, namely we define

$$\langle\tau_{-2}(pt)\rangle_{0,0}^X = 1$$

and

$$\langle\tau_m(\gamma_1)\tau_{-1-m}(\gamma_2)\rangle_{0,0}^X = (-1)^{\max(m,-1-m)}\int_X \gamma_1\cdot\gamma_2, \quad m\in\mathbb{Z}.$$

All other Gromov-Witten invariants that contain a negative power of a cotangent line are defined to be zero.

Motivated by our previous results, we conjecture the following vanishing identities and multinomial value property for Gromov-Witten invariants, which we have checked in various cases. We will discuss them in details in a forthcoming paper.

Conjecture 3.1. *Let $\gamma_i,\lambda_i\in H^*(X)$ and $k\geq 2g-3+a+b$. Then*

$$\sum_{h=0}^{g}\sum_{j\in\mathbb{Z}}(-1)^j\langle\langle\tau_j(T_m)\prod_{i=1}^{a}\tau_{p_i}(\gamma_i)\rangle\rangle_h\langle\langle\tau_{k-j}(T^m)\prod_{i=1}^{b}\tau_{q_i}(\lambda_i)\rangle\rangle_{g-h}=0.$$

Conjecture 3.1 is a direct generaliztion of Theorem 2.4. For example, when $a=b=0$, it becomes

$$\langle\langle\tau_{2k}(1)\rangle\rangle_g-\sum_{d,m}t_d^m\langle\langle\tau_{d+2k-1}(T_m)\rangle\rangle_g+\sum_{h=0}^{g}\sum_{j=0}^{2k-2}(-1)^j\langle\langle\tau_j(T_m)\rangle\rangle_h\langle\langle\tau_{2k-2-j}(T^m)\rangle\rangle_{g-h}=0$$

for $k\geq g$.

Conjecture 3.2. *Let $k>g$. Then*

$$\sum_{j=0}^{2k}(-1)^j\langle\langle\tau_j(T_m)\tau_{2k-j}(T^m)\rangle\rangle_g^X=0.$$

We also have

$$\frac{1}{2}\sum_{j=0}^{2g-2}(-1)^j\langle\langle\tau_j(T_m)\tau_{2g-2-j}(T^m)\rangle\rangle_{g-1}=\langle\langle\mathrm{ch}_{2g-1}(\mathbb{E})\rangle\rangle_g.$$

Conjecture 3.3. *Let $a\geq b$. Then*

$$\left|\left\langle\tau_a(T_m)\tau_b(T^m)\prod_{j=1}^{n}\tau_{p_j}(\gamma_j)\right\rangle_{g,\beta}^X\right|\geq\left|\left\langle\tau_{a+1}(T_m)\tau_{b-1}(T^m)\prod_{j=1}^{n}\tau_{p_j}(\gamma_j)\right\rangle_{g,\beta}^X\right|.$$

Appendix A. Verification for Lemma 1.1

We include the derivation of Lemma 1.1 here just for reader's convenience. We refer the interested reader to our previous papers [7] and [8] for other interesting properties of the n-point functions.

Witten's KdV coefficient equation [10] implies the following ODE satisfied by n-point functions.

$$y\frac{\partial}{\partial y}\left((y+\sum_{j=1}^{n}x_j)^2F_g(y,x_1,\dots,x_n)\right)$$

$$= \frac{y}{8}(y+\sum_{j=1}^{n}x_j)^4F_{g-1}(y,x_1,\ldots,x_n)+\frac{y}{2}(y+\sum_{j=1}^{n}x_j)F_g(y,x_1,\ldots,x_n)$$

$$+\frac{y}{2}\sum_{\underline{n}=I\coprod J}\left(\left(y+\sum_{i\in I}x_i\right)\left(\sum_{i\in J}x_i\right)^3+2\left(y+\sum_{i\in I}x_i\right)^2\left(\sum_{i\in J}x_i\right)^2\right)F_h(y,x_I)F_{g-h}(x_J)$$

$$-\frac{1}{2}\left(y+\sum_{j=1}^{n}x_j\right)^2F_g(y,x_1,\ldots,x_n)$$

We will verify that the functions $F_g(y,x_1,\ldots,x_n)$ recursively defined in Lemma 1.1 satisfy the above ODE. The proof goes by induction on g and n, namely we assume $F_h(y,x_1,\ldots,x_k)$ satisfies Witten's ODE if either $h<g$ or $k<n$. Since Lemma 1.1 holds obviously for $g=0$ or $n=1$ (see [7]), we have the initial case of the induction.

Let LHS and RHS denote the left-hand side and right-hand side of the Witten's ODE. We have

$$(2g+n)LHS=\frac{y\left(y+\sum_{j=1}^{n}x_j\right)^4}{4}F_{g-1}(y,x_1,\ldots,x_n)$$
$$+\frac{y\left(y+\sum_{j=1}^{n}x_j\right)^3}{12}\frac{\partial}{\partial y}\left(\left(y+\sum_{j=1}^{n}x_j\right)^2F_{g-1}(y,x_1,\ldots,x_n)\right)$$
$$+y\sum_{\underline{n}=I\coprod J}\left(y+\sum_{i\in I}x_i\right)^2\left(\sum_{i\in J}x_i\right)^2F_h(y,x_I)F_{g-h}(x_J)$$
$$+\left(y+\sum_{j=1}^{n}x_j\right)y\sum_{\underline{n}=I\coprod J}\frac{\partial}{\partial y}\left(\left(y+\sum_{i\in I}x_i\right)^2F_h(y,x_I)\right)\left(\sum_{i\in J}x_i\right)^2F_{g-h}(x_J)$$

By induction, we substitute the differential terms using Witten's ODE and get

$$(2g+n)LHS=\frac{y\left(y+\sum_{j=1}^{n}x_j\right)^4}{4}F_{g-1}(y,x_{\underline{n}})$$
$$+\frac{\left(y+\sum_{j=1}^{n}x_j\right)^3}{12}\left(\frac{y}{8}\left(y+\sum_{j=1}^{n}x_j\right)^4F_{g-2}(y,x_{\underline{n}})+\frac{y}{2}\left(y+\sum_{j=1}^{n}x_j\right)F_{g-1}(y,x_1,\ldots,x_n)\right.$$
$$+\frac{y}{2}\sum_{\underline{n}=I\coprod J}\left(\left(y+\sum_{i\in I}x_i\right)\left(\sum_{i\in J}x_i\right)^3+2\left(y+\sum_{i\in I}x_i\right)^2\left(\sum_{i\in J}x_i\right)^2\right)F_h(y,x_I)F_{g-1-h}(x_J)$$
$$\left.-\frac{1}{2}\left(y+\sum_{j=1}^{n}x_j\right)^2F_{g-1}(y,x_1,\ldots,x_n)\right)+y\sum_{\underline{n}=I\coprod J}\left(y+\sum_{i\in I}x_i\right)^2\left(\sum_{i\in J}x_i\right)^2F_h(y,x_I)F_{g-h}(x_j)$$
$$+\left(y+\sum_{j=1}^{n}x_i\right)\sum_{\underline{n}=I\coprod J}\left(\frac{y}{8}\left(y+\sum_{i\in I}x_i\right)^4F_{h-1}(y,x_I)+\frac{y}{2}\left(y+\sum_{i\in I}x_i\right)F_h(y,x_I)\right.$$

$$+\frac{y}{2}\sum_{I=I'\coprod I''}\left(\left(y+\sum_{i\in I'}x_i\right)\left(\sum_{i\in I''}x_i\right)^3+2\left(y+\sum_{i\in I'}x_i\right)^2\left(\sum_{i\in I''}x_i\right)^2\right)F(y,x'_I)F(x''_I)$$

$$-\frac{1}{2}\left(y+\sum_{i\in I}x_i\right)^2F_h(y,x_I)\Bigg)F_{g-h}(x_J)\left(\sum_{i\in J}x_i\right)^2$$

Let's introduce some symbols to simplify notations

$$A_g^{a,b}=\sum_{h=0}^{g}\sum_{\underline{n}=I\coprod J}\left(y+\sum_{i\in I}x_i\right)^a\left(\sum_{i\in J}x_i\right)^bF_h(y,x_I)F_{g-h}(x_J),$$

$$B_g^{a,b,c}=\sum_{h=0}^{g}\sum_{\underline{n}=I\coprod J\coprod K}\left(y+\sum_{i\in I}x_i\right)^a\left(\sum_{i\in J}x_i\right)^b\left(\sum_{i\in K}x_i\right)^cF_h(y,x_I)F_{g-h}(x_J).$$

Note that $B_g^{a,b,c}=B_g^{a,c,b}$.

After carefully collecting terms, we arrive at

$$(2g+n)LHS=\left(\frac{y\left(y+\sum_{j=1}^n x_j\right)^4}{4}-\frac{\left(y+\sum_{j=1}^n x_j\right)^4\left(\sum_{j=1}^n x_j\right)}{24}\right)F_{g-1}(y,x_1,\dots,x_n)$$

$$+\frac{y\left(y+\sum_{j=1}^n x_j\right)^7}{96}F_{g-2}(y,x_1,\dots,x_n)+\left(y-\frac{\sum_{j=1}^n x_j}{2}\right)A_g^{2,2}+\frac{y}{2}A_g^{1,3}$$

$$+\frac{y}{24}A_{g-1}^{1,6}+\frac{5y}{24}A_{g-1}^{2,5}+\frac{3y}{8}A_{g-1}^{3,4}+\frac{5y}{12}A_{g-1}^{4,3}+\frac{5y}{24}A_{g-1}^{5,2}$$

$$+\frac{y}{2}B_g^{1,2,4}+\frac{y}{2}B_g^{1,3,3}+\frac{5y}{2}B_g^{2,2,3}+yB_g^{3,2,2}.$$

Substitute the recursion formula for $F_g(x_1,\dots,x_n)$ to the right-hand side. We have

$$(2g+n)RHS=\frac{y}{8}\left(y+\sum_{j=1}^n x_j\right)^4\left(\frac{\left(y+\sum_{j=1}^n x_j\right)^3}{12}F_{g-2}(y,x_1,\dots,x_n)\right.$$

$$\left.+\frac{1}{y+\sum_{j=1}^n x_j}\sum_{\underline{n}=I\coprod J}\left(y+\sum_{i\in J}x_i\right)^2\left(\sum_{i\in J}x_i\right)^2F_h(y,x_I)F_{g-1-h}(x_J)\right)$$

$$+\frac{y}{4}\left(y+\sum_{j=1}^n x_j\right)^4F_{g-1}(y,x_1,\dots,x_n)+\frac{y}{2}\left(y+\sum_{j=1}^n x_j\right)\left(\frac{\left(y+\sum_{j=1}^n x_j\right)^3}{12}F_{g-1}(y,x_1,\dots,x_n)\right.$$

$$\left.+\frac{1}{y+\sum_{j+1}^n x_j}\sum_{h=0}^{g}\sum_{\underline{n}=I\coprod J}\left(y+\sum_{i\in I}x_i\right)^2\left(\sum_{i\in J}x_i\right)^2F_h(y,x_I)F_{g-h}(x_J)\right)$$

$$+\frac{y}{2}\sum_{\underline{n}=I\coprod J}\left(\left(y+\sum_{i\in I}x_i\right)\left(\sum_{i\in J}x_i\right)^3+2\left(y+\sum_{i\in I}x_i\right)^2\left(\sum_{i\in J}x_i\right)^2\right)$$

$$\times \sum_{h=0}^{g} (2h+|I|)\, F_h(y,x_I)F_{g-h}(x_J) \quad \text{(apply Lemma 1.1 to simplify)}$$

$$+ \frac{y}{2} \sum_{\underline{n}=I\coprod J} \left(\left(y + \sum_{i\in I} x_i \right) \left(\sum_{i\in J} x_i \right)^3 + 2 \left(y + \sum_{i\in I} x_i \right)^2 \left(\sum_{i\in J} x_i \right)^2 \right)$$

$$\times \sum_{h=0}^{g} F_h(y,x_I)(2g-2h+|J|-1)F_{g-h}(x_J) \quad \text{(apply Lemma 1.1 to simplify)}$$

$$+ \frac{y}{2} \sum_{h=0}^{g} \sum_{\underline{n}=I\coprod J} \left(\left(y + \sum_{i\in I} x_i \right) \left(\sum_{i\in J} x_i \right)^3 + 2 \left(y + \sum_{i\in I} x_i \right)^2 \left(\sum_{i\in J} x_i \right)^2 \right) F_h(y,x_I)F_{g-h}(x_J)$$

$$- \frac{1}{2} \left(y + \sum_{j=1}^{n} x_j \right)^2 \left(\frac{\left(y+\sum_{j=1}^n x_j\right)^3}{12} F_{g-1}(y,x_1,\ldots,x_n) + \right.$$

$$\left. \frac{1}{y+\sum_{j=1}^n x_j} \sum_{h=0}^{g} \sum_{\underline{n}=I\coprod J} \left(y + \sum_{i\in I} x_i \right)^2 \left(\sum_{i\in J} x_i \right)^2 F_h(y,x_I)F_{g-h}(x_J) \right)$$

After carefully collecting terms, we arrive at

$$(2g+n)RHS = \left(\frac{y\left(y+\sum_{j=1}^n x_j\right)^4}{4} - \frac{\left(y+\sum_{j=1}^n x_j\right)^4 \left(\sum_{j=1}^n x_j\right)}{24} \right) F_{g-1}(y,x_1,\ldots,x_n)$$

$$+ \frac{y\left(y+\sum_{j=1}^n x_j\right)^7}{96} F_{g-2}(y,x_1,\ldots,x_n) + \left(y - \frac{\sum_{j=1}^n x_j}{2} \right) A_g^{2,2} + \frac{y}{2} A_g^{1,3}$$

$$+ \frac{y}{24} A_{g-1}^{1,6} + \frac{5y}{24} A_{g-1}^{2,5} + \frac{3y}{8} A_{g-1}^{3,4} + \frac{5y}{12} A_{g-1}^{4,3} + \frac{5y}{24} A_{g-1}^{5,2}$$

$$+ \frac{y}{2} B_g^{1,2,4} + \frac{y}{2} B_g^{1,3,3} + \frac{5y}{2} B_g^{2,2,3} + yB_g^{3,2,2}.$$

So we verified $LHS = RHS$.

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