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Groups and Lattices
Graduate Course Winter 2008,
Zhejiang University,
Hangzhou, China

Groups and Lattices: main goals

This course will be an introduction to lattices and their isometry groups. The level will be at instructional level for graduate students who have had basic theory of groups, rings and fields.

A lattice is a finitely generated free abelian group with a rational-valued symmetric bilinear form. An isometry of a lattice is a homomorphism of the lattice onto itself which preserves the bilinear form. The set of isometries forms a group under composition of maps.

Example: In the plane \( \mathbb{R}^2 \) with the usual inner product, take the square lattice \( \mathbb{Z}^2 \). Its isometry group is dihedral of order 8. If we take the honeycomb lattice \( \mathbb{Z}(\sqrt{2}, 0) + \mathbb{Z}(-\frac{1}{\sqrt{2}}, \sqrt{3}) \), the isometry group is dihedral of order 12. Both these lattices are integral. The first one is odd and the second one is even.

The lattices in this course will almost always be positive definite. Isometry groups of positive definite lattices are finite groups, so we have an interaction of arithmetic with finite group theory and combinatorics.

The course will begin with general theory of lattices and integral representations of groups.

We shall examine certain lattices and their isometry groups in detail. This list includes root lattices and Weyl groups; the Leech lattice and the twelve sporadic groups found within its isometry group (the Mathieu groups, the groups of Higman-Sims, McLaughlin, Hall-Janko, Suzuki and Conway); the Barnes-Wall lattices in dimensions \( 2^d \) and their groups of shape \( 2_+^{1+2d} \Omega^+ (2d, 2) \).

Related topics, like codes, sphere packing, covering, etc. will be discussed along the way. We will stress group theoretic viewpoints.

Groups and Lattices: a course outline

General properties of lattices. Integral lattices and their duals and discriminant groups. Comparisons of lattices with sublattices, neighbors. Codes.

Existence and uniqueness theorems for lattices. The theta function. Some arithmetic. The modular group \( SL(2, \mathbb{Z}) \).
Isometry groups. Isometries of order 2. Extraspecial groups. Rational representations of finite groups and invariant lattices.

Lattices of special importance and their isometry groups. Certain lattices of ranks up to 8. Root lattices. The Leech lattice of dimension 24 and its isometry group of order $2^{24}3^95^67^211\cdot13\cdot23$. Covering groups. Twelve sporadic simple groups.

The series of Barnes-Wall lattices, of dimensions $2^d$. Sublattices and overlattices of these. Associated finite groups. The Reed-Muller codes and affine geometry over the field of 2 elements.

Sphere packing, covering. Relations to hyperbolic lattices. Modular forms and arithmetic. Extremal lattices.

Prerequisites

Graduate-level basic courses in group theory, field theory and rings, plus some representation theory of finite groups. Students should know most of the material below, and fill in what is missing as soon as possible.

Sylow theorems, actions of a group on a set, orbits, multiple transitivity, systems of imprimitivity. Abelian, nilpotent and solvable groups. Simple groups. The symmetric groups, $Alt_n$, the general linear group, $PSL(n,F)$.

Principal ideal domains (PID) and their finitely generated modules. The equivalence of matrices over such rings (by row operations which add a multiple of one row to another, permute rows and multiply rows by a unit from the ring; similarly for columns). The theory of fundamental invariants (Smith invariants). For us, this means mostly the ring of integers, but sometimes polynomial rings, certain rings of integers like $\mathbb{Z}[\sqrt{-1}]$, $\mathbb{Z}[\frac{-1+\sqrt{-3}}{2}]$, etc.

Tensors and Hom. The symmetric algebra and exterior algebra of a module over a commutative ring. Multilinear forms, especially bilinear forms and the cases of symmetric and alternating forms. Canonical forms for bilinear forms (e.g., Sylvester’s theorem over the real numbers; direct sums of hyperbolic planes and null spaces for alternating forms). The group of isometries of a bilinear form. Witt’s theorem (sufficient conditions for an isometry between subspaces to extend to an isometry on the space).

Wedderburn theory for simple rings with ascending chain condition. These are matrix rings over division algebras.

Some familiarity with basic representation theory of finite groups (irreducible modules, character theory, orthogonality relations, etc.).

Very Helpful

We will use some material which is a bit advanced. I can sketch what we
need and expect to the students to fill in the rest by themselves.

Root systems and Weyl groups (these come up in the theory of Lie algebras). I strongly suggest becoming familiar with the root systems and their symmetry groups, especially types ADE. These are listed in the book of Humphreys or in an appendix to Bourbaki, Groupes et Algebres de Lie, Chapitres 4,5,6.

A little algebraic number theory, mainly rings of integers of quadratic and cyclotomic fields.

Elementary homological algebra (Ext-functors, long exact sequences, group cohomology, etc.).

Familiarity with computer algebra, e.g. Maple, Gap, Magma, etc.