Unitarity cuts with massive propagators and algebraic expressions for coefficients

Ruth Britto[1] and Bo Feng[2,3]

[1]*Institute for Theoretical Physics, University of Amsterdam, Valckenierstraat 65, 1018 XE Amsterdam, The Netherlands*
[2]*Center of Mathematical Science, Zhejiang University, Hangzhou, China*
[3]*Blackett Laboratory & The Institute for Mathematical Sciences, Imperial College, London, SW7 2AZ, United Kingdom*

In the first part of this paper, we extend the *d*-dimensional unitarity cut method of Anastasiou *et al.* to cases with massive propagators. We present formulas for integral reduction with which one can obtain coefficients of all pentagon, box, triangle and massive bubble integrals. In the second part of this paper, we present a detailed study of the phase space integration for unitarity cuts. We carry out spinor integration in generality and give algebraic expressions for coefficients, intended for automated evaluation.

I. INTRODUCTION

With the approach of the Large Hadron Collider experiments, accurate descriptions of particle physics will require knowledge of one-loop cross sections. Computational complexity increases dramatically with the number of legs, even at the amplitude level. It is desirable to find a simple and fast algorithm for these computations. There has been notable progress in the last couple of years [1–11].

The unitarity method introduced in Ref. [12] seeks to compute amplitudes by applying a unitarity cut to an amplitude on one hand, and its expansion in a basis of master integrals on the other [13]. With knowledge of the basis and the general structure of the coefficients in the expansion, the coefficients can be constrained.

The holomorphic anomaly [14] reduces the problem of phase space integration to one of algebraic manipulation, namely, evaluating residues of a complex function. By applying this operation within the unitarity method, coefficients can be extracted systematically. In this manner, a method was introduced to evaluate any finite four-dimensional unitarity cut and systematically derive compact expressions for the coefficients [6,7].

When working with four-dimensional cuts, one loses information of possible rational terms that are cut-free in four dimensions. However, unitarity methods can find rational terms as well, if we carry out the integral in $d = (4 - 2\epsilon)$ dimensions to higher orders in ϵ [15]. This program was developed in Refs. [9,16–18].

Recently, a general *d*-dimensional unitarity cut method was developed for one-loop amplitudes [19,20]. In this method, coefficients are extracted by first separating and performing a four-dimensional integral by a technique of choice, and then identifying the integral over the remaining $d - 4$ dimensions with a coefficient using recursive dimensional shift identities. In principle one can work out the coefficients to all orders in ϵ. In fact, for cases with only massless propagators, this method is a complete alternative to Passarino-Veltman reduction.

In the first part of this paper (Secs. II and III) we generalize the work of [19,20] to cases where propagators have nonzero, nonuniform masses. We present formulas for integral reduction from which one can obtain coefficients of all scalar pentagon, box, triangle and massive bubble integrals. Pentagons in arbitrary dimensions are truly independent master integrals, though in four dimensions they can be expressed in terms of boxes up to $\mathcal{O}(\epsilon)$. Amplitudes allowing massive propagators will most generally have tadpole and massless-bubble master integrals as well. Unitarity methods cannot access this information directly, but other physical considerations pin it down.

In the second part (Secs. IV), we further simplify the technique of spinor integration over phase space by carrying out some intermediate steps in generality. These steps include expressing the integrand as a total derivative and a possible Feynman-parameter integral. We give algebraic expressions for the coefficients of the cut-containing basis integrals. Our formulas should be suitable for programming. To this end we include some possibly helpful identities in the Appendix, as well as a discussion of the issue of multiple poles.

We close with a discussion (Sec. V) of the tree-level input and possible alternative approaches. We compare our approach of spinor integration to the procedure of Ossola, Papadopoulos and Pittau [2] of evaluating integrands at particular values of loop momentum and solving a system of linear equations. Both our method and theirs algebraic substitution at the integrand level. We claim that the complexity is equivalent.

II. UNITARITY CUTS WITH MASSIVE PROPAGATORS

Here we develop the program of double-cut phase space integration in $(4 - 2\epsilon)$ dimensions [9,16–20] by generalizing to the case of propagators with nonzero and different masses.

One difference from the massless case is in the basis of master integrals. If all propagators are massless, the basis consists of scalar pentagons, boxes, triangles and bubbles.

But if some propagators are massive, there are tadpoles and massless bubbles (i.e., the case $K^2 = 0$ for bubbles). The tadpoles are [16]

$$I_1 \sim m^{2-2\epsilon} \frac{\Gamma[1+\epsilon]}{\epsilon(\epsilon-1)}, \qquad (1)$$

$$= \frac{m^2}{\epsilon} + m^2(1 - \gamma - 2\log(m)) + \mathcal{O}(\epsilon), \qquad (2)$$

while for massless bubbles we find

$$I_2(K=0) \sim \frac{M_1^{2-2\epsilon} - M_2^{2-2\epsilon}}{M_1^2 - M_2^2} \frac{\Gamma[1+\epsilon]}{\epsilon(1-\epsilon)} \qquad (3)$$

$$= \frac{1}{\epsilon} + (1-\gamma) - \frac{2}{M_1^2 - M_2^2}(M_1^2 \log(M_1) - M_2^2 \log(M_2))$$
$$+ \mathcal{O}(\epsilon)$$

$$\text{or} \quad \frac{1}{\epsilon} - (\gamma + 2\log M_1) + \mathcal{O}(\epsilon) \quad \text{if } M_1 = M_2. \qquad (4)$$

It is evident from the lack of all momentum dependence that unitarity methods alone can never detect these two functions.

However, as discussed in Ref. [16], it is possible in some cases to address this difficulty by considering the known divergent behavior of the amplitude. For massless gauge theory amplitudes, the conditions are such that quadratic divergences cancel and the remaining (logarithmic) divergence matches the known value. With these conditions we can fix coefficients at leading order in ϵ. In more general cases, we might need some additional information. This could come, for example, from taking the heavy mass limit of one propagator [21]. We shall not discuss the tadpoles and massless bubbles further in this paper.

A. Cuts of scalar integrals

We define the *n*-point scalar function with nonuniform masses as follows[1]:

$$I_n(M_1, M_2, m_1, \ldots, m_{n-2}) \equiv \int \frac{d^{4-2\epsilon}p}{(2\pi)^{4-2\epsilon}} \frac{1}{(p^2 - M_1^2)((p-K)^2 - M_2^2)\prod_{j=1}^{n-2}((p-P_j)^2 - m_j^2)}. \qquad (5)$$

This definition is written with a view towards taking the unitarity cut. The masses of the cut propagators are M_1 and M_2, and the momentum flowing through the cut is K. The other momenta P_j and masses m_j are the ones necessary to complete the scalar function, and not necessarily in cyclic order.

Our calculations are done in the "four-dimensional helicity" (FDH) scheme, i.e. all external momenta K_i are 4 dimensional and only the internal momentum p is $(4 - 2\epsilon)$ dimensional. Thus it is useful to write $p = \tilde{\ell} + \vec{\mu}$, where $\tilde{\ell}$ is 4 dimensional and $\vec{\mu}$ is (-2ϵ) dimensional.

$$\int \frac{d^{4-2\epsilon}p}{(2\pi)^{4-2\epsilon}} = \int \frac{d^4\tilde{\ell}}{(2\pi)^4} \int \frac{d^{-2\epsilon}\ell_\epsilon}{(2\pi)^{-2\epsilon}} = \int \frac{d^4\tilde{\ell}}{(2\pi)^4} \frac{(4\pi)^\epsilon}{\Gamma(-\epsilon)} \int d\mu^2(\mu^2)^{-1-\epsilon}.$$

Then the scalar function as defined in Eq. (5) becomes

$$I_n(M_1, M_2, m_1, \ldots, m_{n-2}) = \frac{(4\pi)^\epsilon}{\Gamma(-\epsilon)} \int d\mu^2(\mu^2)^{-1-\epsilon} \int \frac{d^4\tilde{\ell}}{(2\pi)^4}$$

$$\times \frac{1}{(\tilde{\ell}^2 - M_1^2 - \mu^2)((\tilde{\ell} - K)^2 - M_2^2 - \mu^2)\prod_{j=1}^{n-2}((\tilde{\ell} - P_j)^2 - m_j^2 - \mu^2)}. \qquad (6)$$

Following the setup of Ref. [19], we decompose the 4 momentum into a null component and a component proportional to the cut momentum K.

$$\tilde{\ell} = \ell + zK, \qquad \ell^2 = 0,$$
$$\Rightarrow \int d^4\tilde{\ell} = \int dz d^4\ell \delta^+(\ell^2)(2\ell \cdot K). \qquad (7)$$

We define the "signature of the cut," $\Delta[K, M_1, M_2]$, as

$$\Delta[K, M_1, M_2] \equiv (K^2)^2 + (M_1^2)^2 + (M_2^2)^2 - 2K^2 M_1^2 - 2K^2 M_2^2 - 2M_1^2 M_2^2, \qquad (8)$$

and a dimensionless parameter u to be

$$u \equiv \frac{4K^2\mu^2}{\Delta[K, M_1, M_2]}. \qquad (9)$$

Then

$$\int d\mu^2(\mu^2)^{-1-\epsilon} \rightarrow \left(\frac{\Delta[K, M_1, M_2]}{4K^2}\right)^{-\epsilon} \int_0^1 du\, u^{-1-\epsilon}.$$

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[1]Note that, for ease of presentation, we are omitting the prefactor $i(-1)^{n+1}(4\pi)^{D/2}$ (which was included, for example, in Ref. [16]).

Since $\frac{(4\pi)^\epsilon}{(2\pi)^4\Gamma(-\epsilon)}\left(\frac{\Delta[K,M_1,M_2]}{4K^2}\right)^{-\epsilon}$ is a universal factor on both sides (basis and amplitude) of the cut calculation, we neglect it henceforth.

Thus, we have rewritten the integral as

$$I_n(M_1, M_2, m_1, \ldots, m_{n-2}) = \int_0^1 duu^{-1-\epsilon} \int dz d^4\ell \delta^+(\ell^2)(2\ell \cdot K)$$

$$\times \frac{1}{(\tilde{\ell}^2 - M_1^2 - \mu^2)((\tilde{\ell} - K)^2 - M_2^2 - \mu^2)\prod_{j=1}^{n-2}((\tilde{\ell} - P_j)^2 - m_j^2 - \mu^2)}, \tag{10}$$

where μ^2 is related to u through Eq. (9).

B. Kinematics and the domain of integration

In this subsection we determine the integration domain.

Assuming $K^2 \neq 0$, we choose the frame where $\vec{K} = (K, 0, 0, 0)$, $\tilde{\ell} = (x, y, 0, 0)$. Then the second cut propagator is $\tilde{\ell} - \vec{K} = (x - K, y, 0, 0)$. From the on-shell conditions

$$x^2 - y^2 = M_1^2 + \mu^2, \qquad (x - K)^2 - y^2 = M_2^2 + \mu^2,$$

we solve for x and y to find

$$x = \frac{K^2 + M_1^2 - M_2^2}{2K}, \qquad y = \pm\sqrt{x^2 - M_1^2 - \mu^2}. \tag{11}$$

The requirement that y has a real solution is the following constraint:

$$\mu^2 \leq \frac{\Delta[K, M_1, M_2]}{4K^2}, \tag{12}$$

with the definition of $\Delta[K, M_1, M_2]$ given in Eq. (8). The condition (12) also requires the right hand side to be positive, which can be arranged by working in a region with sufficiently large cut momentum, $K > M_1 + M_2$. Then the physical constraint (12) restricts u to lie in a unit interval:

$$u \in [0, 1]. \tag{13}$$

After using (7) to decompose the vector $\tilde{\ell} = \ell + zK$, the lightlike condition $\ell^2 = 0$ becomes $(x - zK)^2 = y^2$, or

$$z^2K^2 - z(K^2 + M_1^2 - M_2^2) + (M_1^2 + \mu^2) = 0.$$

Solving this equation, we find

$$z = \frac{(K^2 + M_1^2 - M_2^2) \pm \sqrt{\Delta[K, M_1, M_2] - 4K^2\mu^2}}{2K^2}. \tag{14}$$

In the kinematic region of this cut $K > 0$. Then the positive-light-cone condition $\delta^+(\ell^2)$ is equivalent to $x - zK > 0$. Consequently, exactly one of the two solutions (14) for z is selected. Specifically,

$$z = \frac{(K^2 + M_1^2 - M_2^2) - \sqrt{\Delta[K, M_1, M_2] - 4K^2\mu^2}}{2K^2}, \tag{15}$$

which we rewrite as

$$z = \frac{a - b\sqrt{1 - u}}{2}, \tag{16}$$

where we have defined two useful parameters:

$$a \equiv \frac{K^2 + M_1^2 - M_2^2}{K^2}, \qquad b \equiv \frac{\sqrt{\Delta[K, M_1, M_2]}}{K^2}. \tag{17}$$

In the massless limit $a = b = 1$.

C. First steps: Separating the four-dimensional integral

Now we are ready to discuss the cut integral. Consider the two delta functions of the cut propagators

$$\delta(\tilde{\ell}^2 - M_1^2 - \mu^2)\delta((\tilde{\ell} - K)^2 - M_2^2 - \mu^2).$$

Inside the integral, we perform the following manipulations:

$$\delta(\tilde{\ell}^2 - M_1^2 - \mu^2)\delta((\tilde{\ell} - K)^2 - M_2^2 - \mu^2)$$
$$= \delta(\tilde{\ell}^2 - M_1^2 - \mu^2)\delta(K^2 - 2\tilde{\ell} \cdot K + M_1^2 - M_2^2)$$
$$= \delta(z^2K^2 + z(2\ell \cdot K) - M_1^2 - \mu^2)\delta((1 - 2z)K^2 - 2\ell \cdot K + M_1^2 - M_2^2)$$
$$= \delta(z(1 - z)K^2 + z(M_1^2 - M_2^2) - M_1^2 - \mu^2)$$
$$\times \delta((1 - 2z)K^2 - 2\ell \cdot K + M_1^2 - M_2^2).$$

In the last step we used the second delta function to find $2\ell \cdot K = (1 - 2z)K^2 + M_1^2 - M_2^2$. We will make this substitution in the measure (7). At this point the first delta function is independent of ℓ. The cut part of the integral now takes the form

$$\int_0^1 duu^{-1-\epsilon} \int dz((1 - 2z)K^2 + M_1^2 - M_2^2)\delta(z(1 - z)K^2$$
$$+ z(M_1^2 - M_2^2) - M_1^2 - \mu^2) \int d^4\ell \delta^+(\ell^2)\delta((1 - 2z)K^2$$
$$- 2\ell \cdot K + M_1^2 - M_2^2).$$

The second line is the four-dimensional phase space integration, which can be performed in various ways, as discussed in Ref. [19]. We can integrate out z using the delta function in the first line. Here we need to account for the factor

$$\frac{\partial}{\partial z}(z(1-z)K^2 + z(M_1^2 - M_2^2) - M_1^2 - \mu^2)$$
$$= (1-2z)K^2 + (M_1^2 - M_2^2),$$

which serves to cancel the first factor of the first line of Eq. (18). Finally, we arrive at the expression

$$\int_0^1 du\, u^{-1-\epsilon} \int d^4\ell\, \delta^+(\ell^2)\delta((1-2z)K^2 - 2\ell\cdot K$$
$$+ M_1^2 - M_2^2), \qquad (18)$$

where z is related to u by Eq. (16).

III. RECURSION AND REDUCTION FORMULAS WITH MASS

In this section we compute the massive analogs of the recursion and reduction formulas for master integrals presented in Ref. [19] and derived in detail in Ref. [20]. It is not hard to check that the results of this section reproduce the massless results when we set all $m_i = 0$ [and hence $a = b = 1$ in Eq. (17)].

We refer to Ref. [20] rather than review all details of the setup here. However, we do need to remind the reader that the results for cuts of these basis integrals were derived with spinor integration, in which the massless four-dimensional vector ℓ is rewritten as

$$\ell = t\lambda\tilde{\lambda}, \qquad (19)$$

and the measure transforms as

$$\int d^4\ell\, \delta^{(+)}(\ell^2)(\bullet) = \int_0^\infty dt\, t \int_{\tilde{\lambda}=\bar{\lambda}} \langle \lambda, d\lambda\rangle[\tilde{\lambda}, d\tilde{\lambda}](\bullet). \qquad (20)$$

Here t ranges over the positive real line, and λ, $\tilde{\lambda}$ are homogeneous spinors, also written, respectively, as $|\ell\rangle$, $|\ell]$ in many expressions involving spinor products. The first step in spinor integration is to integrate over the variable t simply by solving the delta function of the *second* cut propagator. In this section, we sketch the beginnings of certain derivations by writing the integrand before and after this t integration—which, we emphasize, is not true integration.

Spinor integration proceeds by writing the integrand as a total derivative plus delta functions using "holomorphic anomaly" formulas, and finally identifying the contributions of delta functions as residues of a complex function. For an exposition of this technique, we refer the reader to Refs. [6,7].

A. Bubble

The cut bubble is exactly the integral described in the previous section without any additional factors. We just need to do the four-dimensional integral.

$$\int dz\, d^4\ell\, \delta^+(\ell^2)(2\ell\cdot K)\delta(z(1-z)K^2 + z(M_1^2 - M_2^2) - M_1^2 - \mu^2)\delta((1-2z)K^2 - 2\ell\cdot K + M_1^2 - M_2^2)$$
$$= \int dz\, d^4\ell\, \delta^+(\ell^2)((1-2z)K^2 + M_1^2 - M_2^2)\delta(z(1-z)K^2 + z(M_1^2 - M_2^2) - M_1^2 - \mu^2)$$
$$\times \delta((1-2z)K^2 - 2\ell\cdot K + M_1^2 - M_2^2).$$

After phase space integration we are left with

$$\frac{(1-2z)K^2 + M_1^2 - M_2^2}{K^2} = b\sqrt{1-u}, \qquad (21)$$

where we have put in z given by Eq. (16). So, the cut is

$$C[I_2(M_1, M_2; K)] = b\int_0^1 du\, u^{-1-\epsilon}\sqrt{1-u}, \qquad (22)$$

where b is given in Eq. (17). This factor of b is where this expression differs from the massless case.

Recursion relation: Now we define a sequence of functions indexed by nonnegative integers n:

$$\text{Bub}^{(n)} \equiv \int_0^1 du\, u^{-1-\epsilon} u^n \sqrt{1-u}. \qquad (23)$$

Note that the physical cut is related to the zeroth function by a factor of b:

$$C[I_2(M_1, M_2; K)] = b\, \text{Bub}^{(0)}. \qquad (24)$$

Because the definition (23) is identical to the massless case, we derive exactly the same recursion formula as in Refs. [19,20]:

$$\text{Bub}^{(n)} = F_{2\to2}^{(n)}\, \text{Bub}^{(0)}, \qquad (25)$$

where the form factor is

$$F_{2\to2}^{(n)} = \frac{(-\epsilon)_{3/2}}{(n-\epsilon)_{3/2}}. \qquad (26)$$

Here $(x)_n = \Gamma(x+n)/\Gamma(x)$ is the Pochhammer symbol.

Written in a form suitable for reading master integrands, we have the result

$$b \int_0^1 du u^{-1-\epsilon} u^n \sqrt{1-u} = F_{2\to 2}^{(n)} C[I_2(M_1, M_2; K)]. \quad (27)$$

B. Triangle

The integrand to start with is

$$\frac{\delta(\tilde{\ell}^2 - M_1^2 - \mu^2)\delta((\tilde{\ell} - K_1)^2 - M_2^2 - \mu^2)}{((\tilde{\ell} + K_3)^2 - m_1^2 - \mu^2)}.$$

After t integration we get

$$\left((1 - 2z) + \frac{M_1^2 - M_2^2}{K_1^2}\right)\frac{1}{\langle\ell|K_1|\ell]\langle\ell|Q|\ell]}$$

$$= b\sqrt{1-u}\frac{1}{\langle\ell|K_1|\ell]\langle\ell|Q|\ell]},$$

where

$$K_1 \cdot Q = \frac{K_1^2 + M_1^2 - M_2^2}{K_1^2}(K_1 \cdot K_3) + (K_3^2 + M_1^2 - m_1^2),$$

$$Q^2 = \frac{(2K_1^2(K_3^2 - M_1^2 - m_1^2) + (K_1^2 + M_1^2 - M_2^2)(2K_1 \cdot K_3))^2 + (1-u)\Delta_{3;m=0}\Delta[K_1, M_1, M_2]}{4(K_1^2)^3},$$

where

$$\Delta_{3;m=0} \equiv 4((K_1 \cdot K_3)^2 - K_1^2 K_3^2), \quad (31)$$

which we recognize as the signature of a triangle with massless propagators.

We can now see that

$$\Delta_3 = \frac{(1-u)\Delta_{3;m=0}\Delta[K_1, M_1, M_2]}{(K_1^2)^2} = b^2(1-u)\Delta_{3;m=0}.$$

$$(32)$$

It is interesting to see that Δ_3 is built from the factors $\Delta[K_1, M_1, M_2]$, the signature of the cut, and $\Delta_{3;m=0}$, the signature of the triangle.

Now we define

$$Z \equiv -\frac{(2K_1 \cdot Q)K_1^2}{\sqrt{\Delta_{3;m=0}\Delta[K_1, M_1, M_2]}}. \quad (33)$$

Then we have

$$C[I_3(M_1, M_2, m_1; K_1, K_3)]$$

$$= \int_0^1 du u^{-1-\epsilon}(-b\sqrt{1-u})\frac{1}{\sqrt{\Delta_3}}\ln\left(\frac{Z + \sqrt{1-u}}{Z - \sqrt{1-u}}\right)$$

$$= \int_0^1 du u^{-1-\epsilon}\left(-\frac{1}{\sqrt{\Delta_{3;m=0}}}\right)\ln\left(\frac{Z + \sqrt{1-u}}{Z - \sqrt{1-u}}\right). \quad (34)$$

This is the same expression as in the massless case, but now with a different Z.

$$Q = \left((1 - 2z) + \frac{M_1^2 - M_2^2}{K_1^2}\right)K_3$$

$$+ \frac{K_3^2 + M_1^2 - m_1^2 + z(2K_1 \cdot K_3)}{K_1^2}K_1. \quad (28)$$

The four-dimensional integral gives

$$-\frac{1}{\sqrt{\Delta_3}}\left((1 - 2z) + \frac{M_1^2 - M_2^2}{K_1^2}\right)\ln\left(\frac{-(2K_1 \cdot Q) + \sqrt{\Delta_3}}{-(2K_1 \cdot Q) - \sqrt{\Delta_3}}\right),$$

$$(29)$$

where

$$\Delta_3 = 4((K_1 \cdot Q)^2 - K_1^2 Q^2). \quad (30)$$

The ingredients for Eq. (30) are

Recursion/reduction relation: We define the integrals

$$\text{Tri}^{(n)}(Z) \equiv \int_0^1 du u^{-1-\epsilon} u^n \ln\left(\frac{Z + \sqrt{1-u}}{Z - \sqrt{1-u}}\right), \quad (35)$$

where the parameter Z is defined as in Eq. (33). The physical cut integral is

$$C[I_3(M_1, M_2, m_1; K_1, K_3)] = -\frac{1}{\sqrt{\Delta_{3;m=0}}}\text{Tri}^{(0)}(Z). \quad (36)$$

In this case, our cut triangle functions (35) do depend on the cut-propagator masses M_1 and M_2 via Z. Apart from the generalized definition of Z, the formula is the same as in the massless case. Therefore, we derive the same recursion and reduction formulas as in Ref. [20], namely

$$\text{Tri}^{(n)}(Z) = F_{3\to 3}^{(n)}(Z)\text{Tri}^{(0)}(Z) + \tilde{F}_{3\to 2}^{(n)}(Z)\text{Bub}^{(0)}, \quad (37)$$

where the two form factors are given by

$$F_{3\to 3}^{(n)}(Z) = \frac{-\epsilon}{n - \epsilon}(1 - Z^2)^n, \quad (38)$$

$$\tilde{F}_{3\to 2}^{(n)}(Z) = \frac{(-\epsilon)_{3/2}}{n - \epsilon}\sum_{k=1}^n \frac{2Z(1 - Z^2)^{n-k}}{(k - \epsilon)_{1/2}}. \quad (39)$$

These are functions of the variable Z, defined for a given triangle by Eq. (33).

Equation (37) is not yet in the most applicable form. We return to the language of physical cuts by including the factor $-1/\sqrt{\Delta_{3,m=0}}$ from Eq. (36). The recursion/reduction formula that we need is thus:

$$\int_0^1 du\, u^{-1-\epsilon} u^n \left[-\frac{1}{\sqrt{\Delta_{3;m=0}}} \ln\left(\frac{Z+\sqrt{1-u}}{Z-\sqrt{1-u}}\right) \right]$$

$$= F_{3\to3}^{(n)}(Z) C[I_3(M_1, M_2, m_1; K_1, K_3)]$$

$$+ F_{3\to2}^{(n)}(K_1, K_3) C[I_2(M_1, M_2; K_1)], \qquad (40)$$

where

$$F_{3\to2}^{(n)}(K_1, K_3)) = -\frac{1}{b\sqrt{\Delta_{3;m=0}}} \tilde{F}_{3\to2}^{(n)}(Z). \qquad (41)$$

C. Box

The integrand to start with is

$$\frac{\delta(\tilde{\ell}^2 - M_1^2 - \mu^2)\delta((\tilde{\ell}-K)^2 - M_2^2 - \mu^2)}{((\tilde{\ell}-P_1)^2 - m_1^2 - \mu^2)((\tilde{\ell}-P_2)^2 - m_2^2 - \mu^2)}.$$

We define another useful mass-dependent parameter:

$$a_i \equiv \frac{P_i^2 + M_1^2 - m_i^2}{K^2}. \qquad (42)$$

After t integration we get

$$b\sqrt{1-u}\frac{1}{\langle \ell|Q_1|\ell]\langle \ell|Q_2|\ell]},$$

where

$$Q_i = (-b\sqrt{1-u})P_i + \left(a_i - \frac{P_i \cdot K}{K^2}(a - b\sqrt{1-u})\right)K. \qquad (43)$$

Now the procedure is the same as in the massless case, but with this more general definition of Q_i.

Define

$$R_i = Q_i|_{u=0} \qquad (44)$$

$$= -bP_i + \left(a_i - \frac{P_i \cdot K}{K^2}(a - b)\right)K. \qquad (45)$$

The physical cut is

$$C[I_4(M_1, M_2, m_1, m_2; K, P_1, P_2)]$$

$$= \int_0^1 du\, u^{-1-\epsilon} \frac{b}{2K^2} \frac{1}{\sqrt{B-Au}}$$

$$\times \ln\left(\frac{D-Cu+\sqrt{1-u}\sqrt{B-Au}}{D-Cu-\sqrt{1-u}\sqrt{B-Au}}\right), \qquad (46)$$

where

$$A = -\frac{b^4}{K^2} \det\begin{pmatrix} P_1^2 & P_1 \cdot P_2 & P_1 \cdot K \\ P_1 \cdot P_2 & P_2^2 & P_2 \cdot K \\ P_1 \cdot K & P_2 \cdot K & K^2 \end{pmatrix},$$

$$C = \frac{b^2}{K^2} \det\begin{pmatrix} P_1 \cdot P_2 & P_1 \cdot K \\ P_2 \cdot K & K^2 \end{pmatrix},$$

$$B = -\det\begin{pmatrix} R_1^2 & R_1 \cdot R_2 \\ R_1 \cdot R_2 & R_2^2 \end{pmatrix}, \qquad (47)$$

$$D = R_1 \cdot R_2.$$

Here again, the form of Eq. (46) differs from the one in massless case only by the factor of b.

Recursion/reduction relation: We define

$$\text{Box}^{(n)}(A, B, C, D) \equiv \int_0^1 du\, u^{-1-\epsilon} \frac{u^n}{\sqrt{B-Au}}$$

$$\times \ln\left(\frac{D-Cu+\sqrt{1-u}\sqrt{B-Au}}{D-Cu-\sqrt{1-u}\sqrt{B-Au}}\right).$$

The physical cut is related to the zeroth function in Eq. (48) by

$$C[I_4(M_1, M_2, m_1, m_2; K, P_1, P_2)]$$

$$= \frac{b}{2K^2} \text{Box}^{(0)}(A, B, C, D) \qquad (48)$$

if A, B, C, D are defined as in Eq. (47).

In the table in Eq. (49) we have listed which kinds of triangles a box with given cut would reduce to (the propagator mass must also be correctly identified):

Box Cut K	P_1	P_2	Triangle One's (K_1, K_3)	Triangle Two's (K_1, K_3)
K_1	K_{12}	$-K_4$	(K_1, K_{34})	(K_1, K_4)
K_2	K_{23}	$-K_1$	(K_2, K_{41})	(K_2, K_1)
K_3	K_{34}	$-K_2$	(K_3, K_{12})	(K_3, K_2)
K_4	K_{41}	$-K_3$	(K_4, K_{23})	(K_4, K_3)
K_{12}	K_1	$-K_4$	(K_{34}, K_2)	(K_{12}, K_4)
K_{23}	K_2	$-K_1$	(K_{41}, K_3)	(K_{23}, K_1)

$$(49)$$

Boxes are related to triangles and bubbles in the reduction formulas. In these relations we make use of a quantity combining variables of the box and the associated triangles to the table in Eq. (49):

$$C_{Z_i} \equiv (Z_i^2 - 1)C + D. \tag{50}$$

$$\text{Box}^{(n)}(A, B, C, D) = F_{4 \to 4}^{(n)}(A, B) \, \text{Box}^{(0)}(A, B, C, D)$$
$$+ \tilde{F}_{4 \to 3}^{(n)}(A, B, C, D; Z_1) \, \text{Tri}^{(0)}(Z_1)$$
$$+ \tilde{F}_{4 \to 3}^{(n)}(A, B, C, D; Z_2) \, \text{Tri}^{(0)}(Z_2)$$
$$+ \tilde{F}_{4 \to 2}^{(n)}(A, B, C, D; Z_i) \, \text{Bub}^{(0)}, \tag{51}$$

where the form factors are given by

$$F_{4 \to 4}^{(n)}(A, B) = \frac{(-\epsilon)_{1/2}}{(n - \epsilon)_{1/2}} \left(\frac{B}{A}\right)^n, \tag{52}$$

$$\tilde{F}_{4 \to 3}^{(n)}(A, B, C, D; Z_i) = -\frac{C_{Z_i}}{(n - \epsilon)_{1/2} A Z_i}$$
$$\times \sum_{k=1}^{n} \frac{1}{(k - 1/2 - \epsilon)_{1/2}}$$
$$\times \left(\frac{B}{A}\right)^{n-k} F_{3 \to 3}^{(k-1)}(Z_i), \tag{53}$$

$$\tilde{F}_{4 \to 2}^{(n)}(A, B, C, D; Z_i) = -\frac{\Gamma(n - \epsilon)}{\Gamma(n + 1/2 - \epsilon)} \frac{1}{A} \sum_{k=1}^{n} \frac{1}{(k - 1/2 - \epsilon)_{1/2}} \left(\frac{B}{A}\right)^{n-k} \left(\frac{C_{Z_1}}{Z_1} \tilde{F}_{3 \to 2}^{(k-1)}(Z_1) + \frac{C_{Z_2}}{Z_2} \tilde{F}_{3 \to 2}^{(k-1)}(Z_2)\right). \tag{54}$$

Again Eq. (51) is not the final formula we are after. To get the proper physical result for identifying integrands, we need to replace the kinematic factor $b/2K^2$. The result is

$$\int_0^1 du\, u^{-1-\epsilon} u^n \left[\frac{b}{2K^2 \sqrt{B - Au}} \ln\left(\frac{D - Cu + \sqrt{1 - u}\sqrt{B - Au}}{D - Cu - \sqrt{1 - u}\sqrt{B - Au}}\right)\right]$$
$$= F_{4 \to 4}^{(n)}(A, B) C[I_4(M_1, M_2, m_1, m_2; K, P_1, P_2)] + \sum_{i=1}^{2} F_{4 \to 3}^{(n)}(A, B, C, D; Z_i) C[I_3(M_1, M_2, m_1^{(i)}; K_1^{(i)}, K_3^{(i)})]$$
$$+ F_{4 \to 2}^{(n)}(A, B, C, D; Z_i) C[I_2(M_1, M_2; K)], \tag{55}$$

where for the triangles, $K_1^{(i)}$ and $K_3^{(i)}$ are given by the table in Eq. (49), the mass $m_1^{(1)}$ must also be interpreted correctly according to the same table, and the form factors are

$$F_{4 \to 3}^{(n)}(A, B, C, D; Z_i) = -\frac{b\sqrt{\Delta_3^{(i)}}}{2K^2} \tilde{F}_{4 \to 3}^{(n)}(A, B, C, D; Z_i), \qquad F_{4 \to 2}^{(n)}(A, B, C, D; Z_i) = \frac{1}{2K^2} \tilde{F}_{4 \to 2}^{(n)}(A, B, C, D; Z_i). \tag{56}$$

D. Pentagon

The integrand is

$$\frac{\delta(\tilde{\ell}^2 - M_1^2 - \mu^2)\delta((\tilde{\ell} - K)^2 - M_2^2 - \mu^2)}{((\tilde{\ell} - P_1)^2 - m_1^2 - \mu^2)((\tilde{\ell} - P_2)^2 - m_2^2 - \mu^2)((\tilde{\ell} - P_3)^2 - m_3^2 - \mu^2)}.$$

The physical cut is

$$C[I_5(M_1, M_2, m_1, m_2, m_3; K, P_1, P_2, P_3)] = \int_0^1 du\, u^{-1-\epsilon} \int \langle \ell d\ell \rangle [\ell d\ell] \frac{b\sqrt{1-u}\langle \ell|K|\ell]}{(K^2)^2 \langle \ell|Q_1|\ell]\langle \ell|Q_2|\ell]\langle \ell|Q_3|\ell]},$$

where

$$Q_i = -(b\sqrt{1-u})P_i + \frac{P_i^2 + M_1^2 - m_i^2 - 2z(K \cdot P_i)}{K^2} K. \tag{57}$$

The total integral (apart from the universal prefactor) is

$$b \int_0^1 du\, u^{-1-\epsilon}\sqrt{1-u} \int \langle \ell d\ell \rangle [\ell d\ell] \frac{\langle \ell|K|\ell]}{(K^2)^2 \langle \ell|Q_1|\ell]\langle \ell|Q_2|\ell]\langle \ell|Q_3|\ell]}. \tag{58}$$

Apart from the factor of b and the modified definition of Q_i, the analysis proceeds as in the massless case [20]. Thus we can cite the result directly as

$$C[I_5(M_1, M_2, m_1, m_2, m_3; K, P_1, P_2, P_3)]$$

$$= -b \int_0^1 du\, u^{-1-\epsilon} \frac{\sqrt{1-u}}{(K^2)^2} \left(\frac{S[Q_3, Q_2, Q_1, K]}{4\sqrt{(Q_3 \cdot Q_2)^2 - Q_3^2 Q_2^2}} \ln \frac{Q_3 \cdot Q_2 - \sqrt{(Q_3 \cdot Q_2)^2 - Q_3^2 Q_2^2}}{Q_3 \cdot Q_2 + \sqrt{(Q_3 \cdot Q_2)^2 - Q_3^2 Q_2^2}} \right.$$

$$+ \frac{S[Q_3, Q_1, Q_2, K]}{4\sqrt{(Q_3 \cdot Q_1)^2 - Q_3^2 Q_1^2}} \ln \frac{Q_3 \cdot Q_1 - \sqrt{(Q_3 \cdot Q_1)^2 - Q_3^2 Q_1^2}}{Q_3 \cdot Q_1 + \sqrt{(Q_3 \cdot Q_1)^2 - Q_3^2 Q_1^2}}$$

$$\left. + \frac{S[Q_2, Q_1, Q_3, K]}{4\sqrt{(Q_2 \cdot Q_1)^2 - Q_2^2 Q_1^2}} \ln \frac{Q_2 \cdot Q_1 - \sqrt{(Q_2 \cdot Q_1)^2 - Q_2^2 Q_1^2}}{Q_2 \cdot Q_1 + \sqrt{(Q_2 \cdot Q_1)^2 - Q_2^2 Q_1^2}} \right), \qquad (59)$$

where $S[Q_3, Q_2, Q_1, K]$ is a rational function defined as follows:

$$S[Q_2, Q_1, Q_3, K] = \frac{T_1}{T_2}, \qquad (60)$$

with

$$T_1 = -8 \det \begin{pmatrix} Q_3 \cdot K & Q_2 \cdot K & Q_1 \cdot K \\ Q_2 \cdot Q_3 & Q_2^2 & Q_2 \cdot Q_1 \\ Q_1 \cdot Q_3 & Q_2 \cdot Q_1 & Q_1^2 \end{pmatrix},$$

$$T_2 = -4 \det \begin{pmatrix} Q_3^2 & Q_2 \cdot Q_3 & Q_1 \cdot Q_3 \\ Q_2 \cdot Q_3 & Q_2^2 & Q_2 \cdot Q_1 \\ Q_1 \cdot Q_3 & Q_2 \cdot Q_1 & Q_1^2 \end{pmatrix}. \qquad (61)$$

Let us make a few comments on the behavior of pentagon cuts. There are three terms. Each term looks like a box signature multiplied by the factor $S[\bullet]/(2K^2)$. It is significant that each function $S[\bullet]$ has the same denominator $T_2[\bullet]$, which does not depend on the order of the first three arguments. This can be considered as another signature of a cut-pentagon integral. This feature makes the reduction simple. Where we see a factor of u^n, we just need to write $u^n S = P(u) + A \sum_{i=1}^3 S_i[\bullet]$ where $P(u)$ is a polynomial in u and A is constant in u.[2] The A term will be the pentagon coefficient, while $P(u)$ indicates reduction to boxes. In the massless case, pentagons contribute to terms of $\mathcal{O}(\epsilon)$, so they can be neglected. However, in cases where propagators are massive or we wish to compute to higher orders in ϵ, their contribution must be included.

IV. FORMULAS FOR COEFFICIENTS FROM DOUBLE CUTS

Systematic extraction of coefficients of master integrals from four-dimensional spinor integration techniques has been described in Refs. [6,7], building on earlier techniques reviewed in Refs. [22,23]. When applied to a specific amplitude in practice, there are choices to make regarding how to "split" the integrand in partial fractions as well as choosing arbitrary spinors on which the final answer does not depend.

In this section we present canonical choices to further simplify the method, aiming to automatize the heart of this procedure to allow for easy implementation into a computer program.

For brevity, we do not review the entire spinor integration technique here; instead we refer the reader to the explanations given in Refs. [6,7]. The steps that concern us here come after the coordinate change and t integration mentioned at the beginning of Sec. III. At this point we have an integrand whose terms are rational functions of spinor products and homogeneous in the spinor integration variables. The following steps are to split the integrand into partial fractions (using Schouten identities), followed by identification of master integrals and integration over a single Feynman parameter. Finally we apply the holomorphic anomaly to complete the spinor integration by extracting residues.

Here we are able to give explicit algebraic functions for coefficients. In the Appendix we give a more detailed discussion of how to evaluate these functions in practice.

A summary of the results this section may be found in Sec. IV D.

A. Canonical splitting

Recall that our starting point is the product of the two (on-shell) tree-level amplitudes from each side of the unitarity cut. In spinor notation, this integrand takes the general form[3]

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[2] It is essential that because all three $S_i[\bullet]$ have same denominator T_2, after reduction we have the same A multiplying all three of the $S_i[\bullet]$.

[3] In this section we deal specifically with the case of massless propagators. The generalization to the massive case is straightforward, as in the previous sections.

$$\sum \frac{C \prod_{i=1}^{n_1} \langle \tilde{a}_i | \ell | \tilde{b}_j]}{\prod_{j=1}^{n_2} ((\tilde{\ell} - P_j)^2 - \mu^2)}.$$

Here C is some expression that does not depend on any integration variable, and we have used the relation (7), $\tilde{\ell} = \ell + zK$, to rewrite the numerator. As we emphasized in the beginning of Sec. III, the t integration is trivial because of the second delta function, $\delta((1 - 2z)K^2 - 2\ell \cdot K)$. Thus we can write the result of this step immediately:

$$\frac{(1 - 2z)K^2}{\langle \ell | K | \ell]^2} \sum \left(- \frac{(1 - 2z)K^2}{\langle \ell | K | \ell]} \right)^{n_1}$$
$$\times \frac{\langle \ell | K | \ell]^{n_2} C \prod_{i=1}^{n_1} \langle \tilde{a}_i \ell \rangle [\ell \tilde{b}_i]}{\prod_{j=1}^{n_2} \langle \ell | Q_j | \ell]}.$$

So we see that, after integrating over t, the integrand for phase space integration is a sum of terms of the following form:

$$I_{\text{term}} = \frac{G(\lambda) \prod_{j=1}^{n+k-2} [a_j \ell]}{\langle \ell | K | \ell]^n \prod_{i=1}^{k} \langle \ell | Q_i | \ell]}. \qquad (62)$$

Here $G(\lambda)$ is some monomial in the holomorphic spinor only; hence it factorizes as $\prod \langle \ell c_i \rangle$ times a constant in ℓ. The functions a_j may depend on λ as well, for example, as $[a_j| = \langle \lambda | Q]$. But a_j is the quantity we will need in order to split the term further and extract the residues from multiple poles.

We aim to describe all coefficients in terms of the quantities K, Q_i, a_j, and $G(\lambda)$.

The term (62) depends as well on n and k. The exponent n is related to the type of master integral involved. Terms with $n = 0$ contribute only to boxes and pentagons. Terms with $n = 1$ contribute to triangles in addition, and terms with $n \geq 2$ contribute to all of these plus bubbles. If $n = 0$ we can multiply both numerator and denominator by $\langle \ell | K | \ell]$, so we shall always assume $n \geq 1$.

The first step in our program is to isolate poles by splitting I_{term} using the following partial fraction spinor identity:

$$\frac{[a \ell]}{\langle \ell | Q_1 | \ell] \langle \ell | Q_2 | \ell]} = \frac{[a | Q_1 | \ell \rangle}{\langle \ell | Q_2 Q_1 | \ell \rangle \langle \ell | Q_1 | \ell]}$$
$$+ \frac{[a | Q_2 | \ell \rangle}{\langle \ell | Q_1 Q_2 | \ell \rangle \langle \ell | Q_2 | \ell]}. \qquad (63)$$

Because there are different factors in the denominator, there are different sequences of splitting leading to differ-

ent but equivalent expressions. Here we choose a canonical sequence:

(1) First we keep $\langle \ell | K | \ell]^n$ untouched and split the factors $\langle \ell | Q_i | \ell]$ among themselves. At the end of this step each term takes the form $\frac{\tilde{G}(\lambda)}{\langle \ell | K | \ell]^n \langle \ell | Q_i | \ell]}$.

(2) Then we split $\langle \ell | K | \ell]$ from $\langle \ell | Q_i | \ell]$ as often as necessary to get two types of terms, $\frac{\tilde{G}_m(\lambda)}{\langle \ell | K | \ell]^m}$ and $\frac{\tilde{F}_i(\lambda)}{\langle \ell | K | \ell] \langle \ell | Q_i | \ell]}$.

(3) Finally we will be left with

$$\frac{G(\lambda) \prod_{j=1}^{n+k-2} [a_j \ell]}{\langle \ell | K | \ell]^n \prod_{i=1}^{k} \langle \ell | Q_i | \ell]}$$
$$= \sum_{k=2}^{n} G_k(\lambda) \frac{\prod_{j=1}^{k-2} [b_j \ell]}{\langle \ell | K | \ell]^k}$$
$$+ \sum_{i=1}^{k} F_i(\lambda) \frac{1}{\langle \ell | K | \ell] \langle \ell | Q_i | \ell]}. \qquad (64)$$

Our task is to find expressions for $G_k(\lambda)$, $F_i(\lambda)$, and b_j in the most compact and simple form. The result is

$$F_i(\lambda) = \left(\frac{G(\lambda) \prod_{s=1}^{n+k-2} [a_s | Q_i | \ell \rangle}{\langle \ell | K Q_i | \ell \rangle^{n-1} \prod_{t=1, t \neq i}^{k} \langle \ell | Q_t Q_i | \ell \rangle} \right), \qquad (65)$$

$$G_p(\lambda) = \sum_{i=1}^{k} \frac{G(\lambda) \prod_{s=1}^{k-1} [a_s | Q_i | \ell \rangle}{\prod_{t=1, t \neq i}^{k} \langle \ell | Q_t Q_i | \ell \rangle} \frac{\prod_{l=k}^{n-k-p} [a_l | K | \ell \rangle}{\langle \ell | Q_i K | \ell \rangle^{n+1-p}}, \qquad (66)$$
$$p = 2, \ldots, n,$$

$$b_j = a_{j+n+k-p}. \qquad (67)$$

While $F_i(\lambda)$ finds its simplest and most compact form in Eq. (65), $G_p(\lambda)$ in Eq. (66) may not be the simplest. For example, it can be shown that all terms with $1/\langle \ell | K | \ell]^n$ can be summed into a single term. In general there are several terms with $1/\langle \ell | K | \ell]^a$ with $2 \leq a < n$. As we discuss in Sec. V C, if we know that $n \leq 4$, we can work them out explicitly. However, in Sec. IV C we shall use a slightly different method to deal with these rational terms.

The identities necessary for carrying out steps (1) and (2) are the following, which may be proved by induction, making use of the basic splitting identity (63).

$$\frac{\prod_{j=1}^{k-1} [a_j \ell]}{\prod_{i=1}^{k} \langle \ell | Q_i | \ell]} = \sum_{i=1}^{k} \frac{1}{\langle \ell | Q_i | \ell]} \frac{\prod_{j=1}^{k-1} [a_j | Q_i | \ell \rangle}{\prod_{m=1, m \neq i}^{k} \langle \ell | Q_m Q_i | \ell \rangle}, \qquad (68)$$

$$\frac{\prod_{j=1}^{n-1} [a_j \ell]}{\langle \ell | K | \ell]^n \langle \ell | Q | \ell]} = \frac{\prod_{j=1}^{n-1} [a_j | Q | \ell \rangle}{\langle \ell | K Q | \ell \rangle^{n-1}} \frac{1}{\langle \ell | K | \ell] \langle \ell | Q | \ell]} + \sum_{p=0}^{n-2} (-)^{n-p} \frac{\prod_{j=1}^{n-p-2} [a_j | Q | \ell \rangle [a_{n-p-1} | K | \ell \rangle \prod_{t=n-p}^{n-1} [a_t \ell]}{\langle \ell | K | \ell]^{p+2} \langle \ell | Q K | \ell \rangle^{n-p-1}}. \qquad (69)$$

One necessary condition for the form (68) is that all the Q_i are different, which is satisfied for generic momenta.

B. Box and triangle coefficients

In this subsection we derive a closed expression for box coefficients and systematize the derivation of triangle coefficients.

As we know from Sec. III C, box integrals and their coefficients may be labeled by momenta P_i, P_j along with the cut momentum K. From these momenta we have defined Q_i, Q_j as in Eq. (43).

The terms we need to consider are of the following form:

$$I^{(i)} = \int F_i(\lambda) \frac{1}{\langle\ell|K|\ell\rangle\langle\ell|Q_i|\ell\rangle}$$
$$= \int_0^1 dx \int \langle\ell d\ell\rangle[d\ell\partial_\ell]\left(\frac{F_i(\lambda)[\eta\ell]}{\langle\ell|R|\ell\rangle\langle\ell|R|\eta]}\right),$$
$$R = xQ_i + (1-x)K.$$

As discussed in the previous subsection, there will be two types of poles: those from $\langle\ell|KQ|\ell\rangle$ contribute to triangles, and those from $\langle\ell|Q_jQ_i|\ell\rangle$ contribute to boxes. Let us see precisely how these arise.

We can construct two massless momenta from Q_i and K:

$$P^{(i)}_{1,2} = Q_i + x^{(i)}_{1,2}K, \qquad (70)$$

where

$$x^{(i)}_{1,2} = \frac{-2Q_i \cdot K \pm \sqrt{\Delta^{(i)}}}{2K^2},$$
$$\Delta^{(i)} = (2Q_i \cdot K)^2 - 4Q_i^2K^2. \qquad (71)$$

We make the choice $\eta = P^{(i)}_1$ and find that

$$I^{(i)} = \int_0^1 dx \int \langle\ell d\ell\rangle[d\ell\partial_\ell]\left(\frac{F_i(\lambda)[P^{(i)}_1\ell]}{\langle\ell P^{(i)}_2\rangle[P^{(i)}_2 P^{(i)}_1]}\right.$$
$$\left.\times \frac{(x^{(i)}_1 - x^{(i)}_2)}{\langle\ell|R|\ell\rangle(x(x^{(i)}_1 + 1) - 1)}\right). \qquad (72)$$

The poles in this expression come from $F_i(\lambda)$ and $\langle\ell P^{(i)}_2\rangle$. Note as well that since we have chosen $\eta = P^{(i)}_1$, the factor $[P^{(i)}_1\ell]$ appears in the numerator and therefore there is *no contribution* from the pole at $\langle\ell P^{(i)}_1\rangle$, which one might naively expect from a factor of $\langle\ell|KQ_i|\ell\rangle$ in the denominator of $F_i(\lambda)$.

Apply a partial fraction expansion to the x-dependent factors in the denominator of the integral. The result is

$$I^{(i)} = \int_0^1 dx \int \langle\ell d\ell\rangle[d\ell\partial_\ell]\left(\frac{F_i(\lambda)}{\langle\ell|Q_iK|\ell\rangle}\right.$$
$$\left.\times \left(-\frac{(x^{(i)}_1 + 1)}{x(x^{(i)}_1 + 1) - 1} + \frac{\langle\ell|Q_i - K|\ell]}{x\langle\ell|Q_i - K|\ell] + \langle\ell|K|\ell]}\right)\right),$$
$$\qquad (73)$$

where we have used the fact that

$$\langle\ell|Q_iK|\ell\rangle = \frac{\langle\ell P^{(i)}_1\rangle[P^{(i)}_1 P^{(i)}_2]\langle\ell P^{(i)}_2\rangle}{(x^{(i)}_1 - x^{(i)}_2)}. \qquad (74)$$

First term of (73)*:* Let us consider the two terms of Eq. (73) separately. The first is equal to

$$I^{(i)}_1 \equiv \int_0^1 dx \int \langle\ell d\ell\rangle[d\ell\partial_\ell]\left(\frac{F_i(\lambda)}{\langle\ell|Q_iK|\ell\rangle}\left(-\frac{(x^{(i)}_1 + 1)}{x(x^{(i)}_1 + 1) - 1}\right)\right) = \int \langle\ell d\ell\rangle[d\ell\partial_\ell]\left(\frac{F_i(\lambda)}{\langle\ell|Q_iK|\ell\rangle}(-\ln(-x^{(i)}_1))\right)$$
$$= \int \langle\ell d\ell\rangle[d\ell\partial_\ell]\left(\frac{G(\lambda)\prod_{s=1}^{n+k-2}[a_s|Q_i|\ell\rangle}{\langle\ell|KQ_i|\ell\rangle^n \prod_{t=1,t\neq i}^k \langle\ell|Q_tQ_i|\ell\rangle} \ln(-x^{(i)}_1)\right).$$

In the last line we made the substitution (65).

To complete the calculation for $I^{(i)}_1$, we need to take residue of all poles except the one $\langle\ell P^{(i)}_1\rangle$ from factor $\langle\ell|KQ_i|\ell\rangle$. This seems difficult to do directly. However, notice that the whole expression is holomorphic. Using the result that the sum of residues of all poles of a holomorphic function is zero, we get

$$I^{(i)}_1 = -\left(\frac{G(\lambda)\prod_{s=1}^{n+k-2}[a_s|Q_i|\ell\rangle}{\langle\ell|KQ_i|\ell\rangle^n \prod_{t=1,t\neq i}^k \langle\ell|Q_tQ_i|\ell\rangle} \ln(-x^{(i)}_1)\right)\Bigg|_{\text{residue of } \langle\ell P^{(i)}_1\rangle}. \qquad (75)$$

These residues contribute to triangle coefficients. In the Appendix we show how to evaluate the residue of a multiple pole. It can be seen that the expression (75) defines the algebraic function for the coefficients with input a_i and Q_i.

Second term of (73)*:* The second term of (73) is defined by

$$I^{(i)}_2 \equiv \int_0^1 dx \int \langle\ell d\ell\rangle[d\ell\partial_\ell]\left(\frac{F_i(\lambda)}{\langle\ell|Q_iK|\ell\rangle} \frac{\langle\ell|Q_i - K|\ell]}{x\langle\ell|Q_i - K|\ell] + \langle\ell|K|\ell]}\right), \qquad (76)$$

$$= \int_0^1 dx \int \langle \ell d\ell \rangle [d\ell \partial_\ell] \left(-\frac{G(\lambda) \prod_{s=1}^{n+k-2}[a_s|Q_i|\ell\rangle}{\langle\ell|KQ_i|\ell\rangle^n \prod_{t=1,t\neq i}^k \langle\ell|Q_tQ_i|\ell\rangle} \frac{\langle\ell|Q_i - K|\ell]}{x\langle\ell|Q_i - K|\ell] + \langle\ell|K|\ell]} \right). \quad (77)$$

Again, we do not take the residue of $\langle\ell P_1^{(i)}\rangle$ from the factor $\langle\ell|KQ_i|\ell\rangle$.

There are two kinds of poles in Eq. (77): one is a simple pole from $\langle\ell|Q_jQ_i|\ell\rangle$ and one is a possible multiple pole from $\langle\ell P_2^{(i)}\rangle^n$ within $\langle\ell|KQ_i|\ell\rangle^n$. The former contributes to boxes; the latter contributes to triangles. Let us consider the multiple poles first.

When we replace $|\ell] \to |P_2^{(i)}]$, the spinor dependence cancels out in the integrand, giving

$$\frac{\langle\ell|Q_i - K|\ell]}{x\langle\ell|Q_i - K|\ell] + \langle\ell|K|\ell]} \to \frac{(x_2^{(i)} + 1)}{x(x_2^{(i)} + 1) - 1}.$$

We integrate over the Feynman parameter and find that the residue is

$$I_2^{(i)}|_{\langle\ell P_2^{(i)}\rangle} = \left(-\frac{G(\lambda) \prod_{s=1}^{n+k-2}[a_s|Q_i|\ell\rangle}{\langle\ell|KQ_i|\ell\rangle^n \prod_{t=1,t\neq i}^k \langle\ell|Q_tQ_i|\ell\rangle} \ln(-x_2^{(i)}) \right) \Bigg|_{\text{residue of } \langle\ell P_2^{(i)}\rangle}. \quad (78)$$

Like Eq. (75), it will contribute to triangles.

Now we move to the simple poles in Eq. (77). For these simple poles we can evaluate the integral over the Feynman parameter first and get

$$I_2^{(i)}|_{\text{simple poles}} = \int \langle\ell d\ell \rangle [d\ell \partial_\ell] \left(\frac{G(\lambda) \prod_{s=1}^{n+k-2}[a_s|Q_i|\ell\rangle}{\langle\ell|KQ_i|\ell\rangle^n \prod_{t=1,t\neq i}^k \langle\ell|Q_tQ_i|\ell\rangle} \ln\frac{\langle\ell|K|\ell]}{\langle\ell|Q_i|\ell]} \right). \quad (79)$$

Now we need to compute the residue from the poles in $\langle\ell|Q_jQ_i|\ell\rangle$. Assume without loss of generality that $i < j$, and construct two massless momenta as

$$P_{1,2}^{(ij)} = Q_j + y_{1,2}^{(ij)}Q_i, \qquad (i < j), \quad (80)$$

where

$$y_{1,2}^{(ij)} = \frac{-2Q_i \cdot Q_j \pm \sqrt{\Delta^{(ij)}}}{2Q_i^2}, \quad (81)$$

$$\Delta^{(ij)} = (2Q_i \cdot Q_j)^2 - 4Q_i^2 Q_j^2.$$

To simplify our expressions, we define the following function:

$$F_{i,j}(\ell) = \frac{G(\lambda) \prod_{s=1}^{n+k-2}[a_s|Q_i|\ell\rangle}{\langle\ell|KQ_i|\ell\rangle^n \prod_{t=1,t\neq i,j}^k \langle\ell|Q_tQ_i|\ell\rangle}. \quad (82)$$

To sum the contributions from these two simple poles $\langle\ell P_1^{(ij)}\rangle$ and $\langle\ell P_2^{(ij)}\rangle$, notice that the $P_{1,2}^{(ij)}$ differ only by a sign in front of the square root. Thus we can expand

$$F_{i,j}(P_1^{(ij)}) = F_{i,j}^{(S)} + F_{i,j}^{(A)}, \qquad F_{i,j}(P_2^{(ij)}) = F_{i,j}^{(S)} - F_{i,j}^{(A)}. \quad (83)$$

Putting this back into the expression (79) for the residue, it is straightforward to derive that the contribution from these two poles in $I_2^{(i)}$ is

$$-\frac{1}{\sqrt{\Delta^{(ij)}}} \left(F_{i,j}^{(S)} \ln\frac{\langle P_1^{(ij)}|K|P_1^{(ij)}]}{\langle P_1^{(ij)}|Q_i|P_1^{(ij)}]} \frac{\langle P_2^{(ij)}|Q_i|P_2^{(ij)}]}{\langle P_2^{(ij)}|K|P_2^{(ij)}]} \right.$$

$$\left. + F_{i,j}^{(A)} \ln\frac{\langle P_1^{(ij)}|K|P_1^{(ij)}]}{\langle P_1^{(ij)}|Q_i|P_1^{(ij)}]} \frac{\langle P_2^{(ij)}|K|P_2^{(ij)}]}{\langle P_2^{(ij)}|Q_i|P_2^{(ij)}]} \right).$$

To proceed further, we notice that the same simple-pole factor $\langle\ell|Q_jQ_i|\ell\rangle$ shows up in the $I^{(j)}$ term [or $F_j(\lambda)$ in Eq. (64)]. We can do a similar calculation to get residues in $I_2^{(j)}$ in terms the function $F_{j,i}(\lambda)$. One can easily check that $F_{j,i}(P_{1,2}^{(ij)}) = F_{i,j}(P_{1,2}^{(ij)})$, by noticing that

$$F_{j,i}(P_{1,2}^{(ij)}) = \frac{G(P_{1,2}^{(ij)}) \prod_{s=1}^{n+k-2}[a_s P_{2,1}^{(ij)}]}{\langle P_{1,2}^{(ij)}|K|P_{2,1}^{(ij)}]^n \prod_{t=1,t\neq j,i}^k \langle P_{1,2}^{(ij)}|Q_t|P_{2,1}^{(ij)}]}$$

$$= F_{i,j}(P_{1,2}^{(ij)}). \quad (84)$$

Therefore the sum of contributions from residues associated to $\langle\ell|Q_jQ_i|\ell\rangle = -\langle\ell|Q_iQ_j|\ell\rangle$ in $I_2^{(i)}$ and $I_2^{(j)}$ is

$$-\frac{1}{\sqrt{\Delta^{(ij)}}} \left(F_{i,j}^{(S)} \ln\frac{\langle P_1^{(ij)}|Q_j|P_1^{(ij)}]}{\langle P_1^{(ij)}|Q_i|P_1^{(ij)}]} \frac{\langle P_2^{(ij)}|Q_i|P_2^{(ij)}]}{\langle P_2^{(ij)}|Q_j|P_2^{(ij)}]} \right.$$

$$\left. + F_{i,j}^{(A)} \ln\frac{\langle P_1^{(ij)}|Q_j|P_1^{(ij)}]}{\langle P_1^{(ij)}|Q_i|P_1^{(ij)}]} \frac{\langle P_2^{(ij)}|Q_j|P_2^{(ij)}]}{\langle P_2^{(ij)}|Q_i|P_2^{(ij)}]} \right), \quad (85)$$

$$= -\frac{1}{\sqrt{\Delta^{(ij)}}} \left(F_{i,j}^{(S)} \ln\frac{y_1^{(ij)}}{y_2^{(ij)}} + F_{i,j}^{(A)} \ln\frac{Q_j^2}{Q_i^2} \right), \quad (86)$$

$$= -\frac{1}{\sqrt{\Delta^{(ij)}}}\left(\frac{F_{i,j}(P_1^{(ij)}) + F_{i,j}(P_2^{(ij)})}{2} \ln\frac{-2Q_i \cdot Q_j + \sqrt{\Delta^{(ij)}}}{-2Q_i \cdot Q_j - \sqrt{\Delta^{(ij)}}}\right.$$

$$\left. + \frac{F_{i,j}(P_1^{(ij)}) - F_{i,j}(P_2^{(ij)})}{2} \ln\frac{Q_j^2/K^2}{Q_i^2/K^2}\right). \qquad (87)$$

This is the result we are looking for. The first term is the box contribution. To fix sign conventions, recall that the double cut is given by[4]

$$\frac{(1-2z)}{K^2}\int\langle\ell d\ell\rangle[\ell d\ell]\frac{1}{\langle\ell|Q_1|\ell\rangle\langle\ell|Q_2|\ell\rangle}$$

$$= \frac{(1-2z)}{K^2}\int_0^1 dx \frac{1}{R^2}$$

$$= \frac{(1-2z)}{K^2}\frac{-1}{\sqrt{\Delta^{(ij)}}}\ln\frac{-2Q_i \cdot Q_j + \sqrt{\Delta^{(ij)}}}{-2Q_i \cdot Q_j - \sqrt{\Delta^{(ij)}}}.$$

Therefore the box coefficient is given by

$$C_{\mathrm{box};ij} = \frac{K^2}{(1-2z)}\left(\frac{F_{i,j}(P_1^{(ij)}) + F_{i,j}(P_2^{(ij)})}{2}\right), \qquad (88)$$

where $F_{i,j}$ is given by Eq. (84).

It is important to realize that in fact Eq. (88) will be a rational function of u rather than a polynomial, because it contains results for pentagons as well. To separate the box coefficient, we need to write

$$C_{\mathrm{box};ij}(u) = H(u) + \sum_{i\in\mathrm{pentagons}} A_i P_i, \qquad (89)$$

where $H(u)$ is a polynomial in u, P_i is the pentagon cut given by (59) and A_i is constant in u. Thus $H(u)$ will be the true box coefficient, for which we can apply the recursion and reduction formulas, while A_i is the true pentagon coefficient.

The second term in Eq. (87) is the final piece needed for triangles. When we rewrite it as

$$-\frac{1}{2}\ln\frac{Q_i^2}{K^2}\left(\frac{G(\lambda)\prod_{s=1}^{n+k-2}[a_s|Q_i|\ell\rangle}{\langle\ell|KQ_i|\ell\rangle^n \prod_{t=1,t\neq i}^k\langle\ell|Q_tQ_i|\ell\rangle}\right)\bigg|_{\text{residue of }\langle\ell|Q_jQ_i|\ell\rangle}$$

$$-\frac{1}{2}\ln\frac{Q_j^2}{K^2}\left(\frac{G(\lambda)\prod_{s=1}^{n+k-2}[a_s|Q_j|\ell\rangle}{\langle\ell|KQ_j|\ell\rangle^n \prod_{t=1,t\neq j}^k\langle\ell|Q_tQ_j|\ell\rangle}\right)\bigg|_{\text{residue of }\langle\ell|Q_iQ_j|\ell\rangle},$$

we see that the first term is minus half of the residue contribution of $\langle\ell|Q_jQ_i|\ell\rangle$ inside $F_i(\lambda)$ ($I^{(i)}$) and the second term is minus half of the residue contribution of $\langle\ell|Q_iQ_j|\ell\rangle$ inside $F_j(\lambda)$ ($I^{(j)}$). For the first term, when we sum up all simple-pole contributions for $F_i(\lambda)$, we will be left with the residue of the possible multiple pole $\langle\ell|KQ_i|\ell\rangle^n$. That is to say (using the fact that $Q_i^2/K^2 = x_1^{(i)}x_2^{(i)}$)

$$\frac{1}{2}\ln(x_1^{(i)}x_2^{(i)})\left(\frac{G(\lambda)\prod_{s=1}^{n+k-2}[a_s|Q_i|\ell\rangle}{\langle\ell|KQ_i|\ell\rangle^n \prod_{t=1,t\neq i}^k\langle\ell|Q_tQ_i|\ell\rangle}\right)\bigg|_{\text{residue of }\langle\ell|KQ_i|\ell\rangle}. \qquad (90)$$

Summing the three triangle contributions (90) with Eqs. (75) and (78) we finally reach

$$-\frac{1}{2}\left(\frac{G(\lambda)\prod_{s=1}^{n+k-2}[a_s|Q_i|\ell\rangle}{\langle\ell|KQ_i|\ell\rangle^n \prod_{t=1,t\neq i}^k\langle\ell|Q_tQ_i|\ell\rangle}\ln\frac{(-x_1^{(i)})^2}{(x_1^{(i)}x_2^{(i)})}\right)\bigg|_{\text{residue of }\langle\ell P_1^{(i)}\rangle}$$

$$-\frac{1}{2}\left(\frac{G(\lambda)\prod_{s=1}^{n+k-2}[a_s|Q_i|\ell\rangle}{\langle\ell|KQ_i|\ell\rangle^n \prod_{t=1,t\neq i}^k\langle\ell|Q_tQ_i|\ell\rangle}\ln\frac{(-x_2^{(i)})^2}{(x_1^{(i)}x_2^{(i)})}\right)\bigg|_{\text{residue of }\langle\ell P_2^{(i)}\rangle}.$$

Notice that $\ln(x_1^{(i)}/x_2^{(i)})$ is the signature of triangles. To compare, we recall that for triangles we have

$$(1-2z)\int\langle\ell d\ell\rangle[\ell d\ell]\frac{1}{\langle\ell|K|\ell\rangle\langle\ell|Q_i|\ell\rangle} = (1-2z)\frac{-1}{\sqrt{\Delta^{(i)}}}\ln\frac{-2Q_i \cdot K + \sqrt{\Delta^{(i)}}}{-2Q_i \cdot K - \sqrt{\Delta^{(i)}}}.$$

Therefore the triangle coefficient is given by

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[4]In this part, we compare to the case where propagators are massless. For the massive case, we need to adjust the definition of Q_i and include a factors of b, as explained in Sec. III.

$$C_{\text{tri};i} = \frac{\sqrt{\Delta^{(i)}}}{2(1-2z)} \left\{ \left(\frac{G(\lambda) \prod_{s=1}^{n+k-2} [a_s|Q_i|\ell\rangle}{\langle\ell|KQ_i|\ell\rangle^n \prod_{t=1,t\neq i}^k \langle\ell|Q_tQ_i|\ell\rangle} \right) \right|_{\langle\ell P_1^{(i)}\rangle}$$

(91)

$$- \left(\frac{G(\lambda) \prod_{s=1}^{n+k-2} [a_s|Q_i|\ell\rangle}{\langle\ell|KQ_i|\ell\rangle^n \prod_{t=1,t\neq i}^k \langle\ell|Q_tQ_i|\ell\rangle} \right) \Bigg|_{\langle\ell P_2^{(i)}\rangle} \right\}.$$

(92)

It is important to observe that both box and triangle coefficients are expressed as the difference of residues at the two poles $P_1^{(i)}$ and $P_2^{(i)}$. This is the origin of the "signature" square roots of box and triangle integrals. As we will see shortly, for bubble coefficients, we need to add up the residues at the two poles $P_1^{(i)}$ and $P_2^{(i)}$. The addition will get rid of the square root signature, but we are left with $(1-2z)$, which is related to bubbles.

C. Rational part

Now we address bubble coefficients. They are given by the first term of Eq. (64) with $G_k(\lambda)$ given by Eq. (66). As we have mentioned, for general n and k these are not the simplest and most compact expressions. If we constrain $n \leq 4$, it is possible to use them directly. Here we use a more general approach.

Our starting point is Eq. (62). We write it here as

$$\lim_{\tau\to 0} \frac{G(\lambda) \prod_{j=1}^{n+k-2} [a_j\ell\rangle}{\prod_{s=0}^{n-1} \langle\ell|K+\tau s\eta|\ell] \prod_{i=1}^k \langle\ell|Q_i|\ell]},$$

(93)

where $\eta^2 = 0$ and τ is a parameter which we eventually take to zero.

Because no denominator factor appears more than once, we can use Eq. (68) directly to reach

$$\sum_{s=1}^{n-1} \frac{1}{\langle\ell|K|\ell]\langle\ell|K+\tau s\eta|\ell]} \frac{G(\lambda) \prod_{j=1}^{n+k-2} [a_j|K+\tau s\eta|\ell\rangle}{\prod_{s'=1,s'\neq s}^{n-1} \langle\ell|(K+\tau s'\eta)(K+\tau s\eta)|\ell\rangle \prod_{i=1}^k \langle\ell|Q_i(K+\tau s\eta)|\ell\rangle} + \sum_{i=1}^k \frac{1}{\langle\ell|K|\ell]\langle\ell|Q_i|\ell]}$$

$$\times \frac{G(\lambda) \prod_{j=1}^{n+k-2} [a_j|Q_i|\ell\rangle}{\prod_{s'=1,}^{n-1} \langle\ell|(K+\tau s'\eta)Q_i|\ell\rangle \prod_{i'=1,i'\neq i}^k \langle\ell|Q_{i'}Q_i|\ell\rangle}.$$

(94)

When we take the $\tau \to 0$ limit, the second line becomes the contribution to triangles and boxes that was discussed in the previous subsection. The first line gives the bubble contribution. It appears that each term in the first line diverges in the $\tau \to 0$ limit. To proceed, define

$$\tilde{K}(s) = K + \tau s\eta.$$

Then we find that each term in the first line of Eq. (94) can be rewritten as

$$\frac{1}{\langle\ell|\tilde{K}(s)-\tau s\eta|\ell]\langle\ell|\tilde{K}(s)|\ell]} \frac{G(\lambda) \prod_{j=1}^{n+k-2} [a_j|\tilde{K}(s)|\ell\rangle}{\tau^{n-2} \langle\ell|\eta\tilde{K}(s)|\ell\rangle^{n-2} \prod_{s'=1,s'\neq s}^{n-1} (s'-s) \prod_{i=1}^k \langle\ell|Q_i\tilde{K}(s)|\ell\rangle}.$$

Now we can expand this expression as a power series in τ, keeping those terms that will survive the limit $\tau \to 0$. Specifically, we substitute

$$\frac{1}{\langle\ell|\tilde{K}(s)-\tau s\eta|\ell]\langle\ell|\tilde{K}(s)|\ell]} = \sum_{h=0}^{n-2} \frac{\tau^h s^h \langle\ell|\eta|\ell]^h}{\langle\ell|\tilde{K}(s)|\ell]^{2+h}} + \mathcal{O}(\tau^{n-1}).$$

Apply the familiar spinor integration procedure to replace one integrand by a total derivative using the identity

$$[\ell d\ell] \left(\frac{[\eta\ell]^n}{\langle\ell|P|\ell]^{n+2}} \right) = [d\ell\partial_\ell] \left(\frac{1}{(n+1)} \frac{1}{\langle\ell|P|\eta]} \frac{[\eta\ell]^{n+1}}{\langle\ell|P|\ell]^{n+1}} \right).$$

(95)

The result is then the sum of residues of

$$\frac{G(\lambda) \prod_{j=1}^{n+k-2} [a_j|\tilde{K}(s)|\ell\rangle}{\tau^{n-2} \langle\ell|\eta\tilde{K}(s)|\ell\rangle^{n-1} \prod_{s'=1,s'\neq s}^{n-1} (s'-s) \prod_{i=1}^k \langle\ell|Q_i\tilde{K}(s)|\ell\rangle} \left(\sum_{h=0}^{n-2} \frac{\tau^h s^h \langle\ell|\eta|\ell]^{h+1}}{(h+1)\langle\ell|\tilde{K}(s)|\ell]^{1+h}} \right).$$

(96)

For the multiple pole, $\langle\ell|\eta\tilde{K}(s)|\ell\rangle = \langle\ell\eta\rangle[\eta|\tilde{K}(s)|\ell\rangle$. Since the numerator contains a factor of $[\eta\ell]^{h+1}$, we do not take the residue of $\langle\ell\eta\rangle$.

Finally, after defining the quantity

$$R[\tilde{K}(s), \{Q_i\}, \eta] = \sum_{\text{poles except } \eta} \text{Res}\left(\frac{G(\lambda) \prod_{j=1}^{n+k-2}[a_j|\tilde{K}(s)|\ell\rangle}{\tau^{n-2}\langle\ell|\eta\tilde{K}(s)|\ell\rangle^{n-1} \prod_{s'=1, s'\neq s}^{n-1}(s'-s) \prod_{i=1}^{k}\langle\ell|Q_i\tilde{K}(s)|\ell\rangle}\left(\sum_{h=0}^{n-2}\frac{\tau^h s^h \langle\ell|\eta|\ell]^{h+1}}{(h+1)\langle\ell|\tilde{K}(s)|\ell]^{1+h}}\right)\right),$$

(97)

we can write the bubble coefficients as

$$C_{\text{bubble}} = \frac{1}{\sqrt{1-u}} \sum_{s=1}^{n-1} R[\tilde{K}(s), \{Q_i\}, \eta]|_{\tau\to 0}. \quad (98)$$

The limit is taken by expanding and truncating the series.

D. Summary of results

Conventions:

We start from an integrand of the form

$$I_{\text{term}} = \frac{G(\lambda) \prod_{j=1}^{n+k-2}[a_j\ell]}{\langle\ell|K|\ell]^n \prod_{i=1}^{k}\langle\ell|Q_i|\ell]} \quad (99)$$

and present here an expression to be integrated over the final (-2ϵ) dimensions as discussed in Sec. II C.

Box coefficients:

$$C_{\text{box};ij} = \frac{K^2}{\sqrt{1-u}}\left(\frac{F_{i,j}(P_1^{(ij)}) + F_{i,j}(P_2^{(ij)})}{2}\right), \quad (100)$$

$$P_{1,2}^{(ij)} = Q_j + y_{1,2}^{(ij)} Q_i, \qquad (i < j), \quad (101)$$

$$y_{1,2}^{(ij)} = \frac{-2Q_i \cdot Q_j \pm \sqrt{\Delta^{(ij)}}}{2Q_i^2}, \quad (102)$$

$$\Delta^{(ij)} = (2Q_i \cdot Q_j)^2 - 4Q_i^2 Q_j^2,$$

$$F_{i,j}(\ell) = \frac{G(\lambda) \prod_{s=1}^{n+k-2}[a_s|Q_i|\ell\rangle}{\langle\ell|KQ_i|\ell\rangle^n \prod_{t=1, t\neq i,j}^{k}\langle\ell|Q_t Q_i|\ell\rangle}. \quad (103)$$

Triangle coefficients:

$$C_{\text{tri};i} = \frac{\sqrt{\Delta^{(i)}}}{2\sqrt{1-u}}\left[\left(\frac{G(\lambda) \prod_{s=1}^{n+k-2}[a_s|Q_i|\ell\rangle}{\langle\ell|KQ_i|\ell\rangle^n \prod_{t=1, t\neq i}^{k}\langle\ell|Q_t Q_i|\ell\rangle}\right)\Big|_{\langle\ell P_1^{(i)}\rangle}\right.$$
$$\left. - \left(\frac{G(\lambda) \prod_{s=1}^{n+k-2}[a_s|Q_i|\ell\rangle}{\langle\ell|KQ_i|\ell\rangle^n \prod_{t=1, t\neq i}^{k}\langle\ell|Q_t Q_i|\ell\rangle}\right)\Big|_{\langle\ell P_2^{(i)}\rangle}\right],$$

(104)

$$P_{1,2}^{(i)} = Q_i + x_{1,2}^{(i)} K, \quad (105)$$

$$x_{1,2}^{(i)} = \frac{-2Q_i \cdot K \pm \sqrt{\Delta^{(i)}}}{2K^2}, \quad (106)$$
$$\Delta^{(i)} = (2Q_i \cdot K)^2 - 4Q_i^2 K^2.$$

Bubble coefficients:

$$C_{\text{bubble}} = \frac{1}{\sqrt{1-u}} \sum_{s=1}^{n-1} R[\tilde{K}(s), \{Q_i\}, \eta]|_{\tau\to 0}. \quad (107)$$

The limit is taken by expanding and truncating the series.

$$R[\tilde{K}(s), \{Q_i\}, \eta] = \sum_{\text{poles except } \eta} \text{Res}\left(\frac{G(\lambda) \prod_{j=1}^{n+k-2}[a_j|\tilde{K}(s)|\ell\rangle}{\tau^{n-2}\langle\ell|\eta\tilde{K}(s)|\ell\rangle^{n-1} \prod_{s'=1, s'\neq s}^{n-1}(s'-s) \prod_{i=1}^{k}\langle\ell|Q_i\tilde{K}(s)|\ell\rangle}\left(\sum_{h=0}^{n-2}\frac{\tau^h s^h \langle\ell|\eta|\ell]^{h+1}}{(h+1)\langle\ell|\tilde{K}(s)|\ell]^{1+h}}\right)\right),$$

(108)

$$\tilde{K}(s) = K + \tau s \eta. \quad (109)$$

For a detailed description of evaluation, see the Appendix.

V. DISCUSSION

In this section, we first comment on the input needed for this integration program, and then discuss the alternative approaches of generalized unitarity cuts and the program of Ossola, Papadopoulos, and Pittau [2].

A. Tree-level input

The input needed for the unitary cut method is a collection of (on-shell) tree-level amplitudes, which can be cal-

culated by recursion relations. However, results with spurious poles (which generally arise in the most compact expressions) can make it hard to apply our method. To avoid this difficulty, we can apply spinor identities to regroup the terms into an expression free of spurious poles.

Let us demonstrate this in one example. The five point function with two massive scalars are given by[5]

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[5]The sign of the second term is different from the corresponding formula in Ref. [24]. We have checked, using the method of Ref. [24], that the sign in this earlier formula was a typo.

$$A(\ell_1, 1^+, 2^+, 3^-, \ell_2) = -\frac{\langle 3|\ell_2(1+2)\ell_1|1]^2}{\langle 12\rangle\langle 23\rangle((\ell_1+k_1)^2 - \mu^2)((\ell_2+k_3)^2 - \mu^2)[3|(1+2)\ell_1|1]} + \frac{\mu^2[12]^3}{K_{123}^2[23][3|(1+2)\ell_1|1]}$$

$$= -\frac{\langle 3|\ell_2(1+2)\ell_1|1]^2}{\langle 12\rangle\langle 23\rangle\langle 1|\ell_1|1]\langle 3|\ell_2|3][3|(1+2)\ell_1|1]} + \frac{\mu^2[12]^3}{K_{123}^2[23][3|(1+2)\ell_1|1]}.$$

The spurious pole in the first term can be split from the others by the identity

$$\frac{\langle 3|\ell_2(1+2)|\ell_1|1]}{\langle 1|\ell_1|1][3|(1+2)|\ell_1|1]} = \frac{\langle 3|\ell_2(1+2)|1\rangle}{[3|(1+2)|1\rangle\langle 1|\ell_1|1]} + \frac{\langle 3|\ell_2(1+2)(1+2)|3]}{\langle 1|(1+2)|3][3|(1+2)|\ell_1|1]}.$$

Then

$$A(\ell_1, 1^+, 2^+, 3^-, \ell_2) = -\frac{\langle 3|\ell_2(1+2)\ell_1|1]\langle 3|\ell_2|2]}{\langle 12\rangle\langle 23\rangle[32]\langle 3|\ell_2|3]\langle 1|\ell_1|1]} + \frac{\langle 3|\ell_2(1+2)\ell_1|1][12]}{\langle 12\rangle\langle 23\rangle[23][3|(1+2)\ell_1|1]} + \frac{\mu^2[12]^3}{K_{123}^2[23][3|(1+2)\ell_1|1]}.$$

Using

$$\langle 3|\ell_2(1+2)\ell_1|1]K_{123}^2 = \langle 3|\ell_2(1+2)\ell_1|1](K_{12}^2 - \langle 3|(1+2)|3])$$

$$= \langle 3|\ell_2(1+2)\ell_1|1]K_{12}^2 + \langle 3|(1+2)\ell_2|3][3|(1+2)\ell_1|1] + K_{12}^2\langle 3|\ell_1|1]\langle 3|\ell_2|3],$$

we can add the last two terms together and see that the spurious pole $[3|(1+2)\ell_1|1]$ has been canceled (as it must). Thus we finally get

$$A(\ell_1, 1^+, 2^+, 3^-, \ell_2) = \frac{\langle 3|\ell_2(1+2)\ell_1|1]\langle 3|\ell_2|2]}{\langle 12\rangle K_{23}^2\langle 1|\ell_1|1]\langle 3|\ell_2|3]}$$

$$+ \frac{[12]\langle 3|(1+2)\ell_2|3\rangle}{\langle 12\rangle K_{23}^2 K_{123}^2}. \qquad (110)$$

A similar calculation gives

$$A_5(\ell_1^+, 1^+, 2^-, 3^+, \ell_2^-) = \frac{\langle 2|\ell_1|1]\langle 2|\ell_2|3]^2}{\langle 12\rangle K_{23}^2\langle 3|\ell_2|3]\langle 1|\ell_1|1]}$$

$$+ \frac{[31]\langle 2|\ell_1|1]\langle 2|\ell_2|3]}{K_{12}^2 K_{23}^2\langle 3|\ell_2|3]}$$

$$- \frac{\langle 2|\ell_2 K_{123}|2\rangle[13]^2}{K_{12}^2 K_{23}^2 K_{123}^2}, \qquad (111)$$

where spurious poles have been canceled.

B. The quadruple cut

In four dimensions, the quadruple cut [5] is a powerful tool for getting box coefficients because the integral is completely localized. Quadruple cuts may be applied here as well since we have separated the (-2ϵ)-dimensional and four-dimensional integrations. We apply them to boxes and pentagons.

Again, we write in the language of spinor integration (see the brief remarks at the beginning of Sec. III), but this is absolutely not essential to the technique of quadruple cuts.

Let us start with the quadruple cut (denoted here by C^q) of a box. It is given by

$$C_4^q = \int_0^1 du\, u^{-1-\epsilon} \int d^4\ell\, \delta^+(\ell^2)\delta((1-2z)K^2 - 2\ell\cdot K$$

$$+ M_1^2 - M_2^2)\delta((\tilde{\ell} - P_1)^2 - M_3^2 - \mu^2)$$

$$\times \delta((\tilde{\ell} - P_2)^2 - M_4^2 - \mu^2)$$

$$= \int_0^1 du\, u^{-1-\epsilon} \int d^4\ell\, \delta^+(\ell^2)\delta((1-2z)K^2 - 2\ell\cdot K$$

$$+ M_1^2 - M_2^2)\delta(P_1^2 + M_1^2 - M_3^2 - z(2K\cdot P_1)$$

$$- 2\ell\cdot P_1)\delta(P_2^2 + M_1^2 - M_4^2 - z(2K\cdot P_2) - 2\ell\cdot P_2).$$

After t integration, which sets $t = -((1-2z)K^2 + M_1^2 - M_2^2)/\langle\ell|K|\ell]$, we have

$$C_4^q = \int_0^1 du\, u^{-1-\epsilon} \int \langle\ell d\ell\rangle[\ell d\ell]\frac{(1-2z)K^2 + M_1^2 - M_2^2}{\langle\ell|K|\ell]^2}$$

$$\times \delta\left(\frac{K^2\langle\ell|Q_1|\ell]}{\langle\ell|K|\ell]}\right)\delta\left(\frac{K^2\langle\ell|Q_2|\ell]}{\langle\ell|K|\ell]}\right),$$

where

$$Q_i = -\left((1-2z) + \frac{M_1^2 - M_2^2}{K^2}\right)P_1$$

$$+ \frac{P_1^2 + M_1^2 - M_{i+2}^2 - z(2K\cdot P_1)}{K^2}K. \qquad (112)$$

To solve the remaining two delta functions, we need to find two momenta constructed from $q_i = Q_2 + x_iQ_1$, $i = 1, 2$ such that $q_i^2 = 0$. This is the same construction as in (81). Then the two solutions for the two delta functions are given by[6]

$$\ell_1 = |q_1\rangle|q_2], \qquad \ell_2 = |q_2\rangle|q_1]. \qquad (113)$$

Notice that these solutions are complex, as usual for qua-

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[6]These are the ℓ_3, ℓ_4 of Ref. [2].

druple cuts. There is a universal Jacobian factor, which in general is a function of u. Putting it all together we have

$$C_4^q = \int_0^1 du u^{-1-\epsilon} J(u), \qquad (114)$$

where $J(u)$ is the Jacobian and should be symmetric in K, P_1, P_2.

Next we consider the quadruple cut of pentagons. The calculation is similar, so we shall be brief. After t integration we have

$$C_5^q = \int_0^1 du u^{-1-\epsilon} \int \langle \ell d\ell \rangle [\ell d\ell] \frac{(1-2z)K^2 + M_1^2 - M_2^2}{\langle \ell|K|\ell] K^2 \langle \ell|Q_3|\ell]}$$
$$\times \delta\left(\frac{K^2 \langle \ell|Q_1|\ell]}{\langle \ell|K|\ell]}\right) \delta\left(\frac{K^2 \langle \ell|Q_2|\ell]}{\langle \ell|K|\ell]}\right),$$

with the Q_i defined in Eq. (112).

Now we need to sum up the contributions of the two solutions. The Jacobian of the two solutions is the same. We end up with[7]

$$C_5^q = \int_0^1 du u^{-1-\epsilon} \frac{J(u)}{K^2} \frac{1}{2} \left(\frac{\langle q_1|K|q_2]}{\langle q_1|Q_3|q_2]} + \frac{\langle q_2|K|q_1]}{\langle q_2|Q_3|q_1]}\right)$$
$$= \int_0^1 du u^{-1-\epsilon} \frac{J(u)}{2K^2} S[Q_2, Q_1, Q_3, K], \qquad (115)$$

where $S[Q_2, Q_1, Q_3, K]$ was defined in Eq. (67). We see again that the relative factor between boxes and pentagons is $S[\bullet]/(2K^2)$, just as we learned from the double cut.

In general, we perform the t-integral first and then add the two solutions. This gives a rational function $R(u)$. Then we need to split

$$R(u) = P(u) + \sum_{Q_3} a_{Q_3} \frac{S[Q_2, Q_1, Q_3, K]}{2K^2}, \qquad (116)$$

where $P(u)$ is a polynomial that gives the box contribution, and a_{Q_3} are constants in u that give corresponding coefficients of pentagons. This decomposition is the same as the one given by Eq. (89).

After getting box and pentagon coefficients from quadruple cuts, one might continue by applying triple cuts to target specific triangle coefficients, and then finally use the usual double cut for the bubble part only. Triple cuts have been used to get particular one-loop coefficients in Refs. [25–28], and recently a nice paper [11] has described a general procedure to compute triple cuts in arbitrary dimensions. One can try to systematically study the triple cuts of Ref. [11] along the lines presented in this paper. It is easy to see that the delta function there (plus possible derivatives of delta function) corresponds to our multiple pole.

C. Comparison with OPP method

Recently, a computation by Ossola, Papadopoulos, and Pittau (OPP) [2] has been attracting attention. It seems to be a very simple reduction method, which can be performed at the integrand level. The key point of the OPP method is that knowing the general form of spurious terms, one can solve algebraically for the coefficients of physical and spurious terms from knowing the initial data. The work we have presented here is in the same spirit. In our method, it is through splitting into partial fractions that we are able to identify contributions to the various basis integrals and find the functions for these coefficients.

We remark that in Ref. [2], it is claimed that in the renormalizable gauge, there is an upper bound on the number of spurious terms. For example, there are six for triangles, eight for bubbles. The upper bound in the OPP method will correspond to the upper bound of n in our factor $\langle \ell|K|\ell]^n$. Our experience suggests that we will always find $n \leq 4$. The reason is the following. Before the t integration we have

$$\frac{\prod_{i=1}^{N_1}[a_i|\tilde{\ell}|b_i\rangle}{\prod_{j=1}^{N_2}((\tilde{\ell} - P_j)^2 - \mu^2)},$$

where the power of $\tilde{\ell}$ in the numerator N_1 has an upper bound, $N_1 \leq N_2 + 2$. The reason is that in renormalizable gauge, the expression derived from Feynman rules has the property that the degree of $\tilde{\ell}$ in numerator is less than or equal to that in the denominator. After peeling off the two cut propagators, we have $N_1 \leq N_2 + 2$. Now we count the powers of $1/\langle \ell|K|\ell]$ after t integration. Numerators contribute $-N_1$ while denominators contribute N_2. From $\int t dt$ and the delta function, we get an additional -2. In all, we have $(1/\langle \ell|K|\ell])^{-N_1-2+N_2}$, thus $n = N_1 + 2 - N_2 \leq 4$.

Assuming that indeed $n \leq 4$, the correspondence between our method and the OPP method becomes clearer. It is easy to see that box and pentagon coefficients have the same expressions. For triangle, we have multiple poles up to order 4. From our discussion in the Appendix, it can be seen that a pole of order n can be traded for a simple pole with an $(n-1)$-th derivative. In our case, with $n \leq 4$ we have up to third derivatives. This is similar to $j = 1, 2, 3$ in Ref. . For bubble coefficients, we have up to three terms, and each one has multiple poles up to order three, so we would expect $3 \times 2 = 6$ contributions, compared to the eight spurious terms of the OPP method. Our impression is that the spurious terms of OPP are hidden as multiple poles as well as simple poles in our language. The algebraic complexity of the two methods should be equivalent.

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[7]Here we have inserted the factor $1/2$ since we need to sum the contributions of two solutions. This is the same expression used in original quadruple cut [5], where on one side, we sum up two solutions and divide by two, while on the other side, we have just the Jacobi factor for the basis of box.

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APPENDIX A: EVALUATION OF RESIDUES

In this Appendix, we discuss how to evaluate the various functions we defined in Sec. IV to get formulas for coefficients of the basis integrals. Our aim is to make the formulas programmable, so we go into some detail.

1. Box coefficients

The box (and pentagon) coefficients are given by Eq. (88),

$$C_{\text{box};ij} = \frac{K^2}{(1 - 2z)}\left(\frac{F_{i,j}(P_1^{(ij)}) + F_{i,j}(P_2^{(ij)})}{2}\right), \quad \text{(A1)}$$

where

$$F_{i,j}(P_{1,2}^{(ij)}) = \frac{G(P_{1,2}^{(ij)})\prod_{s=1}^{n+k-2}[a_s P_{2,1}^{(ij)}]}{\langle P_{1,2}^{(ij)}|K|P_{2,1}^{(ij)}]^n \prod_{t=1,t\neq j,i}^{k}\langle P_{1,2}^{(ij)}|Q_t|P_{2,1}^{(ij)}]}.$$

Let us rewrite the coefficient as

$$C_{\text{box};ij} = \frac{K^2}{2(1 - 2z)}\frac{N}{D}, \quad \text{(A2)}$$

where[8]

$$\frac{N}{D} = F_{i,j}(P_1^{(ij)}) + F_{i,j}(P_2^{(ij)}),$$

$$N = G(P_1^{(ij)})\prod_{s=1}^{n+k-2}[a_s P_2^{(ij)}]\langle P_2^{(ij)}|K|P_1^{(ij)}]^n$$

$$\times \prod_{t=1,t\neq j,i}^{k}\langle P_2^{(ij)}|Q_t|P_1^{(ij)}] + G(P_2^{(ij)})$$

$$\times \prod_{s=1}^{n+k-2}[a_s P_1^{(ij)}]\langle P_1^{(ij)}|K|P_2^{(ij)}]^n \prod_{t=1,t\neq j,i}^{k}\langle P_1^{(ij)}|Q_t|P_2^{(ij)}],$$

$$D = \langle P_1^{(ij)}|K|P_2^{(ij)}]^n \prod_{t=1,t\neq j,i}^{k}\langle P_1^{(ij)}|Q_t|P_2^{(ij)}]\langle P_2^{(ij)}|K|P_1^{(ij)}]^n$$

$$\times \prod_{t=1,t\neq j,i}^{k}\langle P_2^{(ij)}|Q_t|P_1^{(ij)}].$$

For D we use the following identity:

$$\langle\eta_1|S|\eta_2]\langle\eta_2|S|\eta_1] = (2\eta_1 \cdot S)(2\eta_2 \cdot S) - S^2(2\eta_1 \cdot \eta_2). \quad \text{(A3)}$$

Then,

$$\langle P_1^{(ij)}|K|P_2^{(ij)}]\langle P_2^{(ij)}|K|P_1^{(ij)}]$$

$$= (2K \cdot (Q_j + y_1^{(ij)}Q_i))(2K \cdot (Q_j + y_1^{(ij)}Q_i)) + \frac{K^2\Delta^{(ij)}}{Q_i^2}$$

$$\equiv T[Q_i, Q_j, K].$$

The function $T[Q_i, Q_j, K]$ in this formula is T_2/Q_i^2, with the definition of T_2 given in (61). Using this definition we have

$$D = T[Q_i, Q_j, K]^n \prod_{t=1,t\neq j,i}^{k} T[Q_i, Q_j, Q_t]. \quad \text{(A4)}$$

It is easy to see that each term in N may be written as a product of the form

$$[a_s P_2]\langle P_2|Q|P_1\rangle\langle P_1 b_s\rangle$$
$$= [a_s|P_2 Q P_1|b_s\rangle = [a_s|(Q_j + y_2 Q_i)Q(Q_j + y_1 Q_i)|b_s\rangle.$$

Thus we see that the evaluation of box coefficients (A2) is straightforward.

2. The residue of multiple poles

For triangles and bubbles, we need to know how to get residue of multiple poles. There are several ways to do this. The first one is to use the splitting method discussed in Ref. [7].

The second method is to shift momentum parametrically and then take the limit where the parameter goes to zero. We illustrate this method with the example of a double pole. We denote the small parameter by τ.

$$\frac{1}{\langle\ell\eta\rangle^2}\frac{\prod_{j=1}^{k}\langle\ell a_j\rangle}{\prod_{j=1}^{k}\langle\ell b_j\rangle} \rightarrow \frac{1}{\langle\ell\eta\rangle\langle\ell(\eta + \tau\alpha)\rangle}\frac{\prod_{j=1}^{k}\langle\ell a_j\rangle}{\prod_{j=1}^{k}\langle\ell b_j\rangle} = \frac{1}{\langle\eta(\eta + \tau\alpha)\rangle}\frac{\prod_{j=1}^{k}\langle\eta a_j\rangle}{\prod_{j=1}^{k}\langle\eta b_j\rangle} - \frac{1}{\langle\eta(\eta + \tau\alpha)\rangle}\frac{\prod_{j=1}^{k}\langle(\eta + \tau\alpha)a_j\rangle}{\prod_{j=1}^{k}\langle(\eta + \tau\alpha)b_j\rangle}$$

$$= \frac{1}{\tau\langle\eta\alpha\rangle}\frac{\prod_{j=1}^{k}\langle\eta a_j\rangle}{\prod_{j=1}^{k}\langle\eta b_j\rangle}\left(1 - \frac{\prod_{j=1}^{k}(1 + \tau\frac{\langle\alpha a_j\rangle}{\langle\eta a_j\rangle})}{\prod_{j=1}^{k}(1 + \tau\frac{\langle\alpha b_j\rangle}{\langle\eta b_j\rangle})}\right) \rightarrow \frac{1}{\langle\eta\alpha\rangle}\frac{\prod_{j=1}^{k}\langle\eta a_j\rangle}{\prod_{j=1}^{k}\langle\eta b_j\rangle}\left(-\sum_{j=1}^{k}\frac{\langle\alpha a_j\rangle}{\langle\eta a_j\rangle} + \sum_{j=1}^{k}\frac{\langle\alpha b_j\rangle}{\langle\eta b_j\rangle}\right)$$

$$= \frac{\prod_{j=1}^{k}\langle\eta a_j\rangle}{\prod_{j=1}^{k}\langle\eta b_j\rangle}\left(\sum_{j=1}^{k}\frac{\langle a_j b_j\rangle}{\langle\eta a_j\rangle\langle\eta b_j\rangle}\right).$$

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[8] Our functions $|P_1^{(ij)}\rangle|P_2^{(ij)}]$ and $|P_2^{(ij)}\rangle|P_1^{(ij)}]$ showing up in N and D are the ℓ_3, ℓ_4 of Ref. [2].

We see that this is the correct result. Compared to the first method, we have more terms and one extra auxiliary spinor η. However, in this method, the symmetry among a_i's and b_i's is explicit.

Now we can use the above idea to get the general expression for multiple poles:

$$\frac{1}{\langle\ell\eta\rangle^n}\frac{\prod_i\langle\ell a_i\rangle}{\prod_j\langle\ell b_j\rangle} \to \frac{1}{\prod_{s=0}^{n-1}\langle\ell(\eta+s\tau\alpha)\rangle}\frac{\prod_i\langle\ell a_i\rangle}{\prod_j\langle\ell b_j\rangle} = \sum_{s_0=0}^{n-1}\frac{1}{\prod_{s=0,s\neq s_0}^{n-1}\langle(\eta+\tau s_0\alpha)(\eta+s\tau\alpha)\rangle}\frac{\prod_i\langle(\eta+\tau s_0\alpha)a_i\rangle}{\prod_j\langle(\eta+\tau s_0\alpha)b_j\rangle}$$

$$= \sum_{s_0=0}^{n-1}\frac{1}{\tau^{n-1}\langle\eta\alpha\rangle^{n-1}\prod_{s=0,s\neq s_0}^{n-1}(s-s_0)}\frac{\prod_i\langle(\eta+\tau s_0\alpha)a_i\rangle}{\prod_j\langle(\eta+\tau s_0\alpha)b_j\rangle}$$

$$= \sum_{s_0=0}^{n-1}\frac{1}{\tau^{n-1}\langle\eta\alpha\rangle^{n-1}\prod_{s=0,s\neq s_0}^{n-1}(s-s_0)}\frac{\prod_i\langle\eta a_i\rangle(1+\tau s_0\frac{\langle\alpha a_i\rangle}{\langle\eta a_i\rangle})}{\prod_j\langle\eta b_j\rangle(1+\tau s_0\frac{\langle\alpha b_j\rangle}{\langle\eta b_j\rangle})}$$

$$= \frac{1}{\tau^{n-1}\langle\eta\alpha\rangle^{n-1}}\frac{\prod_i\langle\eta a_i\rangle}{\prod_j\langle\eta b_j\rangle}\left(\sum_{s_0=0}^{n-1}\frac{1}{\prod_{s=0,s\neq s_0}^{n-1}(s-s_0)}\frac{\prod_i(1+\tau s_0\frac{\langle\alpha a_i\rangle}{\langle\eta a_i\rangle})}{\prod_j(1+\tau s_0\frac{\langle\alpha b_j\rangle}{\langle\eta b_j\rangle})}\right).$$

We see that what we need to do is to expand the expression inside the parentheses as a series in τ, keep only the terms up to order τ^{n-1}. Although this expression has one auxiliary spinor η, the final result does not depend on it. Without writing out intermediate steps, we reach the following result:

$$\frac{1}{\langle\eta\alpha\rangle^{n-1}}\frac{\prod_i\langle\eta a_i\rangle}{\prod_j\langle\eta b_j\rangle}\left(\sum_{s_0=0}^{n-1}\frac{s_0^{n-1}}{\prod_{s=0,s\neq s_0}^{n-1}(s-s_0)}\sum_{N_1+\sum_{j=1}^k m_j=n-1}\left(\prod_{j=1}^k(-)^{m_j}\left(\frac{\langle\alpha b_j\rangle}{\langle\eta b_j\rangle}\right)^{m_j}\right)\left(\sum_{1\leq i_1<i_2\ldots<i_{N_1}\leq k+n-2}\prod_{q=1}^{N_1}\frac{\langle\alpha a_{i_q}\rangle}{\langle\eta a_{i_q}\rangle}\right)\right). \quad \text{(A5)}$$

The third method is very similar to the second method. Upon noticing that

$$\frac{d}{d\tau^{n-1}}\frac{1}{\langle\ell\eta-\tau s\rangle} = \frac{(n-1)!\langle\ell s\rangle^{n-1}}{\langle\ell\eta-\tau s\rangle^n}, \quad \text{(A6)}$$

we can rewrite

$$\frac{1}{\langle\ell\eta\rangle^n}\frac{\prod_i\langle\ell a_i\rangle}{\prod_j\langle\ell b_j\rangle} \to \frac{1}{\langle\ell\eta-\tau s\rangle^n}\frac{\prod_i\langle\ell a_i\rangle}{\prod_j\langle\ell b_j\rangle} = \frac{d}{d\tau^{n-1}}\left(\frac{1}{\langle\ell\eta-\tau s\rangle}\frac{1}{(n-1)!\langle\ell s\rangle^{n-1}}\frac{\prod_i\langle\ell a_i\rangle}{\prod_j\langle\ell b_j\rangle}\right).$$

Now we take the simple residue of $\langle\ell\eta-\tau s\rangle$. That is, we set $|\ell\rangle=|\eta-\tau s\rangle$, take the derivative, and finally find that the residue is

$$\text{Residue} = \frac{d}{d\tau^{n-1}}\left(\frac{1}{(n-1)!\langle\eta s\rangle^{n-1}}\frac{\prod_i\langle(\eta-\tau s)a_i\rangle}{\prod_j\langle(\eta-\tau s)b_j\rangle}\right)_{\tau\to 0}. \quad \text{(A7)}$$

It should not be hard to check that Eq. (A5) is the same as Eq. (A7) if we identify $\alpha=s$. As a demonstration, let us check the case of a double pole:

$$\frac{1}{\langle\ell\eta-\tau s\rangle^2}\frac{\prod_{j=1}^k\langle\ell a_j\rangle}{\prod_{j=1}^k\langle\ell b_j\rangle} \to \frac{d}{d\tau}\left(\frac{1}{\langle\ell s\rangle}\frac{1}{\langle\ell\eta-\tau s\rangle}\frac{\prod_{j=1}^k\langle\ell a_j\rangle}{\prod_{j=1}^k\langle\ell b_j\rangle}\right)\Big|_{\text{residue}} = \frac{d}{d\tau}\left(\frac{1}{\langle\eta-\tau s s\rangle}\frac{\prod_{j=1}^k\langle\eta-\tau s a_j\rangle}{\prod_{j=1}^k\langle\eta-\tau s b_j\rangle}\right)\Big|_{\tau\to 0}$$

$$= \frac{1}{\langle\eta s\rangle}\frac{\prod_{j=1}^k\langle\eta a_j\rangle}{\prod_{j=1}^k\langle\eta b_j\rangle}\left(\sum_j\frac{\langle s b_j\rangle}{\langle\eta b_j\rangle} - \sum_j\frac{\langle s a_j\rangle}{\langle\eta a_j\rangle}\right).$$

This is exactly the same result given by the shifting technique.

Knowing how to evaluate the residue of multiple poles, we can discuss how to evaluate coefficients for triangles and bubbles given by Eqs. (104) and (107). We use the third method. For example, for Eq. (104) we can shift $K\to K-\tau s$. Then using Eq. (A6) we can transfer it to the residue of a simple pole. We can further simplify the sum of these two simple poles using the same technique discussed in our previous subsection for box coefficients. After that we can take the derivative and the $\tau\to 0$ limit by truncating the power series.

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