Well-posedness of initial value problem for Schrödinger-Boussinesq system

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Abstract

In this paper, we study the well-posedness of the initial value problem for the Schrödinger-Boussinesq system. By exploiting the Strichartz estimates for the linear Schrödinger operator, we establish the local and global well-posedness of initial value problem for the Schrödinger-Boussinesq system with the initial data in low regularity spaces.

Key words and phrases: Schrödinger-Boussinesq system, initial value problem, well-posedness.

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1 Introduction

It is well known that the nonlinear Schrödinger (NLS) equation models a wide range of physical phenomena including self-focusing of optical beams in nonlinear media, the modulation of monochromatic waves, propagation of Langmuir waves in plasmas, etc. The nonlinear Schrödinger equations play an important role in many areas of applied physics, such as non-relativistic quantum mechanics, laser beam propagation, Bose-Einstein condensates, and so on (see [20]). The initial value problem (IVP) or the initial-boundary value problem (IBVP) for the nonlinear Schrödinger equations on $\mathbb{R}^n$ have been extensively studied in the last two decades (e.g., see [3]-[4], [1], [7], [17]).

The Boussinesq-type equations are essentially a class of models appearing in physics and fluid mechanics. The so-called Boussinesq equation was originally derived by Boussinesq to describe two-dimensional irrotational flows of an inviscid liquid in a uniform rectangular channel. It also arises in a large range of physical phenomena including the propagation of ion-sound waves in a plasma and nonlinear lattice waves. The study on the IVP for various generalizations of the Boussinesq equation has recently attracted considerable attention from many mathematicians and physicists (see [11], [13]).

This paper concerns with the initial value problem for the Schrödinger-Boussinesq system

$$\begin{cases} iu_t + \Delta u = uv + \alpha|u|^2 u, \\ v_{tt} - \Delta v + \Delta^2 v = \Delta|u|^2 \end{cases} \tag{1.1}$$

with the initial data

$$t = 0: \quad u = u_0(x), \quad v = v_0(x), \quad v_t = v_1(x), \tag{1.2}$$

where $u = u(x,t)$ and $v = v(x,t)$ are complex and real-valued functions of $(x,t) \in (\mathbb{R}^n, \mathbb{R}^+)$ respectively, $u_0(x)$ is a given complex value function, $v_0(x)$ and $v_1(x)$ are two given real value functions, α is a real parameter.

The system (1.1) of the Schrödinger-Boussinesq equations is considered as a model of interactions between short and intermediate long waves, which is derived in describing the dynamics of Langmuir soliton formation and interaction in a plasma (see [14]-[16], [22]) and diatomic lattice system (see [21]), etc. The Schrödinger-Boussinesq system also appears in the study of interaction of solitons in optics. The solitary wave solutions and integrability of nonlinear Schrödinger-Boussinesq equations has been considered by several authors (see [14]-[15]) and the references therein. The IVP for various generalizations of nonlinear

Schrödinger-Boussinesq equations on $\mathbb{R}^n$ have been extensively studied (see [6], [5], [12], [18], [10]). In [6], Guo and Shen established the existence and uniqueness theorem of the global solution of the Cauchy problem for dissipative Schrödinger-Boussinesq equations in H^k(integer $k \geq 4$) with $n = 3$. For the initial-boundary value problem for the damped and dissipative Schrödinger-Boussinesq equations, Guo and Chen [5] and Li and Chen [10] investigated the existence of global attractors and the finiteness of the Hausdorff and the fractal dimensions of the attractor for one-dimensional case ($n = 1$) and multidimensional case ($n \leq 3$), respectively. Linares and Navas [12] considered the IVP for the following one-dimensional Schrödinger-Boussinesq equation

$$\begin{cases} iu_t + \partial_x^2 u = uv + \alpha|u|^2 u, \\ v_{tt} - \partial_x^2 v + \partial_x^4 v = \partial_x^2(\beta|v|^{p-1}v + |u|^2) \end{cases} \tag{1.3}$$

and established the local and global well-posedness results in the spaces $L^2(\mathbb{R}) \times L^2(\mathbb{R}) \times H^{-1}(\mathbb{R})$ and $H^1(\mathbb{R}) \times H^1(\mathbb{R}) \times L^2(\mathbb{R})$, provided that β is a positive (or negative) constant and the initial data is sufficiently small, where $p > 1$ and α is a real number. Ozawa and Tsutaya [18] studied the IVP for the following schrödinger-improved Boussinesq equations

$$\begin{cases} iu_t + \Delta u = uv, \\ v_{tt} - \Delta v - \Delta v_{tt} = \Delta |u|^2 \end{cases} \tag{1.4}$$

and proved that the IVP is locally well-posed in $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ ($n = 1, 2, 3$) and globally well-posed in $H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \bigcap \dot{H}^{-1}(\mathbb{R}^n)$ ($n = 1, 2$).

In this paper, we will investigate the well-posedness on the IVP (1.1)-(1.2), more precisely speaking, we will establish the local well-posedness in $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \times H^{-2}(\mathbb{R}^n)$ and the global well-posedness in $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \times H^{-1}(\mathbb{R}^n)$ for the IVP (1.1)-(1.2). Moreover, we also study the local and global well-posedness for the IVP (1.1)-(1.2) in the space $H^s(\mathbb{R}) \times H^s(\mathbb{R}) \times H^{s-2}(\mathbb{R})$ ($0 < s < 1$). Here we would like to point out that the method employed in the present paper is quite different from usual way used in other papers. Instead of working with the IVP (1.1)-(1.2), we will consider an equivalent integral equation (only) about the unknown function u. By investigating this integral equation, we can establish the local and global well-posedness of the solution u of the integral equation. And then, the local and global well-posedness of v can be obtained by studying the corresponding integral equation corresponding to the second equation in (1.1). This is different from other works (e.g., see [6], [5], [12], [18], [10]).

The paper is organized as follows. In Section 2, we state some notations and give some preliminaries. Section 3 is devoted to establishing the local well-posedness of the IVP (1.1)-(1.2) in $L^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)\times H^{-2}(\mathbb{R}^n)$, while Section 4 is devoted to establishing the local and global well-posedness in the space $H^1(\mathbb{R}^n)\times H^1(\mathbb{R}^n)\times H^{-1}(\mathbb{R}^n)$. Finally, in Section 5 we study the local and global well-posedness of the IVP (1.1)-(1.2) in fractional sobolev spaces.

2 Preliminaries

In this section, we give some preliminaries.

2.1 Notations

Throughout this paper, we will use the following notations:

• The Fourier transform of f is denoted by

$$\hat{f}(\xi)=\int_{\mathbb{R}^n} e^{-ix\xi}f(x)dx. \tag{2.1}$$

• The Fourier inverse transform is denoted by

$$\check{f}(x)=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\xi}f(\xi)d\xi. \tag{2.2}$$

• $L^p(\mathbb{R}^n)$ $(1\le p\le\infty)$ denotes the usual space of all $L^p(\mathbb{R}^n)$-functions on $\mathbb{R}^n$ with L^p-norm.

• H^s denotes the s-th order Sobolev space on $\mathbb{R}^n$ with the norm

$$\|f\|_{H^s}=\|(I-\Delta)^{\frac{s}{2}}f\|_{L^2}=\|(1+|\xi|^2)^{\frac{s}{2}}\hat{f}\|_{L^2}, \tag{2.3}$$

where s is a real number and I is unitary operator.

• The Riesz potential of order $-s$ is denoted by

$$D_x^s=c_s(|\xi|^s\hat{f}(\xi))^{\vee}. \tag{2.4}$$

• The L^p-L^q norms are denoted as

$$\begin{cases} \|f\|_{L_T^pL_x^q}=\left(\int_0^T\|f(\cdot,t)\|_{L^q}^p dt\right)^{\frac{1}{p}}, \\ \|f\|_{L_x^pL_T^q}=\left(\int_{R^n}\left(\int_0^T|f(\cdot,t)|^q dt\right)^{\frac{p}{q}}dx\right)^{\frac{1}{p}}. \end{cases} \tag{2.5}$$

2.2 Method used in this paper

We can simplify the problem (1.1)-(1.2) by writing explicitly the solution of

$$v(x,t) = \frac{\partial}{\partial t}W(t)v_0(x) + W(t)v_1(x) + \int_0^t W(t-\tau)\Delta|u|^2 d\tau \tag{2.6}$$

to get the decoupled integro-differential equation

$$iu_t + \Delta u = u\left(\frac{\partial}{\partial t}W(t)v_0(x)\right) + u(W(t)v_1(x)) + u\int_0^t W(t-\tau)\Delta|u|^2 d\tau + \alpha|u|^2 u \tag{2.7}$$

with the initial data

$$u(x,0) = u_0(x), \tag{2.8}$$

where

$$\frac{\partial}{\partial t}W(t)v_0(x) = \frac{1}{(2\pi)^n}\int_{R^n} e^{ix\xi}\hat{v}_0(\xi)\cos|\xi|(1+|\xi|^2)^{\frac{1}{2}}t d\xi, \tag{2.9}$$

$$W(t)v_1(x) = \frac{1}{(2\pi)^n}\int_{R^n} e^{ix\xi}\hat{v}_1(\xi)\frac{\sin|\xi|(1+|\xi|^2)^{\frac{1}{2}}t}{|\xi|(1+|\xi|^2)^{\frac{1}{2}}}d\xi \tag{2.10}$$

and

$$\int_0^t W(t-\tau)\Delta|u|^2 d\tau = -\frac{1}{(2\pi)^n}\int_0^t\int_{R^n} e^{ix\xi}\frac{\sin|\xi|(1+|\xi|^2)^{\frac{1}{2}}(t-\tau)}{(1+|\xi|^2)^{\frac{1}{2}}}|\xi|\mathcal{F}(|u|^2)d\xi d\tau. \tag{2.11}$$

It follows from (2.7)-(2.8) that

$$u(t) = S(t)u_0 - i\int_0^t S(t-\tau)[F_0(u(\tau)) + F_1(u(\tau)) + F_2(u(\tau)) + F_3(u(\tau))]d\tau, \tag{2.12}$$

where

$$\begin{cases} F_0(u(t)) = u\left(\frac{\partial}{\partial t}W(t)v_0(x)\right), & F_1(u(t)) = u(W(t)v_1(x)), \\ F_2(u(t)) = u\int_0^t W(t-\tau)\Delta|u|^2 d\tau, & F_3(u(t)) = \alpha|u|^2 u, \end{cases} \tag{2.13}$$

and

$$S(t)u_0 = e^{it\Delta}u_0 = (e^{-it|\xi|^2}\hat{u}_0)^{\vee} \tag{2.14}$$

is the unitary group associated to the linear Schrödinger equation

$$iu_t + \Delta u = 0. \tag{2.15}$$

The method used in this paper is as follows. Instead of working with the problem (1.1)-(1.2), we use its equivalent integral equation (2.12). Then we use $L^p - L^q$ estimates or Strichartz estimates to prove our results by the contraction mapping principle. These type of estimates were first established by Strichartz [19] for the solution of the linear Schrödinger equation (2.15) with the initial data (2.8), i.e., the solution of the following initial value problem

$$\begin{cases} iu_t + \Delta u = 0, \\ u(x,0) = u_0(x). \end{cases} \tag{2.16}$$

He proved that the solution of the problem (2.16) satisfies the estimate

$$\left(\int_R \int_{R^n} |S(t)u_0(x)|^{\frac{2(n+2)}{n}} dxdt \right)^{\frac{n}{2(n+2)}} \leq c\|u_0\|_{L^2}. \tag{2.17}$$

Generalization of these estimates have been obtained by several authors. For instance, Ginigre and Velo [3] and Kenig, Ponce and Vega [8]. Since the energy for the solution u of the IVP (1.1)-(1.2) is conserved, we obtain the global well-posedness in $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \times H^{-1}(\mathbb{R}^n)$. From the global result in $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \times H^{-1}(\mathbb{R}^n)$, the global result in $H^s(\mathbb{R}) \times H^s(\mathbb{R}) \times H^{s-2}(\mathbb{R})$ followed.

2.3 Preliminaries

Next we briefly recall some known results on smoothing effect estimates on free Schrödinger evolution group.

Consider the IVP (2.16) and denote its solution by $u(x,t) = S(t)u_0$, where $S(t)$ is defined by (2.14).

In order to state some results on smoothing effect estimates on free Schrödinger evolution group, we need the following definition.

Definition 2.1 *A pair (q,r) is called to be admissible, if*

$$\frac{2}{q} = n\left(\frac{1}{2} - \frac{1}{r}\right) \tag{2.18}$$

and

$$2 \leq r \leq \frac{2n}{n-2} \quad (2 \leq r \leq \infty \text{ if } n=1; \quad 2 \leq r < \infty \text{ if } n=2). \tag{2.19}$$

In this paper, we will use the following well-known Strichartz estimates.

Lemma 2.1 *If (q_1, r_1) and (q_2, r_2) are admissible, then*

$$\|S(t)u_0\|_{L_T^{q_1} L_x^{r_1}} \leq c\|u_0\|_{L^2}, \tag{2.20}$$

$$\left\| \int_0^t S(t-\tau)G(\cdot,\tau)d\tau \right\|_{L_T^{q_1} L_x^{r_1}} \leq c\|G\|_{L_T^{q_2'} L_x^{r_2'}}, \tag{2.21}$$

$$\left\|\int_0^t S(t-\tau)G(\cdot,\tau)d\tau\right\|_{L_T^{q_1}L_x^{r_1}} \leq cT^{(\frac{1}{q_2'}-\frac{1}{2})}\|G\|_{L_T^{r_2'}L_x^2}, \tag{2.22}$$

where

$$\frac{1}{r_2}+\frac{1}{r_2'}=1 \quad \text{and} \quad \frac{1}{q_2}+\frac{1}{q_2'}=1. \tag{2.23}$$

On the other hand,

Lemma 2.2 *The solution u of the IVP (2.16) satisfies the following Kato smoothing effect*

$$\sup_{x\in\mathbb{R}}\left\{\int_{\mathbb{R}}\left|D_x^{\frac{1}{2}}S(t)g\right|^2 dt\right\}^{\frac{1}{2}} \leq c\|g\|_{L^2}. \tag{2.24}$$

See Cazenave [1] or Ginibre and Velo [4] for the proof of estimates (2.20) and (2.21), see Corcho-Linares [2] for the proof of (2.22), see Kenig, Ponce and Vega [8] for the proof of (2.24).

Lemma 2.3 *Let $s \in \mathbb{R}$. For all $t \geq 0$, we have*

$$\left\|\frac{\partial}{\partial t}(W(t)v_0)\right\|_{H^s} \leq \|v_0\|_{H^s} \quad (\text{where } v_0 \in H^s), \tag{2.25}$$

$$\|W(t)v_1\|_{H^s} \leq 2(t+1)\|v_1\|_{H^{s-2}} \quad (\text{where } v_1 \in H^{s-2}), \tag{2.26}$$

$$\left\|\frac{\partial^2}{\partial^2 t}(W(t)v_0)\right\|_{H^s} \leq \|v_0\|_{H^{s+2}} \quad (\text{where } v_0 \in H^{s+2}), \tag{2.27}$$

$$\left\|\frac{\partial}{\partial t}(W(t)v_1)\right\|_{H^s} \leq \|v_1\|_{H^s} \quad (\text{where } v_1 \in H^s). \tag{2.28}$$

Proof. In fact,

$$\begin{aligned}
\left\|\frac{\partial}{\partial t}(W(t)v_0)\right\|_{H^s} &= \left\|(1+|\xi|^2)^{\frac{s}{2}}\mathcal{F}\left(\frac{\partial}{\partial t}(W(t)v_0)\right)\right\|_{L^2} \\
&= \|(1+|\xi|^2)^{\frac{s}{2}}\cos|\xi|(1+|\xi|^2)^{\frac{1}{2}}t\hat{v}_0(\xi)\|_{L^2} \\
&\leq \|(1+|\xi|^2)^{\frac{s}{2}}\hat{v}_0(\xi)\| = \|v_0\|_{H^s}.
\end{aligned}$$

This gives the proof of (2.25).

On the other hand,

$$
\begin{aligned}
\|W(t)v_1\|_{H^s}^2 &= \|(1+|\xi|^2)^{\frac{s}{2}}\mathcal{F}(W(t)v_1)\|_{L^2}^2 \\
&= \int_{\mathbb{R}^n}\left|(1+|\xi|^2)^{\frac{s}{2}}\frac{\sin|\xi|(1+|\xi|^2)^{\frac{1}{2}}t}{|\xi|(1+|\xi|^2)^{\frac{1}{2}}}\hat{v}_1(\xi)\right|^2 d\xi \\
&= \int_{|\xi|\le 1}(1+|\xi|^2)^s\left|\frac{\sin|\xi|(1+|\xi|^2)^{\frac{1}{2}}t}{|\xi|(1+|\xi|^2)^{\frac{1}{2}}}\right|^2|\hat{v}_1(\xi)|^2 d\xi + \\
&\quad \int_{|\xi|>1}(1+|\xi|^2)^s\left|\frac{\sin|\xi|(1+|\xi|^2)^{\frac{1}{2}}t}{|\xi|(1+|\xi|^2)^{\frac{1}{2}}}\right|^2|\hat{v}_1(\xi)|^2 d\xi \\
&= \int_{|\xi|\le 1}(1+|\xi|^2)^s\left|\frac{(1+|\xi|^2)^2}{(1+|\xi|^2)^2}\frac{\sin|\xi|(1+|\xi|^2)^{\frac{1}{2}}t}{|\xi|(1+|\xi|^2)^{\frac{1}{2}}}\right|^2|\hat{v}_1(\xi)|^2 d\xi + \\
&\quad \int_{|\xi|>1}(1+|\xi|^2)^s\frac{1+|\xi|^2}{|\xi|^2(1+|\xi|^2)^2}|\sin|\xi|(1+|\xi|^2)^{\frac{1}{2}}t|^2|\hat{v}_1(\xi)|^2 d\xi \\
&\le 4t^2\int_{|\xi|\le 1}(1+|\xi|^2)^{s-2}|\hat{v}_1(\xi)|^2 d\xi + \\
&\quad \int_{|\xi|>1}(1+|\xi|^2)^{s-2}\left(1+\frac{1}{|\xi|^2}\right)|\hat{v}_1(\xi)|^2 d\xi \\
&\le 4t^2\|v_1\|_{H^{s-2}}^2 + 2\int_{|\xi|>1}(1+|\xi|^2)^{s-2}|\hat{v}_1(\xi)|^2 d\xi \\
&\le (4t^2+2)\|v_1\|_{H^{s-2}}^2 \le 4(t+1)^2\|v_1\|_{H^{s-2}}^2.
\end{aligned}
$$

This proves the estimate (2.26).

The proof of (2.27) and (2.28) is similar to that of (2.25), so we omit the proof. Thus, the proof of Lemma 2.3 is completed. ∎

In order to estimate the nonlinear terms with fractional derivatives, we need the following commutators estimates established by Kenig, Ponce and Vega [9].

Lemma 2.4 *Let $\alpha\in(0,1), \alpha_1,\alpha_2\in(0,\alpha)$ and $p,p_1,p_2,q,q_1,q_2\in(1,\infty)$. If they satisfy*

$$\alpha_1+\alpha_2=\alpha,\quad \frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2},\quad \frac{1}{q}=\frac{1}{q_1}+\frac{1}{q_2}, \tag{2.29}$$

then it holds that

$$\|D_x^\alpha(fg)-fD_x^\alpha g-gD_x^\alpha f\|_{L_x^pL_T^q}\le c\|D_x^{\alpha_1}f\|_{L_x^{p_1}L_T^{q_1}}\|D_x^{\alpha_2}g\|_{L_x^{p_2}L_T^{q_2}} \tag{2.30}$$

and

$$\|D_x^\alpha(fg)-fD_x^\alpha g-gD_x^\alpha f\|_{L^p}\le c\|g\|_{L^\infty}\|D_x^\alpha f\|_{L^p}, \tag{2.31}$$

moreover, (2.30) is still true for the case that $\alpha_1=0$ and $q_1=\infty$.

See the Appendix in Kenig, Ponce and Vega [9] for the proof.

The operators $\frac{\partial}{\partial t}W(t)$ and $W(t)$ also satisfy the estimates of $L^p - L^q$ type similar to those of the solution of the linear Schrödinger equation. In this case, the proof is very complicated. Fortunately, using the oscillatory integrals theory developed by Kenig, Ponce and Vega [8], Linares have obtained these estimates (see [11]).

Lemma 2.5 *For $f \in L^2(\mathbb{R})$, it holds that*

$$\left(\int_0^T \left\|\frac{\partial}{\partial t}W(t)f\right\|_{L^\infty}^4 dt\right)^{\frac{1}{4}} \le c(1+T^{\frac{1}{4}})\|f\|_{L^2}, \tag{2.32}$$

$$\left(\int_0^T \|W(t)\partial_x f\|_{L^\infty}^4 \, dt\right)^{\frac{1}{4}} \le c(1+T^{\frac{1}{4}})\|f\|_{H^{-1}} \tag{2.33}$$

and

$$\left(\int_0^T \|W(t)\partial_x^2 f\|_{L^\infty}^4 dt\right)^{\frac{1}{4}} \le c\|f\|_{L^2}. \tag{2.34}$$

See Lemmas 2.4-2.6 in Linares [11].

We now state the Kato smoothing effect estimates.

Lemma 2.6 *It holds that*

$$\sup_{x\in\mathbb{R}}\left\{\int_0^T \left|D_x^{\frac{1}{2}}\frac{\partial}{\partial t}W(t)v_0\right|^2 dt\right\}^{\frac{1}{2}} \le (1+T^{\frac{1}{2}})\|v_0\|_{L^2}, \tag{2.35}$$

$$\sup_{x\in\mathbb{R}}\left\{\int_0^T |D_x^{\frac{1}{2}}W(t)\partial_x v_1|^2 dt\right\}^{\frac{1}{2}} \le (1+T^{\frac{1}{2}})\|v_1\|_{H^{-1}} \tag{2.36}$$

and

$$\sup_{x\in\mathbb{R}}\left\{\int_0^T |D_x^{\frac{1}{2}}W(t)\partial_x^2 v_1|^2 dt\right\}^{\frac{1}{2}} \le (1+T^{\frac{1}{2}})\|v_1\|_{L^2}. \tag{2.37}$$

The proof of Lemma 2.6 has been given in Linares [11] and Kenig, Ponce and Vega [8].

3 Local well-posedness in $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \times H^{-2}(\mathbb{R}^n)$

Define the mapping

$$\Phi(u)(t) = S(t)u_0 - i\int_0^t S(t-\tau)[F_0(u(\tau)) + F_1(u(\tau)) + F_2(u(\tau)) + F_3(u(\tau))]d\tau, \tag{3.1}$$

where $F_i(u(\tau))$ $(i=0,1,2,3)$ are defined by (2.13). For any fixed $T>0$, we introduce the function space

$$X(T) \triangleq C([0,T];L^2(\mathbb{R}^n)) \bigcap L^{\frac{8}{n}}([0,T];L^4(\mathbb{R}^n))$$

equipped with the norm defined by

$$\|u\|_{X(T)} \triangleq \|u\|_{L_T^\infty L_x^2} + \|u\|_{L_T^{\frac{8}{n}} L_x^4}, \quad \forall\, u \in X(T).$$

It is not difficult to show that $X(T)$ is a Banach space. For $R>0$, let $B_R(T)$ be the closed ball of radius R centered at the origin in $X(T)$, namely,

$$B_R(T) \triangleq \{u \in X(T) |\, \|u\|_{X(T)} \le R\}.$$

In what follows, we show that Φ has a unique fixed point in $B_R(T)$ by appropriately choosing R and T.

Lemma 3.1 *(I) $\alpha \neq 0$: Assume that $u_0, v_0 \in L^2(\mathbb{R})$, $v_1 \in H^{-2}(\mathbb{R})$, then $\Phi : B_R(T) \longmapsto B_R(T)$ is a strictly contractive mapping;*

(II) $\alpha = 0$: Assume that $u_0, v_0 \in L^2(\mathbb{R}^n)$, $v_1 \in H^{-2}(\mathbb{R}^n)$, then $\Phi : B_R(T) \longmapsto B_R(T)$ is a strictly contractive mapping, where n takes its values in $\{1,2,3\}$, i.e., $n=1,2,3$.

Proof. Step 1. Using (2.20) and noting the group's properties, we obtain

$$\|S(t)u_0\|_{X(T)} = \|S(t)u_0\|_{L_T^\infty L_x^2} + \|S(t)u_0\|_{L_T^{\frac{8}{n}} L_x^4} \le c_0 \|u_0\|_{L^2}. \tag{3.2}$$

Step 2. We next estimate the integral part in (3.1).

Taking $(q_1,r_1)=(\infty,2)$, $(q_2,r_2)=(\frac{8}{n},4)$ and using (2.21) yields

$$\left\| \int_0^t S(t-\tau) F_0(u(\tau)) d\tau \right\|_{L_T^\infty L_x^2} \le c \|F_0(u)\|_{L_T^{\frac{8}{8-n}} L_x^{\frac{4}{3}}}. \tag{3.3}$$

Taking $(q_1,r_1)=(\frac{8}{n},4)$, $(q_2,r_2)=(\frac{8}{n},4)$ and using (2.21) again leads to

$$\left\| \int_0^t S(t-\tau) F_0(u(\tau)) d\tau \right\|_{L_T^{\frac{8}{n}} L_x^4} \le c \|F_0(u)\|_{L_T^{\frac{8}{8-n}} L_x^{\frac{4}{3}}}. \tag{3.4}$$

By Hölder inequality and (2.25), we have

$$\begin{aligned} \|F_0(u)\|_{L_T^{\frac{8}{8-n}} L_x^{\frac{4}{3}}} &= \left\| u \tfrac{\partial}{\partial t}(W(t)v_0) \right\|_{L_T^{\frac{8}{8-n}} L_x^{\frac{4}{3}}} \le \|u\|_{L_T^{\frac{8}{n}} L_x^4} \left\| \tfrac{\partial}{\partial t}(W(t)v_0) \right\|_{L_T^{\frac{4}{4-n}} L_x^2} \\ &\le T^{\frac{4-n}{4}} \|u\|_{L_T^{\frac{8}{n}} L_x^4} \|v_0\|_{L^2} \le T^{\frac{4-n}{4}} \|u\|_{X(T)} \|v_0\|_{L^2}. \end{aligned} \tag{3.5}$$

Combining (3.3), (3.4) and (3.5) gives

$$\left\|\int_0^t S(t-\tau)F_0(u(\tau))d\tau\right\|_{X(T)} \le c_1 T^{\frac{4-n}{4}}\|v_0\|_{L^2}\|u\|_{X(T)}. \tag{3.6}$$

Using the same method as that of proof of (3.3)-(3.4), we can prove

$$\left\|\int_0^t S(t-\tau)F_1(u(\tau))d\tau\right\|_{L_T^\infty L_x^2} + \left\|\int_0^t S(t-\tau)F_1(u(\tau))d\tau\right\|_{L_T^{\frac{8}{n}} L_x^4} \le c\|F_1(u)\|_{L_T^{\frac{8}{8-n}} L_x^{\frac{4}{3}}}. \tag{3.7}$$

By Hölder inequality and (2.26), we have

$$\begin{aligned} \|F_1(u)\|_{L_T^{\frac{8}{8-n}} L_x^{\frac{4}{3}}} &= \|u(W(t)v_1)\|_{L_T^{\frac{8}{8-n}} L_x^{\frac{4}{3}}} \le \|u\|_{L_T^{\frac{8}{n}} L_x^4}\|W(t)v_1\|_{L_T^{\frac{4}{4-n}} L_x^2} \\ &\le 2T^{\frac{4-n}{4}}(T+1)\|u\|_{L_T^{\frac{8}{n}} L_x^4}\|v_1\|_{H^{-2}} \le 2T^{\frac{4-n}{4}}(T+1)\|u\|_{X(T)}\|v_1\|_{H^{-2}}. \end{aligned} \tag{3.8}$$

Thus, combining (3.7) and (3.8) yields

$$\left\|\int_0^t S(t-\tau)F_1(u(\tau))d\tau\right\|_{X(T)} \le c_2 T^{\frac{4-n}{4}}(T+1)\|v_1\|_{H^{-2}}\|u\|_{X(T)}. \tag{3.9}$$

Similar to (3.7), we have

$$\left\|\int_0^t S(t-\tau)F_2(u(\tau))d\tau\right\|_{L_T^\infty L_x^2} + \left\|\int_0^t S(t-\tau)F_2(u(\tau))d\tau\right\|_{L_T^{\frac{8}{n}} L_x^4} \le c\|F_2(u)\|_{L_T^{\frac{8}{8-n}} L_x^{\frac{4}{3}}}. \tag{3.10}$$

Using Hölder inequality, Minkowski inequality and (2.26) gives

$$\begin{aligned} \|F_2(u)\|_{L_T^{\frac{8}{8-n}} L_x^{\frac{4}{3}}} &= \left\|u\int_0^t W(t-\tau)\Delta|u|^2 d\tau\right\|_{L_T^{\frac{8}{8-n}} L_x^{\frac{4}{3}}} \\ &\le \|u\|_{L_T^{\frac{8}{n}} L_x^4}\left\|\int_0^t W(t-\tau)\Delta|u|^2 d\tau\right\|_{L_T^{\frac{4}{4-n}} L_x^2} \\ &\le T^{\frac{4-n}{4}}\|u\|_{L_T^{\frac{8}{n}} L_x^4}\left\|\int_0^t W(t-\tau)\Delta|u|^2 d\tau\right\|_{L_T^\infty L_x^2} \\ &\le T^{\frac{4-n}{4}}\|u\|_{L_T^{\frac{8}{n}} L_x^4}\sup_{t\in[0,T]}\left\|\int_0^t W(t-\tau)\Delta|u|^2 d\tau\right\|_{L^2} \\ &\le 2T^{\frac{4-n}{4}}(T+1)\|u\|_{L_T^{\frac{8}{n}} L_x^4}\||u|^2\|_{L_T^1 L_x^2} \\ &\le 2T^{\frac{4-n}{2}}(T+1)\|u\|_{L_T^{\frac{8}{n}} L_x^4}\|u\|^2_{L_T^{\frac{8}{n}} L_x^4} \\ &\le cT^{\frac{4-n}{2}}(T+1)\|u\|^3_{X(T)}. \end{aligned} \tag{3.11}$$

Combining (3.10) and (3.11) leads to

$$\left\|\int_0^t S(t-\tau)F_2(u(\tau))d\tau\right\|_{X(T)} \le c_3 T^{\frac{4-n}{2}}(T+1)\|u\|^3_{X(T)}. \tag{3.12}$$

Similar to (3.7) again, we have

$$\left\|\int_0^t S(t-\tau)F_3(u(\tau))d\tau\right\|_{L_T^\infty L_x^2} + \left\|\int_0^t S(t-\tau)F_3(u(\tau))d\tau\right\|_{L_T^{\frac{8}{n}} L_x^4} \le c\|F_3(u)\|_{L_T^{\frac{8}{8-n}} L_x^{\frac{4}{3}}}. \tag{3.13}$$

Thanks to Hölder inequality,

$$\begin{aligned}\left\|\int_0^t S(t-\tau)F_3(u(\tau))d\tau\right\|_{X(T)} &\le c|\alpha|\||u|^2u\|_{L_T^{\frac{8}{8-n}} L_x^{\frac{4}{3}}} \le c|\alpha|\|u\|_{L_T^{\frac{8}{n}} L_x^4}\||u|^2\|_{L_T^{\frac{4}{4-n}} L_x^2} \\ &\le c|\alpha|T^{\frac{2-n}{2}}\|u\|^3_{L_T^{\frac{8}{n}} L_x^4} \le c|\alpha|T^{\frac{2-n}{2}}\|u\|^3_{X(T)}.\end{aligned} \tag{3.14}$$

Step 3. Combining (3.1), (3.2), (3.6), (3.9), (3.12) and (3.14) yields

$$\begin{aligned}\|\Phi(u)\|_{X(T)} \le\ & c_0\|u_0\|_{L^2} + c_1T^{\frac{4-n}{4}}\|v_0\|_{L^2}\|u\|_{X(T)} + c_2T^{\frac{4-n}{4}}(T+1)\|v_1\|_{H^{-2}}\|u\|_{X(T)} + \\ & c_3(T^{\frac{6-n}{2}} + T^{\frac{4-n}{2}} + |\alpha|T^{\frac{2-n}{2}})\|u\|^3_{X(T)}.\end{aligned} \tag{3.15}$$

Letting $R = 4c_0\|u_0\|_{L^2}$ and choosing T so small that

$$c_1T^{\frac{4-n}{4}}\|v_0\|_{L^2} + c_2T^{\frac{4-n}{4}}(T+1)\|v_1\|_{H^{-2}} + c_3(T^{\frac{6-n}{2}} + T^{\frac{4-n}{2}} + |\alpha|T^{\frac{2-n}{2}})R^2 \le \frac{3}{4}. \tag{3.16}$$

In fact, for two cases under consideration: (I) $\alpha \neq 0$ and $n = 1$; (II) $\alpha = 0$ and $n = 1, 2, 3$, we can always choose small T such that (3.16) holds. Thus, we obtain from (3.15) that

$$\|\Phi(u)\|_{X(T)} \le R.$$

This implies that the mapping Φ maps $B_R(T)$ into $B_R(T)$.

Step 4. In what follows, we prove that when T is suitably small, Φ is a contractive mapping of $B_R(T)$.

In fact, for u and $\tilde{u}$ being in $B_R(T)$, we have

$$\Phi(u)(t) - \Phi(\tilde{u})(t) = -i\int_0^t S(t-\tau)\mathcal{G}(\tau)d\tau, \tag{3.17}$$

where

$$\mathcal{G}(\tau) = F_0(u(\tau)) - F_0(\tilde{u}(\tau)) + F_1(u(\tau)) - F_1(\tilde{u}(\tau)) + F_2(u(\tau)) - F_2(\tilde{u}(\tau)) + F_3(u(\tau)) - F_3(\tilde{u}(\tau)).$$

Similar to (3.15), we get

$$\begin{aligned}\|\Phi(u) - \Phi(\tilde{u})\|_{X(T)} \le\ & c_1T^{\frac{4-n}{4}}\|v_0\|_{L^2}\|u-\tilde{u}\|_{X(T)} + c_2T^{\frac{4-n}{4}}(T+1)\|v_1\|_{H^{-2}}\|u-\tilde{u}\|_{X(T)} + \\ & c_3(T^{\frac{6-n}{2}} + T^{\frac{4-n}{2}} + |\alpha|T^{\frac{2-n}{2}})\|u-\tilde{u}\|_{X(T)}[\|u\|^2_{X(T)} + \|u\|_{X(T)}\|\tilde{u}\|_{X(T)} + \|\tilde{u}\|^2_{X(T)}].\end{aligned}$$

For the cases under consideration: (I) $\alpha \neq 0$ and $n = 1$; (II) $\alpha = 0$ and $n = 1, 2, 3$, we can always choose T so small that (3.16) and the following inequality hold

$$c_1 T^{\frac{4-n}{4}} \|v_0\|_{L^2} + c_2 T^{\frac{4-n}{4}} (T+1)\|v_1\|_{H^{-2}} + 3c_3 (T^{\frac{6-n}{2}} + T^{\frac{4-n}{2}} + |\alpha| T^{\frac{2-n}{2}}) R^2 \leq \frac{1}{2}. \quad (3.18)$$

Thus, we obtain

$$\|\Phi(u) - \Phi(\tilde{u})\|_{X(T)} \leq \frac{1}{2} \|u - \tilde{u}\|_{X(T)}.$$

This implies that Φ is a strict contraction mapping on $B_R(T)$, provided that T satisfies (3.16) and (3.18). Thus, the proof of Lemma 3.1 is finished. ■

When $\alpha \neq 0$, using the part (I) in Lemma 3.1, we may prove the following theorem.

Theorem 3.1 *Assume $\alpha \neq 0$, $u_0, v_0 \in L^2(\mathbb{R}), v_1 \in H^{-2}(\mathbb{R})$. Then there exists a positive constant $T = T(|\alpha|, \|u_0\|_{L^2}, \|v_0\|_{L^2}, \|v_1\|_{H^{-2}})$ such that the IVP (2.7)-(2.8) has a unique solution $u = u(x,t)$ on the strip $\mathbb{R} \times [0,T]$ and the solution satisfies the following properties*

$$u \in C([0,T]; L^2(\mathbb{R})) \bigcap L^8([0,T]; L^4(\mathbb{R})), \quad (3.19)$$

for an admissible pair (q, r)

$$\|u\|_{L^q_T L^r_x} < \infty, \quad (3.20)$$

and the mapping $(u_0, v_0, v_1) \longmapsto u(t)$ from $L^2(\mathbb{R}) \times L^2(\mathbb{R}) \times H^{-2}(\mathbb{R})$ into the space given by (3.19) is locally Lipschitz. Moreover, the function $v = v(x,t)$ defined by (2.6) satisfies

$$v \in C([0,T]; L^2(\mathbb{R})) \bigcap C^1([0,T]; H^{-2}(\mathbb{R})). \quad (3.21)$$

Proof. Thanks to the contraction mapping principle and the Lemma 3.1, there exists a unique $u \in B_R(T)$ such that $\Phi(u) = u$.

We now prove (3.20).

Noting the fact $u = \Phi(u)$, in a way similar to the estimate on $\|\Phi(u)\|_{X(T)}$ we have

$$\begin{aligned} \|u\|_{L^q_T L^r_x} \leq\ & c_0 \|u_0\|_{L^2} + c_1 T^{\frac{3}{4}} \|v_0\|_{L^2} \|u\|_{X(T)} + c_2 T^{\frac{3}{4}} (T+1) \|v_1\|_{H^{-2}} \|u\|_{X(T)} + \\ & c_3 (T^{\frac{5}{2}} + T^{\frac{3}{2}} + |\alpha| T^{\frac{1}{2}}) \|u\|^3_{X(T)} < \infty, \end{aligned}$$

where (q, r) is an admissible pair.

We next investigate the property of $v(t)$ defined by (2.6).

Using Minkowski inequality, Lemma 2.3 and Hölder inequality, for fixed $t \in [0, T]$ we obtain from (2.6) that

$$\begin{aligned}
\|v(t)\|_{L^2} &\le \left\| \frac{\partial}{\partial t} W(t) v_0(x) \right\|_{L^2} + \|W(t)v_1(x)\|_{L^2} + \left\| \int_0^t W(t-\tau)\partial_x^2 |u|^2 d\tau \right\|_{L^2} \\
&\le \|v_0\|_{L^2} + 2(T+1)\|v_1\|_{H^{-2}} + 2(T+1)\int_0^T \|u\|_{L^4}^2 dt \\
&\le \|v_0\|_{L^2} + 2(T+1)\|v_1\|_{H^{-2}} + 2T^{\frac{3}{4}}(T+1)\|u\|^2_{L_T^8 L_x^4}.
\end{aligned} \tag{3.22}$$

Notice that

$$v_t(x,t) = \frac{\partial^2}{\partial t^2} W(t) v_0(x) + \frac{\partial}{\partial t} W(t) v_1(x) + \int_0^t \frac{\partial}{\partial t} W(t-\tau)\partial_x^2 |u|^2 d\tau,$$

where

$$\frac{\partial^2}{\partial t^2} W(t) v_0(x) = -\frac{1}{2\pi}\int_{\mathbb{R}} e^{ix\xi}\hat{v}_0(\xi)|\xi|(1+|\xi|^2)^{\frac{1}{2}} \sin|\xi|(1+|\xi|^2)^{\frac{1}{2}} t d\xi.$$

Similar to (3.22), we have

$$\begin{aligned}
\|v_t(t)\|_{H^{-2}} &\le \|v_0\|_{L^2} + \|v_1\|_{H^{-2}} + \int_0^T \|u\|^2_{L^4} dt \\
&\le \|v_0\|_{L^2} + \|v_1\|_{H^{-2}} + T^{\frac{3}{4}}\|u\|^2_{L_T^8 L_x^4}.
\end{aligned} \tag{3.23}$$

Combining (3.22) and (3.23) gives (3.21) directly. This proves Theorem 3.1. ■

For the case $\alpha = 0$, we have

Theorem 3.2 *Suppose that $\alpha = 0$ and $n = 1, 2, 3$, suppose furthermore that $u_0, v_0 \in L^2(\mathbb{R}^n), v_1 \in H^{-2}(\mathbb{R}^n)$. Then there exists a positive constant $T = T(\|u_0\|_{L^2}, \|v_0\|_{L^2}, \|v_1\|_{H^{-2}})$ such that the IVP (2.7)-(2.8) has a unique solution $u = u(x,t)$ on the strip $\mathbb{R}^n \times [0,T]$ and the solution satisfies the following properties*

$$u \in C([0,T]; L^2(\mathbb{R}^n)) \bigcap L^{\frac{8}{n}}([0,T]; L^4(\mathbb{R}^n)), \tag{3.24}$$

for an admissible pair (q, r)

$$\|u\|_{L_T^q L_x^r} < \infty, \tag{3.25}$$

and the mapping $(u_0, v_0, v_1) \longmapsto u(t)$ from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \times H^{-2}(\mathbb{R}^n)$ into the space given by (3.24) is locally Lipschitz. Moreover, the $v = v(x,t)$ defined by (2.6) satisfies

$$v \in C([0,T]; L^2(\mathbb{R}^n)) \bigcap C^1([0,T]; H^{-2}(\mathbb{R}^n)). \tag{3.26}$$

Proof. The proof is similar to that of Theorem 3.1, so we omit it here. ■

In what follows, we study some regularity properties for the solution of the IVP (1.1)-(1.2). We have

Theorem 3.3 *If* (u,v) *is a solution of the IVP (1.1)-(1.2), and the initial data satisfies*

$$(u_0, v_0, v_1) \in L^2(\mathbb{R}) \times L^2(\mathbb{R}) \times H^{-2}(\mathbb{R}),$$

then

$$D_x^{\frac{1}{2}} u, D_x^{\frac{1}{2}} v \in L^\infty(\mathbb{R}; L^2[0,T]).$$

Proof. Since $(4,\infty)$ and $(6,6)$ are admissible pairs, it follows from Theorem 3.1 that

$$\|u\|_{L_T^4 L_x^\infty} < \infty, \quad \|u\|_{L_T^6 L_x^6} < \infty.$$

Notice that the solution $u = u(x,t)$ of the IVP (1.1)-(1.2) satisfies

$$u(x,t) = S(t)u_0 - i \int_0^t S(t-\tau)(uv + |\alpha||u|^2 u)(\tau) d\tau. \tag{3.27}$$

Using Minkowski inequality and (2.24) gives

$$\begin{aligned} \|D_x^{\frac{1}{2}} u\|_{L_x^\infty L_T^2} &\leq \|D_x^{\frac{1}{2}} S(t) u_0\|_{L_x^\infty L_T^2} + \int_0^t \|D_x^{\frac{1}{2}} S(t-\tau)(uv + |\alpha||u|^2 u)\|_{L_x^\infty L_T^2} d\tau \\ &\leq c\|u_0\|_{L^2} + c\int_0^t \|uv + |\alpha||u|^2 u\|_{L^2} dt \\ &\leq c\|u_0\|_{L^2} + cT^{\frac{3}{4}} \|u\|_{L_T^4 L_x^\infty} \|v\|_{L_T^\infty L_x^2} + c|\alpha| T^{\frac{1}{2}} \|u\|^3_{L_T^6 L_x^6}. \end{aligned}$$

Using Minkowski inequality, Lemma 2.6 and Hölder inequality, we obtain from (2.6) that

$$\begin{aligned} \|D_x^{\frac{1}{2}} v\|_{L_x^\infty L_T^2} &\leq \left\| D_x^{\frac{1}{2}} \frac{\partial}{\partial t} W(t) v_0 \right\|_{L_x^\infty L_T^2} + \|D_x^{\frac{1}{2}} W(t) v_1\|_{L_x^\infty L_T^2} + \int_0^t \|D_x^{\frac{1}{2}} W(t-\tau) \partial_x^2 |u|^2\|_{L_x^\infty L_T^2} d\tau \\ &\leq c(1+T^{\frac{1}{2}})\|v_0\|_{L^2} + c(1+T^{\frac{1}{2}})\|v_1\|_{H^{-2}} + c(1+T^{\frac{1}{2}}) \int_0^T \||u|^2\| dt \\ &\leq c(1+T^{\frac{1}{2}})\|v_0\|_{L^2} + c(1+T^{\frac{1}{2}})\|v_1\|_{H^{-2}} + cT^{\frac{3}{4}}(1+T^{\frac{1}{2}})\|u\|^2_{L_T^8 L_x^4}. \end{aligned}$$

This proves Theorem 3.3. ■

4 Local and global well-posedness in $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \times H^{-1}(\mathbb{R}^n)$

For any fixed $T > 0$, define the function space

$$X(T) \triangleq C([0,T]; H^1(\mathbb{R}^n))$$

equipped with the norm defined by

$$\|u\|_{X(T)} \stackrel{\triangle}{=} \|u\|_{L^\infty_T H^1_x} + \|u\|_{L^{\frac{8}{n}}_T L^4_x} + \|\nabla u\|_{L^{\frac{8}{n}}_T L^4_x}, \quad \forall\, u \in X(T).$$

It is not difficult to show that $X(T)$ is a complete metric space. For any fixed $R > 0$, let $B_R(T)$ be the closed ball of radius R centered at the origin in $X(T)$, namely,

$$B_R(T) \stackrel{\triangle}{=} \{u \in X(T) |\, \|u\|_{X(T)} \le R\}.$$

Introduce the mapping

$$\Phi(u)(t) = S(t)u_0 - i\int_0^t S(t-\tau)[F_0(u(\tau)) + F_1(u(\tau)) + F_2(u(\tau)) + F_3(u(\tau))]d\tau, \quad (4.1)$$

where $F_i(u(\tau))$ $(i = 0, 1, 2, 3)$ are defined by (2.13).

In what follows, we prove that Φ has a unique fixed point in $B_R(T)$ by appropriately choosing R and T. We first prove the following lemma.

Lemma 4.1 *(I) $\alpha \neq 0$: Assume that $u_0, v_0 \in H^1(\mathbb{R})$, $v_1 \in H^{-1}(\mathbb{R})$, then $\Phi : B_R(T) \longmapsto B_R(T)$ is a strictly contractive mapping;*

(II) $\alpha = 0$: Assume that $u_0, v_0 \in H^1(\mathbb{R}^n)$, $v_1 \in H^{-1}(\mathbb{R}^n)$, then $\Phi : B_R(T) \longmapsto B_R(T)$ is a strictly contractive mapping, where n takes its values in $\{1, 2, 3\}$, i.e., $n = 1, 2, 3$.

Proof. Thanks to (2.20),

$$\|S(t)u_0\|_{X(T)} \le c_0\|u_0\|_{H^1}. \quad (4.2)$$

Taking $(q_1, r_1) = (\infty, 2)$, $(q_1, r_1) = (\frac{8}{n}, 4)$, respectively, and $(q_2, r_2) = (\frac{8}{n}, 4)$ and using (2.21) and (2.25) gives

$$\begin{aligned}
&\left\|\nabla\int_0^t S(t-\tau)F_0(u(\tau))d\tau\right\|_{L^\infty_T L^2_x} + \left\|\nabla\int_0^t S(t-\tau)F_0(u(\tau))d\tau\right\|_{L^{\frac{8}{n}}_T L^4_x} \\
&\le c\|\nabla F_0(u)\|_{L^{\frac{8}{8-n}}_T L^{\frac{4}{3}}_x} \\
&= c\left\|\nabla(u\frac{\partial}{\partial t}W(t)v_0)\right\|_{L^{\frac{8}{8-n}}_T L^{\frac{4}{3}}_x} \\
&\le c\left\|\nabla u\frac{\partial}{\partial t}W(t)v_0\right\|_{L^{\frac{8}{8-n}}_T L^{\frac{4}{3}}_x} + c\left\|u\frac{\partial}{\partial t}W(t)\nabla v_0\right\|_{L^{\frac{8}{8-n}}_T L^{\frac{4}{3}}_x} \\
&\le c\|\nabla u\|_{L^{\frac{8}{n}}_T L^4_x}\left\|\frac{\partial}{\partial t}W(t)v_0\right\|_{L^{\frac{4}{4-n}}_T L^2_x} + c\|u\|_{L^{\frac{8}{n}}_T L^4_x}\left\|\frac{\partial}{\partial t}W(t)\nabla v_0\right\|_{L^{\frac{4}{4-n}}_T L^2_x} \\
&\le cT^{\frac{4-n}{4}}\|\nabla u\|_{L^{\frac{8}{n}}_T L^4_x}\|v_0\|_{L^2} + cT^{\frac{4-n}{4}}\|u\|_{L^{\frac{8}{n}}_T L^4_x}\|\nabla v_0\|_{L^2} \\
&\le cT^{\frac{4-n}{4}}\|v_0\|_{H^1}\|u\|_{X(T)}.
\end{aligned} \quad (4.3)$$

Using (2.26), in a way similar to (4.3) we have

$$\begin{aligned}&\left\|\nabla\int_0^t S(t-\tau)F_1(u(\tau))d\tau\right\|_{L_T^\infty L_x^2}+\left\|\nabla\int_0^t S(t-\tau)F_1(u(\tau))d\tau\right\|_{L_T^{\frac{8}{n}}L_x^4}\\&\qquad\le cT^{\frac{4-n}{4}}(T+1)\|v_1\|_{H^{-1}}\|u\|_{X(T)}.\end{aligned}\tag{4.4}$$

Again, similar to (4.3), we get

$$\begin{aligned}&\left\|\nabla\int_0^t S(t-\tau)F_2(u(\tau))d\tau\right\|_{L_T^\infty L_x^2}+\left\|\nabla\int_0^t S(t-\tau)F_2(u(\tau))d\tau\right\|_{L_T^{\frac{8}{n}}L_x^4}\\
&\le c\|\nabla F_2(u)\|_{L_T^{\frac{8}{8-n}}L_x^{\frac{4}{3}}}=c\left\|\nabla(u\int_0^t W(t-\tau)\Delta|u|^2d\tau)\right\|_{L_T^{\frac{8}{8-n}}L_x^{\frac{4}{3}}}\\
&\le c\|\nabla u\|_{L_T^{\frac{8}{n}}L_x^4}\left\|\int_0^t W(t-\tau)\Delta|u|^2d\tau\right\|_{L_T^{\frac{4}{4-n}}L_x^2}+\\
&\quad c\|u\|_{L_T^{\frac{8}{n}}L_x^4}\left\|\int_0^t W(t-\tau)\Delta\nabla|u|^2d\tau\right\|_{L_T^{\frac{4}{4-n}}L_x^2}\\
&\le cT^{\frac{4-n}{4}}\|\nabla u\|_{L_T^{\frac{8}{n}}L_x^4}\left\|\int_0^t W(t-\tau)\Delta|u|^2d\tau\right\|_{L_T^\infty L_x^2}+\\
&\quad cT^{\frac{4-n}{4}}\|u\|_{L_T^{\frac{8}{n}}L_x^4}\left\|\int_0^t W(t-\tau)\Delta\nabla|u|^2d\tau\right\|_{L_T^\infty L_x^2}\\
&\le cT^{\frac{4-n}{2}}(T+1)\|\nabla u\|_{L_T^{\frac{8}{n}}L_x^4}\|u\|^2_{L_T^{\frac{8}{n}}L_x^4}+cT^{\frac{4-n}{4}}(T+1)\|u\|_{L_T^{\frac{8}{n}}L_x^4}\sup_{t\in[0,T]}\int_0^t\|\nabla|u|^2\|_{L^2}d\tau\\
&\le cT^{\frac{4-n}{2}}(T+1)\|\nabla u\|_{L_T^{\frac{8}{n}}L_x^4}\|u\|^2_{L_T^{\frac{8}{n}}L_x^4}+cT^{\frac{4-n}{4}}(T+1)\|u\|_{L_T^{\frac{8}{n}}L_x^4}\int_0^T\|\nabla u\|_{L^4}\|u\|_{L^4}dt\\
&\le cT^{\frac{4-n}{2}}(T+1)\|\nabla u\|_{L_T^{\frac{8}{n}}L_x^4}\|u\|^2_{L_T^{\frac{8}{n}}L_x^4}\\
&\le cT^{\frac{4-n}{2}}(T+1)\|u\|^3_{X(T)}.\end{aligned}\tag{4.5}$$

Similarly, we have

$$\begin{aligned}&\left\|\nabla\int_0^t S(t-\tau)F_3(u(\tau))d\tau\right\|_{L_T^\infty L_x^2}+\left\|\nabla\int_0^t S(t-\tau)F_3(u(\tau))d\tau\right\|_{L_T^{\frac{8}{n}}L_x^4}\\
&\qquad\le c\|\nabla F_3(u)\|_{L_T^{\frac{8}{8-n}}L_x^{\frac{4}{3}}}=c|\alpha|\||u|^2u\|_{L_T^{\frac{8}{8-n}}L_x^{\frac{4}{3}}}\\
&\qquad\le c|\alpha|\|u\nabla u\bar{u}\|_{L_T^{\frac{8}{8-n}}L_x^{\frac{4}{3}}}+c|\alpha|\|u^2\nabla\bar{u}\|_{L_T^{\frac{8}{8-n}}L_x^{\frac{4}{3}}}\\
&\qquad\le c||\alpha|\|\nabla u\|_{L_T^{\frac{8}{n}}L_x^4}\|u^2\|_{L_T^{\frac{4}{4-n}}L_x^2}\\
&\qquad\le c||\alpha|T^{\frac{2-n}{2}}\|u\|^3_{X(T)}.\end{aligned}\tag{4.6}$$

Noting (3.6), (3.9), (3.12), (3.14) and using (4.2)-(4.6), we obtain

$$\begin{aligned}\|\Phi(u)\|_{X(T)} \quad \leq \quad & c_0\|u_0\|_{H^1} + c_1T^{\frac{4-n}{4}}\|v_0\|_{H^1}\|u\|_{X(T)} + c_2T^{\frac{4-n}{4}}(T+1)\|v_1\|_{H^{-1}}\|u\|_{X(T)} + \\ & c_3(T^{\frac{6-n}{2}} + T^{\frac{4-n}{2}} + |\alpha|T^{\frac{2-n}{2}})\|u\|^3_{X(T)}.\end{aligned} \tag{4.7}$$

Similar to (3.16), letting $R = 4c_0\|u_0\|_{H^1}$ and choosing suitably small T leads to

$$c_1T^{\frac{4-n}{4}}\|v_0\|_{H^1} + c_2T^{\frac{4-n}{4}}(T+1)\|v_1\|_{H^{-1}} + c_3(T^{\frac{6-n}{2}} + T^{\frac{4-n}{2}} + |\alpha|T^{\frac{2-n}{2}})R^2 \leq \frac{3}{4}. \tag{4.8}$$

Thus it follows from (4.7) and (4.8) that

$$\|\Phi(u)\|_{X(T)} \leq R.$$

This implies that the mapping Φ maps $B_R(T)$ into $B_R(T)$.

In what follows, we show that Φ: $B_R(T) \longmapsto B_R(T)$ is a strict contraction mapping, provided that T is suitably small.

In fact, for arbitrary $u, \tilde{u} \in B_R(T)$,

$$\Phi(u)(t) - \Phi(\tilde{u})(t) = -i\int_0^t S(t-\tau)\sum_{k=0}^{3}(F_k(u(\tau)) - F_k(\tilde{u}(\tau)))\, d\tau. \tag{4.9}$$

Similar to (4.7), it follows from (4.9) that

$$\begin{aligned}\|\Phi(u) - \Phi(\tilde{u})\|_{X(T)} \leq\ & c_1T^{\frac{4-n}{4}}\|v_0\|_{H^1}\|u-\tilde{u}\|_{X(T)} + c_2T^{\frac{4-n}{4}}(T+1)\|v_1\|_{H^{-1}}\|u-\tilde{u}\|_{X(T)} + \\ & c_3(T^{\frac{6-n}{2}} + T^{\frac{4-n}{2}} + |\alpha|T^{\frac{2-n}{2}})\|u-\tilde{u}\|_{X(T)}[\|u\|^2_{X(T)} + \|u\|_{X(T)}\|\tilde{u}\|_{X(T)} + \|\tilde{u}\|^2_{X(T)}].\end{aligned}$$

Similar to (3.18), we can always choose T so small that (4.8) and the following inequality hold

$$c_1T^{\frac{4-n}{4}}\|v_0\|_{H^1} + c_2T^{\frac{4-n}{4}}(T+1)\|v_1\|_{H^{-1}} + 3c_3(T^{\frac{6-n}{2}} + T^{\frac{4-n}{2}} + |\alpha|T^{\frac{2-n}{2}})R^2 \leq \frac{1}{2}. \tag{4.10}$$

Therefore, we have

$$\|\Phi(u) - \Phi(\tilde{u})\|_{X(T)} \leq \frac{1}{2}\|u-\tilde{u}\|_{X(T)}.$$

This proves Lemma 4.1. ■

In order to prove the global well-posedness of solutions for the problem (1.1)-(1.2), we need the following Lemma.

Lemma 4.2 *Assume that $u_0, v_0 \in H^1(\mathbb{R}^n)$ and $v_1 \in H^{-1}(\mathbb{R}^n)$, where $n = 1, 2, 3$. Then the solution of the problem (1.1)-(1.2) satisfies the following energy equalities*

$$\|u(t)\|_{L^2} = \|u_0\|_{L^2} \tag{4.11}$$

and

$$E(t) = E(0), \tag{4.12}$$

where

$$E(t) = \|\nabla u\|_{L^2}^2 + \frac{1}{2}\|(-\Delta)^{-\frac{1}{2}}v_t\|_{L^2}^2 + \frac{1}{2}\|v\|_{L^2}^2 + \frac{1}{2}\|\nabla v\|_{L^2}^2 + \int_{\mathbb{R}^n} |u|^2 v dx + \frac{\alpha}{2}\|u\|_{L^4}^4.$$

Proof. We only prove the energy equalities for one-dimensional case, i.e., $n = 1$, the proof for other cases is similar.

The proof of (4.11) is easy, here we omit it. In what follows, we prove (4.12).

A direct calculation yields

$$\begin{aligned}\frac{dE}{dt} \quad = \quad & 2Re\left(\int_{\mathbb{R}} u_x \bar{u}_{xt}\right) + \int_{\mathbb{R}} (-\Delta)^{-\frac{1}{2}} v_t (-\Delta)^{-\frac{1}{2}} v_{tt} dx + \int_{\mathbb{R}} v v_t dx + \\ & \int_{\mathbb{R}} v_x v_{xt} dx + \int_{\mathbb{R}} |u|^2 v_t dx + 2Re \int_{\mathbb{R}} u \bar{u}_t v dx + 2\alpha Re \left(\int_{\mathbb{R}} |u|^2 u \bar{u}_t\right) dx.\end{aligned}$$

Using the first equation in (1.1) and integrating by parts gives

$$\begin{aligned}& 2Re\left(\int_{\mathbb{R}} u_x \bar{u}_{xt}\right) + 2Re \int_{\mathbb{R}} u \bar{u}_t v dx + 2\alpha Re \left(\int_{\mathbb{R}} |u|^2 u \bar{u}_t\right) dx \\ & = 2Re\left(\int_{\mathbb{R}} (-\partial_x^2 u + uv + \alpha|u|^2 u)\bar{u}_t dx\right) \\ & = 2Re\left(\int_{\mathbb{R}} i u_t \bar{u}_t dx\right) \\ & = 0.\end{aligned}$$

On the other hand, using the second equation in (1.1) and integrating by parts leads to

$$\begin{aligned}& \int_{\mathbb{R}} (-\partial_x^2)^{-\frac{1}{2}} v_t (-\partial_x^2)^{-\frac{1}{2}} v_{tt} dx + \int_{\mathbb{R}} v v_t dx + \int_{\mathbb{R}} v_x v_{xt} dx + \int_{\mathbb{R}} |u|^2 v_t dx \\ & = \int_{\mathbb{R}} v_t((-\partial_x^2)^{-1} v_{tt} + v - \partial_x^2 v + |u|^2) dx \\ & = \int_{\mathbb{R}} (-\partial_x^2)^{-1} v_t (v_{tt} - \partial_x^2 v + \partial_x^4 v - \partial_x^2 |u|^2) dx \\ & = 0.\end{aligned}$$

Therefore,

$$\frac{dE}{dt} = 0.$$

The proof of Lemma 4.2 is completed. ■

Theorem 4.1 *Suppose that* $\alpha = 0$ *and* $n = 1, 2, 3$*, suppose furthermore that* $u_0, v_0 \in H^1(\mathbb{R}^n)$*,* $v_1 \in H^{-1}(\mathbb{R}^n)$*. Then there exists a positive constant* $T = T(\|u_0\|_{H^1}, \|v_0\|_{H^1}, \|v_1\|_{H^{-1}})$ *such that the IVP (2.7)-(2.8) has a unique solution* $u = u(x,t)$ *on the domain* $\mathbb{R}^n \times [0,T]$ *and the solution satisfies the following properties*

$$u \in C([0,T]; H^1(\mathbb{R}^n)), \tag{4.13}$$

for an admissible pair (q,r)

$$\|u\|_{L_T^q L_x^r} + \|\nabla u\|_{L_T^q L_x^r} < \infty, \tag{4.14}$$

and the mapping $(u_0, v_0, v_1) \longmapsto u(t)$ *from* $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \times H^{-1}(\mathbb{R}^n)$ *into the space defined by (4.13) is locally Lipschitz. Moreover, the function* $v = v(x,t)$ *defined by (2.6) satisfies*

$$v \in C([0,T]; H^1(\mathbb{R}^n)) \bigcap C^1([0,T]; H^{-1}(\mathbb{R}^n)). \tag{4.15}$$

Furthermore, for any given positive T*, the above solution can be extended to the domain* $\mathbb{R}^n \times [0,T]$.

Proof. By Lemma 4.1 and the contraction mapping principle, there exists a unique $u \in B_R(T)$ such that

$$\Phi(u) = u.$$

It is not difficult to show that this solution satisfies

$$\begin{aligned} \|u\|_{L_T^q L_x^r} + \|\nabla u\|_{L_T^q L_x^r} &\leq c_0\|u_0\|_{H^1} + c_1 T^{\frac{4-n}{4}} \|v_0\|_{H^1}\|u\|_{X(T)} + \\ &\quad c_2 T^{\frac{4-n}{4}}(T+1)\|v_1\|_{H^{-1}}\|u\|_{X(T)} + c_3(T^{\frac{6-n}{2}} + T^{\frac{4-n}{2}})\|u\|_{X(T)}^3 \\ &< \infty, \end{aligned}$$

where (q,r) is an admissible pair.

We next prove (4.15).

In fact, noting (2.6) and using Minkowski inequality, Hölder inequality and Lemma

2.3, we obtain

$$
\begin{aligned}
\|v(t)\|_{H^1} &\leq \left\|\frac{\partial}{\partial t}W(t)v_0(x)\right\|_{H^1} + \|W(t)v_1(x)\|_{H^1} + \left\|\int_0^t W(t-\tau)\Delta|u|^2 d\tau\right\|_{H^1} \\
&\leq \|v_0\|_{H^1} + 2(T+1)\|v_1\|_{H^{-1}} + 2(T+1)\int_0^T \||u|^2\|_{H^1} dt \\
&\leq \|v_0\|_{H^1} + 2(T+1)\|v_1\|_{H^{-1}} + 2(T+1)\int_0^T (\||u|^2\|_{L^2} + \|\nabla|u|^2\|_{L^2}) dt \\
&\leq \|v_0\|_{H^1} + 2(T+1)\|v_1\|_{H^{-1}} + 2T^{\frac{4-n}{4}}(T+1)\|u\|^2_{L_T^{\frac{8}{n}}L_x^4} + \\
&\quad 2T^{\frac{4-n}{4}}(T+1)\|\nabla u\|_{L_T^{\frac{8}{n}}L_x^4}\|u\|_{L_T^{\frac{8}{n}}L_x^4}.
\end{aligned}
\tag{4.16}
$$

Notice that

$$
v_t(x,t) = \frac{\partial^2}{\partial t^2}W(t)v_0(x) + \frac{\partial}{\partial t}W(t)v_1(x) + \int_0^t \frac{\partial}{\partial t}W(t-\tau)\Delta|u|^2 d\tau,
$$

where

$$
\frac{\partial^2}{\partial t^2}W(t)v_0(x) = -\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\xi}\hat{v}_0(\xi)|\xi|(1+|\xi|^2)^{\frac{1}{2}}\sin|\xi|(1+|\xi|^2)^{\frac{1}{2}}t d\xi.
$$

Similar to (4.16), we have

$$
\|v_t(t)\|_{H^{-1}} \leq \|v_0\|_{H^1} + \|v_1\|_{H^{-1}} + T^{\frac{4-n}{4}}\|u\|^2_{L_T^{\frac{8}{n}}L_x^4} + T^{\frac{4-n}{4}}\|\nabla u\|_{L_T^{\frac{8}{n}}L_x^4}\|u\|_{L_T^{\frac{8}{n}}L_x^4}. \tag{4.17}
$$

Thus, we have proved that $v \in C([0,T];H^1(\mathbb{R}^n))\bigcap C^1([0,T];H^{-1}(\mathbb{R}^n))$.

For any given positive constant T, we now extend the above solution to the domain $\mathbb{R}^n \times [0,T]$.

Case I: $n=1$

If $n=1$, then it follows from (4.12) that

$$
\begin{aligned}
&\|u_x\|^2_{L^2} + \frac{1}{2}\|(-\partial_x^2)^{-\frac{1}{2}}v_t\|^2_{L^2} + \frac{1}{2}\|v\|^2_{L^2} + \frac{1}{2}\|v_x\|^2_{L^2} \\
&= E(0) - \int_{\mathbb{R}} |u|^2 v dx \\
&\leq E(0) + \int_{\mathbb{R}} ||u|^2 v| dx.
\end{aligned}
$$

Using Cauchy inequality and Gagliardo-Nirenberg inequality gives

$$\begin{aligned}\int_{\mathbb{R}} |u|^2|v|dx &\leq c\|u\|_{L^4}^4 + \frac{1}{4}\|v\|_{L^2}^2 \\ &\leq c\|u\|_{L^2}^3\|u_x\|_{L^2} + \frac{1}{4}\|v\|_{L^2}^2 \\ &\leq \frac{1}{4}\|u_x\|_{L^2}^2 + c\|u\|_{L^2}^6 + \frac{1}{4}\|v\|_{L^2}^2 \\ &\leq \frac{1}{4}\|u_x\|_{L^2}^2 + \frac{1}{4}\|v\|_{L^2}^2 + c\|u_0\|_{L^2}^6.\end{aligned}$$

Thus,

$$\|u_x\|_{L^2}^2 + \|(-\partial_x^2)^{-\frac{1}{2}}v_t\|_{L^2}^2 + \|v\|_{L^2}^2 + \|v_x\|_{L^2}^2 \leq E(0) + c\|u_0\|_{L^2}^6.$$

Case II: $n = 2$

If $n = 2$, then we obtain from (4.12) that

$$\begin{aligned}&\|\nabla u\|_{L^2}^2 + \frac{1}{2}\|(-\Delta)^{-\frac{1}{2}}v_t\|_{L^2}^2 + \frac{1}{2}\|v\|_{L^2}^2 + \frac{1}{2}\|\nabla v\|_{L^2}^2 \\ &= E(0) - \int_{\mathbb{R}^2} |u|^2 v dx \\ &\leq E(0) + \int_{\mathbb{R}^2} |u|^2|v|dx.\end{aligned}$$

Using Hölder inequality, Gagliardo-Nirenberg inequality and Sobolev imbedding theorem (for the case $H^1(\mathbb{R}^2) \subset L^4(\mathbb{R}^2)$) yields

$$\begin{aligned}\int_{\mathbb{R}^2} |u|^2|v|dx &\leq \|v\|_{L^4}\|u\|_{L^{\frac{8}{3}}}^2 \\ &\leq c\|\nabla v\|_{L^2}\|u\|_{L^2}^{\frac{3}{2}}\|\nabla u\|_{L^2}^{\frac{1}{2}} \\ &\leq \frac{1}{4}\|\nabla v\|_{L^2}^2 + c\|u\|_{L^2}^3\|\nabla u\|_{L^2} \\ &\leq \frac{1}{4}\|\nabla v\|_{L^2}^2 + \frac{1}{4}\|\nabla u\|_{L^2}^2 + c\|u\|_{L^2}^6 \\ &\leq \frac{1}{4}\|\nabla v\|_{L^2}^2 + \frac{1}{4}\|\nabla u\|_{L^2}^2 + c\|u_0\|_{L^2}^6.\end{aligned}$$

Hence,

$$\|\nabla u\|_{L^2}^2 + \|(-\Delta)^{-\frac{1}{2}}v_t\|_{L^2}^2 + \|v\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 \leq E(0) + c\|u_0\|_{L^2}^6.$$

Case III: $n = 3$

If $n = 3$, then we obtain from (4.12) that

$$\begin{aligned}\|\nabla u\|_{L^2}^2 + \frac{1}{2}\|(-\Delta)^{-\frac{1}{2}}v_t\|_{L^2}^2 + \frac{1}{2}\|v\|_{L^2}^2 + \frac{1}{2}\|\nabla v\|_{L^2}^2 \\ = E(0) - \int_{\mathbb{R}^3} |u|^2 v dx \\ \leq E(0) + \int_{\mathbb{R}^3} |u|^2 |v| dx.\end{aligned}$$

Using Hölder inequality, Gagliardo-Nirenberg inequality and Sobolev imbedding theorem (for the case $H^1(\mathbb{R}^3) \subset L^6(\mathbb{R}^3)$) leads to

$$\begin{aligned}\int_{\mathbb{R}^3} |u|^2|v| dx &\leq \|v\|_{L^6}\|u\|_{L^{\frac{12}{5}}}^2 \\ &\leq c\|\nabla v\|_{L^2}\|u\|_{L^2}^{\frac{3}{2}}\|\nabla u\|_{L^2}^{\frac{1}{2}} \\ &\leq \frac{1}{4}\|\nabla v\|_{L^2}^2 + c\|u\|_{L^2}^3\|\nabla u\|_{L^2} \\ &\leq \frac{1}{4}\|\nabla v\|_{L^2}^2 + \frac{1}{4}\|\nabla u\|_{L^2}^2 + c\|u\|_{L^2}^6 \\ &\leq \frac{1}{4}\|\nabla v\|_{L^2}^2 + \frac{1}{4}\|\nabla u\|_{L^2}^2 + c\|u_0\|_{L^2}^6.\end{aligned}$$

Thus,

$$\|\nabla u\|_{L^2}^2 + \|(-\Delta)^{-\frac{1}{2}}v_t\|_{L^2}^2 + \|v\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 \leq E(0) + c\|u_0\|_{L^2}^6.$$

We observe from the above inequalities that $\|u\|_{H^1}^2 + \|(-\Delta)^{-\frac{1}{2}}v_t\|_{L^2}^2 + \|v\|_{H^1}^2$ is bound. Therefore we can repeat the argument of local existence of solution and then prove the solution can be extended to the domain $\mathbb{R}^n \times [0,T]$ for any given positive T. Thus, the proof of Theorem 4.1 is finished. ■

Theorem 4.2 *Suppose that $\alpha \neq 0$, suppose furthermore that $u_0, v_0 \in H^1(\mathbb{R})$ and $v_1 \in H^{-1}(\mathbb{R})$. Then there exists a positive constant $T = T(|\alpha|, \|u_0\|_{H^1}, \|v_0\|_{H^1}, \|v_1\|_{H^{-1}})$ such that the IVP (2.7)-(2.8) has a unique solution $u = u(x,t)$ on the strip $\mathbb{R} \times [0,T]$ and the solution satisfies the following properties*

$$u \in C([0,T]; H^1(\mathbb{R})), \tag{4.18}$$

for an admissible pair (q,r)

$$\|u\|_{L_T^q L_x^r} + \|\partial_x u\|_{L_T^q L_x^r} < \infty, \tag{4.19}$$

and the mapping $(u_0, v_0, v_1) \longmapsto u(t)$ from $H^1(\mathbb{R}) \times H^1(\mathbb{R}) \times H^{-1}(\mathbb{R})$ into the space defined by (4.18) is locally Lipschitz. Moreover, the function $v = v(x,t)$ defined by (2.6) satisfies

$$v \in C([0,T]; H^1(\mathbb{R})) \bigcap C^1([0,T]; H^{-1}(\mathbb{R})). \tag{4.20}$$

Furthermore, for any given positive T, the above solution can be extended to the domain $\mathbb{R} \times [0, T]$.

Proof. The proof of Theorem 4.2 is similar to that of Theorem 4.1, here we omit it. ■

The following theorem is a regularity result on the solution of the IVP (1.1)-(1.2).

Theorem 4.3 *If (u, v) is a solution of the IVP (1.1)-(1.2) with the initial data satisfying*

$$(u_0, v_0, v_1) \in H^1(\mathbb{R}) \times H^1(\mathbb{R}) \times H^{-1}(\mathbb{R}),$$

then it holds that

$$u, v, \partial_x u, \partial_x v \in L^4([0, T]; L^\infty(\mathbb{R})).$$
$$D_x^{\frac{3}{2}} u, D_x^{\frac{3}{2}} v \in L^\infty(\mathbb{R}; L^2[0, T]).$$

Proof. Since $(4, \infty)$ is an admissible pair, it follows from Theorem 4.2 that

$$u \in L^4([0, T]; L^\infty(\mathbb{R})).$$

Using Minkowski inequality, Hölder inequality and Lemma 2.5, we obtain from (2.6) that

$$\begin{aligned} \|v\|_{L_T^4 L_x^\infty} &\leq \left\| \frac{\partial}{\partial t} W(t) v_0 \right\|_{L_T^4 L_x^\infty} + \|W(t) v_0\|_{L_T^4 L_x^\infty} + \left\| \int_0^t W(t-\tau) \partial_x^2 |u|^2 d\tau \right\|_{L_T^4 L_x^\infty} \\ &\leq c(1 + T^{\frac{1}{4}}) \|v_0\|_{L^2} + c(1 + T^{\frac{1}{4}}) \|v_1\|_{H^{-1}} + cT^{\frac{3}{4}} \|u\|^2_{L_T^8 L_x^4}. \end{aligned}$$

Similarly, we can show

$$\partial_x v \in L^4([0, T]; L^\infty(\mathbb{R})).$$

On the other hand, using (3.27), (2.20) and Sobolev imbedding Theorem, we have

$$\begin{aligned} \|\partial_x u\|_{L_T^4 L_x^\infty} &\leq c\|\partial_x u_0\|_{L^2} + c \int_0^T \|\partial_x (uv + \alpha |u|^2 u)\|_{L^2} \\ &\leq c\|u_0\|_{H^1} + cT \|u\|_{L_T^\infty H_x^1} \|v\|_{L_T^\infty H_x^1} + cT^{\frac{1}{2}} \|u\|^2_{L_T^4 L_x^\infty} \|u\|_{L_T^\infty H_x^1}. \end{aligned}$$

Noting (2.24) and using Minkowski inequality gives

$$\|D_x^{\frac{3}{2}} u\|_{L_x^\infty L_T^2} \leq \|D_x^{\frac{1}{2}} \partial_x u_0\|_{L_x^\infty L_T^2} + \| \int_0^t D_x^{\frac{1}{2}} S(t-\tau) \partial_x (uv + \alpha |u|^2 u) d\tau \|_{L_x^\infty L_T^2}$$

$$\leq c\|u_0\|_{H^1} + cT \|u\|_{L_T^\infty H_x^1} \|v\|_{L_T^\infty H_x^1} + cT^{\frac{1}{2}} \|u\|^2_{L_T^4 L_x^\infty} \|u\|_{L_T^\infty H_x^1}.$$

By Lemma 2.6, Minkowski inequality and Hölder inequality, it follows from (2.6) that

$$\begin{aligned}
\|D_x^{\frac{3}{2}}v\|_{L_x^\infty L_T^2} &\leq \left\|D_x^{\frac{1}{2}}\frac{\partial}{\partial t}W(t)\partial_x v_0\right\|_{L_x^\infty L_T^2} + \|D_x^{\frac{1}{2}}W(t)\partial_x v_1\|_{L_x^\infty L_T^2} + \\
&\quad \int_0^t \|D_x^{\frac{1}{2}}W(t-\tau)\partial_x^3|u|^2\|_{L_x^\infty L_T^2}d\tau \\
&\leq c(1+T^{\frac{1}{2}})\|v_0\|_{H^1} + c(1+T^{\frac{1}{2}})\|v_1\|_{H^{-1}} + cT^{\frac{3}{4}}(1+T^{\frac{1}{2}})\|u\|_{L_T^4 L_x^\infty}\|u\|_{L_T^\infty H_x^1}.
\end{aligned}$$

This proves Theorem 4.3. ■

5 Local and global well-posedness in $H^s(\mathbb{R}) \times H^s(\mathbb{R}) \times H^s(\mathbb{R})$

For arbitrary fixed $s \in (0,1)$ and $T > 0$, we define the function space

$$X(T) \triangleq C([0,T]; H^s(\mathbb{R}))$$

and equip with the norm

$$\|u\|_{X(T)} \triangleq \|u\|_{L_T^\infty H_x^s} + \|u\|_{L_T^4 L_x^\infty}, \quad \forall\, u \in X(T).$$

It is easy to verify that $X(T)$ is a complete metric space. For any given positive real number $R > 0$, let $B_R(T)$ be a closed ball of radius R centered at the origin in the space $X(T)$, namely,

$$B_R(T) \triangleq \{u \in X(T) |\, \|u\|_{X(T)} \leq R\}.$$

As in Section 4, we introduce the mapping (4.1). We have

Lemma 5.1 *Suppose that $u_0, v_0 \in H^s(\mathbb{R}),\ v_1 \in H^{s-2}(\mathbb{R})$, then $\Phi : B_R(T) \longmapsto B_R(T)$ is a strictly contractive mapping.*

Proof. Step 1. By group properties and (2.20) in Lemma 2.1, we have

$$\|S(t)u_0\|_{X(T)} \leq c_0\|u_0\|_{H^s}. \tag{5.1}$$

Step 2. Taking $(q_2, r_2) = (4, \infty)$ in (2.22) and using Hölder inequality and (2.25), we obtain

$$\begin{aligned}
&\left\|\int_0^t S(t-\tau)F_0(u(\tau))d\tau\right\|_{L_T^\infty L_x^2} + \left\|\int_0^t S(t-\tau)F_0(u(\tau))d\tau\right\|_{L_T^4 L_x^\infty} \\
&\quad\leq cT^{\frac{1}{4}}\left\|u\left(\frac{\partial}{\partial t}W(t)v_0\right)\right\|_{L_x^1 L_T^2} \\
&\quad\leq cT^{\frac{1}{4}}\|u\|_{L_x^2 L_T^2}\left\|\frac{\partial}{\partial t}W(t)v_0\right\|_{L_x^2 L_T^\infty} \\
&\quad\leq cT^{\frac{3}{4}}\|u\|_{L_T^\infty L_x^2}\|v_0\|_{L^2} \\
&\quad\leq cT^{\frac{3}{4}}\|v_0\|_{L^2}\|u\|_{X(T)}.
\end{aligned} \tag{5.2}$$

Similarly,

$$\begin{aligned}
&\left\|\int_0^t S(t-\tau)F_1(u(\tau))d\tau\right\|_{L_T^\infty L_x^2} + \left\|\int_0^t S(t-\tau)F_1(u(\tau))d\tau\right\|_{L_T^4 L_x^\infty} \\
&\quad\leq cT^{\frac{1}{4}}\|u(W(t)v_1)\|_{L_x^1 L_T^2} \\
&\quad\leq cT^{\frac{1}{4}}\|u\|_{L_x^2 L_T^2}\|W(t)v_1\|_{L_x^2 L_T^\infty} \\
&\quad\leq cT^{\frac{3}{4}}(T+1)\|u\|_{L_T^\infty L_x^2}\|v_1\|_{H^{-2}} \\
&\quad\leq cT^{\frac{3}{4}}(T+1)\|v_1\|_{H^{-2}}\|u\|_{X(T)}
\end{aligned} \tag{5.3}$$

and

$$\begin{aligned}
&\left\|\int_0^t S(t-\tau)F_2(u(\tau))d\tau\right\|_{L_T^\infty L_x^2} + \left\|\int_0^t S(t-\tau)F_2(u(\tau))d\tau\right\|_{L_T^4 L_x^\infty} \\
&\quad\leq cT^{\frac{1}{4}}\left\|u\int_0^t W(t-\tau)\partial_x^2|u|^2 d\tau\right\|_{L_x^1 L_T^2} \\
&\quad\leq cT^{\frac{1}{4}}(T+1)\|u\|_{L_x^2 L_T^2}\int_0^T \|u^2\|_{L^2}dt \\
&\quad\leq cT^{\frac{3}{2}}(T+1)\|u\|_{L_T^\infty L_x^2}\|u\|_{L_T^4 L_x^\infty}\|u\|_{L_T^\infty L_x^2} \\
&\quad\leq cT^{\frac{3}{2}}(T+1)\|u\|_{X(T)}^3.
\end{aligned} \tag{5.4}$$

Noting Lemma 2.1 and using Minkowski inequality and Hölder inequality, we have

$$
\begin{aligned}
&\left\|\int_0^t S(t-\tau)F_3(u(\tau))d\tau\right\|_{L_T^\infty L_x^2}+\left\|\int_0^t S(t-\tau)F_3(u(\tau))d\tau\right\|_{L_T^4 L_x^\infty}\\
&\quad\le \int_0^t \|S(t-\tau)F_3(u(\tau))\|_{L_T^\infty L_x^2}d\tau+\int_0^t \|S(t-\tau)F_3(u(\tau))\|_{L_T^4 L_x^\infty}d\tau\\
&\quad\le c|\alpha|\int_0^T \||u|^2u\|_{L^2}dt\le c|\alpha|\int_0^T \|u\|_{L^\infty}^2\|u\|_{L^2}dt\le c|\alpha|T^{\frac12}\|u\|_{L_T^\infty L_x^2}\|u\|_{L_T^4 L_x^\infty}^2\\
&\quad\le c|\alpha|T^{\frac12}\|u\|_{X(T)}^3.
\end{aligned}
\tag{5.5}
$$

Step 3. We next continue to estimate

$$
\left\|D_x^s\left(\int_0^t S(t-\tau)F_i(u(\tau))d\tau\right)\right\|_{L_T^\infty L_x^2}\qquad (i=0,1,2,3).
$$

Using Minkowski inequality, we have

$$
\begin{aligned}
&\left\|D_x^s\left(\int_0^t S(t-\tau)F_0(u(\tau))d\tau\right)\right\|_{L_T^\infty L_x^2}=\left\|\int_0^t S(t-\tau)D_x^s\left(u\frac{\partial}{\partial t}W(t)v_0\right)d\tau\right\|_{L_T^\infty L_x^2}\\
&\quad\le\left\|\int_0^t S(t-\tau)(D_x^s\left(u\frac{\partial}{\partial t}W(t)v_0\right)-uD_x^s\frac{\partial}{\partial t}W(t)v_0-\frac{\partial}{\partial t}W(t)v_0D_x^su)d\tau\right\|_{L_T^\infty L_x^2}\\
&\qquad+\left\|\int_0^t S(t-\tau)uD_x^s\left(\frac{\partial}{\partial t}W(t)v_0\right)d\tau\right\|_{L_T^\infty L_x^2}+\left\|\int_0^t S(t-\tau)\frac{\partial}{\partial t}W(t)v_0D_x^sud\tau\right\|_{L_T^\infty L_x^2}\\
&\quad\triangleq I_1+I_2+I_3.
\end{aligned}
\tag{5.6}
$$

Estimate of I_1: Using Hölder inequality and Lemmas 2.1, 2.3 and 2.4, we obtain

$$
\begin{aligned}
I_1&\le c\left\|D_x^s\left(u\frac{\partial}{\partial t}W(t)v_0\right)-uD_x^s\frac{\partial}{\partial t}W(t)v_0-\frac{\partial}{\partial t}W(t)v_0D_x^su\right\|_{L_T^1L_x^2}\\
&\le c\|u\|_{L_T^4L_x^\infty}\left\|D_x^s\frac{\partial}{\partial t}W(t)v_0\right\|_{L_T^{\frac43}L_x^2}\\
&\le cT^{\frac34}\|u\|_{L_T^4L_x^\infty}\|D_x^sv_0\|_{L^2}\\
&\le cT^{\frac34}\|v_0\|_{H^s}\|u\|_{X(T)}.
\end{aligned}
\tag{5.7}
$$

Estimate of I_2: Taking $(q_1,r_1)=(\infty,2),\ (q_2,r_2)=(4,\infty)$ in (2.21) and using Hölder inequality and Lemma 2.3 , we get

$$
\begin{aligned}
I_2&\le\left\|uD_x^s\left(\frac{\partial}{\partial t}W(t)v_0\right)\right\|_{L_T^{\frac43}L_x^1}\le cT^{\frac34}\|u\|_{L_T^\infty L_x^2}\left\|\frac{\partial}{\partial t}W(t)D_x^sv_0\right\|_{L^2}\\
&\le cT^{\frac34}\|u\|_{L_T^\infty L_x^2}\|v_0\|_{H^s}\le cT^{\frac34}\|v_0\|_{H^s}\|u\|_{X(T)}.
\end{aligned}
\tag{5.8}
$$

Estimate of I_3: Similar to the estimate of I_2,

$$I_3 \leq T^{\frac{3}{4}} \left\| \frac{\partial}{\partial t} W(t) v_0 \right\|_{L^2} \|D_x^s u\|_{L_T^\infty L_x^2} \leq c T^{\frac{3}{4}} \|v_0\|_{H^s} \|u\|_{X(T)}. \tag{5.9}$$

Combining (5.7)-(5.9), we obtain from (5.6) that

$$\left\| D_x^s \left(\int_0^t S(t-\tau) F_0(u(\tau)) d\tau \right) \right\|_{L_T^\infty L_x^2} \leq c T^{\frac{3}{4}} \|v_0\|_{H^s} \|u\|_{X(T)}. \tag{5.10}$$

Similar to (5.10),

$$\left\| D_x^s \left(\int_0^t S(t-\tau) F_1(u(\tau)) d\tau \right) \right\|_{L_T^\infty L_x^2} \leq c T^{\frac{3}{4}} (T+1) \|v_1\|_{H^{s-2}} \|u\|_{X(T)}. \tag{5.11}$$

Taking $(q_2, r_2) = (\infty, 2)$ in (2.21) and using Minkowski inequality, we have

$$\begin{aligned}
&\left\| D_x^s \left(\int_0^t S(t-\tau) F_2(u(\tau)) d\tau \right) \right\|_{L_T^\infty L_x^2} = \left\| \int_0^t S(t-\tau) D_x^s F_2(u(\tau)) d\tau \right\|_{L_T^\infty L_x^2} \\
&\quad \leq c \|D_x^s F_2(u)\|_{L_T^1 L_x^2} = c \left\| D_x^s \left(u \int_0^t W(t-\tau) \partial_x^2 |u|^2 d\tau \right) \right\|_{L_T^1 L_x^2} \\
&\quad \leq c \left\| D_x^s \left(u \int_0^t W(t-\tau) \partial_x^2 |u|^2 d\tau \right) - u \int_0^t W(t-\tau) \partial_x^2 D_x^s |u|^2 d\tau \right. \\
&\qquad \left. - D_x^s u \int_0^t W(t-\tau) \partial_x^2 |u|^2 d\tau \right\|_{L_T^1 L_x^2} + c \left\| u \int_0^t W(t-\tau) \partial_x^2 D_x^s |u|^2 d\tau \right\|_{L_T^1 L_x^2} + \\
&\qquad c \|D_x^s u \int_0^t W(t-\tau) \partial_x^2 |u|^2 d\tau \|_{L_T^1 L_x^2} \\
&\quad \triangleq J_1 + J_2 + J_3.
\end{aligned} \tag{5.12}$$

Estimate of J_1: Noting (2.31) and using Hölder inequality, we obtain

$$\begin{aligned}
J_1 &\leq c \|u\|_{L_T^4 L_x^\infty} \left\| \int_0^t W(t-\tau) \partial_x^2 D_x^s |u|^2 d\tau \right\|_{L_T^{\frac{4}{3}} L_x^2} \\
&\leq c T^{\frac{3}{4}} \|u\|_{L_T^4 L_x^\infty} \left\| \int_0^t W(t-\tau) \partial_x^2 D_x^s |u|^2 d\tau \right\|_{L_T^\infty L_x^2} \\
&\leq c T^{\frac{3}{4}} (T+1) \|u\|_{L_T^4 L_x^\infty} \|D_x^s |u|^2\|_{L_T^1 L_x^2} \\
&\leq c T^{\frac{3}{4}} (T+1) \|u\|_{L_T^4 L_x^\infty}^2 \|D_x^s u\|_{L_T^{\frac{4}{3}} L_x^2} \\
&\leq c T^{\frac{3}{2}} (T+1) \|u\|_{X(T)}^3.
\end{aligned} \tag{5.13}$$

Similarly,

$$J_2 \leq c T^{\frac{3}{2}} (T+1) \|u\|_{X(T)}^3. \tag{5.14}$$

We now estimate J_3.

Using Hölder inequality and Lemma 2.5, we get

$$\begin{aligned}
J_3 &\leq c\left\|\|D_x^s u\|_{L_T^\infty L_x^2}\|\int_0^t W(t-\tau)\partial_x^2|u|^2 d\tau\right\|_{L_T^1 L_x^\infty} \\
&\leq c\|D_x^s u\|_{L_T^\infty L_x^2}\int_0^t \|W(t-\tau)\partial_x^2|u|^2\|_{L_T^1 L_x^\infty} d\tau \\
&\leq c\|D_x^s u\|_{L_T^\infty L_x^2}\int_0^t \left(\int_0^T \|W(t-\tau)\partial_x^2|u|^2\|_{L^\infty} dt\right) d\tau \\
&\leq cT^{\frac{3}{4}}\|D_x^s u\|_{L_T^\infty L_x^2}\int_0^t \left(\int_0^T \|W(t-\tau)\partial_x^2|u|^2\|_{L^\infty}^4 dt\right)^{\frac{1}{4}} d\tau \\
&\leq cT^{\frac{3}{4}}\|D_x^s u\|_{L_T^\infty L_x^2}\int_0^T \||u|^2\|_{L^2} dt \\
&\leq cT^{\frac{3}{2}}\|D_x^s u\|_{L_T^\infty L_x^2}\|u\|_{L_T^4 L_x^\infty}\|u\|_{L_T^\infty L_x^2} \\
&\leq cT^{\frac{3}{2}}\|u\|_{X(T)}^3.
\end{aligned} \tag{5.15}$$

Using (5.13)-(5.15), we obtain from (5.12) that

$$\left\|D_x^s\left(\int_0^t S(t-\tau)F_2(u(\tau))d\tau\right)\right\|_{L_T^\infty L_x^2} \leq c(T^{\frac{5}{2}}+T^{\frac{3}{2}})\|u\|_{X(T)}^3. \tag{5.16}$$

On the other hand, using Lemma 2.1 and Minkowski inequality, we have

$$\begin{aligned}
&\left\|D_x^s\left(\int_0^t S(t-\tau)F_3(u(\tau))d\tau\right)\right\|_{L_T^\infty L_x^2} = \left\|\int_0^t S(t-\tau)D_x^s F_3(u(\tau))d\tau\right\|_{L_T^\infty L_x^2} \\
&\leq \int_0^t \|S(t-\tau)D_x^s F_3(u(\tau))\|_{L_T^\infty L_x^2} d\tau \\
&\leq \int_0^T \|D_x^s F_3(u)\|_{L^2} dt = c|\alpha|\int_0^T \|D_x^s(|u|^2 u)\|_{L^2} dt \\
&\leq c|\alpha|\int_0^T \|D_x^s(|u|^2 u) - uD_x^s|u|^2 - |u|^2 D_x^s u\|_{L^2} dt \\
&\quad + c|\alpha|\int_0^T \|uD_x^s|u|^2\|_{L^2} dt + c|\alpha|\int_0^T \||u|^2 D_x^s u\|_{L^2} dt \\
&\triangleq K_1 + K_2 + K_3.
\end{aligned} \tag{5.17}$$

Estimate of K_1: Using Lemma 2.4 and Hölder inequality, we get

$$\begin{aligned}
K_1 &\leq c|\alpha|\int_0^T \|u\|_{L^\infty}\|D_x^s|u|^2\|_{L^2} dt \\
&\leq c|\alpha|\int_0^T \|u\|_{L^\infty}^2\|D_x^s u\|_{L^2} dt \\
&\leq c|\alpha|T^{\frac{1}{2}}\|u\|_{L_T^4 L_x^\infty}^2\|D_x^s u\|_{L_T^\infty L^2} \\
&\leq c|\alpha|T^{\frac{1}{2}}\|u\|_{X(T)}^3.
\end{aligned} \tag{5.18}$$

Estimate of K_2: Using Lemma 2.4 and Hölder inequality again, we have

$$\begin{aligned} K_2 &\le c|\alpha| \int_0^T \|u\|_{L^\infty} \|D_x^s |u|^2\|_{L^2} dt \\ &\le c|\alpha| T^{\frac{1}{2}} \|u\|^2_{L_T^4 L_x^\infty} \|D_x^s u\|_{L_T^\infty L^2} \\ &\le c|\alpha| T^{\frac{1}{2}} \|u\|^3_{X(T)}. \end{aligned} \tag{5.19}$$

Estimate of K_3: By Hölder inequality, we obtain

$$\begin{aligned} K_3 &\le c|\alpha| \int_0^T \|u\|^2_{L^\infty} \|D_x^s u\|_{L^2} dt \\ &\le c|\alpha| T^{\frac{1}{2}} \|u\|^2_{L_T^4 L_x^\infty} \|D_x^s u\|_{L_T^\infty L_x^2} \\ &\le c|\alpha| T^{\frac{1}{2}} \|u\|^3_{X(T)}. \end{aligned} \tag{5.20}$$

Then, combining (5.18)-(5.20), we obtain from (5.17) that

$$\left\| D_x^s \left(\int_0^t S(t-\tau) F_3(u(\tau)) d\tau \right) \right\|_{L_T^\infty L_x^2} \le c|\alpha| T^{\frac{1}{2}} \|u\|^3_{X(T)}. \tag{5.21}$$

Step 4. Therefore, the above estimates give

$$\begin{aligned} \|\Phi(u)\|_{X(T)} \le\ & c_0\|u_0\|_{H^s} + c_1 T^{\frac{3}{4}} \|v_0\|_{H^s} \|u\|_{X(T)} + c_2 T^{\frac{3}{4}} (T+1) \|v_1\|_{H^{s-2}} \|u\|_{X(T)} \\ & + c_3 (T^{\frac{5}{2}} + T^{\frac{3}{2}} + |\alpha| T^{\frac{1}{2}}) \|u\|^3_{X(T)}. \end{aligned} \tag{5.22}$$

Letting $R = 4c_0\|u_0\|_{H^s}$ and choosing T so small that

$$c_1 T^{\frac{3}{4}} \|v_0\|_{H^s} + c_2 T^{\frac{3}{4}} (T+1) \|v_1\|_{H^{s-2}} + c_3 (T^{\frac{5}{2}} + T^{\frac{3}{2}} + |\alpha| T^{\frac{1}{2}}) R^2 \le \frac{3}{4}, \tag{5.23}$$

we have

$$\|\Phi(u)\|_{X(T)} \le R.$$

This implies that Φ maps $B_R(T)$ into $B_R(T)$.

Step 5. We next show that, when T is small enough, $\Phi : B_R(T) \longmapsto B_R(T)$ is a strictly contractive mapping.

In fact, for arbitrary $u, \tilde{u} \in B_R(T)$,

$$\Phi(u)(t) - \Phi(\tilde{u})(t) = -i \int_0^t S(t-\tau) \sum_{j=0}^{3} [F_j(u(\tau)) - F_j(\tilde{u}(\tau))] d\tau,$$

where $F_j(u(t))$ $(j = 0, 1, 2, 3)$ are defined by (2.13).

On the one hand, similar to (5.2) we have

$$\left\|\int_0^t S(t-\tau)(F_0(u(\tau))-F_0(\tilde{u}(\tau)))d\tau\right\|_{L_T^\infty L_x^2}+\left\|\int_0^t S(t-\tau)(F_0(u(\tau))-F_0(\tilde{u}(\tau)))d\tau\right\|_{L_T^4 L_x^\infty}$$
$$\leq cT^{\frac{3}{4}}\|v_0\|_{L^2}\|u-\tilde{u}\|_{L_T^\infty L_x^2}$$
$$\leq cT^{\frac{3}{4}}\|v_0\|_{L^2}\|u-\tilde{u}\|_{X(T)} \tag{5.24}$$

and

$$\left\|\int_0^t S(t-\tau)(F_1(u(\tau))-F_1(\tilde{u}(\tau)))d\tau\right\|_{L_T^\infty L_x^2}+\left\|\int_0^t S(t-\tau)(F_1(u(\tau))-F_1(\tilde{u}(\tau)))d\tau\right\|_{L_T^4 L_x^\infty}$$
$$\leq cT^{\frac{3}{4}}(T+1)\|v_1\|_{H^{-2}}\|u-\tilde{u}\|_{L_T^\infty L_x^2}$$
$$\leq cT^{\frac{3}{4}}(T+1)\|v_1\|_{H^{-2}}\|u-\tilde{u}\|_{X(T)}. \tag{5.25}$$

On the other hand, similar to (5.4) we obtain

$$\left\|\int_0^t S(t-\tau)(F_2(u(\tau))-F_2(\tilde{u}(\tau)))d\tau\right\|_{L_T^\infty L_x^2}+\left\|\int_0^t S(t-\tau)(F_2(u(\tau))-F_2(\tilde{u}(\tau)))d\tau\right\|_{L_T^4 L_x^\infty}$$
$$\leq cT^{\frac{1}{4}}\left\|u\int_0^t W(t-\tau)\partial_x^2|u|^2d\tau-\tilde{u}\int_0^t W(t-\tau)\partial_x^2|\tilde{u}|^2d\tau\right\|_{L_x^1 L_T^2}$$
$$\leq cT^{\frac{1}{4}}\left\|(u-\tilde{u})\int_0^t W(t-\tau)\partial_x^2|u|^2d\tau\right\|_{L_x^1 L_T^2}+cT^{\frac{1}{4}}\left\|\tilde{u}\int_0^t W(t-\tau)\partial_x^2(|u|^2-|\tilde{u}|^2)d\tau\right\|_{L_x^1 L_T^2}$$
$$\leq cT^{\frac{3}{2}}(T+1)\|u-\tilde{u}\|_{L_T^\infty L_x^2}\|u\|_{L_T^4 L_x^\infty}\|u\|_{L_T^\infty L_x^2}$$
$$+cT^{\frac{3}{2}}(T+1)\|\tilde{u}\|_{L_T^\infty L_x^2}(\|u\|_{L_T^4 L_x^\infty}+\|\tilde{u}\|_{L_T^4 L_x^\infty})\|u-\tilde{u}\|_{L_T^\infty L_x^2}$$
$$\leq cT^{\frac{3}{2}}(T+1)\|u-\tilde{u}\|_{X(T)}(\|u\|_{X(T)}^2+\|u\|_{X(T)}\|\tilde{u}\|_{X(T)}+\|\tilde{u}\|_{X(T)}^2). \tag{5.26}$$

Moreover, similar to (5.5), we get

$$\begin{aligned}
&\left\|\int_0^t S(t-\tau)(F_3(u(\tau))-F_3(\tilde{u}(\tau)))d\tau\right\|_{L_T^\infty L_x^2}+\left\|\int_0^t S(t-\tau)(F_3(u(\tau))-F_3(\tilde{u}(\tau)))d\tau\right\|_{L_T^4 L_x^\infty}\\
&\quad\le c|\alpha|\int_0^T \left\||u|^2u-|\tilde{u}|^2\tilde{u}\right\|_{L^2}dt\\
&\quad\le c|\alpha|\int_0^T \left\|u(|u|^2-|\tilde{u}|^2)\right\|_{L^2}dt+c|\alpha|\int_0^T \left\||\tilde{u}|^2(u-\tilde{u})\right\|_{L^2}dt\\
&\quad\le c|\alpha|\int_0^T \|u-\tilde{u}\|_{L^\infty}(\|u\|_{L^\infty}+\|\tilde{u}\|_{L^\infty})\|u\|_{L^2}dt+c|\alpha|\int_0^T \|\tilde{u}\|_{L^\infty}^2\|u-\tilde{u}\|_{L^2}dt\\
&\quad\le c|\alpha|T^{\frac{1}{2}}\|u-\tilde{u}\|_{L_T^4L_x^\infty}[\|u\|_{L_T^4L_x^\infty}\|u\|_{L_T^\infty L_x^2}+\|\tilde{u}\|_{L_T^4L_x^\infty}\|u\|_{L_T^\infty L_x^2}]+\\
&\quad\quad c|\alpha|T^{\frac{1}{2}}\|\tilde{u}\|_{L_T^4L_x^\infty}^2\|u-\tilde{u}\|_{L_T^\infty L_x^2}\\
&\quad\le c|\alpha|T^{\frac{1}{2}}\|u-\tilde{u}\|_{X(T)}\left(\|u\|_{X(T)}^2+\|u\|_{X(T)}\|\tilde{u}\|_{X(T)}+\|\tilde{u}\|_{X(T)}^2\right).
\end{aligned}\tag{5.27}$$

In a way similar to (5.10), we can prove

$$\left\|D_x^s\left(\int_0^t S(t-\tau)(F_0(u(\tau))-F_0(\tilde{u}(\tau)))d\tau\right)\right\|_{L_T^\infty L_x^2}\le cT^{\frac{3}{4}}\|v_0\|_{H^s}\|u-\tilde{u}\|_{X(T)}\tag{5.28}$$

and

$$\left\|D_x^s\left(\int_0^t S(t-\tau)(F_1(u(\tau))-F_1(\tilde{u}(\tau)))d\tau\right)\right\|_{L_T^\infty L_x^2}\le cT^{\frac{3}{4}}(T+1)\|v_1\|_{H^{s-2}}\|u-\tilde{u}\|_{X(T)}.\tag{5.29}$$

Similar to (5.16),

$$\begin{aligned}
&\left\|D_x^s\left(\int_0^t S(t-\tau)(F_2(u(\tau))-F_2(\tilde{u}(\tau)))d\tau\right)\right\|_{L_T^\infty L_x^2}\\
&\quad\le c(T^{\frac{5}{2}}+T^{\frac{3}{2}})\|u-\tilde{u}\|_{X(T)}(\|u\|_{X(T)}^2+\|u\|_{X(T)}\|\tilde{u}\|_{X(T)}+\|\tilde{u}\|_{X(T)}^2).
\end{aligned}\tag{5.30}$$

Similar to (5.21),

$$\begin{aligned}
&\left\|D_x^s\left(\int_0^t S(t-\tau)(F_3(u(\tau))-F_3(\tilde{u}(\tau)))d\tau\right)\right\|_{L_T^\infty L_x^2}\\
&\quad\le c|\alpha|T^{\frac{1}{2}}\|u-\tilde{u}\|_{X(T)}(\|u\|_{X(T)}^2+\|u\|_{X(T)}\|\tilde{u}\|_{X(T)}+\|\tilde{u}\|_{X(T)}^2).
\end{aligned}\tag{5.31}$$

Combining these estimates yields

$$\begin{aligned}
\|\Phi(u)-\Phi(\tilde{u})\|_{X(T)}&\le c_1T^{\frac{3}{4}}\|v_0\|_{H^s}\|u-\tilde{u}\|_{X(T)}+c_2T^{\frac{3}{4}}(T+1)\|v_1\|_{H^{s-2}}\|u-\tilde{u}\|_{X(T)}\\
&\quad+c_3(T^{\frac{5}{2}}+T^{\frac{3}{2}}+|\alpha|T^{\frac{1}{2}})\|u-\tilde{u}\|_{X(T)}(\|u\|_{X(T)}^2+\|u\|_{X(T)}\|\tilde{u}\|_{X(T)}+\|\tilde{u}\|_{X(T)}^2).
\end{aligned}\tag{5.32}$$

Choosing T so small that (5.23) and the following inequality hold

$$c_1 T^{\frac{3}{4}}\|v_0\|_{H^s} + c_2 T^{\frac{3}{4}}(T+1)\|v_1\|_{H^{s-2}} + 3c_3(T^{\frac{5}{2}} + T^{\frac{3}{2}} + |\alpha| T^{\frac{1}{2}})R^2 \leq \frac{1}{2}, \tag{5.33}$$

we have

$$\|\Phi(u) - \Phi(\tilde{u})\|_{X(T)} \leq \frac{1}{2}\|u - \tilde{u}\|_{X(T)}.$$

This proves the lemma. ■

Theorem 5.1 *Let $s \in (0,1)$ be a fixed real number and suppose that $(u_0, v_0, v_1) \in H^s(\mathbb{R}) \times H^s(\mathbb{R}) \times H^{s-2}(\mathbb{R})$. Then there exists a positive constant $T = (|\alpha|, \|u_0\|_{H^s}, \|v_0\|_{H^s}, \|v_1\|_{H^{s-2}})$ such that the IVP (2.7)-(2.8) has a unique solution $u = u(x,t)$ on the domain $\mathbb{R} \times [0,T]$ and the solution satisfies the following property*

$$u \in C([0,T]; H^s(\mathbb{R})) \tag{5.34}$$

with

$$\|u\|_{L_T^q L_x^r} + \|D_x^s u\|_{L_T^q L_x^r} < \infty, \tag{5.35}$$

where $r \in [2, +\infty]$ and

$$\frac{2}{q} = \frac{1}{2} - \frac{1}{r}.$$

Moreover, the mapping $(u_0, v_0, v_1) \longmapsto u(t)$ from $H^s(\mathbb{R}) \times H^s(\mathbb{R}) \times H^{s-2}(\mathbb{R})$ into the space defined by (5.34) is locally Lipschitz, and the function $v = v(x,t)$ defined by (2.6) satisfies

$$v \in C([0,T]; H^s(\mathbb{R})) \bigcap C^1([0,T]; H^{s-2}(\mathbb{R})). \tag{5.36}$$

Furthermore, for any given positive T, the above solution can be extended to the domain $\mathbb{R} \times [0,T]$.

Proof. Step 1. By Lemma 5.1 and the contraction mapping principle, there exists a unique $u \in B_R(T)$ such that

$$\Phi(u) = u.$$

It is easy to show that this solution satisfies

$$\begin{aligned}\|u\|_{L_T^q L_x^r} + \|D_x^s u\|_{L_T^q L_x^r} \leq\ & c_0\|u_0\|_{H^s} + c_1 T^{\frac{3}{4}}\|v_0\|_{H^s}\|u\|_{X(T)} + c_2 T^{\frac{3}{4}}(T+1)\|v_1\|_{H^{s-2}}\|u\|_{X(T)} + \\ & c_3(T^{\frac{5}{2}} + T^{\frac{3}{2}} + |\alpha| T^{\frac{1}{2}})\|u\|_{X(T)}^3 < \infty,\end{aligned}$$

where $\frac{2}{q} = \frac{1}{2} - \frac{1}{r}$.

Step 2. We next show that

$$v \in C^1([0,T];H^s(\mathbb{R}))\bigcap C^1([0,T];H^{s-2}(\mathbb{R})).$$

In fact, noting (2.6) and using Minkowski inequality, Hölder inequality and Lemmas 2.3-2.4, we have

$$\begin{aligned}
\|v(t)\|_{H^s} \quad &\leq \quad \left\|\frac{\partial}{\partial t}W(t)v_0\right\|_{H^s} + \|W(t)v_1\|_{H^s} + \left\|\int_0^t W(t-\tau)\partial_x^2|u|^2 d\tau\right\|_{H^s} \\
&\leq \quad \|v_0\|_{H^s} + 2(T+1)\|v_1\|_{H^{s-2}} + 2(T+1)\int_0^T \||u|^2\|_{H^s}dt \\
&\leq \quad \|v_0\|_{H^s} + 2(T+1)\|v_1\|_{H^{s-2}} + \\
&\quad\quad 2(T+1)\int_0^T \||u|^2\|_{L^2}dt + 2(T+1)\int_0^T \|D_x^s|u|^2\|_{L^2}dt \qquad (5.37)\\
&\leq \quad \|v_0\|_{H^s} + 2(T+1)\|v_1\|_{H^{s-2}} + \\
&\quad\quad 2(T+1)\int_0^T \|u\|_{L^\infty}\|u\|_{L^2}dt + 6(T+1)\int_0^T \|u\|_{L^\infty}\|D_x^s u\|_{L^2}dt \\
&\leq \quad \|v_0\|_{H^s} + 2(T+1)\|v_1\|_{H^{s-2}} + 2T^{\frac{3}{4}}(T+1)\|u\|_{L_T^4L_x^\infty}\|u\|_{L_T^\infty L_x^2} + \\
&\quad\quad 6T^{\frac{3}{4}}(T+1)\|u\|_{L_T^4L_x^\infty}\|D_x^s u\|_{L_T^\infty L_x^2}.
\end{aligned}$$

Notice that

$$v_t(x,t) = \frac{\partial^2}{\partial t^2}W(t)v_0(x) + \frac{\partial}{\partial t}W(t)v_1(x) + \int_0^t \frac{\partial}{\partial t}W(t-\tau)\Delta|u|^2 d\tau,$$

where

$$\frac{\partial^2}{\partial t^2}W(t)v_0(x) = -\frac{1}{2\pi}\int_{\mathbb{R}} e^{ix\xi}\hat{v}_0(\xi)|\xi|(1+|\xi|^2)^{\frac{1}{2}}\sin|\xi|(1+|\xi|^2)^{\frac{1}{2}}t d\xi.$$

Similar to (5.37), we can prove

$$\|v_t(t)\|_{H^{s-2}} \leq \|v_0\|_{H^s} + \|v_1\|_{H^{s-2}} + T^{\frac{3}{4}}\|u\|_{L_T^4L_x^\infty}(\|u\|_{L_T^\infty L_x^2} + \|D_x^s u\|_{L_T^\infty L_x^2}). \qquad (5.38)$$

Thus, we have proved the following fact

$$v \in C([0,T];H^s(\mathbb{R}))\bigcap C^1([0,T];H^{s-2}(\mathbb{R})).$$

Step 3. For any given positive constant T, we now extend the above solution to the domain $\mathbb{R}\times[0,T]$.

Assume that the maximal time T^* of existence of the solution $u = u(x,t)$ is finite. Noting the fact that the solution satisfies the integral equation $\Phi(u) = u$ and the estimate (5.22), we have

$$\|u\|_{X(T)} \leq c_0\|u_0\|_{H^s} + \theta(T)\|u\|_{X(T)}, \quad \forall\, T \in [0,T^*), \qquad (5.39)$$

where $\theta(T)$ is a positive constant satisfying

$$\theta(T) \le c_1 T^{\frac{3}{4}}\|v_0\|_{H^s} + c_2 T^{\frac{3}{4}}(T+1)\|v_1\|_{H^{s-2}} + c_3(T^{\frac{5}{2}} + T^{\frac{3}{2}} + |\alpha| T^{\frac{1}{2}})\|u\|_{L_T^\infty H_x^s}\|u\|_{L_T^4 L_x^\infty}.$$

By Theorems 4.1-4.3, we get

$$\|u\|_{L_T^\infty H_x^s} \le \|u\|_{L_T^\infty H_x^1} \le c \quad \text{and} \quad \|u\|_{L_T^4 L_x^\infty} \le c.$$

Thus, we choose suitable $\tilde{T} \in [0, T^*)$ such that

$$\theta(\tilde{T}) \le \frac{1}{2}.$$

Obviously, $\tilde{T}$ depends on $|\alpha|$, $\|v_0\|_{H^s}$ and $\|v_1\|_{H^{s-2}}$. Then it follows from (5.39) that

$$\|u\|_{X(T')} \le 2c_0\|u_0\|_{H^s} \tag{5.40}$$

for any fixed $T' \in [0, \tilde{T}]$.

If $\tilde{T} = T^*$, then it is obvious that the solution $u = u(x,t)$ of IVP (2.7), (2.8) can be extended to the domain $\mathbb{R} \times [0, T^* + \varepsilon]$ and the solution satisfies

$$\sup_{t \in [0, T^*+\varepsilon]} \|u(t)\|_{H^s} \le 2c_0\|u_0\|_{H^s},$$

where ε is a positive constant. This contradicts the definition of T^*. Therefore, we may assume that

$$0 < \tilde{T} < T^*.$$

Let $m \in \mathbb{N}$ satisfy $T^* \le m\tilde{T}$ and replace $\tilde{T}$ by $\tilde{T} = \frac{T^*}{m}$. We now consider the IVP for the following equation

$$i\omega_t + \frac{1}{2}\Delta\omega = \omega\left(\frac{\partial}{\partial t}W(t)v_0(x)\right) + \omega(W(t)v_1(x)) + \omega\int_0^t W(t-\tau)\Delta|\omega|^2 d\tau + \alpha|\omega|^2\omega,$$

with the initial data

$$\omega(x, \tilde{T}) = u(x, \tilde{T}).$$

The uniqueness of the solution yields that the function

$$\omega(x,t) = \begin{cases} u(x,t), & t \in [0, \tilde{T}], \\ \omega(x,t); & t \in [\tilde{T}, 2\tilde{T}]. \end{cases} \tag{5.41}$$

is a solution of IVP (2.7), (2.8) in the domain $\mathbb{R} \times [0, 2\tilde{T}]$. On the other hand, thanks to Theorems 4.1-4.3, the norm of $\|u\|_{L_T^4 L_x^\infty}$ and $\|u\|_{L_T^\infty H_x^s}$ is bounded for any given positive T. Therefore, we repeat the same procedure and obtain

$$\begin{aligned} \|u\|_{X(2\tilde{T})} &\le \max\{2c_0\|u_0\|_{H^s}, 2c_0\|u(\tilde{T})\|_{H^s}\} \\ &\quad \max\{2c_0\|u_0\|_{H^s}, 4c_0^2\|u_0\|_{H^s}\}. \end{aligned}$$

Repeating this process m times gives

$$\|u\|_{X(T^*)} \le \max\{2c_0\|u_0\|_{H^s}, 4c_0^2\|u_0\|_{H^s}, \cdots, (2c_0)^m\|u_0\|_{H^s}\}.$$

This contradicts the definition of T^*, hence $T^* = \infty$. This proves Theorem 5.1. ■

The following theorem is on the regularity of the solution of the IVP (1.1)-(1.2).

Theorem 5.2 *If (u, v) is a solution of the IVP (1.1)-(1.2) with the initial data satisfying*

$$(u_0, v_0, v_1) \in H^s(\mathbb{R}) \times H^s(\mathbb{R}) \times H^{s-2}(\mathbb{R}),$$

then it holds that

$$u,\ v,\ D_x^s u,\ D_x^s v \in L^4([0,T]; L^\infty(\mathbb{R})) \tag{5.42}$$

and

$$D_x^{s+\frac{1}{2}} u,\ D_x^{s+\frac{1}{2}} v \in L^\infty(R; L^2[0,T]). \tag{5.43}$$

Proof. It follows from the proof of Theorem 4.3 that

$$u,\ v \in L^4([0,T]; L^\infty(\mathbb{R})).$$

We next show that

$$D_x^s u,\ D_x^s v \in L^4([0,T]; L^\infty(\mathbb{R})).$$

Noting (2.31) and using Minkowski inequality, we have

$$\begin{aligned} \|D_x^s(uv)\|_{L^2} &\le \|D_x^s(uv) - uD_x^s v - vD_x^s u\|_{L^2} + \|uD_x^s v\|_{L_x^2} + \|vD_x^s u\|_{L^2} \\ &\le c\|u\|_{L_x^\infty}\|D_x^s v\|_{L^2} + \|v\|_{L_x^\infty}\|D_x^s u\|_{L^2} \end{aligned} \tag{5.44}$$

and

$$\begin{aligned} \|D_x^s(|u|^2)\|_{L^2} &\le \|D_x^s(u\bar{u}) - uD_x^s\bar{u} - \bar{u}D_x^s u\|_{L^2} + \|uD_x^s\bar{u}\|_{L^2} + \|\bar{u}D_x^s u\|_{L^2} \\ &\le c\|u\|_{L_x^\infty}\|D_x^s u\|_{L^2}. \end{aligned} \tag{5.45}$$

By Minkowski inequality, (2.31) and (5.44), we obtain

$$\begin{aligned} \|D_x^s(|u|^2u)\|_{L^2} &\le \|D_x^s(u^2\bar{u}) - u^2D_x^s\bar{u} - \bar{u}D_x^s(u^2)\|_{L^2} + \|u^2D_x^s\bar{u}\|_{L^2} + \|\bar{u}D_x^s(u^2)\|_{L^2} \\ &\le c\|u\|_{L^\infty}^2\|D_x^s u\|_{L^2} + \|u\|_{L^\infty}\|D_x^s(u^2)\|_{L^2} \le c\|u\|_{L^\infty}^2\|D_x^s u\|_{L^2}. \end{aligned} \tag{5.46}$$

Using (3.27), Minkowski inequality, (2.20), (5.44), (5.46) and Hölder inequality, we get

$$\begin{aligned}
\|D_x^s u\|_{L_T^4 L_x^\infty} &\leq c\|u_0\|_{H^s} + c\int_0^T \|D_x^s(uv)\|_{L^2}dt + c\int_0^T \|D_x^s(|u|^2 u)\|_{L^2}dt \\
&\leq c\|u_0\|_{H^s} + cT^{\frac{3}{4}}\|u\|_{L_T^4 L_x^\infty}\|D_x^s v\|_{L_T^\infty L_x^2} + \\
&\quad cT^{\frac{3}{4}}\|v\|_{L_T^4 L_x^\infty}\|D_x^s u\|_{L_T^\infty L_x^2} + cT^{\frac{1}{2}}\|u\|^2_{L_T^4 L_x^\infty}\|D_x^s u\|_{L_T^\infty L_x^2}.
\end{aligned} \tag{5.47}$$

This proves

$$D_x^s u \in L^4([0,T]; L^\infty(\mathbb{R})).$$

Noting (2.6) and using Minkowski inequality, Lemma 2.5, (5.45) and Hölder inequality, we have

$$\begin{aligned}
\|D_x^s v\|_{L_T^4 L_x^\infty} &\leq c(1+T^{\frac{1}{4}})\|v_0\|_{H^s} + c(1+T^{\frac{1}{4}})\|v_1\|_{H^{s-2}} + c\int_0^T \|D_x^s(|u|^2)\|_{L^2}dt \\
&\leq c(1+T^{\frac{1}{4}})\|v_0\|_{H^s} + c(1+T^{\frac{1}{4}})\|v_1\|_{H^{s-2}} + cT^{\frac{3}{4}}\|u\|_{L_T^4 L_x^\infty}\|D_x^s u\|_{L_T^\infty L_x^2}.
\end{aligned} \tag{5.48}$$

The above inequality implies

$$D_x^s v \in L^4([0,T]; L^\infty(\mathbb{R})).$$

In what follows, we prove (5.43).

Noting (2.24) and using Minkowski inequality, (5.44) and (5.46), we obtain from (3.27) that

$$\begin{aligned}
\|D_x^{s+\frac{1}{2}} u\|_{L_x^\infty L_T^2} &\leq c\|D_x^s u_0\|_{L^2} + c\int_0^T \|D_x^s(uv + |\alpha||u|^2 u)\|_{L^2}dt \\
&\leq c\|u_0\|_{H^s} + cT^{\frac{3}{4}}\|u\|_{L_T^4 L_x^\infty}\|D_x^s v\|_{L_T^\infty L_x^2} + \\
&\quad cT^{\frac{3}{4}}\|v\|_{L_T^4 L_x^\infty}\|D_x^s u\|_{L_T^\infty L_x^2} + cT^{\frac{1}{2}}\|u\|^2_{L_T^4 L_x^\infty}\|D_x^s u\|_{L_T^\infty L_x^2}.
\end{aligned} \tag{5.49}$$

This yields

$$D_x^{s+\frac{1}{2}} u \in L^\infty(\mathbb{R}; L^2[0,T]).$$

Noting (2.6) again and using Minkowski inequality, Lemma 2.6, (5.45) and Hölder inequality, we have

$$\begin{aligned}
\|D_x^{s+\frac{1}{2}} v\|_{L_x^\infty L_T^2} &\leq c(1+T^{\frac{1}{2}})\|v_0\|_{H^s} + c(1+T^{\frac{1}{2}})\|v_1\|_{H^{s-2}} + c(1+T^{\frac{1}{2}})\int_0^T \|D_x^s(|u|^2)\|_{L^2}dt \\
&\leq c(1+T^{\frac{1}{2}})\|v_0\|_{H^s} + c(1+T^{\frac{1}{2}})\|v_1\|_{H^{s-2}} + cT^{\frac{3}{4}}(1+T^{\frac{1}{2}})\|u\|_{L_T^4 L_x^\infty}\|D_x^s u\|_{L_T^\infty L_x^2}.
\end{aligned} \tag{5.50}$$

Thus, the proof of Theorem 5.2 is completed. ■

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References

[1] T. Cazenave, Semilinear Schrödinger equations, Courant Lecture Notes in Mathematics, 2003.

[2] A.J. Corcho and F. Linares, *Well-posedness for the Schrödinger-Debye equation*, Contemp. Math. **362** (2004), 113-131.

[3] J. Ginibre and G. Velo, *On the class of nonlinear Schrödinger equations*, J. Func. Anal. **32** (1979), 1-71.

[4] J. Ginibre and G. Velo, *The global Cauchy problem for the nonlinear Schröinger equation revisited*, Ann. Inst. Henri Poincaré, Analyse non linéaire **2** (1985), 309-327.

[5] B.L. Guo and F.X. Chen, *Finite dimensional behavior of global attractors for weakly damped nonlinear Schrödinger-Boussinesq equations*, Phys. D **93** (1996), 101-118.

[6] B.L. Guo and L.J. Shen, *The global solution of initial value problem for nonlinear Schrödinger-Boussinesq equation in 3-dimensions*, Acta Math. Appl. Sinica **6** (1990), 11-21.

[7] T. Kato, *On nonlinear Schröinger equations*, Ann. Inst. Henri Poincaré, Phys. Theor. **46** (1987), 113-129.

[8] C.E. Kenig, G. Ponce and L. Vega, *Oscilatory integrals and regularity of dispersive equations*, Indiana Univ. Math. J. **40** (1991), 33-69.

[9] C.E. Kenig, G. Ponce and L. Vega, *Well-posedness and scatting results for the generalied Korteweg-deVries equation via the contraction principle*, Comm. Pure Appl. Math. **46** (1993), 527-620.

[10] Y.S. Li and Q.Y. Chen, *Finite dimensional global attractor for dissipative Schrödinger-Boussinesq equations*, J. Math. Anal. Appl. **205** (1997), 107-132.

[11] F. Linares, *Global existence of small solutions for a generalized Boussinesq equation*, J. Differential Equations **106** (1993), 257-293.

[12] F. Linares and A. Navas, *On the Schrödinger-Boussinesq Equations*, Adv. Differential Equations **9** (2004), 159-176.

[13] Y. Liu, *Decay and scattering of small solutions of a generalized Boussinesq equation*, J. Func. Anal. **147** (1997), 51-68.

[14] V.G. Makhankov, *On stationary solutions of the Schrödinger equation with a self-consistent potential satisfying Boussinesq's equations*, Phys. Lett. A **50** (1974), 42-44.

[15] V.G. Makhankov, *Dynamics of classical solitons (in non-integrable systems)*, Phys. Rep. **35** (1978), 1-128.

[16] J.P. Nguenang and T.C. Kofané, *Nonlinear excitations in a compressible quantum Heisenberg chain*, Physica D, **147** (2000), 311-335.

[17] T. Ogawa and Y. Tsutsumi, *Blow-up solutions for the nonlinear Schröinger equation with quartic potential and periodic boundary conditions*, Lect. Notes Math. **1450** (1989), 236-251 .

[18] T. Ozawa and K. Tsutaya, *On the Cauchy problem for Schrödinger-improved Boussinesq equations*, Adv. Stud. Pure Math., to appear.

[19] R.S. Strichartz, *Restriction of Fourier transform to quadratic surfaces and decay of solution of wave equation*, Duke Math. J. **44** (1977), 705-714.

[20] C. Sulem and P. Sulem, *The Nonlinear Schröinger Equation. Self-Focusing and Wave Collapse*, Applied Mathematical Sciences **139**, Springer-Verlag, New York, 1999.

[21] N. Yajima and J. Satsuma, *Soliton solutions in a diatomic lattice system*, Prog. Theor. Phys. **62** (1979), 370-378.

[22] V.E. Zakharov, *Collapse of Langmuir waves*, Sov. Phys. JETP **35** (1972), 908-914.