Lifespan of Classical Solutions to Quasi-linear Hyperbolic Systems with Small BV Normal Initial Data

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Abstract

In this paper, we first give a lower bound of the lifespan and some estimates of classical solutions to the Cauchy problem for general quasi-linear hyperbolic systems, whose characteristic fields are not weakly linearly degenerate and the inhomogeneous terms satisfy Kong's matching condition. After that, we investigate the lifespan of the classical solution to the Cauchy problem and give a sharp limit formula. In this paper, we only require that the initial data are sufficiently small in the L^1 sense and the BV sense.

Key Words: Quasi-linear hyperbolic system; Weakly linear degeneracy; Matching condition; Normalized Coordinates; Blow-up; Lifespan.

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1 Introduction and main results

Consider the following quasi-linear hyperbolic system of first order

$$\frac{\partial u}{\partial t} + A(u)\frac{\partial u}{\partial x} = B(u), \tag{1.1}$$

where $u = (u_1, \cdots, u_n)^T$ are the unknown vector-valued functions of (t, x), $A(u) = (a_{ij}(u))$ is an $n \times n$ matrix and $B(u) = (B_1(u), B_2(u), \cdots, B_n(u))^T$ are $n-$dimensional vector-valued functions.

By hyperbolicity, for any given u on the domain under consideration, $A(u)$ has n real eigenvalues $\lambda_1(u), \cdots, \lambda_n(u)$ and a complete system of left (resp. right) eigenvectors $l_1(u), \cdots, l_n(u)$ (*resp.* $r_1(u), \cdots, r_n(u)$). In this paper, we assume that (1.1) is a strictly hyperbolic system, i.e.,

$$\lambda_1(u) < \lambda_2(u) < \cdots < \lambda_n(u). \tag{1.2}$$

Without loss of generality, we suppose that on the domain under consideration

$$l_i(u)r_j(u) \equiv \delta_{ij}, \quad r_i^T(u)r_i(u) \equiv 1 \ \ (i, j = 1, \cdots, n),$$

where δ_{ij} stands for the Kronecker's symbol.

The following definitions come from Kong [7].

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Definition 1.1 *The $i-th$ characteristic $\lambda_i(u)$ is* **weakly linearly degenerate***, if, along the $i-th$ characteristic trajectory $u = u^{(i)}(s)$ passing through $u = 0$, defined by $\frac{du^{(i)}(s)}{ds} = r_i(u^{(i)}(s)),\ s = 0 : u = 0$, we have*

$$\nabla\lambda_i(u)r_i(u) \equiv 0, \ \ \forall\ |s| \ \ small, \tag{1.3}$$

namely

$$\lambda_i(u^{(i)}(s)) \equiv \lambda(0), \ \ \forall\ |s| \ \ small. \tag{1.4}$$

If all characteristics $\lambda_i(u)$ $(i = 1, 2, \cdots, n)$ are weakly linearly degenerate, then the system (1.1) is called **weakly linearly degenerate**.

Definition 1.2 *The inhomogeneous term $B(u)$ is called to be satisfied* **the matching condition***, if, along the $i-th$ characteristic trajectory $u = u^{(i)}(s)$ passing through $u = 0$, $B(u) \equiv 0$, i.e.,*

$$B(u^{(i)}(s)) \equiv 0, \ \ \forall\ |s| \ \ small. \tag{1.5}$$

Definition 1.3 *If there exists a sufficiently smooth invertible transformation $u = u(\tilde{u})$ $(u(0) = 0)$ such that in the $\tilde{u}-$ space, for each $i = 1, 2, \cdots, n$, the $i-th$ characteristic trajectory passing through $\tilde{u} = 0$ coincides with the $\tilde{u}-$axis at least for $|\tilde{u}_i|$ small, namely*

$$\tilde{r}_i(\tilde{u}_i e_i) \equiv e_i, \ \ \forall\ |\tilde{u}_i| \ \ small, \tag{1.6}$$

where $e_i = (0, \cdots, 0, \overset{(i)}{1}, 0, \cdots, 0)^T$. Such a transformation is called **a normalized transformation** *and the corresponding unknown variables $\tilde{u} = (\tilde{u}_1, \tilde{u}_2, \cdots, \tilde{u}_n)^T$ are called* **normalized variables** *or* **normalized coordinates**.

If the system (1.1) is strictly hyperbolic, then there always exists the normalized transformation (cf. [8]). In this paper, for the sake of simplicity, we assume that the unknown variables u are already normalized variables. That is to say,

$$r_i(u_i e_i) \equiv e_i. \tag{1.7}$$

It is easy to see

$$r_i(0) = e_i, \ \ l_i(0) = e_i^T. \tag{1.8}$$

At the same time, (1.4) and (1.5) can be deduced to

$$\lambda_i(u_i e_i) \equiv 0 \tag{1.9}$$

and

$$B(u_i e_i) = 0 \tag{1.10}$$

respectively.

We consider the Cauchy problem of the hyperbolic system (1.1) with the following initial data

$$t = 0 : u(0, x) = f(\epsilon, x), \tag{1.11}$$

where $f(\epsilon, x)$ is a C^1 vector-valued function of $\epsilon,\ x$ such that

$$f(0,x) \equiv 0, \quad \frac{\partial^2 f}{\partial\epsilon\partial x}(\epsilon,\cdot) \in (C^r[0,\epsilon_1])^n, \ 0 < r \le 1, \tag{1.12}$$

where $\epsilon \in [0, \epsilon_1]$, ϵ_1 is a sufficiently small positive constant. Then we know that

$$\lim_{\epsilon \to 0^+} \frac{f(\epsilon,x)}{\epsilon} = \frac{\partial f}{\partial \epsilon}(0,x) \triangleq \psi(x) \in \left(C^1(\mathbb{R})\right)^n. \tag{1.13}$$

For the case that the initial data $f(\epsilon, x)$ satisfies the following decay property: there exists a constant $\mu > 0$ such that

$$\varrho \triangleq \sup_{x\in\mathbb{R}} \left\{ (1+|x|)^{1+\mu} \left(|f(\epsilon,x)| + \left| \frac{\partial f}{\partial x}(\epsilon,x) \right| \right) \right\} < +\infty \tag{1.14}$$

is sufficiently small, by means of the normalized coordinates Li et al proved that the Cauchy problem (1.1) and (1.11) admits a unique global classical solution, provided that the system (1.1) is weakly linearly degenerate (see [12]-[15] and [8]). Kong and Yang [11] studied the asymptotic behavior of the classical solution. In their works, the condition $\mu > 0$ is essential. If $\mu = 0$, a counterexample was constructed by Kong [7] showing that the classical solution may blow up in a finite time, even when the system (1.1) is weakly linearly degenerate.

For the quasi-linear strictly hyperbolic system with linearly degenerate characteristic fields, A. Bressan [1] proved the global existence of classical solution with initial data of small BV norm. If the characteristic fields are weakly linearly degenerate, Zhou [19] proved the global existence of classical solution with initial data of small L^1 norm and BV norm. Dai and Kong [4] and Dai [2] studied the asymptotic behavior of the classical solution.

When system (1.1) is not weakly linearly degenerate, there exists a nonempty set $J \subseteq \{1, 2, \cdots, n\}$ such that $\lambda_i(u)$ is not weakly linearly degenerate if and only if $i \in J$.

Noting (1.4), we observe that for any fixed $i \in J$, either there exists an integer $\alpha_i \ge 0$ such that

$$\left.\frac{d^l \lambda_i\left(u^{(i)}(s)\right)}{ds^l}\right|_{s=0} = 0 \quad (l = 1, \cdots, \alpha_i), \quad \text{but} \quad \left.\frac{d^{\alpha_i+1} \lambda_i\left(u^{(i)}(s)\right)}{ds^{\alpha_i+1}}\right|_{s=0} \neq 0, \tag{1.15}$$

or

$$\left.\frac{d^l \lambda_i\left(u^{(i)}(s)\right)}{ds^l}\right|_{s=0} = 0 \quad (l = 1, 2, \cdots). \tag{1.16}$$

In the case that (1.16) holds, we define $\alpha_i = +\infty$.

For the normalized coordinates, conditions (1.15) and (1.16) simply reduce to

$$\frac{\partial^l \lambda_i}{\partial u_i^l}(0) = 0 \ \ (l = 1, \cdots, \alpha_i), \ \ \text{but} \ \ \frac{\partial^{\alpha_i+1} \lambda_i}{\partial u_i^{\alpha_i+1}}(0) \neq 0$$

and

$$\frac{\partial^l \lambda_i}{\partial u_i^l}(0) = 0 \ \ (l = 1, 2, \cdots)$$

respectively.

Our first goal in this paper is to give the following uniform *a priori* estimates of the classical solution to the Cauchy problem (1.1) and (1.11).

Theorem 1.1 *Suppose that the system (1.1) is strictly hyperbolic, $A(u), B(u)$ is suitably smooth in a neighborhood of $u = 0$ and $B(u)$ satisfies the matching condition, suppose furthermore that the initial data (1.11) satisfies*

$$\int_{-\infty}^{+\infty} \left|\frac{\partial f}{\partial x}(\epsilon, x)\right| dx \le K_1\epsilon, \quad \int_{-\infty}^{+\infty} |f(\epsilon, x)| dx \le \frac{K_2}{M+1}\epsilon, \tag{1.17}$$

where ϵ is a sufficiently small positive constant and K_1, K_2 and $M \triangleq \sup_{x\in\mathbb{R}} \left|\frac{\partial\psi}{\partial x}(x)\right|$ are constants independent of ϵ. Suppose finally that system (1.1) is not weakly linearly degenerate and

$$\alpha = \min\{\alpha_i \mid i \in J\} < \infty, \tag{1.18}$$

where α_i is defined by (1.15)-(1.16). Then, on the existence domain $[0, T] \times \mathbb{R}$ of the C^1 solution $u = u(t, x)$, there exist positive constants $K_3,\ K_4,\ K_5,\ K_6$ independent of $\epsilon,\ M,\ T$ such that

$$V_1(T),\ \tilde{V}_1(T) \le K_3(\epsilon + \epsilon^{\alpha+2}T),\ W_1(T),\ \tilde{W}_1(T),\ U_\infty(T),\ V_\infty(T) \le K_3\epsilon, \tag{1.19}$$

where

$$T\epsilon^{\alpha+\frac{3}{2}} \le K_4 \tag{1.20}$$

and

$$W_\infty(T) \le K_5\epsilon, \tag{1.21}$$

where

$$T\epsilon^{\alpha+1} \le K_6. \tag{1.22}$$

In (1.19) and (1.21), $V_1(T),\ \tilde{V}_1(T),\ W_1(T),\ \tilde{W}_1(T),\ U_\infty(T),\ V_\infty(T),\ W_\infty(T)$ are defined as follows: For any fixed $T \ge 0$,

$$U_\infty(T) = \sup_{0\le t\le T} \sup_{x\in\mathbb{R}} |u(t, x)|, \quad V_\infty(T) = \sup_{0\le t\le T} \sup_{x\in\mathbb{R}} |v(t, x)|,$$

$$W_\infty(T) = \sup_{0\le t\le T} \sup_{x\in\mathbb{R}} |w(t, x)|,$$

$$V_1(T) = \sup_{0\le t\le T} \int_{-\infty}^{+\infty} |v(t, x)| dx, \quad W_1(T) = \sup_{0\le t\le T} \int_{-\infty}^{+\infty} |w(t, x)| dx,$$

$$\tilde{V}_1(T) = \max_{i\ne j} \sup_{\tilde{C}_j} \int_{\tilde{C}_j} |v_i(t, x)| dt, \quad \tilde{W}_1(T) = \max_{i\ne j} \sup_{\tilde{C}_j} \int_{\tilde{C}_j} |w_i(t, x)| dt,$$

where $|\cdot|$ stands for the Euclidean norm in $\mathbb{R}^n$, $v = (v_1, \cdots, v_n)^T$ and $w = (w_1, \cdots, w_n)^T$ in which $v_i = l_i(u)u$ and $w_i = l_i(u)u_x$ are defined by (2.1) in §2, $\tilde{C}_j$ stands for any given j-th characteristic on the domain $[0, T] \times \mathbb{R}$.

Remark 1.1 *By (1.21)-(1.22), we know that the life span of the classical solution $\tilde{T}(\epsilon) \ge K_6\epsilon^{-(\alpha+1)}$. It is obvious that (1.14) implies (1.17). Therefore, Theorem 1.1 is a generalization of corresponding results of Li et al [15] and Kong [8] where the decay initial data was considered.*

For the critical case, i.e., in (1.18), $\alpha = +\infty$, from Theorem 1.1 and its proof in §3, we can easily get the following corollary.

Corollary 1.1 *Assume that the assumptions except (1.18) in Theorem 1.1 hold. In (1.18), we assume that* $\alpha = +\infty$. *Then, for any given integer* $N \geq 1$, *there exists* $\epsilon_0 = \epsilon_0(N) > 0$ *so small that for any fixed* $\epsilon \in (0, \epsilon_0]$, *the lifespan* $\tilde{T}(\epsilon)$ *of the* C^1 *solution* $u = u(t,x)$ *to the Cauchy problem (1.1) and (1.11) satisfies*

$$\tilde{T}(\epsilon) \geq C_N \epsilon^{-N},$$

where C_N *is a positive constant independent of* ϵ.

Next we consider the blow-up of the classical solution to the Cauchy problem of the hyperbolic system (1.1) with the initial data (1.11). If the hyperbolic system (1.1) is not weakly linearly degenerate, Li et al [15] and Kong [8] estimated the lifespan of classical solution to the Cauchy problem (1.1) with the special initial data $u(0,x) = \epsilon\phi(x)$ which satisfies the following decay property: there exists a constant $\mu > 0$ such that

$$\varrho \stackrel{\triangle}{=} \sup_{x\in\mathbb{R}} \{(1+|x|)^{1+\mu}(|\phi(x)| + |\phi'(x)|)\} < +\infty \tag{1.23}$$

and the zero or matching inhomogeneous term $B(u)$.

Our second goal is to investigate the lifespan of classical solution to the Cauchy problem (1.1) and (1.11) when the system (1.1) is not weakly linearly degenerate.

Theorem 1.2 *Suppose that the assumptions in Theorem 1.1 hold. Let*

$$J_1 = \{ i \mid i \in J,\, \alpha_i = \alpha \} \neq \emptyset. \tag{1.24}$$

If there exists $i_0 \in J_1$ *and a point* $x_0 \in \mathbb{R}$ *such that*

$$\frac{\partial^{\alpha+1}\lambda_{i_0}}{\partial u_{i_0}^{\alpha+1}}(0)\psi_{i_0}^{\alpha}\psi'_{i_0}(x_0) < 0, \tag{1.25}$$

where $\psi(x) \in \left(C^1(\mathbb{R})\right)^n$ *is defined in (1.13), then there exists* $\epsilon_0 > 0$ *so small that for any fixed* $\epsilon \in (0, \epsilon_0]$, *the first order derivatives of the* C^1 *solution* $u = u(t,x)$ *to the Cauchy problem (1.1) and (1.11) must blow up in a finite time and the lifespan* $\tilde{T}(\epsilon)$ *of* $u = u(t,x)$ *satisfies*

$$\lim_{\epsilon\to 0^+} \left(\epsilon^{\alpha+1}\tilde{T}(\epsilon)\right)^{-1} = \max_{i\in J_1}\sup_{x\in\mathbb{R}} \left(-\frac{1}{\alpha!}\frac{\partial^{\alpha+1}\lambda_i}{\partial u_i^{\alpha+1}}(0)\psi_i^{\alpha}(x)\,\psi'_i(x) \right). \tag{1.26}$$

Remark 1.2 *It is obvious that the decay property (1.23) implies (1.17). Therefore, Theorem 1.2 is a generalization of responding results of Li et al [15] and Kong [8] and results of L. Hörmander [5], John [6], Liu [16] where the decay initial data and the compactly supported initial data are considered respectively.*

Remark 1.3 *For the hyperbolic system (1.1) with constant multiple characteristic fields, we can obtain the similar results in Theorem 1.1 and Theorem 1.2 if we prove them as in this paper and in [2]-[3].*

Remark 1.4 *Similar to Kong and Li [10], if along $i-th$ characteristic $x = x_i(t, y)$, $w_i(t, x_i(t, y)) = l_i(u)u_x(t, x_i(t, y))$ blow up at the lifespan $\tilde{T}(\epsilon)$, then we have*

$$w_i(t, x_i(t, y)) = O((\tilde{T}(\epsilon) - t)^{-1}), \quad \textit{when } \ t \to \tilde{T}(\epsilon)^-.$$

Remark 1.5 *For the conservation laws, shock will appear (see Kong [9]).*

This paper is organized as follows. In §2, we recall John's formula on the decomposition of waves with some supplements for the hyperbolic system (1.1). Then we give some uniform *a priori* estimates for the Cauchy problem (1.1) and (1.11) and prove Theorem 1.1 in §3. In §4, we obtain some important uniform estimates by making use of an invertible characteristics' transformation of the hyperbolic system (1.1). Finally, we investigate the lifespan of the classical solution to the Cauchy problem (1.1) and (1.11) and give the proof of Theorem 1.2 in §5.

2 Preliminaries and Decomposed Formulas of Waves

For the sake of completeness, in this section we briefly recall John's formula on the decomposition of waves with some supplements for the hyperbolic system (1.1), which play an important role in our proof.

Let

$$v_i = l_i(u)u, \quad w_i = l_i(u)u_x \quad (i = 1, \cdots, n) \tag{2.1}$$

and

$$b_i(u) = l_i(u)B(u) \quad (i = 1, 2, \cdots, n). \tag{2.2}$$

Then we have

$$u = \sum_{k=1}^{n} v_k r_k(u), \quad u_x = \sum_{k=1}^{n} w_k r_k(u) \tag{2.3}$$

and

$$B(u) = \sum_{k=1}^{n} b_k(u) r_k(u). \tag{2.4}$$

Let

$$\frac{d}{d_i t} = \frac{\partial}{\partial t} + \lambda_i(u)\frac{\partial}{\partial x} \tag{2.5}$$

be the directional derivative along the i-th characteristic. We have (see [13]-[15] or [8])

$$\frac{dv_i}{d_i t} = \sum_{j,k=1}^{n} \beta_{ijk}(u)v_j w_k + \sum_{j,k=1}^{n} \nu_{ijk}(u)v_j b_k(u) + b_i(u) \triangleq F_i(t, x) \tag{2.6}$$

and

$$\frac{dw_i}{d_i t} = \sum_{j,k=1}^{n} \gamma_{ijk}(u)w_j w_k + \sum_{j,k=1}^{n} \sigma_{ijk}(u)w_j b_k(u) + (b_i(u))_x \triangleq G_i(t, x), \tag{2.7}$$

where

$$\beta_{ijk}(u) = (\lambda_k(u)-\lambda_i(u))l_i(u)\nabla r_j(u)r_k(u), \tag{2.8}$$
$$\nu_{ijk}(u) = -l_i(u)\nabla r_j(u)r_k(u), \tag{2.9}$$
$$\gamma_{ijk}(u) = (\lambda_k(u)-\lambda_j(u))l_i(u)\nabla r_j(u)r_k(u) - \nabla\lambda_j(u)r_k(u)\delta_{ij}, \tag{2.10}$$
$$\sigma_{ijk}(u) = l_i(u))(\nabla r_k(u)r_j(u) - \nabla r_j(u)r_k(u)). \tag{2.11}$$

Equivalently we also get

$$\frac{\partial v_i}{\partial t} + \frac{\partial(\lambda_i(u)v_i)}{\partial x} = \sum_{j,k=1}^n \tilde{\beta}_{ijk}(u)v_jw_k + \sum_{j,k=1}^n \nu_{ijk}(u)v_jb_k(u) + b_i(u) \triangleq \tilde{F}_i(t,x), \tag{2.12}$$

$$d[v_i(dx-\lambda_i(u)dt)] = \left[\sum_{j,k=1}^n \tilde{\beta}_{ijk}(u)v_jw_k + \sum_{j,k=1}^n \nu_{ijk}(u)v_jb_k(u) + b_i(u)\right] dt\wedge dx \triangleq \tilde{F}_i(t,x)dt\wedge dx \tag{2.13}$$

and

$$\frac{\partial w_i}{\partial t} + \frac{\partial(\lambda_i(u)w_i)}{\partial x} = \sum_{j,k=1}^n \tilde{\gamma}_{ijk}(u)w_jw_k + \sum_{j,k=1}^n \sigma_{ijk}(u)w_jb_k(u) + (b_i(u))_x \triangleq \tilde{G}_i(t,x), \tag{2.14}$$

$$d[w_i(dx-\lambda_i(u)dt)] = \left[\sum_{j,k=1}^n \tilde{\gamma}_{ijk}(u)w_jw_k + \sum_{j,k=1}^n \sigma_{ijk}(u)w_jb_k(u) + (b_i(u))_x\right] dt\wedge dx \triangleq \tilde{G}_i(t,x)dt\wedge dx, \tag{2.15}$$

where

$$\tilde{\beta}_{ijk}(u) = \beta_{ijk}(u) + \nabla\lambda_i(u)r_k(u)\delta_{ij}, \tag{2.16}$$
$$\tilde{\gamma}_{ijk}(u) = \gamma_{ijk}(u) + \frac{1}{2}[\nabla\lambda_j(u)r_k(u)\delta_{ij} + \nabla\lambda_k(u)r_j(u)\delta_{ik}]. \tag{2.17}$$

From (2.8), (2.10) and (2.16)-(2.17), we see that

$$\beta_{iji}(u) \equiv 0, \ \tilde{\gamma}_{ijj}(u)\equiv 0, \quad \forall\, i,j\in\{1,2,\cdots,n\}, \quad \forall\, |u| \ \ small, \tag{2.18}$$
$$\gamma_{ijj}(u) \equiv 0, \ \tilde{\beta}_{iji}(u)\equiv 0, \quad \forall\, j\neq i, \quad \forall\, |u| \ \ small. \tag{2.19}$$

As we already assume that u are the normalized coordinates, making use of (1.7), the following relations hold (see [8]):

$$\beta_{ijj}(u_je_j) \equiv 0, \quad \nu_{ijj}(u_je_j)\equiv 0, \quad \sigma_{ijj}(u_je_j)\equiv 0, \ \ |u_j| \text{ small}, \ \forall\, i,j, \tag{2.20}$$
$$\tilde{\beta}_{ijj}(u_je_j) \equiv 0, \qquad \forall\, |u_j| \text{ small}, \ \forall\, i\neq j. \tag{2.21}$$

When the inhomogeneous term $B(u)$ satisfies the matching condition, then in the normalized coordinates u (see [8]),

$$b_i(u) = \sum_{j\neq k} b_{ijk}(u)u_ju_k, \quad \forall\, |u| \text{ small}, \ \forall\, i\in\{1,2,\cdots,n\}, \tag{2.22}$$
$$(b_i(u))_x = \sum_{k=1}^n \tilde{b}_{ik}(u)w_k, \tag{2.23}$$

where $b_{ijk}(u)$ is a C^1 function and $\tilde{b}_{ik}(u) = \sum_{l=1}^{n} \frac{\partial b_i(u)}{\partial u_l} r_{kl}(u)$ satisfies that

$$\tilde{b}_{ik}(u_k e_k) \equiv 0, \quad \forall\, |u_k| \text{ small}, \ \forall\, k \in \{1, \cdots, n\}. \tag{2.24}$$

3 Uniform Estimates—Proof of Theorem 1.1

In this section, we shall establish some uniform estimates under the assumptions in Theorem 1.1 and give the proof of Theorem 1.1.

First we recall some basic L^1 estimates. They are essentially due to Schartzman [17], [18] and Zhou [19].

Lemma 3.1 *Let $\phi = \phi(t,x) \in C^1$ satisfy*

$$\phi_t + (\lambda(t,x)\phi)_x = F(t,x), \ \ 0 \le t \le T, x \in \mathbb{R}, \ \ \phi(0,x) = g(x),$$

where $\lambda \in C^1$. Then

$$\int_{-\infty}^{+\infty} |\phi(t,x)| dx \le \int_{-\infty}^{+\infty} |g(x)| dx + \int_0^T \int_{-\infty}^{+\infty} |F(s,x)| ds dx, \ \ \forall\, t \le T,$$

provided that the right hand side of the inequality is bounded.

Lemma 3.2 *Let $\phi = \phi(t,x)$ and $\psi = \psi(t,x)$ be C^1 functions satisfying*

$$\phi_t + (\lambda(t,x)\phi)_x = F(t,x), \ \ 0 \le t \le T, x \in \mathbb{R}, \ \ \phi(0,x) = g_1(x),$$

and

$$\psi_t + (\mu(t,x)\phi)_x = G(t,x), \ \ 0 \le t \le T, x \in \mathbb{R}, \ \ \phi(0,x) = g_2(x),$$

respectively, where λ, $\mu \in C^1$ such that there exists a positive constants δ_0 independent of T verifying

$$\mu(t,x) - \lambda(t,x) \ge \delta_0, \ \ 0 \le t \le T, \ \ x \in \mathbb{R}.$$

Then

$$\begin{aligned} \int_0^T \int_{-\infty}^{+\infty} |\phi(t,x)||\psi(t,x)| dx dt \quad \le \quad & C \left(\int_{-\infty}^{+\infty} |g_1(x)| dx + \int_0^T \int_{-\infty}^{+\infty} |F(t,x)| dx dt \right) \times \\ & \left(\int_{-\infty}^{+\infty} |g_2(x)| dx + \int_0^T \int_{-\infty}^{+\infty} |G(t,x)| dx dt \right), \end{aligned}$$

provided that the two factors on the right hand side of the inequality is bounded.

By the existence and uniqueness of local C^1 solution to the Cauchy problem, in order to prove Theorem 1.1, it suffices to establish *a prior* estimates on the C^0 norm of u and $\frac{\partial u}{\partial x}$ on the existence domain of C^1 solution $u = u(t,x)$.

By (1.2), there exist positive constants δ_0, δ_1 and δ such that

$$|\lambda_i(u) - \lambda_j(v)| \ge \delta_0, \ \ |\lambda_i(u) - \lambda_i(v)| \le \delta_1, \quad \forall\, |u|, |v| \le \delta, \quad \forall\, i \ne j. \tag{3.1}$$

For the time being it is supposed that on the existence domain $[0,T]\times\mathbb{R}$ of the C^1 solution $u=u\,(t,x)$ we have

$$|u(t,x)|\le K_7\epsilon, \tag{3.2}$$

where K_7 is a positive constant independent of $\epsilon,\ t,\ x$. At the end of the proof of Theorem 1.1, we shall explain that this hypothesis is reasonable. Then, (3.1) hold if we take $\delta=K_7\epsilon$.

Introduce

$$\begin{aligned} Q_W(T) &= \sum_{j\neq k}\int_0^T\int_{\mathbb{R}}|w_j(t,x)||w_k(t,x)|dtdx,\\ Q_{VW}(T) &= \sum_{j\neq k}\int_0^T\int_{\mathbb{R}}|v_j(t,x)||w_k(t,x)|dtdx,\\ Q_V(T) &= \sum_{j\neq k}\int_0^T\int_{\mathbb{R}}|v_j(t,x)||v_k(t,x)|dtdx. \end{aligned}$$

As we already assume that u are the normalized coordinates, by (1.7) it can be easily seen that

$$\sum_{i\neq j}|u_i|\le C_1\sum_{i\neq j}|v_i|,\ \text{for fixed}\ \ j;\ \sum_{i\neq j}|u_iw_j|\le C_1\sum_{i\neq j}|v_iw_j|;\quad \sum_{i\neq j}|u_iu_j|\le C_1\sum_{i\neq j}|v_iv_j|.$$

Here and hereafter $C_j\ (j=1,2,\cdots)$ stand for some positive constants independent of $\epsilon,\ M,\ T$.

It follows from (2.12) and (2.18)-(2.24) that

$$\begin{aligned} \tilde{F}_i(t,x) &= \sum_{j,k=1}^n\tilde{\beta}_{ijk}(u)v_jw_k+\sum_{j,k=1}^n\nu_{ijk}(u)v_jb_k(u)+b_i(u)\\ &= \sum_{j\neq k}\tilde{\beta}_{ijk}(u)v_jw_k+\sum_{j=1}^n(\tilde{\beta}_{ijj}(u)-\tilde{\beta}_{ijj}(u_je_j))v_jw_j+\tilde{\beta}_{iii}(u_ie_i)v_iw_i\\ &\quad+\sum_{j,k=1}^n\nu_{ijk}(u)v_j\sum_{p\neq q}b_{kpq}(u)u_pu_q+\sum_{p\neq q}b_{ipq}(u)u_pu_q. \end{aligned}$$

On the other hand,

$$\tilde{\beta}_{iii}(u_ie_i)=\nabla\lambda_i(u_ie_i)r_i(u_ie_i)=\frac{\partial\lambda_i}{\partial u_i}(u_ie_i).$$

Therefore, we have

$$|\tilde{F}_i(t,x)|\le C_2\left[\sum_{j\neq k}|v_jw_k|+\sum_{j\neq k}|v_jv_k|+|u_i|^\alpha|v_iw_i|\right]. \tag{3.3}$$

By (2.14) and (2.18)-(2.24), we have

$$\begin{aligned} \tilde{G}_i(t,x) &= \sum_{j,k=1}^n\tilde{\gamma}_{ijk}(u)w_jw_k+\sum_{j,k=1}^n\sigma_{ijk}(u)w_jb_k(u)+(b_i(u))_x\\ &= \sum_{j\neq k}\tilde{\gamma}_{ijk}(u)w_jw_k+\sum_{j,k=1}^n\sigma_{ijk}(u)w_j\sum_{p\neq q}b_{kpq}(u)u_pu_q\\ &\quad+\sum_{k=1}^n(\tilde{b}_{ik}(u)-\tilde{b}_{ik}(u_ke_k))w_k. \end{aligned}$$

Then we get

$$|\tilde{G}_i(t,x)| \le C_3 \left[\sum_{j\neq k} |v_j w_k| + \sum_{j\neq k} |w_j w_k| \right]. \tag{3.4}$$

By (2.12), (2.14), (3.3)-(3.4), it follows from Lemma 3.2 that

$$\begin{aligned} Q_W(T) &\le C_4 \left(W_1(0) + \int_0^T \int_{\mathbb{R}} |\tilde{G}(t,x)| dt dx \right)^2 \\ &\le C_4 (W_1(0) + Q_W(T) + Q_{VW}(T))^2 \\ &\le C_4 (\epsilon + Q_W(T) + Q_{VW}(T))^2, \end{aligned} \tag{3.5}$$

$$\begin{aligned} Q_V(T) &\le C_4 \left(V_1(0) + \int_0^T \int_{\mathbb{R}} |\tilde{F}(t,x)| dt dx \right)^2 \\ &\le C_4 (V_1(0) + Q_V(T) + Q_{VW}(T) + |U_\infty(T)|^{\alpha+1} W_1(T)\cdot T)^2 \\ &\le C_4 \left(\frac{\epsilon}{M+1} + Q_V(T) + Q_{VW}(T) + |U_\infty(T)|^{\alpha+1} W_1(T)\cdot T \right)^2, \end{aligned} \tag{3.6}$$

$$\begin{aligned} Q_{VW}(T) &\le C_4 \left(V_1(0) + \int_0^T \int_{\mathbb{R}} |\tilde{F}(t,x)| dt dx \right) \left(W_1(0) + \int_0^T \int_{\mathbb{R}} |\tilde{G}(t,x)| dt dx \right) \\ &\le C_4 (V_1(0) + Q_V(T) + Q_{VW}(T) + |U_\infty(T)|^{\alpha+1} W_1(T)\cdot T) \\ &\quad \cdot (W_1(0) + Q_W(T) + Q_{VW}(T)) \\ &\le C_4 \left(\frac{\epsilon}{M+1} + Q_V(T) + Q_{VW}(T) + |U_\infty(T)|^{\alpha+1} W_1(T)\cdot T \right) \\ &\quad \cdot (\epsilon + Q_W(T) + Q_{VW}(T)), \end{aligned} \tag{3.7}$$

where $\tilde{F} = (\tilde{F}_1, \tilde{F}_2, \cdots, \tilde{F}_n)^T$, $\tilde{G} = (\tilde{G}_1, \tilde{G}_2, \cdots, \tilde{G}_n)^T$.

We assume that the $j-$th characteristic $\tilde{C}_j$ intersects $t=0$ with point A, intersects $t=T$ with point B. We draw an $i-$th characteristic $\tilde{C}_i$ from B downward and intersects $t=0$ with point C. We rewrite (2.15) as

$$d(|w_i(t,x)|(dx - \lambda_i(u)dt)) = sgn(w_i)\tilde{G}_i dt dx, \ \ a.e.$$

and integrate it in the region ABC to get

$$\begin{aligned} \left| \int_{\tilde{C}_j} |w_i(t,x)|(\lambda_j(u) - \lambda_i(u)) dt \right| &\le \int_A^C |w_i(0,x)| dx + \int\int_{ABC} |\tilde{G}_i| dt dx \\ &\le C_5 (W_1(0) + Q_W(T) + Q_{VW}(T)). \end{aligned}$$

Noting (3.1), it follows that

$$\int_{\tilde{C}_j} |w_i(t,x)| dt \le C_6 (W_1(0) + Q_W(T) + Q_{VW}(T)) \le C_7 (\epsilon + Q_W(T) + Q_{VW}(T)),$$

hence

$$\tilde{W}_1(T) \le C_8 (\epsilon + Q_W(T) + Q_{VW}(T)). \tag{3.8}$$

In a similar way, we can deduce from (2.13) that

$$\tilde{V}_1(T) \le C_9 \left[\frac{\epsilon}{M+1} + Q_V(T) + Q_{VW}(T) + |U_\infty(T)|^{\alpha+1} W_1(T)\cdot T \right]. \tag{3.9}$$

It follows from (2.12) and Lemma 3.1 that

$$\begin{aligned}\int_{-\infty}^{+\infty}|v_i(T,x)|dx &\leq \int_{-\infty}^{+\infty}|v_i(0,x)|dx+\int_0^T\int_{-\infty}^{+\infty}|\tilde{F}_i(t,x)|dtdx\\ &\leq C_{10}V_1(0)+C_2\int_0^T\int_{-\infty}^{+\infty}\left[\sum_{j\neq k}|v_jw_k|+\sum_{j\neq k}|v_jv_k|+|u_i|^\alpha|v_iw_i|\right]dtdx\\ &\leq C_{11}\left[\frac{\epsilon}{M+1}+Q_V(T)+Q_{VW}(T)+|U_\infty(T)|^{\alpha+1}W_1(T)\cdot T\right].\end{aligned}$$

That is to say,

$$V_1(T)\leq C_{11}\left[\frac{\epsilon}{M+1}+Q_V(T)+Q_{VW}(T)+|U_\infty(T)|^{\alpha+1}W_1(T)\cdot T\right]. \tag{3.10}$$

In a similar way, it follows from (2.14) and Lemma 3.1 that

$$W_1(T)\leq C_{12}\left[\epsilon+Q_W(T)+Q_{VW}(T)\right]. \tag{3.11}$$

It can be easily seen that

$$U_\infty(T),\ \ V_\infty(T)\leq C_{13}\sup_{0\leq t\leq T}\int_{-\infty}^{+\infty}|u_x(t,x)|dx\leq C_{14}W_1(T). \tag{3.12}$$

Thus, in order to prove (1.19) it suffices to show that we can choose some constants d_i $(i=1,2,3,4,5)$ in such a way that for any fixed T_0 $(0\leq T_0\leq T)$ with $T_0\epsilon^{\alpha+\frac{3}{2}}\leq K_4$ such that

$$\begin{aligned}&V_1(T_0),\ \tilde{V}_1(T_0)\leq 2d_1\epsilon+2d_2\epsilon^{\alpha+2}T_0,\ W_1(T_0)\leq 2d_3\epsilon,\ \tilde{W}_1(T_0)\leq 2d_4\epsilon,\\ &U_\infty(T_0),\ V_\infty(T_0)\leq 2d_5\epsilon,\end{aligned} \tag{3.13}$$

we have

$$\begin{aligned}&V_1(T_0),\ \tilde{V}_1(T_0)\leq d_1\epsilon+d_2\epsilon^{\alpha+2}T_0,\ W_1(T_0)\leq d_3\epsilon,\ \tilde{W}_1(T_0)\leq d_4\epsilon,\\ &U_\infty(T_0),\ V_\infty(T_0)\leq d_5\epsilon.\end{aligned} \tag{3.14}$$

Substituting (3.13) into (3.5)-(3.7), we have

$$\begin{aligned}Q_W(T_0) &\leq C_4(\epsilon+Q_W(T_0)+Q_{VW}(T_0))^2,\\ Q_V(T_0) &\leq C_4\left[\frac{\epsilon}{M+1}+Q_V(T_0)+Q_{VW}(T_0)+(2d_5)^{\alpha+1}(2d_3)K_4\epsilon^{\frac{1}{2}}\right]^2,\\ Q_{VW}(T_0) &\leq C_4\left[\frac{\epsilon}{M+1}+Q_V(T_0)+Q_{VW}(T_0)+(2d_5)^{\alpha+1}(2d_3)K_4\epsilon^{\frac{1}{2}}\right](\epsilon+Q_W(T_0)+Q_{VW}(T_0)).\end{aligned}$$

Denote $a_1=(2d_5)^{\alpha+1}(2d_3)K_4$. It follows that

$$Q_W(T_0)\leq C_4(1+3C_4a_1^2)^2\epsilon^2,\ Q_V(T_0)\leq 2C_4a_1^2\epsilon,\ Q_{VW}(T_0)\leq 2C_4a_1\epsilon^{\frac{3}{2}}, \tag{3.15}$$

provided that ϵ is sufficiently small.

Furthermore, by making use of (3.15), from (3.8)-(3.12), we get

$$\begin{aligned}\tilde{W}_1(T_0) &\leq 2C_8\epsilon,\ \tilde{V}_1(T_0)\leq 2C_9\left[(\frac{1}{M+1}+2C_4a_1^2)\epsilon+(2d_5)^{\alpha+1}(2d_3)\epsilon^{\alpha+2}T_0\right],\\ W_1(T_0) &\leq 2C_{12}\epsilon,\ V_1(T_0)\leq 2C_{11}\left[(\frac{1}{M+1}+2C_4a_1^2)\epsilon+(2d_5)^{\alpha+1}(2d_3)\epsilon^{\alpha+2}T_0\right],\\ U_\infty(T_0),V_\infty(T_0) &\leq 2C_{12}C_{14}\epsilon.\end{aligned}$$

If we take

$$d_3 \geq 2C_{12},\ d_4 \geq 2C_8,\ d_5 \geq 2C_{12}C_{14}$$

and

$$d_1 \geq 2\max\{C_9, C_{11}\}\left[\frac{1}{M+1} + 2C_4 a_1^2\right],\ d_2 \geq 2\max\{C_9, C_{11}\}(2d_5)^{\alpha+1}(2d_3),$$

then we obtain (3.14). Thus, if we take $K_3 = \max\{d_1, d_2, d_3, d_4, d_5\}$, we obtain (1.19).

It follow from (2.7) that

$$w_i(t, x_i(t,y)) = w_i(0,y) + \int_{\tilde{C}_i} G_i(t, x_i(t,y))dt,$$

where $\tilde{C}_i$ is the $i-$th characteristic defined by

$$\frac{dx_i(t,y)}{dt} = \lambda_i(u(t, x_i(t,y))),\ \ t = 0 : x_i(0,y) = y.$$

By (2.7) and (2.18)-(2.24), we have

$$\begin{aligned} G_i(t,x) &= \sum_{j,k=1}^{n} \gamma_{ijk}(u) w_j w_k + \sum_{j,k=1}^{n} \sigma_{ijk}(u) w_j b_k(u) + (b_i(u))_x \\ &= \sum_{j\neq k} \gamma_{ijk}(u) w_j w_k + (\gamma_{iii}(u) - \gamma_{iii}(u_i e_i)) w_i^2 + \gamma_{iii}(u_i e_i) w_i^2 \\ &\quad + \sum_{j,k=1}^{n} \sigma_{ijk}(u) w_j \sum_{p\neq q} b_{kpq}(u) u_p u_q + \sum_{k=1}^{n} (\tilde{b}_{ik}(u) - \tilde{b}_{ik}(u_k e_k)) w_k. \end{aligned}$$

On the other hand,

$$\gamma_{iii}(u_i e_i) = -\nabla\lambda_i(u_i e_i) r_i(u_i e_i) = -\frac{\partial \lambda_i}{\partial u_i}(u_i e_i).$$

Therefore, we get

$$|G_i(t,x)| \leq C_{15}\left[\sum_{j\neq k}(|w_j w_k| + |v_j w_k|) + \sum_{j\neq i} |v_j w_i^2| + |u_i|^\alpha |w_i|^2\right]. \tag{3.16}$$

Then we obtain

$$\begin{aligned} W_\infty(T) &\leq C_{16}\Big[W_\infty(0) + W_\infty(T)\tilde{W}_1(T) + V_\infty(T)\tilde{W}_1(T) + W_\infty(T)\tilde{V}_1(T) \\ &\qquad + W_\infty(T)^2\tilde{V}_1(T) + U_\infty(T)^\alpha W_\infty(T)^2 \cdot T\Big] \\ &\leq C_{17}\Big[\epsilon + W_\infty(T)\tilde{W}_1(T) + V_\infty(T)\tilde{W}_1(T) + W_\infty(T)\tilde{V}_1(T) \\ &\qquad + W_\infty(T)^2\tilde{V}_1(T) + U_\infty(T)^\alpha W_\infty(T)^2 \cdot T\Big]. \end{aligned} \tag{3.17}$$

Thus, in order to prove (1.21) it suffices to show that we can choose some constant d_6 in such a way that, for any fixed T_1 $(0 \leq T_1 \leq T)$ with $T_1\epsilon^{\alpha+1} \leq K_6$,

$$W_\infty(T_1) \leq 2d_6\epsilon, \tag{3.18}$$

we have

$$W_\infty(T_1) \leq d_6\epsilon. \tag{3.19}$$

Substituting (1.19) and (3.18) into (3.17), we have

$$W_\infty(T_1) \le 2C_{17}[1 + K_3^\alpha(2d_6)^2 K_6]\epsilon.$$

Hence, if $d_6 \ge 4C_{17},\ K_6 = \frac{1}{K_3^\alpha(2d_6)^2}$, then we have (3.19). Therefore (1.21) is proved.

It follows from (1.19) that $U_\infty(T) \le K_7\epsilon$ where T satisfies (1.20), provided that ϵ is sufficient small and $K_7 \ge K_3$. Then the hypothesis (3.2) is reasonable. This proves Theorem 1.1. □

4 Some important uniform estimates on classical solutions

On the domain where the classical solution $u = u(t,x)$ of the Cauchy problem (1.1) and (1.11) exists, we denote the i-th characteristic passing through the point $(0,y)$ by $x = \phi^{(i)}(t,y)$, which is defined by

$$\frac{\partial \phi^{(i)}(t,y)}{\partial t} = \lambda_i\left(u\left(t,\phi^{(i)}(t,y)\right)\right), \quad \phi^{(i)}(0,y) = y. \tag{4.1}$$

Let

$$z^{(i)}(t,y) = u\left(t,\phi^{(i)}(t,y)\right). \tag{4.2}$$

For the sake of simplicity, we omit the upper index (i) of $z^{(i)},\ \phi^{(i)}$ etc. in this section. Then from (1.1) we easily have

$$l_i(z)\partial_t z = b_i(z) \tag{4.3}$$

and

$$l_j(z)\partial_y z = \frac{b_j(z) - l_j(z)\partial_t z}{\lambda_j(z) - \lambda_i(z)}(\partial_y\phi), \quad \forall\ j \ne i. \tag{4.4}$$

Theorem 4.1 *Under the assumptions of Theorem 1.1, we know that* $(\phi,z) = (\phi(t,y), z(t,y))$ *is* C^1 *smooth with respect to* (t,y) *on the domain*

$$D(M_1) = \left\{(t,y)\mid 0 \le t < \min\left\{\tilde{T}(\epsilon),\ M_1\epsilon^{-(\alpha+1)}\right\},\ -\infty < y < \infty\right\},$$

provided that ϵ *is sufficiently small, where* M_1 *is any positive constant independent of* ϵ, t, y *and* $\tilde{T}(\epsilon)$ *is the lifespan of the* C^1 *classical solution* $u = u(t,x)$ *to the Cauchy problem (1.1) and (1.11). Moreover, we have* $\phi_{ty} \in C^0$ *and the following estimates hold in the domain* $D(M_1)$*:*

$$\begin{aligned} |\phi_t(t,y)| &\le C_{18},\ |\phi_y(t,y)| \le C_{18},\ |\phi_{ty}(t,y)| \le C_{18}\epsilon, \\ |z(t,y)| &\le C_{18}\epsilon,\ |z_t(t,y)| \le C_{18}\epsilon,\ |z_y(t,y)| \le C_{18}\epsilon. \end{aligned} \tag{4.5}$$

In addition, in the domain $D(M_1)$,

$$\bar{w}_j(t,y) \triangleq \frac{b_j(z) - l_j(z)\frac{\partial z}{\partial t}}{\lambda_j(z) - \lambda_i(z)} \in C^0,\ \ |\bar{w}_j(t,y)| \le C_{19}\epsilon,\ \ j \ne i. \tag{4.6}$$

Remark 4.1 *In the existence domain of the C^1 solution $u = u(t,x)$ to the Cauchy problem (1.1) and (1.11), i.e., in the domain $\left[0, \min\left\{\tilde{T}(\epsilon), M_1\epsilon^{-(\alpha+1)}\right\}\right) \times(-\infty,+\infty)$, from (1.1) and (4.4) we have, along the $i-th$ characteristic $x=\phi(t,y)$ passing the point $(0,y)$,*

$$u(t,\phi(t,y)) \equiv z(t,y)$$

and

$$\begin{aligned}\bar{w}_j(t,y) &= \frac{b_j(u) - l_j(u)\left(u_t + \lambda_i(u)u_x\right)}{\lambda_j(u)-\lambda_i(u)}(t,\phi(t,y))\\ &= \frac{b_j(u) - l_j(u)\left(-A(u)u_x + B(u) + \lambda_i(u)u_x\right)}{\lambda_j(u)-\lambda_i(u)}(t,\phi(t,y))\\ &= w_j(t,\phi(t,y)), \qquad \forall\, j\neq i,\end{aligned}$$

where $u=u(t,x)$ is the C^1 smooth solution to the Cauchy problem (1.1) and (1.11) and $w_j = l_j(u)u_x$ is defined by (2.1). It follows from (4.6) that

$$|w_j(t,\phi(t,y))| \le C_{19}\epsilon,\ j\neq i,\ \ \textit{if}\ t\in\left[0,\min\left\{\tilde{T}(\epsilon), M_1\epsilon^{-1}\right\}\right),\ \ y\in\mathbb{R}. \tag{4.7}$$

Proof. It follows from (4.1) that

$$\phi_{ty}(t,y) = \left(\lambda_i(u(t,\phi(t,y)))\right)_y = \sum_{j=1}^{n}\frac{\partial\lambda_i(u)}{\partial u_j}(u_j)_x\phi_y(t,y),\ \ \phi_y(0,y)=1.$$

Then, we get

$$\ln|\phi_y(t,y)| = \int_0^t\sum_{j=1}^{n}\frac{\partial\lambda_i(u)}{\partial u_j}(u_j)_x(t,\phi(t,y))$$

Before the blow-up time, i.e., the lifespan $\tilde{T}(\epsilon)$, we know that

$$\phi_y(t,y)>0,\ \ 0\le t<\tilde{T}(\epsilon). \tag{4.8}$$

The Cauchy problem (1.1) and (1.11) has a unique C^1 smooth solution $u=u(t,x)$ and the transformation $(t,y)\to(t,x):(t,x)=(t,\phi(t,y))$ is C^1 invertible before the lifespan $\tilde{T}(\epsilon)$. Therefore, $(\phi,z) = (\phi(t,y), z(t,y))$ is C^1 smooth when the time $0\le t<\tilde{T}(\epsilon)$. It is obvious that (4.6) can be deduced from (4.5). Thus, in order to prove Theorem 4.1, it suffices to prove (4.5) when $0\le t<\min\left\{\tilde{T}(\epsilon),\ M_1\epsilon^{-(\alpha+1)}\right\}$. To do so, it is sufficient to give uniform *a priori* estimates of C^1 norm of $z=z(t,y)$ and $\phi=\phi(t,y)$ in the domain $D(M_1)$.

We fix that

$$0<\tau_1=\min\left\{\epsilon^{\alpha+1}\tilde{T}(\epsilon),\ M_1\right\}\le M_1 \tag{4.9}$$

and introduce

$$k=\partial_y\phi,\ \ \tilde{w}_i = l_i(z)\partial_y z = w_i k. \tag{4.10}$$

Assume that

$$z(t,y)=\epsilon\sigma(\tau,y),$$

where we denote

$$\tau = \epsilon^{\alpha+1} t.$$

Introducing the supplemental invariants

$$\zeta_i = l_i\,(\epsilon\sigma)\,\partial_y \sigma, \quad \zeta_j = \epsilon^{\alpha+1} l_j\,(\epsilon\sigma)\,\partial_\tau \sigma \quad (j \neq i), \tag{4.11}$$

by (4.3)-(4.4) we have

$$\epsilon^{\alpha+1}\partial_\tau \sigma = \sum_{j\neq i} \zeta_j r_j(\epsilon\sigma) + \epsilon^{-1} b_i(\epsilon\sigma) r_i(\epsilon\sigma),\ \partial_y \sigma = \sum_{j\neq i} \frac{k(\epsilon^{-1} b_j(\epsilon\sigma) - \zeta_j)}{\lambda_j\,(\epsilon\sigma) - \lambda_i\,(\epsilon\sigma)} r_j\,(\epsilon\sigma) + \zeta_i r_i(\epsilon\sigma). \tag{4.12}$$

We denote $\tilde{\zeta} = (\zeta_1, \cdots, \widehat{\zeta_i}, \cdots, \zeta_n)^t$ and $\tilde{b} = (b_1, \cdots, \widehat{b_i}, \cdots, b_n)^t$ which do not include ζ_i and b_i respectively.

By (1.19) in Theorem 1.1, we have

$$|z(t,y)| = |\epsilon\sigma(t,y)| = |u(t,\phi(t,y))| \leq U_\infty(t) \leq K_3\epsilon, \ \text{ when } \ (t,y) \in D(M_1). \tag{4.13}$$

We now estimate k, ζ_i and $\tilde{\zeta}$. Denote

$$K(T) = \max_{0\leq t\leq T\leq \tau_1 \epsilon^{-(\alpha+1)}} \sup_{y\in\mathbb{R}} |k(t,y)|,$$

$$H^{(i)}(\overline{\tau}) = \max_{0\leq \tau\leq\overline{\tau}<\tau_1\leq M_1} \sup_{y\in\mathbb{R}} |\zeta_i(\tau,y)|,$$

$$\tilde{H}(\overline{\tau}) = \max_{0\leq \tau\leq\overline{\tau}<\tau_1\leq M_1} \sup_{y\in\mathbb{R}} |\tilde{\zeta}(\tau,y)|.$$

It is obvious that

$$K(0) \equiv 1, \quad H^{(i)}(0) = O(1), \quad \tilde{H}(0) = O(1).$$

It follows from (4.1) that

$$\partial_t k = \phi_{ty} = (\lambda_i(u))_y = \nabla\lambda_i(u) u_x k = \nabla\lambda_i(u) \left(\sum_{j=1}^n w_j r_j(u)\right) k. \tag{4.14}$$

On the other hand, by the Hadamard's formula we have

$$\begin{aligned}\nabla\lambda_i(u)r_i(u) &= (\nabla\lambda_i(u)r_i(u) - \nabla\lambda_i(u_ie_i)r_i(u_ie_i)) + \nabla\lambda_i(u_ie_i)r_i(u_ie_i)\\ &= \sum_{j\neq i}\left[\int_0^1 \frac{\partial(\nabla\lambda_i r_i)}{\partial u_j}(su_1,\cdots,su_{i-1},u_i,su_{i+1},\cdots,su_n)ds\right]u_j + \nabla\lambda_i(u_ie_i)r_i(u_ie_i).\end{aligned}$$

Noting (1.7), (1.15) and (1.19), we obtain

$$|\nabla\lambda_i(u)r_i(u)| \leq C_{20}\left(\sum_{j\neq i}|v_j| + \epsilon^\alpha\right).$$

It is obvious that

$$w_i(t,\phi(t,y))k(t,y) = \tilde{w}_i(t,\phi(t,y)) = \epsilon\zeta_i(\tau,y), \ \text{ where } \ \tau = \epsilon^{\alpha+1}t. \tag{4.15}$$

Then, it follows from Theorem 1.1 that

$$\begin{aligned}K(T) &\leq 1 + C_{21}\left[\tilde{W}_1(T)K(T) + \epsilon H^{(i)}(\epsilon^{\alpha+1}T)\left(\tilde{V}_1(T) + \epsilon^{-1}M_1\right)\right]\\ &\leq 1 + C_{22}\left[\epsilon K(T) + \epsilon^2 H^{(i)}(\epsilon^{\alpha+1}T) + M_1 H^{(i)}(\epsilon^{\alpha+1}T)\right].\end{aligned}$$

Therefore, we get

$$K(T) \leq 2 + C_{23} M_1 H^{(i)}(\epsilon^{\alpha+1} T). \tag{4.16}$$

From (2.7), we have

$$\begin{aligned}
\frac{d\tilde{w}_i}{d_i t} &= G_i(t,x)k + w_i \partial_t k(t,y) \\
&= G_i(t,x)k + w_i \nabla\lambda_i(u) \left(\sum_{j=1}^n w_j r_j(u) \right) k \\
&= \Bigg[\sum_{l \neq i} \left(\gamma_{iil}(u) + \gamma_{ili}(u) + \nabla\lambda_i(u) r_l(u) \right) w_l \\
&\quad + \sum_{l=1}^n \sum_{p \neq q} \sigma_{iil}(u) b_{lpq}(u) u_p u_q + (\tilde{b}_{ii}(u) - \tilde{b}_{ii}(u_i e_i)) \Bigg] \tilde{w}_i \\
&\quad + \sum_{j \neq l; j, l \neq i} \gamma_{ijl}(u) w_j w_l k + \sum_{j \neq i} \sum_{l=1}^n \sigma_{ijl}(u) b_l(u) w_j k + \sum_{l \neq i} (\tilde{b}_{il}(u) - \tilde{b}_{il}(u_l e_l)) w_l k \\
&\triangleq a(t, \phi(t,y)) \tilde{w}_i + b(t, \phi(t,y)) k(t,y).
\end{aligned}$$

Thus, we get

$$\begin{aligned}
\tilde{w}_i(t, \phi(t,y)) = \left[\int_0^t b(s, \phi(s,y)) k(t,y) \exp\left(- \int_0^s a(s', \phi(s',y)) ds' \right) ds + \tilde{w}_i(0, \phi(0,y)) \right] \\
\times \exp\left(\int_0^t a(s', \phi(s',y)) ds' \right).
\end{aligned}$$

From Remark 4.1, (4.12)-(4.13) and (2.22), we know that

$$|w_j(t, \phi(t,y))| \leq C_{24} |\partial_t z(t,y)| \leq C_{24} \epsilon^{\alpha+2} |\partial_\tau \sigma(\epsilon^{\alpha+1} t, y)| \leq C_{25} \epsilon (\tilde{H}(\epsilon^{\alpha+1} t) + \epsilon), \quad \forall\, j \neq i. \tag{4.17}$$

Thus, Theorem 1.1 implies that

$$\int_0^t |a(s, \phi(s,y))| ds \leq C_{26} \left[\tilde{W}_1(t) + V_\infty(t) \tilde{V}_1(t) + \tilde{V}_1(t) \right] \leq C_{27} \epsilon$$

and

$$\begin{aligned}
\int_0^t |b(s, \phi(s,y))| ds &\leq C_{28} \left[C_{25} \epsilon (\tilde{H}(\epsilon^{\alpha+1} t) + \epsilon) \tilde{W}_1(t) + V_\infty(t)^2 \tilde{W}_1(t) + \tilde{W}_1(t) V_\infty(t) \right] \\
&\leq C_{29} (\epsilon^2 + \epsilon^2 \tilde{H}(\epsilon^{\alpha+1} t)).
\end{aligned}$$

Therefore, noting (4.15) we obtain

$$H^{(i)}(\bar{\tau}) \leq C_{30} \left[H^{(i)}(0) + \epsilon (1 + \tilde{H}(\bar{\tau})) K(\epsilon^{-(\alpha+1)} \bar{\tau}) \right]. \tag{4.18}$$

(4.4) gives

$$l_j(\epsilon\sigma) \left[\epsilon^{\alpha+1} k \partial_\tau \sigma + (\lambda_j(\epsilon\sigma) - \lambda_i(\epsilon\sigma)) \partial_y \sigma \right] = \epsilon^{-1} b_j(\epsilon\sigma) k, \quad \forall\, j \neq i.$$

Differentiating it with respect to τ and then multiplying $\epsilon^{\alpha+1}$ yields

$$\begin{aligned}
&\epsilon \left(\nabla_u l_j(\epsilon\sigma) (\epsilon^{\alpha+1} \partial_\tau \sigma) \right)^t \left[k (\epsilon^{\alpha+1} \partial_\tau \sigma) + (\lambda_j(\epsilon\sigma) - \lambda_i(\epsilon\sigma)) \partial_y \sigma \right] \\
&+ l_j(\epsilon\sigma) \left[\epsilon^{\alpha+1} k \partial_\tau + (\lambda_j(\epsilon\sigma) - \lambda_i(\epsilon\sigma)) \partial_y \right] (\epsilon^{\alpha+1} \partial_\tau \sigma) \\
&+ l_j(\epsilon\sigma) \left[(\epsilon^{\alpha+1} \partial_\tau k)(\epsilon^{\alpha+1} \partial_\tau \sigma) + \epsilon \left((\nabla_u \lambda_j(\epsilon\sigma) - \nabla_u \lambda_i(\epsilon\sigma)) (\epsilon^{\alpha+1} \partial_\tau \sigma) \right)^t \partial_y \sigma \right] \\
= \;& \epsilon^\alpha \partial_\tau (b_j(\epsilon\sigma) k).
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
& \left[\epsilon^{\alpha+1}k\partial_\tau + (\lambda_j(\epsilon\sigma)-\lambda_i(\epsilon\sigma))\partial_y\right]\zeta_j \\
& -\epsilon\left[k\nabla_u l_j(\epsilon\sigma)(\epsilon^{\alpha+1}\partial_\tau\sigma) + (\lambda_j(\epsilon\sigma)-\lambda_i(\epsilon\sigma))(\nabla_u l_j(\epsilon\sigma)\partial_y\sigma)\right]^t(\epsilon^{\alpha+1}\partial_\tau\sigma) \\
& +\epsilon\left(\nabla_u l_j(\epsilon\sigma)(\epsilon^{\alpha+1}\partial_\tau\sigma)\right)^t\left[k(\epsilon^{\alpha+1}\partial_\tau\sigma) + (\lambda_j(\epsilon\sigma)-\lambda_i(\epsilon\sigma))\partial_y\sigma\right] \\
& +l_j(\epsilon\sigma)\left[(\epsilon^{\alpha+1}\partial_\tau k)(\epsilon^{\alpha+1}\partial_\tau\sigma) + \epsilon\left((\nabla_u\lambda_j(\epsilon\sigma)-\nabla_u\lambda_i(\epsilon\sigma))(\epsilon^{\alpha+1}\partial_\tau\sigma)\right)^t\partial_y\sigma\right] \\
= \ & \epsilon^\alpha\partial_\tau(b_j(\epsilon\sigma)k).
\end{aligned}$$

We can rewrite it in a simple form as follows

$$\begin{aligned}
& \left[\epsilon^{\alpha+1}k\partial_\tau + (\lambda_j(\epsilon\sigma)-\lambda_i(\epsilon\sigma))\partial_y\right]\zeta_j \\
& +\epsilon\left[(\epsilon^{\alpha+1}\partial_\tau\sigma)^t Q^{(1)}(\epsilon\sigma)(\epsilon^{\alpha+1}\partial_\tau\sigma)k + (\epsilon^{\alpha+1}\partial_\tau\sigma)^t Q^{(2)}(\epsilon\sigma)(\partial_y\sigma)\right] \\
= \ & \epsilon^\alpha\partial_\tau(b_j(\epsilon\sigma)k) - \epsilon^{\alpha+1}\partial_\tau k\zeta_j, \qquad (4.19)
\end{aligned}$$

here and hereafter $Q^{(p)}(\epsilon\sigma)$ $(p=1,2,\cdots,9)$ are matrix, column vectors or scalar quantities which are dependent of $\epsilon\sigma$ continuously.

On the other hand, it follows from (4.12) that

$$\begin{aligned}
& (\epsilon^{\alpha+1}\partial_\tau\sigma)^t Q^{(1)}(\epsilon\sigma)(\epsilon^{\alpha+1}\partial_\tau\sigma)k + (\epsilon^{\alpha+1}\partial_\tau\sigma)^t Q^{(2)}(\epsilon\sigma)(\partial_y\sigma) \\
= \ & \left(\sum_{j\neq i}\zeta_j r_j(\epsilon\sigma) + \epsilon^{-1}b_i r_i(\epsilon\sigma)\right)^t Q^{(1)}(\epsilon\sigma)\left(\sum_{l\neq i}\zeta_l r_l(\epsilon\sigma) + \epsilon^{-1}b_i r_i(\epsilon\sigma)\right)k \\
& + \left(\sum_{j\neq i}\zeta_j r_j(\epsilon\sigma) + \epsilon^{-1}b_i r_i(\epsilon\sigma)\right)^t Q^{(2)}(\epsilon\sigma) \\
& \times\left(\sum_{l\neq i}\frac{-\zeta_l}{\lambda_l(\epsilon\sigma)-\lambda_i(\epsilon\sigma)}kr_l(\epsilon\sigma) + \sum_{l\neq i}\frac{\epsilon^{-1}b_l}{\lambda_l(\epsilon\sigma)-\lambda_i(\epsilon\sigma)}kr_l(\epsilon\sigma) + \zeta_i r_i(\epsilon\sigma)\right) \\
= \ & k\left[\tilde{\zeta}^t Q^{(3)}(\epsilon\sigma)\tilde{\zeta} + \epsilon^{-1}Q^{(4)}(\epsilon\sigma)^t\tilde{\zeta}b_i + \epsilon^{-2}Q^{(5)}(\epsilon\sigma)b_i^2 + \epsilon^{-2}Q^{(6)}(\epsilon\sigma)^t\tilde{b}b_i + \epsilon^{-1}\tilde{\zeta}^t Q^{(7)}(\epsilon\sigma)\tilde{b}\right] \\
& +\epsilon^{-1}Q^{(8)}(\epsilon\sigma)b_i\zeta_i + Q^{(9)}(\epsilon\sigma)^t\tilde{\zeta}\zeta_i.
\end{aligned}$$

Therefore, noting (2.22) and (4.13) we have $|b_j(\epsilon\sigma)| \le C_{31}\epsilon^2$ $(\forall\ j=1,2,\cdots,n)$ and then from (4.16) and (4.18) we get, $\forall\ \tau\in[0,\bar{\tau}]\subset[0,\tau_1)$,

$$\begin{aligned}
& \left|(\epsilon^{\alpha+1}\partial_\tau\sigma)^t Q^{(1)}(\epsilon\sigma)(\epsilon^{\alpha+1}\partial_\tau\sigma)(\tau,y)k(\epsilon^{-(\alpha+1)}\bar{\tau},y) + (\epsilon^{\alpha+1}\partial_\tau\sigma)^t Q^{(2)}(\epsilon\sigma)(\partial_y\sigma)(\tau,y)\right| \\
\le \ & C_{32}\left[\left(\tilde{H}(\bar{\tau})^2 + \epsilon\tilde{H}(\bar{\tau}) + \epsilon^2\right)K(\epsilon^{-(\alpha+1)}\bar{\tau}) + \epsilon H^{(i)}(\bar{\tau}) + \tilde{H}(\bar{\tau})H^{(i)}(\bar{\tau})\right]. \qquad (4.20)
\end{aligned}$$

In the proof of (4.16), we have deduced that

$$\begin{aligned}
\int_0^{\bar{\tau}}|\partial_\tau k(t,y)|d\tau &= \int_0^{\epsilon^{-(\alpha+1)}\bar{\tau}}|\partial_t k(t,y)|dt \\
&\le C_{22}\left[\epsilon K(\epsilon^{-(\alpha+1)}\bar{\tau}) + \epsilon^2 H^{(i)}(\bar{\tau}) + M_1 H^{(i)}(\bar{\tau})\right] \\
&\le C_{33}\left(\epsilon + M_1 H^{(i)}(\bar{\tau})\right). \qquad (4.21)
\end{aligned}$$

On the other hand, we easily get

$$\left|\int_0^{\bar{\tau}}(b_j(\epsilon\sigma)k)_\tau d\tau\right| = \left|\int_0^{\epsilon^{-(\alpha+1)}\bar{\tau}}(b_j(\epsilon\sigma)k)_t dt\right| \le C_{34}\epsilon^2 K(\epsilon^{-(\alpha+1)}\bar{\tau}). \qquad (4.22)$$

(4.8) ensures that the family of $i-th$ characteristics do not collapse. Integrating (4.19) and making use of (4.20)-(4.22), we obtain

$$\begin{aligned}\tilde{H}(\bar{\tau}) \quad \le \quad & \tilde{H}(0)+C_{34}\epsilon^{\alpha+2}K(\epsilon^{-(\alpha+1)}\bar{\tau}) \\ & +C_{32}\epsilon\left[\left(\tilde{H}(\bar{\tau})^2+\epsilon\tilde{H}(\bar{\tau})+\epsilon^2\right)K(\epsilon^{-(\alpha+1)}\bar{\tau})+\epsilon H^{(i)}(\bar{\tau})+\tilde{H}(\bar{\tau})H^{(i)}(\bar{\tau})\right]M_1 \\ & +C_{33}\epsilon^{\alpha+1}\tilde{H}(\bar{\tau})\left(\epsilon+M_1H^{(i)}(\bar{\tau})\right). \end{aligned} \tag{4.23}$$

Therefore, we can deduce from (4.16), (4.18) and (4.23) that, $\forall\ T\in[0,\tau_1\epsilon^{-(\alpha+1)}),\ \ \bar{\tau}\in[0,\tau_1)$,

$$K(T)\le 2+2C_{23}C_{30}M_1H^{(i)}(0),\ \ H^{(i)}(\bar{\tau})\le 2C_{30}H^{(i)}(0),\ \ \tilde{H}(\bar{\tau})\le 2\tilde{H}(0), \tag{4.24}$$

provided that ϵ is sufficiently small.

Furthermore, noting (4.12)-(4.13) it follows from (4.24) that

$$|\sigma|\le C_{35},\quad |\partial_\tau\sigma|\le C_{35}\epsilon^{-(\alpha+1)},\quad |\partial_y\sigma|\le C_{35} \tag{4.25}$$

and

$$|z|\le C_{35}\epsilon,\quad |\partial_t z|\le C_{35}\epsilon,\quad |\partial_y z|\le C_{35}\epsilon. \tag{4.26}$$

Therefore, (4.6) and (4.7) hold. Then (4.14)-(4.15) and (4.24) imply that

$$|\phi_y|\le C_{36},\quad |\phi_{ty}|\le C_{36}\epsilon,\quad |\phi_t|=|\lambda_i(z)|\le C_{36}. \tag{4.27}$$

Therefore, the estimates in (4.5) can be deduced from (4.26)-(4.27). Thus, Theorem 4.1 is completely proved. □

5 Estimate of lifespan—Proof of Theorem 1.2

In order to prove Theorem 1.2, i.e., (1.26), as we already assume that u are the normalized coordinates and noting (1.24)-(1.25), it suffices to prove

$$\lim_{\epsilon\longrightarrow 0}(\epsilon^{\alpha+1}\tilde{T}(\epsilon))=M_0, \tag{5.1}$$

where

$$\begin{aligned} M_0 \quad = \quad & \left\{\max_{i\in J_1}\sup_{x\in\mathbb{R}}\left\{-\frac{1}{\alpha!}\frac{d^{\alpha+1}\lambda_i}{ds^{\alpha+1}}(0)\psi_i(x)^\alpha\psi_i'(x)\right\}\right\}^{-1} \\ = \quad & \left\{\max_{i\in\{1,2,\cdots,n\}}\sup_{x\in\mathbb{R}}\left\{-\frac{1}{\alpha!}\frac{d^{\alpha+1}\lambda_i}{ds^{\alpha+1}}(0)\psi_i(x)^\alpha\psi_i'(x)\right\}\right\}^{-1}. \end{aligned} \tag{5.2}$$

In order to prove (5.1), similar to L. Hörmander [5] and Kong [7]-[8], it suffices to show that

(I). for any fixed $M^*\ge M_0$, we have $\tilde{T}(\epsilon)\le M^*\epsilon^{-(\alpha+1)}$, namely $\overline{\lim_{\epsilon\longrightarrow 0}}(\epsilon^{\alpha+1}\tilde{T}(\epsilon))\le M_0$

and

(II). for any fixed $M_*\le M_0-\epsilon^{\frac{1}{2}}$, we have $\tilde{T}(\epsilon)\ge M_*\epsilon^{-(\alpha+1)}$, namely $\underline{\lim}_{\epsilon\longrightarrow 0}(\epsilon^{\alpha+1}\tilde{T}(\epsilon))\ge M_0$.

Let

$$T_*=K_*\epsilon^{-(\alpha+1)},\ \ T^*=M^*\epsilon^{-(\alpha+1)}, \tag{5.3}$$

where K_* is the positive constant K_6 given in (1.22) and M^* is an arbitrary fixed constant satisfying that $M^* \geq M_0$. It is easy to see that

$$0 < T_* < T^* < K_4 \epsilon^{-(\alpha+\frac{3}{2})}. \tag{5.4}$$

Let $x = x_i(t, y)$ $(i = 1, 2, \cdots, n)$ be the $i-th$ characteristic passing through an arbitrary given point $(0, y)$. On any given existence domain $0 \leq t \leq T$ $(T \leq T^*)$ of the C^1 solution $u = u(t, x)$, we consider (2.7) along the $i-th$ characteristic $x = x_i(t, y)$. We can rewrite (2.7) as

$$\frac{dw_i}{d_i t} = a_0(t; i, y) w_i^2 + a_1(t; i, y) w_i + a_2(t; i, y), \tag{5.5}$$

where

$$\begin{aligned} a_0(t; i, y) &= \gamma_{iii}(u), & (5.6)\\ a_1(t; i, y) &= \sum_{j \neq i} (\gamma_{iij}(u) + \gamma_{iji}(u)) w_j + \sum_{j=1}^{n} \sigma_{iji}(u) b_j(u) + (\tilde{b}_{ii}(u) - \tilde{b}_{ii}(u_i e_i)), & (5.7)\\ a_2(t; i, y) &= \sum_{j \neq k; j, k \neq i} \gamma_{ijk}(u) w_j w_k + \sum_{j=1}^{n} \sum_{k \neq i} \sigma_{ijk}(u) b_j(u) w_k + \sum_{k \neq i} \tilde{b}_{ik}(u) w_k, & (5.8) \end{aligned}$$

in which $u = u(t, x_i(t, y))$ and $w_j = w_j(t, x_i(t, y))$ $(j = 1, 2, \cdots, n)$.

Lemma 5.1 *On any given domain $0 \leq t \leq T (\leq T^*)$ of the C^1 solution $u = u(t, x)$, there exist positive constants K_8 independent of ϵ, y and T such that the following estimates hold:*

$$\begin{aligned} \int_0^T |a_1(t; i, y)| dt &\leq K_8 \epsilon, & (5.9)\\ \int_0^T |a_2(t; i, y)| dt &\leq K_8 \epsilon^2, & (5.10)\\ K(i, y; 0, T) &\triangleq \int_0^T |a_2(t; i, y)| dt \cdot \exp\left(\int_0^T |a_1(t; i, y)| dt \right) \leq K_8 \epsilon^2. & (5.11) \end{aligned}$$

Proof. It follows from (2.18)-(2.24) and Theorem 1.1 that

$$\int_0^T |a_1(t; i, y)| dt \leq C_{37} (\tilde{W}_1(T) + V_\infty(T) \tilde{V}_1(T) + \tilde{V}_1(T)) \leq C_{38} \epsilon. \tag{5.12}$$

In Theorem 4.1, we take $M_1 = M^* + 1$. Noting (4.7) and (1.19), we easily see that

$$\begin{aligned} \sum_{j \neq k; j, k \neq i} \int_0^T |w_j w_k|(s, x_i(s, y)) ds &\leq C_{19} (n-2) \epsilon \sum_{k \neq i} \int_0^T |w_k|(s, x_i(s, y)) ds \\ &\leq C_{39} (n-2) \epsilon \sum_{k \neq i} \tilde{W}_1(T) \\ &\leq C_{40} \epsilon^2. \end{aligned}$$

Therefore, noting (2.22), (2.24) and Theorem 1.1, we get

$$\begin{aligned}\int_0^T |a_2(t;i,y)|dt &= \int_0^T \Big| \sum_{j\neq k;j,k\neq i} \gamma_{ijk}(u)w_jw_k + \sum_{j=1}^n \sum_{k\neq i} \sigma_{ijk}(u) \sum_{p\neq q} b_{jpq}(u)u_pu_qw_k \\ &\quad + \sum_{k\neq i}(\tilde{b}_{ik}(u) - \tilde{b}_{ik}(u_ke_k))w_k \Big| dt \\ &\leq C_{41}\left[\sum_{j\neq k;j,k\neq i}\int_0^T |w_jw_k|(s,x_i(s,y))ds + U_\infty(T)^2\tilde{W}_1(T) + U_\infty(T)\tilde{W}_1(T)\right] \\ &\leq C_{42}\epsilon^2. \qquad (5.13)\end{aligned}$$

Then (5.9)-(5.11) can be easily deduced from (5.12)-(5.13). □

Similar to Lemma 1.4.1 in L. Hörmander [5], we have

Lemma 5.2 *Let $z = z(t)$ be a solution in $[0,T]$ of the Riccadi's differential equation:*

$$\frac{dz}{dt} = a_0(t)z^2 + a_1(t)z + a_2(t),$$

where $a_j(t)$ $(j = 0,1,2)$ are continuous and $T > 0$ is a given real number. Let

$$K = \int_0^T |a_2(t)|dt \cdot \exp\left(\int_0^T |a_1(t)|dt\right).$$

If $z(0) > K$, then it follows that

$$\int_0^T |a_0(t)|dt \cdot \exp\left(-\int_0^T |a_1(t)|dt\right) < (z(0) - K)^{-1}.$$

Remark 5.1 *L. Hörmander assumed that $a_0(t) \geq 0$ in Lemma 1.4.1 (see page 230 in [5]). In Lemma 5.2, we do not assume this. It is easy to find that we can prove Lemma 5.2 similar to Lemma 1.4.1 in L. Hörmander [5].*

Next we give the estimate of the lifespan of classical solution to the Cauchy problem (1.1) and (1.11) under the assumptions of Theorem 1.2.

(I) Upper bound of the lifespan— Estimate on $\overline{\lim}_{\epsilon\longrightarrow 0^+}(\epsilon^{\alpha+1}\tilde{T}(\epsilon)) \leq M_0$

It follows from (2.6), (2.18)-(2.24) and (1.19) that, along the $i-th$ characteristic $x = x_i(t,y)$,

$$\begin{aligned}|v_i(i;t,y) - v_i(i;0,y)| &\leq \int_0^t |F_i(s,x_i(s,y))|ds \\ &\leq \int_0^t \Big|\sum_{k\neq i}\sum_{j=1}^n \beta_{ijk}(u)v_jw_k + \sum_{j,k}\nu_{ijk}(u)v_j\sum_{p\neq q} b_{kpq}(u)u_pu_q \\ &\quad + \sum_{p\neq q} b_{ipq}(u)u_pu_q\Big|(s,x_i(s,y))ds \\ &\leq C_{43}\left[V_\infty(t)\tilde{W}_1(t) + V_\infty(t)^2\tilde{V}_1(t) + V_\infty(t)\tilde{V}_1(t)\right] \\ &\leq C_{44}\epsilon^2.\end{aligned}$$

Then, as u are the normalized coordinates and $l_i(0) = e_i$, from (1.19) we easily get, along the $i-th$ characteristic $x = x_i(t, y)$,

$$|u_i(i;t,y) - u_i(i;0,y)| = |u_i(i;t,y) - f_i(\epsilon, y)| \le C_{45}\epsilon^2.$$

Using Hadamard's formula and noting (1.12)-(1.13), from (5.6) we get, along the $i-th$ characteristic $x = x_i(t, y)$,

$$\begin{aligned} a_0(t;i,y) &= \gamma_{iii}(u) = \gamma_{iii}(u_ie_i) + (\gamma_{iii}(u) - \gamma_{iii}(u_ie_i)) \\ &= \gamma_{iii}(u_ie_i) + \sum_{j\neq i}\left[\int_0^1 \frac{\partial\gamma_{iii}}{\partial u_j}(su_1,\cdots,su_{i-1},u_i,su_{i+1},\cdots,su_n)ds\right]u_j \\ &= -\frac{1}{\alpha!}\frac{\partial^{1+\alpha}\lambda_i}{\partial u_i^{1+\alpha}}(0)(u_i)^\alpha + O(\epsilon^{1+\alpha}) \\ &\quad + \sum_{j\neq i}\left[\int_0^1 \frac{\partial\gamma_{iii}}{\partial u_j}(su_1,\cdots,su_{i-1},u_i,su_{i+1},\cdots,su_n)ds\right]u_j \\ &= -\frac{1}{\alpha!}\frac{\partial^{1+\alpha}\lambda_i}{\partial u_i^{1+\alpha}}(0)(\epsilon\psi_i(y))^\alpha + \alpha O(\epsilon^{\alpha+r}) + O(\epsilon^{1+\alpha}) \\ &\quad + \sum_{j\neq i}\left[\int_0^1 \frac{\partial\gamma_{iii}}{\partial u_j}(su_1,\cdots,su_{i-1},u_i,su_{i+1},\cdots,su_n)ds\right]u_j \end{aligned} \tag{5.14}$$

Noting that the initial data satisfies (1.25), we observe that there exist an index $i_0 \in J_1$ and a point $x_0 \in \mathbb{R}$ such that

$$M_0 = \left\{-\frac{1}{\alpha!}\frac{\partial^{\alpha+1}\lambda_{i_0}}{\partial u_{i_0}^{\alpha+1}}(0)\psi_{i_0}(x_0)^\alpha\psi'_{i_0}(x_0)\right\}^{-1}. \tag{5.15}$$

Noting (2.10) and (1.15), we have

$$\frac{\partial^l\gamma_{i_0i_0i_0}}{\partial u_{i_0}^l}(0) = 0 \ \ (l = 0, 1, \cdots, \alpha-1) \ \text{ but } \ \frac{\partial^\alpha\gamma_{i_0i_0i_0}}{\partial u_{i_0}^\alpha}(0) \neq 0.$$

Then (5.15) becomes

$$M_0 = \left\{\frac{1}{\alpha!}\frac{\partial^\alpha\gamma_{i_0i_0i_0}}{\partial u_{i_0}^\alpha}(0)\psi_{i_0}(x_0)^\alpha\psi'_{i_0}(x_0)\right\}^{-1} \triangleq (b\psi'_{i_0}(x_0))^{-1}. \tag{5.16}$$

Without loss of generality, we may suppose that

$$b > 0 \ \ and \ \ \psi'_{i_0}(x_0) > 0. \tag{5.17}$$

Otherwise, changing the sign of u, we can draw the same conclusion.

Noting (1.12)-(1.13), (1.25) and (5.11), we get immediately

$$\begin{aligned} w_{i_0}(0,x_0) &= l_{i_0}(f(\epsilon,x_0))\frac{\partial f}{\partial x}(\epsilon,x_0) \\ &= [l_{i_0}(0) + O(\epsilon)] \times \left[\frac{\partial f}{\partial x}(0,x_0) + \frac{\partial^2 f}{\partial\epsilon\partial x}(0,x_0)\epsilon + O(\epsilon^{1+r})\right] \\ &= \epsilon\psi'_{i_0}(x_0) + O(\epsilon^{1+r}) > K_8\epsilon^2 \ge K(i_0, x_0; 0, T). \end{aligned}$$

Therefore, we immediately observe that Lemma 5.2 (revised version of Lemma 1.4.1 in L. Hörmander [5]) can be applied to the initial value problem for (5.5) with the following initial condition

$$t = 0: \quad w_{i_0} = w_{i_0}(0,x_0) = \epsilon\psi'_{i_0}(x_0) + O(\epsilon^{1+r}) \tag{5.18}$$

and then we obtain

$$\int_0^T |a_0(t;i_0,x_0)|dt \cdot \exp\left(-\int_0^T |a_1(t;i_0,x_0)|dt\right) < (w_{i_0}(0,x_0) - K(i_0,x_0;0,T))^{-1},$$

namely,

$$\exp\left(-\int_0^T |a_1(t;i_0,x_0)|dt\right) \times \int_0^T |a_0(t;i_0,x_0)|\,(w_{i_0}(0,x_0) - K(i_0,x_0;0,T))\,dt < 1. \tag{5.19}$$

Substituting (5.14) into (5.19) and noting (1.19) and the fact that $T \le T^* = M^*\epsilon^{-(1+\alpha)}$, we obtain

$$\overline{\lim_{\epsilon \longrightarrow 0}} \left\{\epsilon^{\alpha+1}T \cdot \frac{1}{\alpha!}\frac{\partial^\alpha \gamma_{i_0i_0i_0}}{\partial u_{i_0}^\alpha}(0)\psi_{i_0}(x_0)^\alpha \psi'_{i_0}(x_0)\right\} \le 1. \tag{5.20}$$

Noting (5.16), from (5.20) we get immediately

$$\overline{\lim_{\epsilon \longrightarrow 0}} (\epsilon^{\alpha+1}\tilde{T}(\epsilon)) \le M_0. \tag{5.21}$$

(5.21) gives an upper bound of the lifespan $\tilde{T}(\epsilon)$.

(II) Lower bound of the lifespan— Estimate on $\underline{\lim}_{\epsilon \longrightarrow 0^+} (\epsilon^{\alpha+1}\tilde{T}(\epsilon)) \ge M_0$

To do so, it suffices to prove that, for any fixed M_* satisfying that

$$0 < M_* < M_0 - \epsilon^{\frac{1}{2}r}, \tag{5.22}$$

we have

$$\tilde{T}(\epsilon) \ge M_*\epsilon^{-(\alpha+1)}, \tag{5.23}$$

provided that $\epsilon > 0$ is small enough. Hence, we only need to establish a uniform *a priori* estimate on C^1 norm of the C^1 solution $u = u(t,x)$ on any given existence domain $0 \le t \le T \le M_*\epsilon^{-(\alpha+1)}$. The uniform *a priori* estimate on C^0 norm of $u = u(t,x)$ has been established in Theorem 1.1. It remain to establish a uniform *a priori* estimate on C^0 norm of the first derivatives of $u = u(t,x)$, namely a uniform *a priori* estimate on C^0 norm of $w = (w_1(t,x), w_2(t,x), \cdots, w_n(t,x))^T$.

In order to estimate $w_i = w_i(t,x)$ on the existence domain $0 \le t \le T$ (where T satisfies $T \le M_*\epsilon^{-(\alpha+1)}$) of the C^1 solution $u = u(t,x)$, we still consider (5.5) along the $i-th$ characteristic $x = x_i(t,y)$ passing through an arbitrary fixed point $(0,y)$. Without loss of generality, we may suppose that

$$\psi'_i(y) \ge 0. \tag{5.24}$$

Otherwise, changing the sign of u, we can draw the same conclusion.

Let

$$a_0^+(t;i,y) = \max\{a_0(t;i,y), 0\}.$$

Noting the fact that $T \le M_* \epsilon^{-(\alpha+1)}$ and using Theorem 1.1, (5.14) and (5.18), we obtain

$$\begin{aligned} & w_i(0,y)\int_0^T a_0^+(t;i,y)dt \\ \le\ & \left(\epsilon\psi_i'(y)+O(\epsilon^{1+r})\right)\Big\{\max\Big\{-\frac{1}{\alpha!}\frac{\partial^{1+\alpha}\lambda_i}{\partial u_i^{1+\alpha}}(0)(\epsilon\psi_i(y))^\alpha,0\Big\}T \\ & +C_{46}\left(\alpha\epsilon^{\alpha+r}T+\epsilon^{1+\alpha}T+\tilde{V}_1(T)\right)\Big\} \\ \le\ & \max\Big\{-\frac{1}{\alpha!}\frac{\partial^{1+\alpha}\lambda_i}{\partial u_i^{1+\alpha}}(0)(\psi_i(y))^\alpha,0\Big\}\psi_i'(y)M_*+C_{47}(\epsilon^r+\epsilon) \\ \le\ & M_0^{-1}M_*+C_{48}\epsilon^r = M_0^{-1}(M_0-\epsilon^{\frac{1}{2}r})+C_{48}\epsilon^r < 1, \end{aligned} \tag{5.25}$$

provided that $\epsilon > 0$ is small enough. On the other hand, noting (5.14) and Theorem 1.1, we get immediately

$$\int_0^T |a_0(t;i,y)|dt \le C_{49}(\epsilon^\alpha T+\alpha\epsilon^{\alpha+r}T+\epsilon^{\alpha+1}T+\epsilon) \le C_{50}M_*\epsilon^{-1} \le C_{51}\epsilon^{-1}. \tag{5.26}$$

Then, noting (5.25)-(5.26) and Lemma 5.1, we obtain

$$\int_0^T a_0^+(t;i,y)dt \times \exp\left(\int_0^T |a_1(t;i,y)|dt\right) < (w_i(0,y)+K(i,y;0,T))^{-1} \tag{5.27}$$

and

$$\int_0^T |a_0(t;i,y)|dt \times \exp\left(\int_0^T |a_1(t;i,y)|dt\right) < (K(i,y;0,T))^{-1}, \tag{5.28}$$

where $T \le M_*\epsilon^{-(\alpha+1)}$.

Noting (5.24) and (5.27)-(5.28), we observe that Lemma 1.4.2 in L. Hörmander [5] can be applied to the initial value problem for equation (5.5) with the following initial condition

$$t = 0 : w_i = w_i(0,y).$$

Then we have

$$\begin{aligned} (w_i(T,x_i(T,y)))^{-1} \ \ge\ & (w_i(0,y)+K(i,y;0,T))^{-1} - \int_0^T a_0^+(t;i,y)dt \\ & \times\exp\left(\int_0^T |a_1(t;i,y)|dt\right), \text{ if } w_i(T,x_i(T,y)) > 0 \end{aligned} \tag{5.29}$$

and

$$\begin{aligned} |w_i(T,x_i(T,y))|^{-1} \ \ge\ & (K(i,y;0,T))^{-1} - \int_0^T |a_0(t;i,y)|dt \\ & \times\exp\left(\int_0^T |a_1(t;i,y)|dt\right), \text{ if } w_i(T,x_i(T,y)) < 0. \end{aligned} \tag{5.30}$$

Noting (5.25)-(5.26) and Lemma 5.1, from (5.29)-(5.30) we get respectively

$$(w_i(T,x_i(T,y)))^{-1} \ge \frac{1}{2}\left(1-\frac{M_*}{M_0}\right)(w_i(0,y)+K(i,y;0,T))^{-1}, \text{ if } w_i(T,x_i(T,y)) > 0 \tag{5.31}$$

and

$$|w_i(T,x_i(T,y))|^{-1} \ge \frac{1}{2}(K(i,y;0,T))^{-1}, \text{ if } w_i(T,x_i(T,y)) < 0. \tag{5.32}$$

Therefore, we have

$$w_i(T, x_i(T, y)) \leq \frac{2}{1 - \frac{M_*}{M_0}}(w_i(0, y) + K(i, y; 0, T)) \leq C_{52}\epsilon^{1-\frac{1}{2}r}, \text{ if } w_i(T, x_i(T, y)) > 0 \quad (5.33)$$

and

$$|w_i(T, x_i(T, y))| \leq 2K(i, y; 0, T) \leq 2K_8\epsilon^2, \text{ if } w_i(T, x_i(T, y)) < 0. \quad (5.34)$$

It follows from (5.33)-(5.34) that

$$|w_i(T, x_i(T, y))| \leq C_{53}\epsilon^{1-\frac{1}{2}r}. \quad (5.35)$$

For each $i \in \{1, 2, \cdots, n\}$ and any $t \in [0, T]$, we can prove similarly that $w_i(t, x_i(t, y))$ satisfies the same estimate. Noting that $(0, y)$ is arbitrary, we have

$$||w(t, x)||_{C^0[0,T]\times\mathbb{R}} \leq C_{54}\epsilon^{1-\frac{1}{2}r},$$

where $T \leq M_*\epsilon^{-(\alpha+1)}$. Hence, (5.23) holds and then

$$\underline{\lim}_{\epsilon \longrightarrow 0^+} (\epsilon^{\alpha+1}\tilde{T}(\epsilon)) \geq M_0. \quad (5.36)$$

The combination of (5.21) and (5.36) gives (1.26). Thus, Theorem 1.2 is proved completely. □

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