Arithmetic
Related to Calabi-Yau

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A Calabi-Yau manifold is a Kähler manifold with trivial canonical line bundle. However, we shall focus on projective manifolds.

Examples: Elliptic curves, Abelian varieties, $K3$, all have very rich arithmetic properties.

String theory predicts that 3-dimensional Calabi-Yau is the hidden space of space-time. These manifolds exhibit a mysterious mirror symmetry and string duality that we shall discuss later.

We will discuss the integrality and modularity properties of the enumerative functions that appeared in such manifolds.

Elliptic curves are one dimensional Calabi-Yau which number theorists are very familiar with.

We will start from two dimension: counting curves and points inside Calabi-Yau manifolds.

Two dimension

Counting curves with nodes on an algebraic surface was observed by Yau-Zaslow to be related to quasi-modular forms. Note that the counting that we used is different from the instanton calculation in Calabi-Yau manifolds.

Notation: Let $\tau \in \mathbf{H}$ the complex upper half plane, and $q = e^{2\pi i\tau}$. Recall that a modular form of weight k for $SL(2, \mathbf{Z})$ is a holomorphic function f on $\mathbf{H}$ satisfying

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau+d)^k f(\tau), \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{Z})$$

and having a Fourier series $f(\tau) = \sum_{n=0}^{\infty} a_n q^n$.

Writing $\sigma_k(n) := \sum_{d|n} d^k$, the Eisenstein series

$$G_k(q) = -\frac{B_k}{2k} + \sum_{n>0} \sigma_{k-1}(n) q^n,$$

are modular forms of weight $k/2$. Here B_k is the k-th Bernoulli number, while $G_2(\tau)$ is a quasimodular form, with $\bar{\tau}$ appeared in the modular transformation.

Let

$$\Delta(q) = q \prod_{k>0} (1 - q^k)^{24} = \eta(q)^{24}$$

where $\eta(q)$ is the Dedekind η function.

Let M be an algebraic surface and C be a holomorphic curve representing a primitive homology class. Let $n_g(r)$ be the number of curves of geometric genus g with r nodes passing through g generic points in the linear system $[C]$.

Denote differential operator

$$D = \frac{1}{2\pi i}\frac{d}{d\tau} = q\frac{d}{d\,q}.$$

The ring of quasimodular forms is closed under differentiation.

For $g = 0$ Yau-Zaslow obtained the following formula for $K3$ surfaces:

$$\sum_{r=0}^{\infty} n_g(r) q^r = \Big(DG_2\Big)^g \cdot q/\Delta(q).$$

In our proof, we assume that rational curves in a generic K3 surface have normal crossing (The assumption was proved by X. Chen later). The counting of rational curves on K3 was a well-known question.

Prior to the work of Yau-Zaslow, the first 6 numbers were obtained by Vainsencher without knowing that it come from $q/\Delta(q)$. Yau-Zaslow formula comes from understanding the moduli space of (Σ_g, L) where Σ_g is an algebraic curve of genus g and L is flat line bundle over Σ_g. The Euler number of the last moduli space can be computed by the Euler number of symmetric products of K3-surfaces.

Goettsche generalized this theorem of Yau-Zaslow to a conjecture over arbitrary algebraic surfaces.

Yau-Zaslow-Goettsche Conjecture:

Generating series of counting curve can be identified with

$$\frac{(DG_2(q)/q)^{\chi(L)}\, B_1(q)^{K_S^2}\, B_2(q)^{L\cdot K_M}}{(\Delta(q)\, D^2G_2(q)/q^2)^{\frac{\chi(\mathcal{O}_S)}{2}}}.$$

where B_1 *and* B_2 *are the two power series derived by Göttsche starting as*

$$B_1(q) = 1 - q - 5q^2 + 30q^3 - 345q^4 + 2961q^5 + \cdots\ ,$$
$$B_2(q) = 1 + 5q + 2q^2 + 35q^3 - 140q^4 + 986q^5 + \cdots\ .$$

Here $\chi(L)$, $\chi(\mathcal{O}_S)$ are the Riemann-Roch numbers and K_S^2 and LK_S are the intersection numbers of the canonical line bundle and the the line bundle L.

It is not known how B_1 and B_2 look like exactly.

Remarks:

1. Bryan-Leung proved the Yau-Zaslow conjecture for $K3$ when the cohomology class represented by the curves is primitive. The general Goettsche-Yau-Zaslow conjecture was established by Ai-Ko Liu by a rather complicated arguments.

2. Ai-Ko Liu later also demonstrated the Harvey-Moore conjecture which claimed that similar formula holds for certain family of Calabi-Yau manifolds with $K3$ fibration. They are also given by modular forms.

3. Many mirror symmetry computations for $K3$ or elliptic fibered Calabi-Yau manifolds are given by modular forms and quasimodular forms. (Klemm, Marino, Hasono, ...)

The mirror manifold of an elliptic curve or a K3 surface is another elliptic curve or K3. This may explain the modularity appeared in the above counting functions. Depending on the lattice $H^{1,1} \cap H^2(\mathbb{Z})$, the number of manifolds mirror to a given K3 surface is related to the class number.

Theorem (Hosono, Lian, Oguiso, Yau). *Suppose the Picard number of a K3 surface S is two and the determinant of its Neron-Severi lattice is $-p$ where p is a prime number. Then the number of the mirror K3 surface of S is given by $\frac{h(p)+1}{2}$ where $h(p)$ is the class number of $\mathbb{Q}(\sqrt{p})$.*

Three dimension.

In dimension higher than two, the counting of curves cannot be expressed by quasi-automorphic forms alone. Some "quantum" perturbation is required.

We start with discussions about the idea of mirror symmetry which relates deformation of complex structures of one CY manifold to counting of curves in its mirror.

B-**model:** Deformation of complex structures via calculation of periods:

- X, a CY threefold
- Ω, a holomorphic three form on X
- Fix a symplectic base α_i, β^i of $H_3(X, \mathbf{Z})$, $i = 0, 1, ..., m = h^{2,1}(X)$.
- Periods of X:

$$\xi_i = \int_{\alpha_i} \Omega, \quad \eta^i = \int_{\beta^i} \Omega$$

- The ξ_i depends on the complex structure of X holomorphically and form a set of projective coordinates on the moduli space of X.

  They satisfy the Picard-Fuchs equations which can be solved by the Forbenius method, where canonical expansions can be found in the large radius limit.

- By local Torelli theorem for Calabi-Yau manifold, there exists a homogeneous function $G(\xi)$ of degree 2 such that

$$\eta^i = \frac{\partial G}{\partial \xi_i}, \quad G(\lambda\xi) = \lambda^2 G(\xi)$$

- The degree zero function

$$F_0(t) = G/\xi_0^2, \quad t_i = \xi_i/\xi_0, \; i \geq 1$$

is the (IIB) *prepotential* of X. The t_i are called *flat coordinates.*

- Making use of the relation $\int \Omega \wedge \Omega = 0$, one can derive

$$(\xi_0, \cdots, \xi_m, \eta^0, \cdots, \eta^m)$$
$$= \xi_0(1, t_1, .., t_m, \frac{\partial F_0}{\partial t_1}, .., \frac{\partial F_0}{\partial t_m}, 2F_0 - t_i \frac{\partial F_0}{\partial t_i})$$

Hence F_0 can be determined by the periods.

The prepotential F_0 determines the B-model of the Calabi-Yau manifold.

Mirror Symmetry: The B-model prepotential F_0 is related to the A-model prepotential of the mirror of the CY manifold, which counts rational curves of the mirror manifold.

The quintic manifold was the first example to demonstrate the power of mirror symmetry through the striking work of Greene-Plesser and Candelas et al.

- Consider quintic hypersurfaces X'_z

$$a_1x_1^5 + \cdots + a_5x_5^5 + a_0x_1\cdots x_5 = 0$$

where $z = \frac{a_1\cdots a_5}{a_0^5}$. It has automorphisms $\Gamma = \mathbf{Z_5}^4$. X'_z/Γ has a crepant resolution X_z.

- $X = X_z$ is a mirror of the quintic.
This CY threefold has $h^{2,1} = 1$ and $h^{1,1} = 101$.
(A quintic threefold Y has $h^{2,1} = 101$ and $h^{1,1} = 1$.)

Quintic Mirror Conjecture (Candelas et al): *For mirror quintics X, the prepotential defined previously is given by*

$$F_0(t) = 5t^3 + \sum_{d \geq 1} K_d e^{2\pi i d t},$$
$$K_d = deg\ [\bar{M}_{0,0}(Y,d)]_{vir} \in \mathbf{Q}.$$

Here $\bar{M}_{0,0}(Y,d)$ is the degree d moduli space of stable maps to Y (Kontsevich) and K_d is the degree of its virtual fundamental class in the Chow group $A_*(\bar{M}_{0,0}(Y,d))$ (Jun Li et al.)

- The proof based on conformal field theory of the Gepner model was due to Greene-Plesser and the calculation was due to Candelas et al. (The argument depends on integration over infinite dimensional space and cannot be considered as mathematical rigorous.)

- Mathematical rigorous proof of Quintic Mirror Conjecture was due independently to Lian-Liu-Yau and Givental (cf. Pandharipande, Bini et al for expositions.)

- Constructions of pairs of toric complete intersections were due to Batyrev and Borisov who speculate them to be mirror pairs. That they are indeed mirror pair was justified by the work of Strominger-Zaslow-Yau based on M-theory.

- Period formula and generalizations of mirror conjecture for CY complete intersection in nonnegative ($c_1 \geq 0$) toric manifolds conjectured by Hosono-Lian-Yau was proved by Lian-Liu-Yau.

Prepotentials

Integrality Conjecture:

F_0 can be computed by the following general recipe:

1. The 3rd derivatives of the (A model) prepotential $F_0(t)$ of a CY threefold X, with mirror Y, should have the form

$$\partial_i \partial_j \partial_k F_0(t) = \int_Y \gamma_i \gamma_j \gamma_k + \sum_{d \in H_2(Y, \mathbf{Z})} \frac{n_d d_i d_j d_k q^d}{1 - q^d}$$

where γ_i is a suitable base of $H^2(Y, \mathbf{Z})$, where the n_d are *integers.*

2. The integers n_d counting the virtual number of rational curves in homology class $d = d_i\gamma^i$ (γ^i dual to γ_i.)

- This implies that the GW invariants $K_d \in \mathbf{Q}$ are related to the n_d by the formula

$$K_d = \sum_{k|d} k^{-3} n_{d/k}.$$

The denominator of K_d is related to automorphisms of genus zero stable maps to Y. This formula implies that this denominator can't be too big: at worst N^3 where N is the largest integer such that $d/N \in H_2(Y, \mathbf{Z})$.

- The Integrality Conjecture also implies that $\partial_i\partial_j\partial_k F_0(t)$ has q-series with integer coefficients.

- The integrality of $\partial_i \partial_j \partial_k F_0(t)$ has been verified in some cases (e.g. Lian-Yau.)

- The q-series usually have small radii of convergence. It is still not known if there is always an analytic continuation to the unit polydisk $|q_i| < 1$.

- In many cases, one can show that $F_0(t)$ satisfied a system of nonlinear (polynomial) PDE.

- For $h^{2,1} = 1$, this is a 10th order ODE. The ODE determines F_0 uniquely up to a degree 3 polynomial in t.

Example

- X Calabi-Yau manifolds which are complete intersection.

- The ODE for X has the form
$$P(\delta, \chi, \rho) = 0.$$

- δ, χ, ρ are certain invariant differential polynomials of $F(t)$ of weights 15,10,4. P is a quasi-homogeneous polynomial of weight 180. There are 37 terms in this polynomial correspond to the 37 partitions of 180 by 15,10,4.

- One can use this ODE to prove that the 'number' of rational curves of degree d in a generic quintic is divisible by 5^3 (at least for $5|d$), a conjecture of Clemens 1981.

Large Radius Limit

Conjecture: If X is not rigid, then it can degenerate into a union of smooth rational varieties intersecting transversally, where the monodromy is maximally unipotent.

- The SYZ fibration of X, whenever known, supports this conjecture.

- Example: CY complete intersections in toric varieties.

- This degeneration is called the *large radius limit* (or large complex structure limit) of X.

- Their existence have been proved in many cases (see Hosono-Lian-Yau, Lian-Todorov-Yau.)

- In a neighborhood of the large radius limit, periods of X are expected to have canonical q-series expansions of the form ($q_i = exp(2\pi\sqrt{-1}t_i)$)

$$w_0(t) \begin{pmatrix} 1 \\ t_i \\ \frac{1}{2}K^0_{ijk}t^j t^k + b_i + \partial_{t_i}F^0_{inst} \\ -\frac{1}{6}K^0_{ijk}t^i t^j t^k - b_i t^i + c + (2F^0_{inst} - t^i\partial_{t_i}F^0_{inst}) \end{pmatrix}$$

How do we compute the cubic polynomial appeared in the above expression of the period? This was proposed by Hosono-Klemm-Theisen-Yau (HKTY). Its choice is based on the Frobenius method. It is mathematically elegant and is important for many later considerations of mirror principle. (The formal variable in the Frobenius method is later interpreted to be the hyperplane class in the equivariant cohomology in the proof of mirror conjecture.)

There is a period which has power series expansion

$$w_0(z) = \sum_{n \geq 0} c(n) z^n \, .$$

It satisfies the generalized hypergeometric system of Gilfand-Kapranov-Zilovauski.

HKTY used the Frobenius argument to solve the Picard-Fuchs equation.

When X is a hypersurface in products of weighted projected space.
Let $c(n_1, \ldots, n_k)$ be written as products of $(l_i(n_1, \ldots, n_k)!)^{\pm 1}$ when l_j are linear forms with integer coefficients.

We define $c(\rho_1, \ldots, \rho_k)$ by replacing $(l_j)!$ by $\Gamma(l_j(\cdots, \rho_j, \cdots) + 1)$.

Let J_i be the pull back of the Kähler form of the i-th projective space. Then $K_{ijk} = \int J_i \wedge J_j \wedge J_k$.

Let

$$\partial_{\rho_i} = \frac{1}{2\pi i}\frac{\partial}{\partial \rho_i}$$

$$D_i^{(2)} = \frac{1}{2} K_{ijk} \partial_{\rho_j} \partial_{\rho_k}$$

$$D^{(3)} = -\frac{1}{6} K_{ijk} \partial_{\rho_i} \partial_{\rho_j} \partial_{\rho_k},$$

then we find

$$\int c_2 \wedge J_i = -24 D^{(2)}\, c(\rho_1, \ldots, \rho_k)|_{(0,\ldots,0)}$$

$$\int c_3 = i \frac{2\pi^3}{\zeta(3)}\, D^{(3)} c(\rho_1, \ldots, \rho_k)\Big|_{(0,\ldots,0)}$$

From this formulae, we can calculate the cubic polynomial in the expression of the period to be:

$$b_i = \frac{1}{24}\int c_2 \wedge J_i$$
$$c = i\frac{\zeta(3)}{2\pi^3}\int c_3$$

This means that the Chern Class of a Calabi-Yau manifold can be expressed in terms of the period of the mirror manifold.

This observation was generalized by A. Libgober in higher dimension by finding

$$\int Q_k(c_1, \cdots, c_k) \wedge J_{i_1} \wedge \cdots \wedge J_{i_{n-k}}$$

$$= \sum k! \frac{\partial^k c(\rho_1, \cdots, \rho_k)}{\partial \rho_{j_1} \cdots \partial \rho_{j_k}} \bigg|_{(0, \cdots, 0)} K_{j_1, \cdots, j_k, i_1, \cdots, i_{n-k}}$$

where Q is the Hirzebruch multiplicative sequence associated to $\frac{1}{\Gamma(1+z)}$.

Mirror Maps

- X, CY complete intersection (corresponding to the line bundles $L_1, .., L_k \in Pic(M)$) in a toric manifold M with $c_1(M) \geq 0$.

- In [Hosono-Lian-Yau 1995], we define the *cohomology valued B-series*

$$B(x) = \sum_{d \in H_2(M,\mathbf{Z})_+} \frac{\prod_i \Gamma(c_1(L_i) + c_1(L_i) \cdot d + 1)}{\prod_{D_a} \Gamma(D_a + D_a \cdot d + 1)} \, e^{x \cdot d + x}.$$

The sum is over the classes d in the Mori cone of M.

The D_a are the toric divisors of M.

x is a vector parameter in $H^2(X, \mathbf{C})$.

The Γ functions should be expanded as power series in the cohomology valued parameters $c_1(L_i), D_a$.

- It was shown that $B(x)$ computes the periods of the mirror CY manifold X, generalizing the classical Frobenius method.

- This generalizes all previously found period formulas for toric CY (Candelas-de la Ossa-Font-Katz-Morrizon, Hosono-Kleem-Theisen-Yau, Batyrev-Vau Straten, Hosono-Lian-Yau)

- $B(t)$ is a solution to a GKZ hypergeometric PDE system.

- Example: for X quintic 3-fold, this reduces to the familiar series (Candelas et al)
$$B(x) = \sum_{d \geq 0} \frac{\Gamma(5H+d+1)}{\Gamma(H+d+1)^5}\ z^{d+H}$$
where H is the hyperplane class and $z = e^{x_1}$, $x = x_1 H$.

- In general, the periods of the mirror CY X are recovered by expanding the function

$$B(x) = B(x_1 H_1 + \cdots + x_m H_m)$$

where the H_i, independent vectors in $H^2(X)$, are the expansion parameters (Frobenius parameters.)

The $x_1, .., x_m$ are interpreted as coordinates on the moduli space of X.

- One finds

$$B(x) = [X] \left(f_0(z) + g_i(z) H_i + O(H^2) \right)$$

where $z_i = e^{x_i}$, and $f_0(z)$ is a holomorphic function, and $g_i(z) = f_0(z) log\ z_i + O(z)$.

- Put

$$q_i := exp(2\pi\sqrt{-1}t_i), \quad t_i := \frac{g_i(z)}{f_0(z)}.$$

Inverting this relations:

$$z_i = z_i(q) = q_i + O(q^2).$$

These are the q-series of the mirror map; the t_i are interpreted as flat coordinates on the moduli space of X.

- These q-series can be thought of as some higher dimensional analogue of the elliptic modular function $j(q)$.

Example: X, quintic 3-fold.

$$z(q) = q - 770q^2 + 171525q^3 - 81623000q^4 \\ - 35423171250q^5 - 54572818340154q^6 - \cdots$$

Conjecture: The q-series of the mirror map are series with integer coefficients.

- This conjecture is known to hold for many toric CY.

- For many toric CYs, this can be seen as a consequence of Dwork's p-adic ODE theory (see Lian-Yau 1994.)

- It can also be shown that the $5th$ root $z(q)^{1/5}$ of $z(q)$ above also has integer coefficients.

Schwarzian Equations for Mirror Maps

- We restrict to CY 3-fold X with one moduli, i.e. $h^{2,1}(X) = 1$.

- We construct a nonlinear ODE for the q-series $z(q)$, generalizing the classical Schwarzian ODE for the elliptic modular function:

$$\{j, t\} = Q(j) j'(t)^2.$$

  where $Q(j)$ is a rational function of j.

- Since $h^{2,1}(X) = 1$, *Transversality* implies that the periods of X satisfies a 4th order homogeneous ODE, the Picard-Fuchs equation.

- *Special Geometry* implies that the PF equation can be transformed into the special form:

$$f'''' + c_2 f'' + c_2' f' + f = 0$$

where

$$c_2 := a_2(z) z'^2 - \frac{15}{2}\left(\frac{z''}{z'}\right)^2 + 5\frac{z^{(3)}}{z'} = a_2 z'^2 + 5\{z(t), t\}$$

$$c_0 := a_0(z) z'^4 + \frac{3}{2}\frac{da_2}{dz} z'^2 z'' - \frac{3}{4} a_2 z''^2 - \frac{135 z''^4}{16 z'^4}$$

$$+ \frac{3}{2} a_2 z' z^{(3)} + \frac{75 z''^2 z^{(3)}}{4 z'^3} - \frac{15 {z^{(3)}}^2}{4 z'^2} - \frac{15 z'' z^{(4)}}{2 z'^2} + \frac{3 z^{(5)}}{2 z'}$$

where $' := \frac{d}{dt}$, and $a_2(z), a_0(z)$ are some functions of z.

- The differential equation above is equivalent to the following pair of coupled ODEs:

$$c_2 = \frac{2K''}{K} - \frac{5}{2}(\frac{K'}{K})^2$$
$$c_0 = \frac{-35\,K'^4}{16\,K^4} + \frac{5\,K'^2\,K''}{K^3} - \frac{5\,K''^2}{4\,K^2} - \frac{2\,K'\,K^{(3)}}{K^2} + \frac{K^{(4)}}{2\,K}$$

where $K(t) = F_0'''(t)$, and F_0 =prepotential function of X.

- By elimination technique, we can explicitly decouple the equations into two nonlinear ODEs for $z(t)$ and $K(t)$ separately.

- The Schwarzian equation in three dimension is therefore perturbed by $K(t)$:

$$2Q(z)z'^2 + \{z, t\} = \frac{2}{5}y'' - \frac{1}{10}y'^2,$$

  where $y = \log K(t)$ and Q can be written as a rational function of z.

Counting curves of higher genus

The prepotentials F_0 can be generalized to F_g to count curves of genus g. It can be defined through the Gromov-Witten invariants for higher genus in the following way:

Assume:
X is a smooth Calabi-Yau threefold
$\beta \in H_2(X, \mathbb{Z})$, and
$\overline{M}_g(X, \beta)$ has virtual dimension 0

Then the **Gromov-Witten invariants** can be defined to be

$$K_\beta^g(X) = \int_{[\overline{M}_g(X,\beta)]^{vir}} 1$$

It is a rational number.

Let

$$F_g = \sum_{0 \neq \beta \in H_2(X,\mathbb{Z})} K_\beta^g e^{2\pi\omega\cdot\beta}.$$

where ω is the Kähler class of X.

Conjecture (Gopakumar-Vafa): *There exist* **integral invariants** n_β^g *such that*

$$\sum_{g=0}^{\infty} F_g u^{2g-2}$$
$$= \sum_{g=0}^{\infty} \sum_{\beta \neq 0} \sum_{k=1}^{\infty} \frac{n_\beta^g}{k} \left(q^{1/2} - q^{-1/2}\right)^{2g-2} e^{2k\pi\omega\cdot\beta},$$

where $q = e^u$, $\beta \in H_2(X,\mathbb{Z})$.

- Gopakumar-Vafa conjecture is based on the compactification of M-theory on Calabi-Yau threefold.

  It reveals how a BPS state in type IIA compactification on Calabi-Yau threefold contributes to topological string theory.

- In mathematics, there are no commonly accepted definition of these invariants.

Verification of GV conjecture for local Calabi-Yau geometry

Let S be a toric, Fano surface inside a Calabi-Yau threefold X. Then the tubular neighborhood of S in X can be approximated by canonical bundle $K_S \to S$.

Consider the following diagram:

$$
\begin{array}{ccccc}
\pi_* ev^* K_S & & & & K_S \\
\downarrow & & & & \downarrow \\
\overline{M}_g(S,\beta) & \xleftarrow{\pi} & \overline{M}_{g,1}(S,\beta) & \xrightarrow{ev} & S
\end{array}
$$

Local Gromov-Witten invariant:

$$K_\beta^g(S) = \int_{[\overline{M}_g(S,\beta)]^{vir}} c_{top}(R^1 \pi_* ev^* K_S)$$

(Lian, Liu, Yau, Vafa et al.)

Theorem (Peng). *For any local toric Calabi-Yau threefold $K_S \to S$,*

- $P_\beta(x) = \sum_{g=0}^{\infty} n_\beta^g(S) x^g \in \mathbb{Z}[x]$.
- *The degree of P_β is the arithmetic genus of the curve representing the given class $\beta \in H_2(S, \mathbb{Z})$.*
- *The leading coefficient of P_β is equal to the Euler Characteristics of the holomorphic line bundle corresponding to devisor β in S except for a possible negative sign depending on β.*

Multi-cover contribution and p-adic argument is combined in the proof of the above theorem. The similar argument is used later in the work of Kontsevich-Schwarz-Vologodksy on studying the integrality of instanton number on the quintic manifold.

Polynomial relation between F_g

The computation of F_g, the topological string partition function, is of fundamental importance.

However, in general, the complexities of computation is growing rapidly as genus grows higher and higher. How to effectively reduce the complexity of computation to a computable level is really a challenging problem.

Except for noncompact Calabi-Yau manifolds, F_g has not been computed for any compact Calabi-Yau manifolds. On the other hand, they have many properties similar to modular forms.

The method of computations for F_g depend on the work of BCOV on holomorphic anomaly equation. They are determined only up to a polynomial and this polynomial is fixed by direct calculation and the method is not satisfactory.

A great deal more need to be done. In comparison with the fact that modular forms form a ring and can be written as quasi-homogeneous polynomial of Eisentein series E_4 and E_6, we look into the ring structure of F_g.

Yamaguchi-Yau found polynomial relation among F_g. This ring structure was used by Huang, Klemm, Quackenbush to compute F_g for $g \leq 51$.

Algorithm (Yamaguchi-Yau)

Step 1

Show genus g partition function F_g can be written as a polynomial of a set of generators by Yukawa coupling

$$\partial_\psi C_{\psi\psi\psi} = \frac{(2\psi^{-1})^3}{5^3}\frac{\psi^2}{1-\psi^5}$$

ψ correspond to the one family parameter of Quintic Calabi-Yau threefold whose mirror manifold is expressed by

$$x_1^5 + x_2^5 + \cdots + x_5^5 - 5\psi x_1 x_2 \cdots x_3 = 0$$

F_g are actually sections of L^{2-2g} where L is the holomorphic line bundle defined by the holomorphic 3-forms.

Step 2

Use Yukawa coupling and Picard-Fuchs equation, we find five power series and a recursive formula to define polynomial of five generators of degree $3g-3$ by appropriately assigning degrees of these five generators. The relation greatly reduced the complexity of computation of Gromov-Witten invariants to polynomial complexity.

Theorem (Yamaguchi-Yau 2004) *The generating function of genus g Gromov-Witten invariants $F_g^{A-model}$ is a degree $(3g-3)$ quasi-homogeneous polynomial of five generators.*

example: Quintic Calabi-Yau threefold

Genus two partition function can be written as

$$\frac{3125}{144} - \frac{15625}{288}v_1 + \frac{125}{24}v_1^2 - \frac{5}{24}v_1^3$$
$$- \frac{3125}{36}v_2 + \frac{25}{6}v_1 v_2 + \frac{350}{9}v_3 - \frac{28795}{144}X$$
$$- \frac{835}{144}v_1 X + \frac{5}{6}v^2 X - \frac{2375}{12}v_2 X + \frac{205}{144}X^2$$
$$- \frac{325}{288}v_1 X^2 + \frac{25}{48}X^3.$$

where V_i and X can be computed in terms of Yukawa couplings.

This theorem works in general for one parameter family of Calabi-Yau threefolds.

It has been used by Huang, Klemm, Quackenbush to make explicit computations for counting higher genus curves in Calabi-Yau threefolds.

Modular forms have appeared in the computations.

Formal toric Calabi-Yau threefold

J. Li-C.-C. Liu-K. Liu-J. Zhou introduced the notion of formal Calabi-Yau threefolds.

A formal toric Calabi-Yau threefold X contains an algebraic torus $(\mathbb{C}^*)^3$ as its open dense subset and $(\mathbb{C}^*)^3$-action can naturally extend to X.

• X^1: one-dimensional $(\mathbb{C}^*)^3$-orbit closures (assume connected)

• X^0: $(\mathbb{C}^*)^3$ fixed points

$p \in X^0$, the action of $(\mathbb{C}^*)^3$ on $\wedge^3 T_p X$ gives an irreducible character $\alpha : (\mathbb{C}^*)^3 \to \mathbb{C}^*$.

α is independent of choice of p due to Calabi-Yau condition and connectedness of X^1.

• $T = \mathrm{Ker}\alpha \cong (\mathbb{C}^*)^2$.

• $T_{\mathbb{R}} \cong U(1)^2$ is the maximal compact subgroup of T.

• $\mathfrak{t}_{\mathbb{R}}^{\vee}$ is the dual of the Lie algebra of $T_{\mathbb{R}}$.

$\mu : X \longrightarrow \mathfrak{t}_{\mathbb{R}}^{\vee}$ be the moment map of the $T_{\mathbb{R}}$-action on X.

The image of X^1 gives a planar trivalent graph Γ, called **toric diagram** of X.

Topological vertex theory:

This is a theory developed by string theorists Aganagic-Klemm-Marino-Vafa, and by mathematicians J. Li, C.-C. Liu, K. Liu, J. Zhou.

- Degenerate X along some devisor related to its toric diagram until all pieces are indecomposable

- Each piece is a **topological vertex** whose generating functions of (open) Gromov-Witten invariants are by duality equal to some Chern-Simons invariants.

- A gluing algorithm can be used to compute Gromov-Witten invariants of toric Calabi-Yau threefold.

- The theory gives complete closed formulas for the generating series of all genera and all degree in terms of Chern-Simons knot invariants, or symmetric functions.

By studying degeneration/gluing algorithm in topological vertex theory, combine multi-cover contribution and p-adic argument

Theorem (Peng). *Gopakumar-Vafa conjecture holds for any formal toric Calabi-Yau threefold and Gopakumar-Vafa invariants vanish at large genera.*

The theorem is not known for compact Calabi-Yau manifolds.

Mirror symmetry and rational points over finite fields.

Let

$$P(x,\psi) = \sum_{i=1}^{5} x_i^5 - 5\psi \prod_{i=1}^{5} x_i = 0$$

be a family of Calabi-Yau quintics. Let $N_r(\psi)$ be the number of rational points in $\mathbf{F}_{p^r}$, and the generating series

$$\zeta(t,\psi) = \exp\left(\sum_{r=1}^{\infty} N_r(\psi)\frac{t^r}{r}\right).$$

• Candelas et al developed a method to express $\zeta(t,\psi)$ in terms of the periods of the mirror family.

• D. Wan made some interesting congruence conjectures about $\zeta(t,\psi)$ and the corresponding generating function of the mirror family.

In order to really understand mirror symmetry for Calabi-Yau manifolds over number field, it seems that the mirror of Frobenius action has to be understood. We know that $H^3(X)$ is mapped to $H^0(X') \oplus H^2(X') \oplus H^4(X') \oplus H^6(X')$ and special lagrangian cycles of X are mapped to algebraic vector bundles V over X'. The latter is then mapped to cohomology by

$$ch(V)\sqrt{Tod(X')}.$$

It is therefore interesting to see whether there is an action mirror to Frobenius action on the algebraic K-groups.

Modularity of Calabi-Yau threefold

There is a general modularity conjecture for rigid Calabi-Yau manifolds, generalizing the Shimura-Taniyama conjecture. Many interesting works have been done along this direction.

Theorem(Dieulefait-Manoharmayum) Every rigid ($h^3 = 2$) Calabi-Yau threefold with good reduction at 3 or at 5 is modular. Probably any rigid CY threefold is modular.

Examples of nonrigid Calabi-Yau manifolds

Let $F \subset \mathbb{P}^4$ be the quintic threefold

$$\sum_{i=1}^{5}(x_i^3 x_{i+1} x_{i+4} + x_i^3 x_{i+2} x_{i+3} - x_i x_{i+1}^2 x_{i+4}^2 - x_i x_{i+2}^2 x_{i+3}^2) = 0$$

F is the total space of a pencil of abelian surfaces cut out by sections of the Horrocks-Mumford bundle on $\mathbb{P}^4$.

(E. Lee) There is a desingularization $\widehat{F}$ of F such that its middle cohomology, up to semisimplification, has the form

$$H^3(\widehat{F}) = V \oplus \mathrm{Ind}\,_{G_{\mathbb{Q}(i)}}^{G_{\mathbb{Q}}} H^1(E)(-1)$$

E is an elliptic curve defined over $\mathbb{Q}(i)$.

V is a rank 2 motive which is modular: the L-series of V (up to Euler factors at bad primes) is the Mellin transform of the unique normalized cusp form f of weight 4, level 5

$$f = q - 4q^2 + 2q^3 + 8q^4 - 5q^5 - 8q^6 + 6q^7 + ...$$

The term $\mathrm{Ind}\,_{G_{\mathbb{Q}(i)}}^{G_{\mathbb{Q}}} H^1(E)(-1)$ comes from a complex-conjugate pair of elliptic ruled surfaces embedded in $\widehat{F}$ defined over $\mathbb{Q}(i)$.

Proof is by counting points on $\widehat{F}$ and applying the Lefschetz fixed-point theorem to the Frobenius automorphism.

Other examples of nonrigid modular CY 3folds: van Geemen-Nygaard, Hulek-Verrill, Livne-Yui, Schuett, Cynk-Meyer and others. In each case H^3 = sum of two-dimensional pieces whose modular forms are known. In general higher-dimensional Galois representations and more complicated automorphic forms might be involved.

Motives and mirror symmetry of a pair of Calabi-Yau threefolds

An admissible weight $Q=(q_1,q_2,q_3,q_4,q_5)$ is a 5-tuple of positive integers such that $gcd(q_i)=1$, each q_i divides $m=q_1+q_2+q_3+q_4+q_5$. Consider the family of degree m Calabi-Yau hypersurfaces X in the weighted projective space $\mathbb{P}^4(Q)$.

There is the Fermat-type hypersurface

$$V(Q): x_1^{m_1}+x_2^{m_2}+x_3^{m_3}+x_4^{m_4}+x_5^{m_5}=0$$

where $m_i=\frac{m}{q_i}$.

To get the mirror family $\tilde{X}$ (Greene-Plesser), take a family of deformations of the Fermat hypersurface, quotient out by a group G of automorphisms which preserves the holomorphic 3-form, and desingularize:

$$G = \{\mathrm{g} = (g_1, g_2, g_3, g_4, g_5) | g_i^{m_i} = 1, \prod_i g_i = 1\} / \{g, g, g, g, g\}$$

In the $(1, 1, 1, 1, 1)$ case

$$\tilde{X} = \{x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 - \psi x_1 x_2 x_3 x_4 x_5 = 0\} / (\mathbb{Z}_5^3)$$

The mirror $\tilde{X}$ flips the Hodge numbers of X: $h^{i,j}(X) = h^{3-i,j}(\tilde{X})$.

Consider also the dual group $\widehat{G}$ and a subset $A(Q) \subset \widehat{G}$:

$$\widehat{G} = \{\mathbf{a} = (a_1, a_2, a_3, a_4, a_5) | a_i \in q_i(\mathbb{Z}/m), \sum a_i = 0\}$$

$$(\mathbf{g}, \mathbf{a}) = \prod_i g_i^{a_i}$$

$$\mathcal{A}(Q) = \{(a_1, a_2, a_3, a_4, a_5) \in \widehat{G} | \text{ each } a_i \neq 0\}$$

For each $a \in \widehat{G}$ define $A = [a]$ to be the $(\mathbb{Z}/m)^*$ orbit of a.

Each $g \in G$ defines a cycle on $V \times V$ given by its graph. For each a define the cycles

$$p_a = \frac{1}{|\widehat{G}|} \sum_{g \in G} (g, a)^{-1} g \in A^*(V \times V)$$

$$p_A = \sum_{a \in A} p_a$$

The $\{p_A\}$ form an orthogonal set of projectors. If $A = [a] \subset \mathcal{A}(Q)$ it defines a *Fermat motive.*

Theorem. (N. Yui and S. Kadir) Let $(X, \tilde{X})$ be a mirror pair of Calabi-Yau orbifolds of admissible weight Q. If p is a good prime and $q = p^k$, then the zeta function of $\tilde{X}$ at the Fermat point is given by

$$Z(\tilde{X}_{\mathbb{F}_q}, t) = \frac{P_3(\tilde{X}_{\mathbb{F}_q}, t)}{(1-t)(1-qt)^{h^{2,1}(X)}(1-q^2t)^{h^{1,2}(X)}(1-q^3t)}$$

$$P_3(\tilde{X}_{\mathbb{F}_q}, t) = \prod_A P_3(\mathcal{M}_A, t)$$

Here the product runs over all Fermat motives A with $(G, A) = 1$. $\mathcal{M}_A$ is the image $H^3(X)^{p_A}$ of $H^3(X)$ under the projector p_A, and P_3 is the characteristic polynomial of the Frobenius action.

This relation between the zeta-functions of X and $\tilde{X}$ leads to a correspondence between monomial types of X (periods) and Fermat motives.