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Modular Calabi-Yau threefolds, by Christian Meyer, American Mathematical Society, Providence, RI, 2005, ix+194 pp., US$59.00, ISBN 978-0-8218-3908-9

In 1977, S.-T. Yau [21] proved the famous Calabi conjecture; as a consequence we know that every compact Kähler manifold with vanishing first Chern class admits a Ricci-flat Kähler metric. This special class of Kähler manifolds is known as the Calabi-Yau manifolds. To include the noncompact case, we may also define a Calabi-Yau manifold as a complex manifold with $SU(n)$ holonomy or as a complex manifold with a global nowhere vanishing holomorphic $(n, 0)$-form. More generally we may define a possibly singular Calabi-Yau variety as a complex variety with trivial canonical line bundle. In this review we consider only compact Calabi-Yau manifolds. Calabi-Yau manifolds have many interesting special properties; for example, the deformation spaces of Calabi-Yau manifolds are proved to be unobstructed. It is conjectured by both mathematicians and string theorists that all Calabi-Yau threefolds can be connected through conifold transitions. A Calabi-Yau manifold is called rigid if it has no infinitesimal complex deformations. Famous examples of Calabi-Yau manifolds include the $K3$ surface, the canonical bundle of $\mathbf{C}P^2$ and the quintic hypersurfaces in $\mathbf{C}P^4$.

Calabi-Yau manifolds are important in superstring theory, the most promising theory to unify the four fundamental forces in Nature. They are the shapes that satisfy the requirement of space for the six hidden spatial dimensions of string theory, which must be contained in a space smaller than our currently observable lengths. String theory asserts that Calabi-Yau manifolds have the remarkable mirror symmetry property which can be used to solve important enumerative problems in algebraic geometry. A famous example of this is the mirror formula of Candelas-de la Ossa-Green-Parkes [3]. (For proofs of this formula from two different points of view, see [15] and [10, 17, 1]; for a comparison of the two points of view, see [4]). String theory and Calabi-Yau manifolds have stimulated several active mathematical research areas such as Gromov-Witten theory, the Strominger-Yau-Zaslow program, Kontsevich's homological mirror symmetry conjecture and recently the arithmetic aspects of Calabi-Yau manifolds. In these developments mirror symmetry from string theory is the central topic. See [11] and [4] for the history of mirror symmetry and its applications in physics and mathematics.

It should be mentioned that the recent development of string duality has motivated many exciting new mathematical results. Mirror symmetry has become one of the dualities among the five string theories. By comparing the mathematical descriptions of these theories, one often reveals quite deep and unexpected mathematical conjectures, many of which are related to Calabi-Yau manifolds. See [16] for an example of large N duality between Chern-Simons theory and Calabi-Yau geometry. The mathematical proofs of these conjectures often help verify the physical theories which cannot be achieved today through traditional experiments. We can say that with stimulations from string theory Calabi-Yau manifolds have become a

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good testing ground for analysis, geometry, algebraic geometry, automorphic forms, number theory and theoretical physics.

On the other hand, elliptic curves have played the most fundamental roles in modern algebraic number theory. The proof of Fermat's last theorem by Wiles and Taylor is to prove the modularity of semistable elliptic curves. Since an elliptic curve is a one-dimensional Calabi-Yau manifold, it is rather natural to see whether such modularity property still holds for higher dimensional Calabi-Yau manifolds. This is actually part of the Fontaine-Mazur-Serre modularity conjecture for Galois representations. Now we first explain the precise modularity conjecture for rigid Calabi-Yau manifolds.

Let X be a Calabi-Yau manifold of dimension d defined over the rational numbers $\mathbf{Q}$. Then it has a model defined over the integers $\mathbf{Z}$. For a variety X defined over a field K, we denote by $\overline{X}$ the same variety considered over the algebraic closure $\overline{K}$. If we fix a model of X over $\mathbf{Z}$, then for all but finitely many primes p, its reductions to the finite fields $\mathbf{F}_p$ are smooth. Such primes p are called good primes. If p does not have this property, then we say that X has bad reduction at p and such a prime p is called a bad prime.

Let Y be a smooth projective variety defined over an algebraically closed field of characteristic p which can be either 0 or positive. For some prime $l \neq p$, there is a cohomology theory, the étale cohomology, which associates to Y certain $\mathbf{Q}_l$-vector spaces $H^i_{\text{ét}}(Y, \mathbf{Q}_l)$ for $i \geq 0$. This cohomology theory has many properties similar to the classical singular cohomology in characteristic 0. In particular the Lefschetz fixed point formula holds: Let $f : Y \to Y$ be a morphism such that the set of fixed points $\mathrm{Fix}(f)$ is finite and $1 - \mathrm{d}f$ is injective, where $\mathrm{d}f$ denotes the differential of f; then

$$\#\,\mathrm{Fix}(f) = \sum_{i=0}^{2d} (-1)^i \,\mathrm{tr}(f^* \mid H^i_{\text{ét}}(Y, \mathbf{Q}_l)).$$

We also have the comparison theorem with singular cohomology: if Y is smooth and projective over $\mathbf{C}$, then there are isomorphisms of $\mathbf{C}$-vector spaces

$$H^i_{\text{ét}}(Y, \mathbf{Q}_l) \otimes_{\mathbf{Q}_l} \mathbf{C} \cong H^i(Y, \mathbf{C}).$$

Let Fr_p denote the geometric Frobenius morphism of the Calabi-Yau manifold X at p. We consider the induced action of Fr_p on the ℓ-adic étale cohomology group $H^3_{\text{ét}}(\overline{X}, \mathbf{Q}_l)$, and define

$$P_{3,p}(z) := \det(1 - z\,\mathrm{Fr}_p \mid H^3_{\text{ét}}(\overline{X}, \mathbf{Q}_l)).$$

If we assume that X is a rigid Calabi-Yau threefold, then the Hodge numbers $h^{2,1}(X_{\mathbf{C}}) = h^{1,2}(X_{\mathbf{C}}) = 0$, and $P_{p,3}(z)$ is an integral polynomial of degree $\deg(P_{3,p}) = 2$ of the form:

$$P_{3,p}(z) = 1 - a_3(p)z + p^3 z^2$$

where $a_3(p)$ is subject to the Riemann Hypothesis, $|a_3(p)| \leq 2p^{3/2}$, as proved by Deligne in [6] and [7]. The L–series $L(H^3_{\text{ét}}(\overline{X}, \mathbf{Q}_l), s)$ is defined by

$$L(H^3_{\text{ét}}(\overline{X}, \mathbf{Q}_l), s) = (*) \prod_p P_{3,p}(p^{-s})^{-1}$$

where p runs over good primes and $(*)$ is the Euler factor corresponding to bad primes. The L–series of a rigid Calabi-Yau threefold X is simply defined to be

$$L(X, s) := L(H^3_{\text{ét}}(\overline{X}, \mathbf{Q}_l), s).$$

Now we recall the notions of modular forms and the associated L–series, following [12]. First recall that the group $\mathrm{SL}(2,\mathbf{Z})$ acts on the complex upper half-plane $\mathbf{H}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \ \tau \mapsto \frac{a\tau + b}{c\tau + d}.$$

Define $\overline{\mathbf{H}} = \mathbf{H} \cup \mathbf{P}^1(\mathbf{Q})$ by identifying the point $(x:1) \in \mathbf{P}^1(\mathbf{Q})$ with $x \in \mathbf{Q} \subset \mathbf{C}$ and the point $(1:0)$ with the point at infinity $i\infty$ along the complex axis. The action of $\mathrm{SL}(2,\mathbf{Z})$ extends to $\overline{\mathbf{H}}$.

For a given integer N, one has the modular subgroup

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbf{Z});\ c \equiv 0 \bmod N \right\}.$$

The quotient $Y_0(N) = \Gamma_0(N)\backslash\mathbf{H}$ is a Riemann surface. Its compactification $X_0(N) = \Gamma_0(N)\backslash\overline{\mathbf{H}}$ is called the modular curve of level N, and the points in $X_0(N) \setminus Y_0(N)$ are called cusps. It is easy to see that the action of $\mathrm{SL}(2,\mathbf{Z})$ permutes those cusps.

A modular form of weight k and level N is a holomorphic function f on $\mathbf{H}$ such that it transforms as

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau), \text{ for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N).$$

Consider the cusp $i\infty$. Writing $q = e^{2\pi i\tau}$, we have the Fourier expansion $f(\tau) = \sum_n b_n q^n$. The cusp $i\infty$ corresponds to $q = 0$ and f is holomorphic at the cusp $i\infty$ if $b_n = 0$ for $n < 0$. Such a function f is called a modular form if it is holomorphic at all cusps. A cusp form is a modular form which vanishes at all cusps. Recall that a Dirichlet character is a homomorphism $\chi : \ (\mathbf{Z}/N\mathbf{Z})^* \longrightarrow \mathbf{C}^*$ of abelian groups. More generally, modular forms with a Dirichlet character χ have a modified transformation behavior given by

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = \chi(d)(c\tau + d)^k f(\tau).$$

The finite dimensional vector space of weight k cusp forms for $\Gamma_0(N)$ is denoted by $S_k(\Gamma_0(N))$. This space can be considered as the space of holomorphic sections of a holomorphic line bundle on the modular curve $X_0(N)$. Similarly the vector space of weight k cusp forms with character χ is denoted by $S_k(\Gamma_0(N), \chi)$. On $S_k(\Gamma_0(N), \chi)$ there are the Hecke operators T_p for prime numbers $p \nmid N$. They generate the Hecke algebra.

The Hecke operators T_p and T_q commute for distinct primes p, q not dividing N, so we have their simultaneous eigenspaces. An eigenform is a simultaneous eigenvector for every element of the Hecke algebra. A Hecke newform is an eigenform that does not come from a space of cusp forms of lower level and is normalized so that the coefficient of q is 1. To a normalised Hecke newform f we can associate an L–function $L(f,s)$ by taking the Fourier expansion $f = \sum_n b_n q^n$ and its Mellin transform

$$L(f,s) = \sum_n b_n n^{-s}.$$

If f is a normalised Hecke newform with respect to the group $\Gamma_0(N)$, then its Fourier coefficients satisfy the properties

$$\begin{aligned} b_{p^r} b_p &= b_{p^{r+1}} + p^{k-1} b_{p^{r-1}} \text{ for } p \text{ prime, } p \nmid N \\ b_{p^r} &= (b_p)^r \text{ for } p \text{ prime, } p \mid N \\ b_n b_m &= b_{nm} \text{ if } (n,m) = 1 \end{aligned}$$

where k is the weight of the form f. It follows that the series $L(f, s)$ has a product expansion

$$L(f,s) = \sum_{n \geq 1} b_n n^{-s} = \prod_p \frac{1}{1 - b_p p^{-s} + \chi(p) p^{k-1-2s}} \tag{1}$$

where $\chi(p) = 0$ for $p \mid N$.

We remark that this can also be done by using ℓ-adic semi-simple Galois representation as in [5] and [19]. One can attach a Galois representation by using the Eichler-Shimura theory or its extensions by Deligne and Serre.

With the above preparations we now can formulate the modularity conjecture for rigid Calabi-Yau threefolds defined over $\mathbf{Q}$.

The modularity conjecture. *Any rigid Calabi-Yau threefold X defined over $\mathbf{Q}$ is modular in the sense that its L–series of X coincides with the Mellin transform of the L–series of a Hecke newform f of weight 4 on $\Gamma_0(N)$. Here N is a positive integer divisible only by the primes of bad reduction. More precisely, we have, up to finite Euler factors,*

$$L(X, s) = L(f, s).$$

One can state a similar conjecture for higher dimensional Calabi-Yau manifolds. Fontaine and Mazur [9] have conjectured that all irreducible odd 2–dimensional Galois representations coming from geometry should be modular up to a Tate twist. The above modularity conjecture may be regarded as a concrete realization of the Fontaine–Mazur conjecture. On the other hand, given X a projective variety of odd dimension n over $\mathbf{Q}$ such that the n-th Betti number $\dim H^n(X_{\mathbf{C}}, \mathbf{C}) = 2$ and $H^n(X_{\mathbf{C}}, \mathbf{C})$ has the Hodge decomposition of type $(n, 0) + (0, n)$, Serre [19] has formulated a modularity conjecture for the residual mod p 2-dimensional Galois representation attached to X for all primes. For recent exciting results about the proofs of these conjectures, see [13]. A related famous conjecture is the Beilinson-Bloch conjecture, which asserts that the order of vanishing of the L-series of the middle cohomology of a Calabi-Yau manifold X is equal to the dimension of cohomologically trivial Chow cycles of dimension 1 on X.

For one-dimensional Calabi-Yau manifolds, the elliptic curves over $\mathbf{Q}$, we may assume they have the Weierstrass form

$$y^2 = x^3 + ax + b, \quad \Delta = 4a^3 + 27b^2 \neq 0$$

where $a, b \in \mathbf{Q}$. For an elliptic curve E and the good primes p, the Lefschetz fixed point formula gives

$$N_p = \#E_p = 1 - a_p + p$$

where

$$a_p = \mathrm{tr}(\mathrm{Fr}_p \mid H^1_{\text{ét}}(\overline{E}_p, \mathbf{Q}_l)).$$

The following theorem of Wiles-Taylor-Breuil-Conrad-Diamond [2] is the famous Taniyama-Shimura-Weil conjecture, which implies Fermat's last theorem:

Theorem 0.1. *Let E be an elliptic curve defined over $\mathbf{Q}$. Then E is modular; i.e., there exists a Hecke newform of weight two with Fourier expansion $f(q) = \sum_n b_n q^n$ whose level N is equal to the conductor of the elliptic curve such that for all primes p of good reduction*

$$a_p = 1 - N_p + p = b_p.$$

A general modularity result for rigid Calabi-Yau threefolds has been proved in [8].

Theorem 0.2. *Let X be a rigid Calabi-Yau threefold defined over $\mathbf{Q}$, and assume that one of the following conditions holds:*

(1) *X has good reduction at 3 and 7 or*
(2) *X has good reduction at 5 or*
(3) *X has good reduction at 3 and the trace of Fr_3 on $H^3_{\text{ét}}(\overline{X}, \mathbf{Q}_l)$ is not divisible by 3.*

Then X is modular. More precisely

$$L(X, s) \mathrel{\mathring{=}} L(f, s)$$

for some weight four modular form, where $\mathring{=}$ means equality up to finitely many Euler factors.

Very recently Dieulefait showed that X is modular if it has good reduction at 3. So far very little is known about the Euler factors associated to the primes of bad reduction. Note that the above theorem is an existence result and does not provide a method to determine the form f explicitly. There is a method due to Faltings, Serre and Livné which is very effective to check whether a modular form gives a right candidate. We refer the reader to [12] and [18] for more advanced discussions on the related topics.

We remark that a number of non-rigid Calabi-Yau varieties have been constructed for which modularity has been established. Most of these known examples are either of Kummer type or contain elliptic ruled surfaces or both. In these cases the middle cohomology of the varieties breaks up into two-dimensional pieces. See [14] for an interesting example from resolving singularities of a quintic Calabi-Yau threefold.

After presenting notations and facts about Calabi-Yau manifolds and their arithmetic results, the book discusses many different constructions and hundreds of examples of rigid and non-rigid Calabi-Yau manifolds and studies their modularity in detail. Many concrete examples of Calabi-Yau manifolds, like the rigid Calabi-Yau in the mirror quintic family and double coverings of $\mathbf{C}P^3$, are constructed. Questions about prime numbers which can occur in the levels of the weight 4 modular forms and Calabi-Yau threefolds with same L–series are also discussed. Detailed tables about these examples of Calabi-Yau's and weight 2 and 4 newforms for $\Gamma_0(N)$ are given in the appendices. This book serves as a good reference for researchers and students who are interested in the subject.

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