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# Riemannian Geometry over different Normed Division Algebras

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## Contents

**Abstract**: We develop a unifed theory to study geometry of manifolds with different holonomy groups. They are classified by (1) real, complex, quaternion or octonion number they are defined over and (2) being special or not. Specialty is an orientation with respect to the corresponding normed algebra $\mathbb{A}$. For example, special Riemannian $\mathbb{A}$-manifolds are oriented Riemannian, Calabi-Yau, Hyperkähler and $G_2$-manifolds respectively.

For vector bundles over such manifolds, we introduce (special) $\mathbb{A}$-connections. They include holomorphic, Hermitian Yang-Mills, Anti-Self-Dual and Donaldson-Thomas connections. Similarly we introduce (special) $\frac{1}{2}\mathbb{A}$-Lagrangian submanifolds as maximally real submanifolds. They include (special) Lagrangian, complex Lagrangian, Cayley and (co-)associative submanifolds.

We also discuss geometric dualities from this viewpoint: Fourier transformations on $\mathbb{A}$-geometry for flat tori and a conjectural SYZ mirror transformation from (special) $\mathbb{A}$-geometry to (special) $\frac{1}{2}\mathbb{A}$-Lagrangian geometry on mirror special $\mathbb{A}$-manifolds.

# 1 Introduction

It is well-known that a Riemannian metric $g$ on a manifold $M$ determines a unique torsion free Riemannian connection on its tangent bundle, called the Levi-Civita connection. For a generic metric $g$, its holonomy group $Hol(g)$ equals $O(m)$ with $m = \dim M$. The size of the holonomy group is inversely proportional to the amount of geometric structures $M$ possesses. For example $Hol(g) \subset U(n)$, with $m = 2n$, is equivalent to $M$ being a Kähler manifold. When we further restrict the holonomy group to $SU(n)$, we obtain a Calabi-Yau manifold and they are the central objects of interest in mirror symmetry. Recently, M-theory suggests that the geometry of seven dimensional manifolds with $Hol(g) \subset G_2$ has even richer geometry. Other holonomy groups, like $Sp(n)$ for hyperkähler manifolds, are also very important in modern geometry. A complete classification of all possible holonomy groups has been obtained by Berger [4] many years ago.

In this paper we are going to study all these geometries from a unified point of view. Namely we analyze geometries as they are defined over $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$, the four normed division algebras $\mathbb{A}$. In a sense this approach is very natural, as *metric geometry* should be defined over *metric algebras.*

A Riemannian manifold is called an $\mathbb{A}$-*manifold* if its holonomy group is inside $G_{\mathbb{A}}(n)$, the group of twisted isomorphisms of $\mathbb{A}^n$. We can identify these manifolds for various $\mathbb{A}$ as Riemannian manifolds, Kähler manifolds, quaternionic Kähler manifolds and $Spin(7)$-manifolds respectively. There is also a notion of an $\mathbb{A}$-orientation and it defines a subgroup $H_{\mathbb{A}}(n)$ in $G_{\mathbb{A}}(n)$ consisting of special twisted isomorphisms of $\mathbb{A}^n$. Corresponding manifolds are called *special* $\mathbb{A}$-*manifolds,* and they are oriented Riemannian manifolds, Calabi-Yau manifolds, hyperkähler manifolds and $G_2$-manifolds respectively. Notice that these are precisely all possible holonomy groups for a Riemannian manifold which is not locally symmetric. In this classification, manifolds with $G_2$ holonomy group are those with richest geometric structures.

Unlike real or complex manifolds, $\mathbb{H}$- and $\mathbb{O}$-manifolds do not have many functions. Their geometries are reflected by submanifolds and bundles over them. To define them in a unified way, we note that holonomy groups $G_{\mathbb{A}}(n)$ and $H_{\mathbb{A}}(n)$ define natural subbundles $\mathbf{g}_{\mathbb{A}}(T_M)$ and $\mathbf{h}_{\mathbb{A}}(T_M)$ in $\Lambda^2 T_M^*$. For example when $\mathbb{A} = \mathbb{C}$, i.e. $M$ is a Kähler manifold, we have $\mathbf{g}_{\mathbb{C}}(T_M) = \Lambda^{1,1}(M)$ and $\mathbf{h}_{\mathbb{C}}(T_M) = \Lambda_0^{1,1}(M)$.

A connection $D_E$ on a vector bundle $E$ over $M$ is called an $\mathbb{A}$-*connection* (resp. *special* $\mathbb{A}$-*connection*) if its curvature tensor $F_E$ lies inside $\mathbf{g}_{\mathbb{A}}(T_M) \otimes ad(E)$ (resp. $\mathbf{h}_{\mathbb{A}}(T_M) \otimes ad(E)$). In the Kähler case, such a connection $D_E$ is a holomorphic connection, i.e. $F_E^{0,2} = F_E^{2,0} = 0$ (resp. Hermitian Yang-Mills connection). Special $\mathbb{A}$-connections, with $\mathbb{A} \neq \mathbb{R}$, are always absolute minimum for the Yang-Mills energy functional, as it will be explained in terms of bundle calibrations.

A natural class of submanifolds in any $\mathbb{A}$-manifold consists of $\mathbb{A}$-submanifolds. However, there is another natural class of submanifolds in the middle dimension, which plays the role of *decomplexifying* $M$, and they are called $\frac{1}{2}\mathbb{A}$-*Lagrangian* *submanifolds*. For example they include Lagrangian submanifolds in Kähler manifolds and Cayley submanifolds in $Spin(7)$-manifolds. Using the $\mathbb{A}$-orientation on a special $\mathbb{A}$-manifold, we also have the notion of *special* $\frac{1}{2}\mathbb{A}$-*Lagrangian submanifolds of type I* or *type II*. They can be identified as special Lagrangian submanifolds (with phase angle 0 or $\pi/2$) in Calabi-Yau manifolds, complex Lagrangian submanifolds in hyperkähler manifolds, associative submanifolds and coassociative submanifolds in $G_2$-manifolds. As in the bundle case, special $\frac{1}{2}\mathbb{A}$-Lagrangian submanifolds are absolute minimum for the volume functional as they are all volume calibrated.

A good notion of a *global decomplexification* of $M$ is a fibration with a section on $M$ by (special) $\frac{1}{2}\mathbb{A}$-Lagrangian submanifolds, possibly singular. For example, in the theory of geometric quantization of symplectic manifolds, a real polarization is merely a smooth Lagrangian fibration with a section.

To define and study the geometry of any (special) $\mathbb{A}$-manifold $M$, we need to couple submanifolds $C$ in $M$ with connections $D_E$ over $C$ and we call any such pair $(C, D_E)$ a *cycle*. We have (i) a *(special)* $\mathbb{A}$-*cycle* consists of an $\mathbb{A}$-submanifold and a (special) $\mathbb{A}$-connection over it and (ii) a *(special)* $\frac{1}{2}\mathbb{A}$-*Lagrangian cycle* consists of a (special) $\frac{1}{2}\mathbb{A}$-Lagrangian submanifold and a special $\mathbb{A}$-connection over it. For instance, in the Kähler case, a $\mathbb{C}$-cycle is a holomorphic bundle over a complex submanifold in $M$, in particular it is a coherent sheaf on $M$. Such a $\mathbb{C}$-cycle is special if the bundle carries a Hermitian Yang-Mills connection. Similarly, a (special) $\mathbb{R}$-Lagrangian cycle is a flat bundle over a (special) Lagrangian submanifold in the Calabi-Yau manifold $M$. The mirror symmetry conjecture says that these two types of geometries can be transformed to each other on mirror Calabi-Yau manifolds.

Such duality transformations play very important roles both in mathematical physics and geometry. A basic ingredient is the Fourier transformation. We will recall how it transforms the $\mathbb{A}$-geometry on a flat torus over $\mathbb{A}$ to the $\mathbb{A}$-geometry on its dual torus. For a special $\mathbb{A}$-manifold $M$ with a special $\frac{1}{2}\mathbb{A}$-Lagrangian fibration and a section, we also discuss briefly the SYZ mirror duality which transforms the (special) $\mathbb{A}$-geometry of $M$ to the (special) $\frac{1}{2}\mathbb{A}$-Lagrangian geometry of its mirror manifold. This can be viewed as a fiberwise Fourier transformation along $\frac{1}{2}\mathbb{A}$-Lagrangian fibrations (see [32], [10], [27], [16], [22] in the Calabi-Yau case, [23] in the hyperkähler case and [1], [11], [18] in the $G_2$-manifolds case).

In the last section, we give several remarks and questions on related aspects of geometries over different normed division algebras $\mathbb{A}$.

## 2 (Special) Riemannian $\mathbb{A}$-manifolds

In this section we define Riemannian manifolds over different normed division algebras. As we will see, *all* possible holonomy groups are naturally aroused from manifolds defined over $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$; unless they are locally symmetric spaces. This gives us a unified way to look at all the seemingly unrelated branches of geometry in mathematics. We first recall some basic facts about normed division algebras $\mathbb{A}$ (see e.g. [12]), then we will define the key notion: twisted isomorphisms.

### 2.1 Normed division algebra, $\mathbb{A}$

**Definition 1** *A normed algebra* $\mathbb{A}$ *is a finite dimensional real algebra with a unit* $1$ *and a norm* $\|\cdot\|$ *satisfying* $\|a \cdot b\| = \|a\| \, \|b\|$ *for any* $a, b \in \mathbb{A}$.

There are exactly four of them, namely the real $\mathbb{R}$, the complex $\mathbb{C}$, the quaternion $\mathbb{H}$ and the octonion (or Cayley) $\mathbb{O}$ numbers. Each can be interpreted as the complexification of the previous one, the so-called Cayley-Dickson process: suppose $\mathbb{A}$ is any algebra with conjugation $*$,[1] we define an algebra structure on $\mathbb{B} = \mathbb{A} \oplus \mathbb{A}$ as follows,

$$\begin{aligned}(a, b)(c, d) &= (ac - db^*, a^* d + cb) \\ (a, b)^* &= (a^*, -b).\end{aligned}$$

This process will construct $\mathbb{C}$ from $\mathbb{R}$ (and we write $\mathbb{R} = \frac{1}{2}\mathbb{C}$) and so on.

The following properties of a normed algebra will be needed in this article: (i) $\langle xy, z\rangle = \langle x, z\bar{y}\rangle$, (ii) $\langle xy, zy\rangle = \langle x, z\rangle |y|^2$ and (iii) $(xy)\, y = x\left(y^2\right)$ for any $x, y, z \in \mathbb{A}$. Furthermore, $\mathbb{A}$ is always a division algebra.

Each time we complexify $\mathbb{A}$, we loss some nice properties: (1) $\mathbb{C}$ is not real, (2) $\mathbb{H}$ is not commutative, (3) $\mathbb{O}$ is not associative and lastly $\mathbb{O} \oplus \mathbb{O}$ is no longer normed. As a result, $\mathbb{H}^n$ is only a bi-module of $\mathbb{H}$ and not a vector space because $\mathbb{H}$ is not a field. Furthermore, $\mathbb{O}$ does not even act on $\mathbb{O}^n$, with $n \geq 2$, because of the non-associativity. We call $\mathbb{A}^n$ a *linear* $\mathbb{A}$*-space* of rank $n$, and its real dimension is $m = 2^a n$ with $a = 0, 1, 2$ or $3$. An inner product on $\mathbb{A}^n$ always refers to one satisfying

$$\langle u \cdot x, v \cdot x\rangle = \langle u, v\rangle |x|^2,$$

for any $u, v \in V$ and $x \in \mathbb{A}$.

To define $\mathbb{A}$-manifolds, the nonlinear analog of linear $\mathbb{A}$-spaces, we first need to define twisted isomorphisms of $\mathbb{A}^n$. Recall on a normed linear $\mathbb{A}$-space $V \cong \mathbb{A}^n$, an automorphism of $V$ is a real linear isomorphism $\phi : V \to V$ satisfying

$$\begin{aligned}\langle \phi(u), \phi(v)\rangle &= \langle u, v\rangle, \\ \phi(vx) &= \phi(v)\, x,\end{aligned}$$

for any $u, v \in V$ and $x \in \mathbb{A}$. For example, in the quaternionic case, an automorphism of $\mathbb{H}^n$ preserves all three complex structures $I$, $J$ and $K$ on it, in fact it preserves the whole $S^2$ (twistor) family of complex structures. From the metric point of view, it is more natural to allow $\phi$ to rotate these complex structures. This brings us to the following definition.

[1] We also denote $*a$ as $\bar{a}$.

**Definition 2** *Suppose $V$ is a normed linear $\mathbb{A}$-space of rank $n$. A $\mathbb{R}$-linear isometry $\phi$ of $V$ is called a twisted isomorphism if there exists $\theta \in SO(\mathbb{A})$ such that*

$$\phi(vx) = \phi(v)\,\theta(x)$$

*for any $v \in V$ and $x \in \mathbb{A}$.*

*We denote the group of twisted isomorphisms of $\bigoplus^n \mathbb{A}$ as $G_{\mathbb{A}}(n)$.*

The following proposition identifies the group of twisted isomorphisms for each $\mathbb{A}$. Recall that when $\mathbb{A} = \mathbb{O}$, we assume that $V$ has octonion dimension one.

**Proposition 3** *The normed algebras and their corresponding groups of twisted isomorphisms are given as follows,*

$$\begin{array}{ccccccc} \mathbb{A} & = & \mathbb{R} & \mathbb{C} & \mathbb{H} & \mathbb{O} \\ G_{\mathbb{A}}(n) & = & O(n) & U(n) & Sp(n)\,Sp(1) & Spin(7)\,. \end{array}$$

Proof: For $\mathbb{A} = \mathbb{R}$, the assertion is trivial because $SO(\mathbb{R})$ is the trivial group. In the complex case, we have $\theta \in SO(\mathbb{C}) = U(1)$, that is there exists $z \in \mathbb{C}$ with $|z| = 1$ satisfying $\theta(x) = zx$ for any $x \in \mathbb{C}$. The requirement $\phi(vx) = \phi(v)\,zx$ for $x = 1$ implies that $z = 1$. That is $\phi$ is complex linear and hence $G_{\mathbb{A}}(n) = GL(n, \mathbb{C}) \cap O(2n) = U(n)$.

In the quaternionic case, we have $\theta \in SO(\mathbb{H}) = Sp(1)\,Sp(1)$, i.e. $\theta(x) = \alpha x \beta$ for some unit quaternions $\alpha, \beta \in S^3 \subset \mathbb{H}$. The requirement $\phi(vx) = \phi(v)\,\alpha x \beta$ with $x = 1$ implies that $\alpha\beta = 1 \in SO(4)$, i.e. $\beta = \alpha^{-1}$. If we define a linear homomorphism $A$ by

$$A(v) = \phi(v)\,\alpha,$$

for any $v \in \mathbb{H}^n$, then $A(vx) = A(v)\,x$ for any $x \in \mathbb{H}$, i.e. $A \in GL(n, \mathbb{H}) \cap O(4n) = Sp(n)$. From this, we have $\phi \in Sp(n)\,Sp(1) = G_{\mathbb{H}}(n)$. For the octonionic case, the identification of $G_{\mathbb{O}}$ with $Spin(7)$ can be found in [30]. ■

## 2.2 Special Riemannian $\mathbb{A}$-manifolds

We begin with the definition of Riemannian manifolds defined over $\mathbb{A}$.

**Definition 4** *A Riemannian manifold $(M, g)$ of is called a Riemannian $\mathbb{A}$-manifold, or simply an $\mathbb{A}$-manifold, if the holonomy group of its Levi-Civita connection is a subgroup of $G_{\mathbb{A}}(n) \subset O(m)$ with $m = \dim M = n \dim \mathbb{A}$.*

From the previous proposition, we know that Riemannian $\mathbb{A}$-manifolds for various $\mathbb{A}$ have holonomy groups insides $O(n)$, $U(n)$, $Sp(n)\,Sp(1)$ and $Spin(7)$ respectively and these manifolds are called Riemannian manifolds, Kähler manifolds, quaternionic Kähler manifolds and $Spin(7)$-manifolds respectively.

In section 6, we will discuss $\mathbb{A}$-manifolds without Riemannian metrics, e.g. complex manifolds.

Next we introduce the notion of an $\mathbb{A}$-orientation for Riemannian $\mathbb{A}$-manifolds. For a real Riemannian manifold $M$, an orientation is a parallel volume form on $M$. Equivalently the holonomy group of $M$ is inside $SO(m) \subset O(m)$. The determinant defines a natural action of $O(m)$ on $\Lambda^m \mathbb{R} \cong \mathbb{R}$ and $SO(m)$ is the isotropic subgroup for any nonzero element in $\mathbb{R}$. In general there is a natural real representation of $G_{\mathbb{A}}(n)$ on $\mathbb{A}$,

$$\lambda_{\mathbb{A}} : G_{\mathbb{A}}(n) \to O(\mathbb{A}),$$

except in the quaternionic case where the action is only defined projectively.

**Definition 5** *For any $g \in G_{\mathbb{A}}(n)$ and $x \in \mathbb{A}$, we define*
*(1) $\lambda_{\mathbb{R}}(g)(x) = x \det(g) \in \mathbb{R}$,*
*(2) $\lambda_{\mathbb{C}}(g)(x) = x \det_{\mathbb{C}}(g) \in \mathbb{C}$,*
*(3) $\lambda_{\mathbb{H}}(g)(x) = x\beta \in \mathbb{H}$ with $g = (\alpha, \beta) \in Sp(n)\, Sp(1)$ and*
*(4) $\lambda_{\mathbb{O}}(g)(x) = g \cdot x \in \mathbb{O}$ via $Spin(7) \subset SO(8)$.*

Note that $G_{\mathbb{A}}(n)$ always acts transitively on the unit sphere in $\mathbb{A}$.

The explanation of the seemingly different looking $\lambda_{\mathbb{H}}$ is as follow: First $\det_{\mathbb{H}}$ can not be defined because of the non-commutativity of $\mathbb{H}$. Even when $\mathbb{A} = \mathbb{C}$, $\det_{\mathbb{C}}$ can be interpreted as giving a decomposition,

$$\begin{aligned} U(n) &\overset{\cong}{\to} SU(n) \times U(1)/\mathbb{Z}_n \\ A &\to \left( A \cdot (\det{}_{\mathbb{C}} A)^{-1/n}, (\det{}_{\mathbb{C}} A)^{1/n} \right) \end{aligned}$$

and the projection to the second factor $(\det{}_{\mathbb{C}} A)^{1/n} \in U(1)/\mathbb{Z}_n$ can be reinterpreted as an element in $U(1)$ by raising to the $n^{th}$-power, thus giving the complex determinant of $A$. The natural analog in the quaternionic case is the identification,

$$G_{\mathbb{H}}(n) = Sp(n)\, Sp(1) \overset{\cong}{\to} Sp(n) \times Sp(1)/\mathbb{Z}_2$$

thus $(\alpha, \beta) \to \beta$ is the direct analog to $(\det{}_{\mathbb{C}} A)^{1/n}$ in the complex case. Later this $\mathbb{Z}_2$ factor will identify special $\mathbb{C}$-Lagrangian submanifolds of type I and type II in any hyperkähler manifolds to the same kind of objects, namely the complex Lagrangian submanifolds, see section 4.2.

**Definition 6** *A twisted isomorphism $g \in G_{\mathbb{A}}(n)$ is called special if $\lambda_{\mathbb{A}}(g)$ fixes $1 \in \mathbb{A}$. That is $g$ is an element in the isotropic subgroup of $1$ in $G_{\mathbb{A}}(n)$, which we denote $H_{\mathbb{A}}(n)$.*

It is not difficult to identify these groups when $\mathbb{A} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. In the octonionic case, it is a classical result (see e.g. [30]) that $H_{\mathbb{O}}(1)$ is $G_2$. Recall that the Lie group $G_2$can be identified as the stabilizer of the natunal action of $Spin(7)$ on $S^7$. Thus we have the following lemma.

**Lemma 7** *The normed algebras and their corresponding groups of special twisted isomorphisms are given as follows,*

$$\begin{array}{ccccccc} \mathbb{A} & = & \mathbb{R} & \mathbb{C} & \mathbb{H} & \mathbb{O} \\ H_{\mathbb{A}}(n) & = & SO(n) & SU(n) & Sp(n) & G_2 \end{array}$$

Remark: An isomorphism of a normed linear $\mathbb{A}$-space is the same as (i) a twisted isomorphism when $\mathbb{A}$ is $\mathbb{R}$ or $\mathbb{C}$ and (ii) a special twisted isomorphism when $\mathbb{A}$ is $\mathbb{H}$ or $\mathbb{O}$.

Next we define the analog of an orientation for Riemannian $\mathbb{A}$-manifold $(M, g)$, i.e. the holonomy group for the Levi-Civita connection is inside $G_{\mathbb{A}}(n)$.

**Definition 8** *A Riemannian $\mathbb{A}$-manifold is called special if its holonomy group is inside $H_{\mathbb{A}}(n)$.*

There is another characterization of special $\mathbb{A}$-manifolds. Using the representation $\lambda_{\mathbb{A}}(n)$ of $G_{\mathbb{A}}(n)$, we obtain a vector bundle $\mathbb{A}_M$ over any Riemannian $\mathbb{A}$-manifold $M$,

$$\mathbb{A} \to \mathbb{A}_M \to M.$$

Then $M$ is special if and only if there is a parallel section for $\mathbb{A}_M$.

In the real case, we have $\mathbb{R}_M = \Lambda^n T_M^*$. In the complex case, we have $\mathbb{C}_M = \Lambda^n T_M^{*(1,0)}$, the canonical line bundle of $M$. In the quaternionic case, the projection of its holonomy group $Sp(n)\,Sp(1)$ to its first and second factor are only well-defined up to $\pm 1$. Suppose that $M$ is spin, $w_2(M) = 0$, then this $\pm 1$ ambiguity can be lifted and we obtain a $Sp(n)$-bundle $V$ and a $Sp(1)$-bundle $S$ over $M$ via the standard representation of $Sp(n)$ and $Sp(1)$ respectively. The inclusion of $Sp(n)\,Sp(1)$ in $SO(4n)$ gives the isomorphism

$$T_M^* \otimes_{\mathbb{R}} \mathbb{C} = V \otimes_{\mathbb{C}} S.$$

We have $\mathbb{H}_M = S$. When $w_2(M) \neq 0$, then $V$ and $S$ only exist locally (see e.g. [5]).

In the octonionic case, we simply have $\mathbb{O}_M = T_M^*$.

The following table lists the possible holonomy groups for all (special) Riemannian $\mathbb{A}$-manifolds and their usual names.

| | Riemannian $\mathbb{A}$-manifolds | Special Riemannian $\mathbb{A}$-manifolds |
|---|---|---|
| $\mathbb{R}$ | $O(n)$<br>(Riemannian manifolds) | $SO(n)$<br>(Oriented Riemannian manifolds) |
| $\mathbb{C}$ | $U(n)$<br>(Kahler manifolds) | $SU(n)$<br>(Calabi-Yau manifolds) |
| $\mathbb{H}$ | $Sp(n)\,Sp(1)$<br>(Quaternionic-Kahler manifolds) | $Sp(n)$<br>(Hyperkahler manifolds) |
| $\mathbb{O}$ | $Spin(7)$<br>($Spin(7)$-manifolds) | $G_2$<br>($G_2$-manifolds) |

This list gives all possible holonomy groups of a (non-locally symmetric) irreducible Riemannian manifold $M$, as classified by Berger [4]. To phrase this differently, Riemannian manifolds with various holonomy groups are classified in terms of a normed division algebra $\mathbb{A}$ and its $\mathbb{A}$-orientability.

## 2.3 Characterizations and properties

For completeness, we briefly describe these (special) $\mathbb{A}$-manifolds and introduce certain natural differential forms on them that we will need later, see [5] or [31] for details. In the real case, (special) Riemannian $\mathbb{R}$-manifolds are simply (oriented) Riemannian manifolds. Orientability of $M$ allows us to determine a square root $\sqrt{\det(g_{ij})}$ consistently, and we obtain a parallel volume form $\nu_M$. Other (special) $\mathbb{A}$-manifolds also admit characterizations in terms of the existence of certain non-degenerate parallel forms. In all cases, harmonicity is already enough.

(1) $\mathbb{C}$-*manifolds* (i.e. *Kähler manifolds*), $Hol(g) \subset U(n)$. Since $U(n) = O(2n) \cap GL(n, \mathbb{C})$, the Levi-Civita connection $\nabla$ preserves an (almost) complex structure $J$, i.e.

$$J^2 = -1 \text{ and } \nabla J = 0.$$

This implies integrability of $J$. Alternatively we can use the Kähler form $\omega$,

$$\omega(v, w) = g(Jv, w) \text{ and } \nabla\omega = 0,$$

to characterize a Kähler manifold. Namely, a Hermitian metric on a complex manifold $M$ is Kähler if and only if $d\omega = 0$. It follows that every projective algebraic manifold inside $\mathbb{CP}^N$ is Kähler.

(2) *Special* $\mathbb{C}$-*manifolds* (i.e. *Calabi-Yau manifolds*), $Hol(g) \subset SU(n)$. By definition, these are Kähler manifolds with a parallel section of the canonical line bundle $K_M = \Lambda^n T_M^{*(1,0)}$, i.e. a parallel holomorphic volume form $\Omega \in$

$\Omega^{n,0}(M)$. By the celebrated result of Yau [36], such a structure always exist on any compact Kähler manifold with topological trivial $K_M$. This follows that a degree $d$ smooth hypersurface in $\mathbb{CP}^{n+1}$ admits a Calabi-Yau metric if and only if $d = n+2$. For instance, when $n = 2$, we have a quartic surface in $\mathbb{CP}^3$, that is a K3 surface.

(3) *Special $\mathbb{H}$-manifolds* (i.e. *hyperkähler manifolds*), $Hol(g) \subset Sp(n)$. Since $Sp(n)$ is the automorphism group of the quaternionic vector space $\mathbb{H}^n$, the Riemannian metric on such a manifold support three Kählerian complex structures $I$, $J$, $K$ satisfying the Hamilton relation

$$I^2 = J^2 = K^2 = IJK = -1.$$

A characterization by Hitchin says that a Riemannian metric on $M$ which is Hermitian with respect to three almost complex structures $I$, $J$ and $K$, satisfying the Hamilton relation and $d\omega_I = d\omega_J = d\omega_K = 0$, then its holonomy group is inside $Sp(n)$. This implies that the distinction between Kähler manifolds and symplectic manifolds no longer exist in the quaternionic case.

If we denote $\Omega_J = \omega_I + i\omega_K$, then $\Omega_J$ is a parallel holomorphic symplectic form on $M$. Using Yau's theorem, every compact Kähler manifold with a holomorphic symplectic form admits a hyperkähler structure.

When $\dim M = 4$, i.e. $n = 1$, hyperkähler manifolds are the same as Calabi-Yau manifolds because of $Sp(1) = SU(2)$. If $M$ is compact then it is either a flat torus of dimension four or a K3 surface. Using Yau's theorem, Fujiki and Beauville show that if $S$ is a compact hyperkähler manifold of dimension four, then the minimal resolution of the symmetric product of $S$ admits a natural hyperkähler structure.

(4) $\mathbb{H}$-*manifold*, (i.e. *quaternionic Kähler manifolds*), $Hol(g) \subset Sp(n)\,Sp(1)$. Such a manifold is similar to a hyperkähler manifold, however, the individual complex structures $I$, $J$ and $K$ can only be defined locally. The four form $\Theta = \omega_I^2 + \omega_J^2 + \omega_K^2$ is nevertheless well-defined and parallel. Examples include quaternionic projective spaces $\mathbb{HP}^n$.

(5) *Special $\mathbb{O}$-manifolds*, (i.e. *$G_2$-manifolds*), $Hol(g) \subset G_2$. Since $G_2 = H_{\mathbb{O}}$ stabilizes $1 \in \mathbb{O}$, it is really a subgroup of $SO(\operatorname{Im}\mathbb{O}) = SO(7)$. That implies that, up to covering, $M = X \times \mathbb{R}$ because of the deRham decomposition (see e.g. [5]). Traditionally a $G_2$-manifold is referred to the *seven* dimensional manifold $X$. The cross product on $\mathbb{O}$, defined as $x \times y = \operatorname{Im}\bar{y}x$, induces a product structure $\times$ on any $G_2$-manifold because $G_2$ is the automorphism group of the normed algebra $\mathbb{O}$. This determines a parallel (positive) three form $\Omega$,

$$\Omega(x, y, z) = \langle x, y \times z\rangle \text{ and } \nabla\Omega = 0$$

on any $G_2$-manifold $X$. Recall that the natural action of $GL(7, \mathbb{R})$ on $\Lambda^3\mathbb{R}^7$ has two open orbits, called positive and negative (see e.g. [13]). Those three forms in the same open orbit as the one given in the following table are called positive. Gray shows that any harmonic positive three form on $X$ determines a $G_2$-manifold structure on it. We will also use the parallel four form $\Theta = *\Omega$ later.

(6) $\mathbb{O}$-*manifolds*, (i.e. *Spin* (7)-*manifolds*), $Hol\,(g) \subset Spin\,(7) \subset SO\,(8)$. Gray shows that an eight dimensional manifold $M$ has holonomy group $Spin\,(7)$ if and only if it admits a harmonic (and hence parallel) self-dual four form $\Theta$. We have $\Theta = \Omega_{G_2} \wedge dx^0 - \Theta_{G_2}$ when our $\mathbb{O}$-manifold $M = X \times \mathbb{R}$ is special. Complete examples of (special) $\mathbb{O}$-manifolds are constructed by Bryant and Salamon [6] and compact examples are constructed by Joyce [14], and recently by Kovalev [17].

In the next table, we list the various parallel forms on $\mathbb{A}$-manifolds, in the Euclidean case:[2]

| | Riemannian $\mathbb{A}$-manifolds | Parallel forms |
|---|---|---|
| $\mathbb{R}$ | Oriented Riem. manifolds | On $\mathbb{R}^n$, $\nu_M = dx^1 dx^2 \cdots dx^n$ |
| $\mathbb{C}$ | Kähler manifolds | On $\mathbb{C}^n$, $\omega = \frac{i}{2}\left(dz^1 d\bar{z}^1 + \cdots + dz^n d\bar{z}^n\right)$ |
| | Calabi-Yau manifolds | On $\mathbb{C}^n$, $\omega$ and $\Omega = dz^1 \wedge \cdots \wedge dz^n$ |
| $\mathbb{H}$ | Quaternionic Kähler mfds | On $\mathbb{H}^n$, $\Theta = \omega_I^2 + \omega_J^2 + \omega_K^2$ |
| | Hyperkähler manifolds | On $\mathbb{H}^n$, $\omega_J = \frac{i}{2}\left(dz^1 d\bar{z}^1 + \cdots + dz^{2n} d\bar{z}^{2n}\right)$ <br> $\Omega_J = \omega_I + i\omega_K = dz^1 dz^2 + \cdots + dz^{2n-1} dz^{2n}$ |
| $\mathbb{O}$ | $Spin\,(7)$-manifolds | On $\mathbb{R}^8$, $\Theta = \Omega_{G_2} \wedge dx^0 - \Theta_{G_2}$ |
| | $G_2$-manifolds | On $\mathbb{R}^7$, $\Omega_{G_2} = dx^{123} - dx^1\left(dy^{23} + dy^{10}\right) + [123]$ <br> $\Theta_{G_2} = *\Omega_{G_2}$ |

Remark: Special Riemannian $\mathbb{O}$-manifolds (with $\mathbb{O} \neq \mathbb{R}$) are Calabi-Yau manifolds, hyperkähler manifolds and $G_2$-manifolds. These spaces are the central objects of interests in string theory, conformal field theory and M-theory in physics. From a mathematical point of view, they share many good geometric properties[3]: (1) their Ricci tensors are all zero, $Ricci = 0$. (2) If $M$ is compact with holonomy group equals $H_{\mathbb{A}}(n)$, then $\pi_1(M)$ is finite. (3) the moduli spaces of these metrics are always smooth, as shown by Bogomolov, Joyce, Tian and Todorov. (4) We can define a *period map* on the moduli space by integrating the parallel form $\Omega$ over topological cycles. Locally the period map determines the moduli space together with its Weil-Peterrson metric. (5) The first Pontriajin number of $M$ is non-negative, moreover it is zero if and only if $M$ is flat.

On real (resp. complex) manifolds, there are many local differentiable (resp. holomorphic) functions, which are used to describe the geometry of these manifolds. However on $\mathbb{H}$- (or $\mathbb{O}$-)manifolds, there are very few such functions.

[2] [123] means terms obtained by permuting the indexes insides the bracket.

[3] Many of these properties are also shared by $Spin\,(7)$-manifolds.

For instance, Gray shows that every quaternionic map is totally geodesic. In particular there are no quaternionic submanifolds in $\mathbb{HP}^N$ other than affine subspaces. Instead the geometries of these manifolds are reflected by their (Yang-Mills calibrated) $\mathbb{A}$-connections and (volume calibrated) $\mathbb{A}$-submanifolds and $\frac{1}{2}\mathbb{A}$-Lagrangian submanifolds.

# 3 Yang-Mills bundles

Suppose $E$ is a Hermitian vector bundle over $M$

$$\mathbb{C}^r \to E \to M.$$

A connection on $E$ gives a first order differential operator

$$D_E : \Omega^k (M, E) \to \Omega^{k+1} (M, E) .$$

Its square is a zeroth order operator, called the curvature tensor $F_E = (D_E)^2 \in \Lambda^2 (M, ad(E))$. If $D_E$ is a flat connection, i.e. $F_E = 0$, then holonomy around any point $x \in M$ gives a representation of the fundamental group,

$$\rho : \pi_1 (M, x) \to U (r) ,$$

and vice versa.

When $M$ is a complex manifold of complex dimension $n$, we can decompose differential forms into $(p, q)$-types,

$$\Lambda^2 (M, \mathbb{C}) = \Lambda^{2,0} (M) + \Lambda^{1,1} (M) + \Lambda^{0,2} (M) .$$

If $E$ is a *holomorphic* bundle over $M$ then it admits a connection with $F_E^{0,2} = F_E^{2,0} = 0$ and vice versa. Roughly speaking, holomorphic structures on $E$ are equivalent to *partial flat* connections on it. When $M$ is also Kähler, we have a further decomposition into primitive components (e.g. [5]),

$$\Lambda^{1,1} (M) = \Lambda_0^{1,1} (M) + \mathbb{C}\omega.$$

Hermitian connections on $E$ satisfying $F_E^{0,2} = F_E^{2,0} = 0$ and $F_E \wedge \omega^{n-1} = 0$ are called *Hermitian Yang-Mills* connections (with zero slope). They have absolute minimum Yang-Mills energy (see section 3.2). By a famous result of Donaldson, Uhlenbeck and Yau, the existence of such connections is equivalent to the Mumford poly-stability for $E$, a natural notion in algebraic geometry which is used in constructing moduli spaces. Note that we can rephrase these two equations as requiring $F_E \in \Lambda_0^{1,1} (M, ad(E))$.

Another familiar class of partial flat connections are ASD connections over an oriented Riemannian four manifold $M$. Using the isomorphism $SO(4) \cong Sp(1) Sp(1)$ we can decompose two forms into self-dual and anti-self-dual components,

$$\begin{aligned} \Lambda^2 (M) &= \Lambda_+^2 (M) + \Lambda_-^2 (M) \\ F &= F^+ + F^- = \frac{F + *F}{2} + \frac{F - *F}{2}. \end{aligned}$$

A Hermitian connection $D_E$ over $M$ is called an *instanton*, or *ASD* connection if the self-dual component of its curvature vanishes, $F_E^+ = 0$. Equivalently we have $F_E \in \Lambda_-^2 (M, ad(E))$. Again an ASD connection always have absolute minimum Yang-Mills energy. Donaldson studies their moduli space in details and obtain many nontrivial results in four dimensional differential topology [7].

We are going to generalize these and define natural classes of partial flat connections over Riemannian $\mathbb{A}$-manifolds.

## 3.1 (Special) $\mathbb{A}$-connections

Suppose that $M$ is a (special) Riemannian $\mathbb{A}$-manifold of dimension $m = 2^a n$, where $a = \dim \mathbb{A}$. Its holonomy group $\mathbf{G}_{\mathbb{A}}(n)$ (resp. $\mathbf{H}_{\mathbb{A}}(n)$) is a subgroup of $O(m)$. We denote their Lie algebra as $\mathbf{g}_{\mathbb{A}}(n)$ and $\mathbf{h}_{\mathbb{A}}(n)$ respectively. We have

$$\mathbf{h}_{\mathbb{A}}(n) \subset \mathbf{g}_{\mathbb{A}}(n) \subset \mathbf{o}(m).$$

Using the natural identification $\mathbf{o}(m) \cong \Lambda^2 \mathbb{R}^{m*}$, we obtain natural subbundles

$$\mathbf{h}_{\mathbb{A}}(T_M) \subset \mathbf{g}_{\mathbb{A}}(T_M) \subset \Lambda^2(M),$$

over a (special) $\mathbb{A}$-manifold $M$. In fact the subbundles $\mathbf{h}_{\mathbb{A}}(T_M) \subset \Lambda^2(M)$ is well-defined even for $\mathbb{A}$-manifolds as long as $\mathbb{A} \neq \mathbb{O}$. This is because $\mathbf{h}_{\mathbb{A}}(n)$ is an ideal in $\mathbf{g}_{\mathbb{A}}(n)$ in these cases. We define two natural classes of partial flat connections on an $\mathbb{A}$-manifold as follows.

**Definition 9** *Suppose $D_E$ is a connection on a vector bundle $E$ over a (special) Riemannian $\mathbb{A}$-manifold $M$. We denote its curvature two form as $F_E \in \Lambda^2(M, ad(E))$. Then*

*(1) $D_A$ a called an $\mathbb{A}$-connection if*

$$F_A \in \mathbf{g}_{\mathbb{A}}(T_M) \otimes ad(E),$$

*(2) $D_A$ is called a special $\mathbb{A}$-connection if*

$$F_A \in \mathbf{h}_{\mathbb{A}}(T_M) \otimes ad(E).$$

We are going to describe these partial flat connections individually. Most of them can be identified with well-known Yang-Mills connections in the literature, as indicated in the following table.

| | $\mathbb{A}$-connections | special $\mathbb{A}$-connections |
|---|---|---|
| $\mathbb{C}$ | $F_E^{0,2} = F_E^{2,0} = 0$ (Holomorphic bundles) | $F_E^{0,2} = F_E^{2,0} = \Lambda F = 0$ (Hermitian Yang-Mills bdls.) |
| $\mathbb{H}$ | $F \in \mathbf{g}_{\mathbb{H}}(T_M) \otimes ad(E)$ | $F_I^{0,2} = F_J^{0,2} = F_K^{0,2} = 0$ (ASD or hyperholomorphic bdls.) |
| $\mathbb{O}$ | $*F_E + \Theta \wedge F_E = 0$ ($Spin(7)$-Donaldson-Thomas bdls.) | $F \wedge \Theta = 0$ ($G_2$-Donaldson-Thomas bdls.) |

When $\mathbb{A}=\mathbb{R}$ we have $\mathbf{o}(n)=\mathbf{h}_{\mathbb{R}}(n)=\mathbf{g}_{\mathbb{R}}(n)$. Therefore a (special) $\mathbb{R}$-connection is simply any connection over $M$.

When $\mathbb{A}=\mathbb{C}$ we have $\mathbf{g}_{\mathbb{C}}(n)=\mathbf{u}(n)$ and $\mathbf{h}_{\mathbb{C}}(n)=\mathbf{su}(n)$. The Lie algebra $\mathbf{u}(n)$ consists of skew-Hermitian matrices. Using the Hermitian inner product on the vector space $V\cong\mathbb{C}^n$ to identify $V$ with $\bar{V}^*$, we obtain an identification $\mathbf{u}(n)=\left(V^*\otimes\bar{V}^*\right)\cap\Lambda^2V_{\mathbb{R}}^*$. Globally on $M$, we get $\mathbf{g}(T_M)\cong\Lambda^{1,1}(M)_{\mathbb{R}}$. Similarly the trace component in $\mathbf{u}(n)$ corresponds to the $\mathbb{R}$-span of the Kähler form $\omega$ in $\Lambda^{1,1}(M)_{\mathbb{R}}$. Thus in the primitive decomposition of two forms on $M$,

$$\Lambda^2(M)=\Lambda_0^{1,1}(M)_{\mathbb{R}}+\mathbb{R}\omega+\left(\Lambda^{2,0}(M)+\Lambda^{0,2}(M)\right)_{\mathbb{R}},$$

we have

$$\begin{aligned}\mathbf{g}(T_M)&\cong\Lambda^{1,1}(M)_{\mathbb{R}}\\ \mathbf{h}(T_M)&\cong\Lambda_0^{1,1}(M)_{\mathbb{R}}.\end{aligned}$$

Therefore a Hermitian connection $D_E$ over $M$ is a $\mathbb{C}$-connection iff it defines a holomorphic structure on $E$ and it is special if $F_E\wedge\omega^{n-1}=0$, i.e. a Hermitian Yang-Mills connection.

When $\mathbb{A}=\mathbb{H}$ we have $\mathbf{g}_{\mathbb{H}}(n)=\mathbf{sp}(n)\,\mathbf{sp}(1)$, $\mathbf{h}_{\mathbb{H}}(n)=\mathbf{sp}(n)$ and

$$T_M^*\otimes_{\mathbb{R}}\mathbb{C}=V\otimes_{\mathbb{C}}S,$$

well-defined up to $\otimes L^{1/2}$, for any $\mathbb{H}$-manifold $M$. In any event, $V\otimes V$ and $S\otimes S$ are always well-defined and we have the following decomposition of two forms on $M$,

$$\Lambda^2(M,\mathbb{C})=Sym^2V\otimes\Lambda^2S+\Lambda^2V\otimes Sym^2S.$$

Note that (i) $S$ being a $Sp(1)$-bundle implies $\Lambda^2S$ is trivial and (ii) $V$ is a symplectic bundle, using its symplectic form $\Omega\in\Lambda^2V^*\cong\Lambda^2V$, we have a further splitting $\Lambda^2V=\Lambda_0^2V+\mathbb{C}$. Hence

$$\Lambda^2(M,\mathbb{C})=Sym^2V+Sym^2S+\Lambda_0^2V\otimes Sym^2S.$$

When $\dim_{\mathbb{R}}M=4$, $Sym^2S$ (resp. $Sym^2V$) coincides with the space of self-dual (resp. anti-self-dual) two forms on $M$ and $\Lambda_0^2V\otimes Sym^2S$ is zero. We will continue to call a connection $D_E$ with its curvature $F_E$ inside $Sym^2V\otimes ad(E)$ an *anti-self-dual connection*, even though it is traditionally called a B-connection.

If we denote the standard representation of $\mathbf{sp}(n)$ as $V_0$, then the Lie algebra $\mathbf{sp}(n)$ is naturally identified with $Sym^2V_0$. This implies that

$$\begin{aligned}\mathbf{g}_{\mathbb{H}}(T_M)&\cong Sym^2V+Sym^2S,\\ \mathbf{h}_{\mathbb{H}}(T_M)&\cong Sym^2V.\end{aligned}$$

In particular a special $\mathbb{H}$-connection is the same as an anti-self-dual connection on $M$. Similar to Hermitian Yang-Mills connections, special $\mathbb{H}$-connections have absolute minimum Yang-Mills energy. On the other hand, on an oriented four manifold, every connection is a $\mathbb{H}$-connection because $\mathbf{sp}(1)+\mathbf{sp}(1)=\mathbf{so}(4)$.

Remark: There is also an identification,

$$Sym^2 V = \Lambda_I^{1,1} \cap \Lambda_J^{1,1} \cap \Lambda_K^{1,1}.$$

Therefore a connection is a special $\mathbb{H}$-connection if and only if it is holomorphic with respect to $I$, $J$ and $K$, and it is sometimes called a *hyperholomorphic* connection. On a hyperkähler manifold $M$, Verbitsky [35] shows that if $D_E$ is a Hermitian-Yang-Mills connection with respect to the Kähler structure $\omega_J$ and $c_1(E)$ and $c_2(E)$ are both $Sp(1)$ invariant cohomology class, then $D_E$ is a special $\mathbb{H}$-connection.

When $\mathbb{A} = \mathbb{O}$ we have $\mathbf{g}_{\mathbb{O}}(n) = \mathbf{spin}(7)$ and $\mathbf{h}_{\mathbb{O}}(n) = \mathbf{g}_2$. When $M$ is a $\mathbb{O}$-manifold, i.e. a $Spin(7)$-manifold, we have a natural decomposition of two forms [31],

$$\Lambda^2(M) = \Lambda^2_{21}(M) + \Lambda^2_7(M).$$

They are characterized as follows: for any $\phi \in \Lambda^2(M)$,

$$\begin{aligned} \phi \in \Lambda^2_{21}(M) & \text{ iff } \phi + *(\Theta \wedge \phi) = 0 \\ \phi \in \Lambda^2_7(M) & \text{ iff } 3\phi = *(\Theta \wedge \phi). \end{aligned}$$

Furthermore, we have

$$\mathbf{g}_O(T_M) \cong \Lambda^2_{21}(M).$$

Therefore the curvature of any $\mathbb{O}$-connection satisfies

$$F_E + *(\Theta \wedge F_E) = 0.$$

This equation, and its $G_2$-analog, are introduced by Donaldson and Thomas in [8].

When $M = X \times S^1$ is a $G_2$-manifold, then corresponding to the reduction from $SO(8)$ to $SO(7)$ we have a decomposition,

$$\Lambda^2(M) = \Lambda^2(X) + \Lambda^1(X) \wedge d\theta$$

where $\theta$ is the angle coordinate on $S^1$. On the other hand we have a similar decomposition of two forms [31] for the seven dimensional manifold $X$,

$$\Lambda^2(X) = \Lambda^2_{14}(X) + \Lambda^2_7(X),$$

with

$$\mathbf{h}_{\mathbb{O}}(T_M) \cong \Lambda^2_{14}(X).$$

A special $\mathbb{O}$-connection on $X$ is again a $G_2$-Donaldson-Thomas connection, i.e.

$$F_E \wedge \Theta = 0,$$

and these are absolute minimums for the Yang-Mills energy.

## 3.2 Yang-Mills calibrations

Suppose $E$ is a Hermitian vector bundle over an oriented Riemannian manifold $M$, with volume form $\nu_M$. The *Yang-Mills energy* of a Hermitian connection $D_A$ is defined as follows,

$$YM(D_A) = \int_M |F_A|^2 v_M.$$

The Euler-Lagrange equation is called the *Yang-Mills equation* and it is given by,

$$D_A^* F_A = 0.$$

Analogous to the volume calibration for minimal submanifolds (see [12] and section 4.4), we have the notion of Yang-Mills calibration for connections, which gives Yang-Mills connections with absolute minimal energy. This is a modification of the $\Omega$-ASD connections introduced by Tian in [34].

On a vector space $V \cong \mathbb{R}^m$ with a fixed volume form $\nu$, each element $\Phi$ in $\Lambda^{m-4}V^*$ defines a quadratic form $q_\Phi$ on $\Lambda^2 V^*$ as follows,

$$q_\Phi : \Lambda^2 V^* \to \mathbb{R}$$
$$q_\Phi(\phi) = \phi \wedge \phi \wedge \Phi / \nu_M.$$

**Definition 10** *Suppose $M$ is an oriented Riemannian manifold $M$ of dimension $m$, a differential form $\Phi \in \Omega^{m-4}(M)$ is called a Yang-Mills calibrating form if*

$$d\Phi = 0$$
$$q_\Phi(\phi) \leq |\phi|^2$$

*for any $\phi \in \Omega^2(M)$.*

**Definition 11** *Suppose $E$ is a Hermitian vector bundle over a manifold $M$ with a Yang-Mills calibrating form $\Phi$. A Hermitian connection $D_A$ on $E$ is called Yang-Mills calibrated by $\Phi$, or simply $\Phi$-calibrated, if its curvature tensor $F_A$ satisfies*[4]

$$q_\Phi(F_A) \leq |F_A|^2.$$

As in the volume calibration case, we have the following fundamental lemma.

**Lemma 12** *If $D_A$ is a $\Phi$-calibrated connection on $E$ and $D_{A'}$ is any other connection, then the Yang-Mills energy of $D_{A'}$ is smaller than or equal to that for $D_A$,*

$$YM(D_{A'}) \geq YM(D_A).$$

*Moreover, if the equality sign holds, then $D_{A'}$ is also $\Phi$-calibrated.*

[4] The quadratic form $q_\Phi$ is extended to $ad(E)$-valued two forms using the Killing form.

Proof:

$$\begin{aligned}YM\left(D_{A'}\right) &= \int_M \left|F_{A'}\right|^2 \nu_M \geq \int_M Tr\left(F_{A'}^2\right) \wedge \Phi \\ &= \int_M Tr\left(F_A^2\right) \wedge \Phi \quad \text{(since } d\Phi = 0\text{)} \\ &= \int_M \left|F_A\right|^2 \nu_M = YM\left(D_A\right).\end{aligned}$$

Hence the result. ■

The following Chern number inequality gives a topological constraint to the existence of $\Phi$-calibrated connections on $E$. It also give an effective way to characterize flat connections. The proof of it is simple and standard.

**Proposition 13** *If $E$ admits a $\Phi$-calibrated connection then we have*

$$\int_M ch\left(E\right) \Phi \leq 0$$

*and the equality sign holds iff $E$ is a flat bundle.*

Proof: This follows immediately from the Chern-Weil formula

$$ch\left(E\right) = \exp\left(\frac{i}{2\pi} F_E\right)$$

and the definition of a $\Phi$-calibrated connection. ■

For any Riemannian $\mathbb{A}$-manifold $M$, there is a natural Yang-Mills calibrating form and connections they calibrate are basically the same as special $\mathbb{A}$-connections over $M$. We list these Yang-Mills calibrating forms are the common names for the corresponding calibrated Yang-Mills connections.

| | Calibrating form, $\Phi$ | Yang-Mills connections |
|---|---|---|
| $\mathbb{R}$-manifold | $\Phi = 0$ | Flat connections |
| $\mathbb{C}$-manifold | $\Phi = \omega^{n-2}$ | Hermitian Yang-Mills connections |
| $\mathbb{H}$-manifold | $\Phi = \Theta^{n-1}$ | Anti-Self-Dual connections |
| $\mathbb{O}$-manifold | $\Phi = \Theta$ | Donaldson-Thomas connections |

For example, when $M$ is an oriented Riemannian four manifold, $\Phi = 1$ is a Yang-Mills calibrating form and connections it calibrates are precisely ASD connections, i.e. $F_A^+ = 0$.

# 4 Minimal submanifolds

On any general Riemannian $\mathbb{A}$-manifolds $M$, we introduce two natural classes of submanifolds: (i) $\mathbb{A}$-submanifolds and (ii) $\frac{1}{2}\mathbb{A}$-Lagrangian submanifolds. $\mathbb{A}$-submanifolds can be defined on any $\mathbb{A}$-manifold, even without a Riemannian metric (see section 6). $\frac{1}{2}\mathbb{A}$-Lagrangian submanifolds can be viewed as the *maximally real submanifolds* in $M$. For example, when $\mathbb{A} = \mathbb{C}$ they are (i) complex submanifolds and (ii) Lagrangian submanifolds in a Kähler manifold.

When $M$ is special, there are two natural subclasses of (ii), called special $\frac{1}{2}\mathbb{A}$-Lagrangian submanifolds of type I and type II. For example, if $M$ is a Calabi-Yau manifold, then they are special Lagrangian submanifolds of phase 0 and phase $\pi/2$.

These $\mathbb{A}$-submanifolds and special $\frac{1}{2}\mathbb{A}$-Lagrangian submanifolds are always absolute volume minimizers, as they are volume calibrated.

## 4.1 $\mathbb{A}$-submanifolds

We first discuss the linear case. Suppose $V$ is a linear $\mathbb{A}$-space. To discuss its linear $\mathbb{A}$-subspaces, we can assume that $\mathbb{A} \neq \mathbb{O}$, since every nontrivial linear $\mathbb{O}$-space is isomorphic to $\mathbb{O}$ itself. A linear $\mathbb{A}$-space is the same as a bi-module over $\mathbb{A}$, which is really an $\mathbb{A}$-vector space when $\mathbb{A}$ equals $\mathbb{R}$ or $\mathbb{C}$. A linear $\mathbb{A}$-subspace of $V$ is then a bi-submodule of $V$, i.e. a real vector subspace in $V$ which is stable under the left and right action of $\mathbb{A}$. To globalize these $\mathbb{A}$-subspace structures to any $\mathbb{A}$-manifold, we need to know that they are stable under twisted isomorphisms.

**Lemma 14** *Suppose $W$ is a linear $\mathbb{A}$-subspace of $\mathbb{A}^n$ and $\phi : \mathbb{A}^n \to \mathbb{A}^n$ is a twisted isomorphism of $\mathbb{A}^n$. Then the image of $\phi$ is also a linear $\mathbb{A}$-subspace of $\mathbb{A}^n$*

Proof: This is obvious when $\mathbb{A}$ is either $\mathbb{R}$ or $\mathbb{C}$ because twisted isomorphisms are the same as vector space isomorphisms. When $\mathbb{A} = \mathbb{H}$, we have $\phi = (\alpha, \beta) \in G_{\mathbb{H}}(n) = Sp(n)\, Sp(1)$. If $\beta = 1$ then $\phi$ is an automorphism of $\mathbb{A}^n$, thus it transforms linear $\mathbb{H}$-subspaces to one another. In any event, the action of $\beta \in Sp(1)$ on $\mathbb{A}^n$ is the diagonal action on the right, thus its also stabilize any linear $\mathbb{H}$-subspace of $\mathbb{A}^n$. Hence we have the result. ■

From the above lemma, we have the following well-defined notion.

**Definition 15** *Let $M$ be any Riemannian $\mathbb{A}$-manifold. A submanifold $C$ of $M$ is called a $\mathbb{A}$-submanifold if for any point $p$ in $C$, its tangent space $T_pC$ is a linear $\mathbb{A}$-subspace of $T_pM$.*

It is easy to see that any $\mathbb{A}$-submanifold is itself a Riemannian $\mathbb{A}$-manifold. In the real case, a $\mathbb{R}$-submanifold is simply an ordinary submanifold. In the complex case, a $\mathbb{C}$-submanifold in a Kähler manifold $M$ is equivalent to a complex submanifold of $M$. It always have absolute minimal volume by the Wirtinger formula, or via calibration theory.

In the quaternionic case, Gray [9] shows that a $\mathbb{H}$-submanifold in a quaternionic Kähler manifold $M$ is always a totally geodesic submanifold. In particular they are rather rare. Note that a submanifold $C$ in a hyperkähler manifold $M$ which is complex with respect to all $I$, $J$ and $K$, i.e. a hyperholomorphic submanifold, is a $\mathbb{H}$-submanifold of $M$.

Remark: If $f$ is an $\mathbb{A}$-isometry of a Riemannian $\mathbb{A}$-manifold $M$, then its fixed point set is always an $\mathbb{A}$-submanifold of $M$.

### 4.2 $\frac{1}{2}\mathbb{A}$-Lagrangian submanifolds

Symplectic geometry is a subject about Lagrangian submanifolds in a symplectic manifold $M$. If $M$ has a compatible Riemannian metric such that the symplectic form is parallel, then $M$ is a Kähler manifold. To rigidity Lagrangian submanifolds using the metric, we need $M$ to be a Calabi-Yau manifold and we consider those Lagrangian submanifolds $C$ which are *special* in the sense that $\operatorname{Im}\Omega|_C = 0$. They are an essential ingredient in the mirror symmetry conjecture as discovered by Strominger, Yau and Zaslow [32]. Recently we also realized that complex Lagrangian submanifolds in hyperkähler manifolds, Cayley submanifold in $Spin(7)$-manifolds, associative and coassociative submanifolds in $G_2$-manifolds all play important roles in conformal field theory, string theory and M-theory, and therefore on the geometry of these manifolds. As we will explain below, all these are (special) $\frac{1}{2}\mathbb{A}$-Lagrangian submanifolds and play the role of decomplexifications of (special) $\mathbb{A}$-manifolds.

First we study the linear case and we begin by reviewing linear Lagrangian subspaces in $\mathbb{C}^n$ with the standard Hermitian complex structure $J$ and the standard symplectic structure $\omega = \Sigma dx^j \wedge dy^j$. A real linear subspace $C$ of dimension $n$ in $\mathbb{C}^n$ is called *Lagrangian* if

$$\omega|_C = 0.$$

Equivalently it satisfies

$$JC \text{ is perpendicular to } C,$$

because

$$\omega(u, v) = g(Ju, v).$$

That is we have an orthogonal decomposition

$$\mathbb{C}^n = C \oplus JC$$

and therefore we can regard a Lagrangian subspace as giving a *real structure* on $\mathbb{C}^n$.

For example $\mathbb{R}^n$, the fixed point set of the complex conjugation, is a Lagrangian subspace in $\mathbb{C}^n$. In fact every Lagrangian submanifold in $\mathbb{C}^n$ can be brought to $\mathbb{R}^n$ by some unitary transformation of $\mathbb{C}^n$ and the set of all Lagrangian subspaces in $\mathbb{C}^n$ is a homogeneous space $U(n)/SO(n)$.

We are going to generalize this to other normed linear $\mathbb{A}$-spaces $V \cong \mathbb{A}^n$. Note that every imaginary element $u$ in $\mathbb{A}$ with unit length, i.e. $u \in S(\operatorname{Im}\mathbb{A})$, defines a complex structure $J_u$ on $V$,

$$J_u(y) = yu,$$

for any $y \in V$. This is because every normed algebra $\mathbb{A}$ is alternative, $(yu)u = y(u^2)$, and this implies that $(J_u)^2 = -1$, i.e. a complex structure on $V$. Now we define a $\frac{1}{2}\mathbb{A}$-Lagrangian linear subspace in $V$ to be a *maximally real* subspace in $V$.

**Definition 16** *Suppose $C$ is a middle dimensional real linear subspace in a normed linear $\mathbb{A}$-space $V$. It is called a $\frac{1}{2}\mathbb{A}$-Lagrangian linear subspace in $V$ if*

$$J_u C \perp C$$

*for any unit element $u \in L$ in some real linear subspace $L \subset \operatorname{Im}\mathbb{A}$ of $\dim L = \frac{1}{2}\dim\mathbb{A}$.*

This definition is equivalent to having an orthogonal decomposition

$$V = C \oplus J_u C,$$

for every $u \in L$ with $|u| = 1$.

**Theorem 17** *If $C$ is a $\frac{1}{2}\mathbb{A}$-Lagrangian linear subspace in $V$ and we write $L \subset \operatorname{Im}\mathbb{A}$ as in the above definition, then $C$ is a $J_v$-complex linear subspace of $V$ for any unit vector $v \in \operatorname{Im}\mathbb{A}$ perpendicular to $L$.*

Proof: This is an empty assertion when $\mathbb{A}$ is either $\mathbb{R}$ or $\mathbb{C}$. When $\mathbb{A} = \mathbb{H}$ with the standard complex structures $I$, $J$, $K$, we can assume, without loss of generality,

$$IC \perp C \text{ and } JC \perp C.$$

Using the associativity of $\mathbb{H}$ and $C$ being of middle dimensional, we have

$$KC = (IJ)C = I(JC) = C.$$

That is $C$ is a complex $K$-linear subspace of $V$, thus proving the assertion.

When $\mathbb{A} = \mathbb{O}$, the above arguments do not work since $\mathbb{O}$ is non-associative. First we claim that we can assume that the $\frac{1}{2}\mathbb{A}$-Lagrangian linear subspace $C$ in $V \cong \mathbb{O}$ contains $\mathbb{R}$. To see this, suppose that $u \in \operatorname{Im}\mathbb{O}$ with $|u| = 1$ satisfies $J_u C \perp C$, i.e.

$$\langle cu, c' \rangle = 0$$

for any $c, c' \in C$. We recall that for any $g \in Spin(7)$ there is an $\theta \in SO(8)$ such that $g(cu) = g(c)\theta(u)$. In particular, $u \in \operatorname{Im}\mathbb{O}$ implies that $\theta(u)$ is also imaginary because

$$0 = \langle u, 1 \rangle = \langle g(1)\theta(u), g(1) \rangle = |g(1)|^2 \langle \theta(u), 1 \rangle.$$

From $\langle cu, c' \rangle = 0$, we obtain

$$\langle g(c)\,\theta(u), g(c') \rangle = 0,$$

with $\theta(u) \in \operatorname{Im}\mathbb{O}$, $|\theta(u)| = 1$. That is

$$J_{\theta(u)} g(C) \perp g(C).$$

Using the fact that $Spin(7)$ acts transitively on the unit sphere in $\mathbb{O}$, we have therefore verified our claim.

Since $\mathbb{R} \subset C$, we have an orthogonal direct sum decomposition,

$$C = \mathbb{R} \oplus D$$

for some three dimension subspace $D \subset \operatorname{Im}\mathbb{O}$.

Second we claim that there is an orthogonal direct sum decomposition

$$\operatorname{Im}\mathbb{O} = D \oplus L.$$

The reason is, for any $d \in D$ and $u \in L$ with $|u| = 1$, we have

$$\langle u, d \rangle = \langle 1, d\bar{u} \rangle = -\langle 1, J_u(d) \rangle = 0$$

because $1 \in C$ which is perpendicular to $J_u(d) \in J_u C$.

Third we want to prove that $C$ is the real span of 1, $i$, $j$ and $k$, possibly after a $G_2$ rotation. Without loss of generality, we can assume that $i, j \in D$, by a $G_2$ rotation if necessary. We write

$$k = ij = d + u \in \operatorname{Im}\mathbb{O}$$

for some element $d \in D$ perpendicular to both $i$ and $j$, and for some $u \in L$. Then

$$\langle u, u \rangle = \langle u, k \rangle - \langle u, d \rangle = \langle u, ij \rangle,$$

because $d$ and $u$ are perpendicular by the second claim. On the other hand,

$$\langle u, ij \rangle = \langle \bar{\imath}u, j \rangle = \langle \bar{\imath}, j\bar{u} \rangle = \langle i, ju \rangle = 0,$$

because $i \in C$ and $ju \in Cu$ are perpendicular to each other. This implies that $u = 0$, or equivalently, $k \in D$. That is $C$ is the real span of 1, $i$, $j$, $k$ and $L$ is the real span of $e$, $ei$, $ej$, $ek$ with $\mathbb{O} = \mathbb{H} \oplus e\mathbb{H}$. This, in particular, gives our theorem. ■

Remark: The converse to the above corollary is not true. For example $C = \mathbb{H} \times \{0\} \subset \mathbb{H}^2$ is $J$-linear but it is not a $\frac{1}{2}\mathbb{A}$-Lagrangian subspace of $V = \mathbb{H}^2$.

From the proof, we see that a $\mathbb{H}$-Lagrangian linear subspace in $\mathbb{O}$ is precisely $\mathbb{H} \times \{0\} \subset \mathbb{H}^2 = \mathbb{O}$ up to the action of $Spin(7)$. Such linear subspaces are studied by Harvey and Lawson and they are called *Cayley subspaces* of $\mathbb{O}$ and they are volume calibrated by the four form $\Theta$ on $\mathbb{O}$ (see [12] for various characterizations of Cayley subspaces). We therefore have the following corollary.

**Corollary 18** *$\mathbb{H}$-Lagrangian linear subspaces in $\mathbb{O}$ are equivalent to Cayley subspaces.*

Moreover we can justify the definition of $\frac{1}{2}\mathbb{A}$-Lagrangian linear subspace as being maximally real subspaces of $V$, in the following lemma.

**Corollary 19** *Suppose that $C$ is a middle dimensional real linear subspace in a normed linear $\mathbb{A}$-space $V$. If there is a real linear subspace $L' \subset \operatorname{Im}\mathbb{A}$ such that*

$$J_u C \perp C$$

*for any unit element $u \in L'$, then $\dim L' \leq \frac{1}{2}\dim\mathbb{A}$.*

For submanifolds in a Kähler manifold, the condition $JC \perp C$ is usually written as $\omega|_C = 0$, namely the Lagrangian condition. We want to do the same thing for all $\frac{1}{2}\mathbb{A}$-Lagrangian linear subspaces.

**Proposition 20** *There is an embedding of $\operatorname{Im}\mathbb{A}$ into the space of two forms on the real vector space $V_{\mathbb{R}}$,*

$$\operatorname{Im}\mathbb{A} \overset{\subsetneq}{\to} \Lambda^2 V_{\mathbb{R}}^*,$$
$$u \to \omega_u$$

*defined by*

$$\omega_u(x \otimes y) = \langle x, yu\rangle.$$

**Lemma 21** *Moreover this embedding intertwine the action of $G_{\mathbb{A}}(n)$ on $\Lambda^2 V_{\mathbb{R}}^*$ via the inclusion $G_{\mathbb{A}}(n) \subset O(m)$, and a natural action of $G_{\mathbb{A}}(n)$ on $\operatorname{Im}\mathbb{A}$ given as follows: When $\mathbb{A} = \mathbb{C}$ or $\mathbb{H}$ the action is given by the composition of $\lambda_{\mathbb{A}}(n)$ and the conjugation; when $\mathbb{A} = \mathbb{O}$, the action is given by $Spin(7) \to SO(\operatorname{Im}\mathbb{O})$, $g \to \theta_g$.*

Proof: First $\omega_u$ is a two form because when $u \in \operatorname{Im}\mathbb{O}$, we have

$$\omega_u(x \otimes x) = \langle x, xu\rangle = |x|^2 \langle 1, u\rangle = 0.$$

Next we prove the compatibility with respect to the actions by $G_{\mathbb{A}}(n)$. In the complex case, it follows from $G_{\mathbb{A}}(n) = U(n)$ acts trivially on $\operatorname{Im}\mathbb{C}$. In the quaternionic case, $(\alpha, \beta) \in Sp(n)\,Sp(1)$ acts on $u \in \operatorname{Im}\mathbb{H}$ and gives $\beta^{-1}u\beta$. We compute,

$$\left\langle \alpha x\beta, \alpha y\beta\left(\beta^{-1}u\beta\right)\right\rangle = \langle x\beta, yu\beta\rangle = \langle x, yu\rangle,$$

and we have the claim in this case. For the octonionic case, for any $g \in Spin(7)$, we have

$$g(yu) = g(y)\,\theta_g(u)$$

for any $y, u \in \mathbb{O}$. We compute

$$\omega_u(x \otimes y) = \langle x, yu\rangle = \langle g(x), g(yu)\rangle = \langle g(x), g(y)\,\theta_g(u)\rangle = \omega_{\theta_g(u)}(g(x) \otimes g(y)),$$

and hence the result. ■

Remark: This proposition implies that, on any Riemannian $\mathbb{A}$-manifold $M$, there is a real vector subbundle over $\Lambda^2(M)$ of rank $\dim\mathbb{A}-1$, denotes $\mathbf{s}_{\mathbb{A}}(T_M)$, such that each fiber is a copy of $\operatorname{Im}\mathbb{A}$.

We can identify $\operatorname{Im}\mathbb{A}\subset\Lambda^2V_{\mathbb{R}}^*\cong\mathbf{o}(m)$ explicitly. Obviously $\operatorname{Im}\mathbb{R}=0$. In the complex case, if we choose $u=i$ then $\omega_u$ is simply the standard Kähler form as can be easily checked. Similarly, in the quaternionic case, $\omega_i$, $\omega_j$ and $\omega_k$ are the standard Kähler forms for the complex structures $I$, $J$ and $K$ respectively. In particular we have the following decomposition,

$$\mathbf{g}_{\mathbb{A}}(n)=\mathbf{h}_{\mathbb{A}}(n)+\operatorname{Im}\mathbb{A},$$

for $\mathbb{A}\neq\mathbb{O}$. In the octonionic case, we have

$$\operatorname{Im}\mathbb{A}\cong\Lambda^2_7\mathbb{O}$$

where

$$\Lambda^2\mathbb{O}=\Lambda^2_{21}\mathbb{O}+\Lambda^2_7\mathbb{O},$$

is the decomposition of $\Lambda^2\mathbb{O}$ into irreducible $Spin(7)$-representations. This can be verified either by identifying the $Spin(7)$representation $\Lambda^2_7\mathbb{O}$ as given by $g\to\theta_g$, or simply by checking the dimensions of these irreducible pieces. Thus we have obtained the first part of the following proposition.

**Proposition 22** *Suppose that $V$ is a normed linear $\mathbb{A}$-space and we denote the image of $\operatorname{Im}\mathbb{A}$ in $\Lambda^2V_{\mathbb{R}}^*$ as $\mathbf{s}_{\mathbb{A}}$. There is a decomposition of $\mathbf{g}_{\mathbb{A}}(n)$-representations,*

$$\mathbf{g}_{\mathbb{A}}(n)=\mathbf{h}_{\mathbb{A}}(n)+\mathbf{s}_{\mathbb{A}},$$

*when $\mathbb{A}\neq\mathbb{O}$ and $\Lambda^2\mathbb{O}=\Lambda^2_{21}\mathbb{O}+\mathbf{s}_{\mathbb{O}}$.*

*Moreover for any $u\in\operatorname{Im}\mathbb{A}$ with unit length, we have a natural complex structure $J_u$ on $V$ and $\omega_u\in\mathbf{s}_{\mathbb{A}}\subset\Lambda^2V_{\mathbb{R}}^*$ satisfying*

$$V=C\oplus J_uC \text{ if and only if } \omega_u|_C=0.$$

Proof: The first half is proven above. The proof for the second half is the same as in the Kähler case, namely for any $x,y\in C$, we have

$$\omega_u(x\wedge y)=\langle x,yu\rangle=\langle x,J_uy\rangle.$$

The claim follows from the middle dimensionality of $C$. ■

As a corollary, we obtain an equivalent definition of a $\frac{1}{2}\mathbb{A}$-Lagrangian linear subspace which resemblance the definition of an ordinary Lagrangian subspace.

**Corollary 23** *Suppose that $C$ is a half dimensional real linear subspace in a normed linear $\mathbb{A}$-space $V$. Then $C$ is a $\frac{1}{2}\mathbb{A}$-Lagrangian linear subspace if and only if there is a linear subspace $L \subset \operatorname{Im} \mathbb{A} \cong \mathbf{s}_{\mathbb{A}} \subset \Lambda^2 V_{\mathbb{R}}^*$ with $\dim L = \frac{1}{2} \dim \mathbb{A}$ such that*

$$\omega|_C = 0$$

*for any $\omega \in L$.*

Using the above proposition and the proof of the earlier theorem which describe explicitly $\mathbb{H}$-Lagrangian linear subspaces in $\mathbb{O}$, we obtain the following corollary.

**Corollary 24** *If $C$ is a real four dimensional linear subspace of $\mathbb{O}$, then it is a $\mathbb{H}$-Lagrangian linear subspace if and only if the homomorphism defined by restricting differential forms,*

$$\Lambda_7^2(\mathbb{O}) \to \Lambda^2(C)$$

*has a four dimensional kernel. Moreover this happens exactly when the image of the above homomorphism is $\Lambda_+^2(C)$.*

The next proposition is basically well-known (see [12] for the proof in the octonion case).

**Proposition 25** *The space of $\frac{1}{2}\mathbb{A}$-Lagrangian linear subspaces in $\mathbb{A}^n$ is a homogeneous space of $G_{\mathbb{A}}(n)$. Explicitly they are $U(n)/SO(n)$, $Sp(n)Sp(1)/U(n)U(1)$ and $Spin(7)/Sp(1)^3$ for $\mathbb{A} = \mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ respectively.*

After all these studies of $\frac{1}{2}\mathbb{A}$-Lagrangian in the linear case, we come to the definition of a $\frac{1}{2}\mathbb{A}$-Lagrangian submanifold.

**Definition 26** *A middle dimensional real submanifold $C$ in a Riemannian $\mathbb{A}$-manifold $M$ is called a $\frac{1}{2}\mathbb{A}$-Lagrangian submanifold of $M$ if there is a vector subbundle[5] $L \subset \mathbf{s}_{\mathbb{A}}(T_M) \subset \Lambda^2(M)$ of $\operatorname{rank}(L) = \frac{1}{2}\dim \mathbb{A}$ such that*

$$\omega|_C = 0$$

*for any smooth section $\omega \in \Gamma(M, L)$.*

Remark: Unlike Lagrangian submanifolds in a Kähler manifold, $\mathbb{C}$-Lagrangian submanifolds in a quaternionic Kähler manifold are not widely studied in the literature. Nonetheless, every surface in an oriented four manifold is a $\mathbb{C}$-Lagrangian submanifold.

[5] In fact we only need the subbundle $L$ to be defined over $C$.

The following table summarizes $\frac{1}{2}\mathbb{A}$-Lagrangian submanifolds in Riemannian $\mathbb{A}$-manifolds and their common names.

| | $\frac{1}{2}\mathbb{A}$-Lagrangian submanifolds |
|---|---|
| $\mathbb{C}$ | $\omega\|_C = 0$<br>(Lagrangian submanifolds) |
| $\mathbb{H}$ | $\omega\|_C = 0$ for all $\omega \in L \subset Sym^2 S$ w/ rank$(L) = 2$ |
| $\mathbb{O}$ | $\omega\|_C = 0$ for all $\omega \in L \subset \Lambda^2_7(M)$ w/ rank$(L) = 4$<br>(Cayley submanifolds) |

## 4.3 Special $\frac{1}{2}\mathbb{A}$-Lagrangian submanifolds

Next we introduce *special* $\frac{1}{2}\mathbb{A}$-Lagrangian submanifolds inside special $\mathbb{A}$-manifolds. As before, we start with the linear theory about special $\frac{1}{2}\mathbb{A}$-Lagrangian linear subspaces in $\mathbb{A}^n$.

For example in the complex case, if $C$ is a Lagrangian linear subspace of $V \cong \mathbb{C}^n$, then there is an $A \in SU(n)$ such that

$$A(C) = e^{i\theta}\mathbb{R}^n$$

for some angle $\theta$, which is usually called the *phase* of $C$. When $C$ is a Lagrangian submanifold in a Calabi-Yau manifold $M$, then the gradient of $\theta$ is the mean curvature vector of $C$ in $M$. In fact, if $\theta$ is constant over $C$, then $C$ is a minimal submanifold in $M$ with absolute minimal volume and it is called a special Lagrangian submanifold in $M$ with phase $\theta$. We want to generalize this concept to other special $\mathbb{A}$-manifolds. First we need the following definitions, which are roughly the *determinants* of $C$.

**Definition 27** *For any $\frac{1}{2}\mathbb{A}$-Lagrangian subspace $C$ in $V \cong \mathbb{A}^n$ we define another $\frac{1}{2}\mathbb{A}$-Lagrangian subspace $\lambda(C)$ in $\mathbb{A}$ as follows:*

*When $\mathbb{A} = \mathbb{C}$, $\Lambda(C)$ is the image of $\Lambda^n C$ under the following composition of natural homomorphisms,*

$$\Lambda^n V_{\mathbb{R}} \to \left(\Lambda^{n,0}(V) \oplus \Lambda^{0,n}(V)\right)_{\mathbb{R}} \cong \Lambda^{n,0}(V) = \Lambda^n(V) \cong \mathbb{C}.$$

*Here the first homomorphism is the orthogonal projection to the Hodge $(p,q)$-decomposition.*

*When $\mathbb{A} = \mathbb{H}$, any $\mathbb{C}$-Lagrangian submanifold is holomorphic with respect to a unique complex structures $\pm J_u$ with $u \in \operatorname{Im}\mathbb{H}$. $\lambda(C)$ is the complex line in $\mathbb{H} \cong \mathbb{C}^2$ corresponding to $u$ under the following identification,*

$$S(\operatorname{Im}\mathbb{H})/\mathbb{Z}_2 \cong S\left(\Lambda^2_+\mathbb{H}\right)/\mathbb{Z}_2 \cong \mathbb{P}_{\mathbb{C}}\left(\mathbb{C}^2\right)/\mathbb{Z}_2.$$

*$\lambda(C) \subset \mathbb{H}$ is only well-defined up to replacing it by its orthogonal complement $\lambda(C)^{\perp} \subset \mathbb{H}$.*

*When $\mathbb{A} = \mathbb{O}$, we simply define $\lambda(C) = C$.*

Remark: In the complex case, $C$ being a Lagrangian in $V$ implies that $\lambda(C)$ is a line in $\Lambda^n V \cong \mathbb{C}$, thus also a $\mathbb{R}$-Lagrangian by trivial reason. It can be checked directly that this is the line in $\mathbb{C}$ with slope $\tan\theta$.

**Definition 28** *A $\frac{1}{2}\mathbb{A}$-Lagrangian linear subspace $C$ in a normed linear $\mathbb{A}$-space $V \cong \mathbb{C}^n$ is called special of type I (resp. type II) if $1 \in \lambda(C) \subset \mathbb{A}$ (resp. $1 \in \lambda(C)^{\perp} \subset \mathbb{A}$).*

Remark: When $\mathbb{A} = \mathbb{H}$, type I and type II $\mathbb{C}$-Lagrangian subspaces are equivalent due to quotient by $\{\pm 1\}$ in $G_{\mathbb{H}}(n) = Sp(n) \times Sp(1) / \pm 1$.

Remark: As in the $\frac{1}{2}\mathbb{A}$-Lagrangian case, the space of special $\frac{1}{2}\mathbb{A}$-Lagrangian linear subspaces in $\mathbb{A}^n$ is a homogeneous space of $H_{\mathbb{A}}(n)$. Explicitly they are the fibers of the following fiber bundles (see [12] for the proof in the octonion case),

$$\begin{array}{lllllll}
\frac{SU(n)}{SO(n)} & \to & \frac{U(n)}{SO(n)} & \to & \frac{U(1)}{SO(1)} & = S^1 \\
\frac{Sp(n)}{U(n)} & \to & \frac{Sp(n)Sp(1)}{U(n)U(1)} & \to & \frac{Sp(1)}{U(1)} / \pm 1 & = S^2 / \pm 1 \\
\frac{G_2}{Sp(1)^2} & \to & \frac{Spin(7)}{Sp(1)^3} & \to & \frac{S^7}{Sp(1)} & = S^4.
\end{array}$$

To define the corresponding notion for submanifolds, we recall that there is a canonical $\mathbb{A}$-bundle $\mathbb{A}_M$ over any Riemannian $\mathbb{A}$-manifold $M$ corresponding to the representation $\lambda_{\mathbb{A}}(n) : G_{\mathbb{A}}(n) \to O(\mathbb{A})$, which is trivial when $M$ is special. We fix a trivialization compatible with the action by $\mathbb{A}$ and let $s$ be the section of this bundle corresponding to $1 \in \mathbb{A}$. From the above linear considerations, if $C$ is a $\frac{1}{2}\mathbb{A}$-Lagrangian submanifold in $M$, the $\lambda(C)$ is a subbundle of $\mathbb{A}_M$ restricted to $C$.

**Definition 29** *A $\frac{1}{2}\mathbb{A}$-Lagrangian submanifold $C$ in a special Riemannian $\mathbb{A}$-manifold $M$ is called special of type I (resp. type II) if $s \in \lambda(C) \subset \mathbb{A}_M$ (resp. $s \in \lambda(C)^{\perp} \subset \mathbb{A}_M$).*

Note that $\mathbb{C}$-Lagrangians of type I and type II are the same.

We have the following characterizations of special $\frac{1}{2}\mathbb{A}$-Lagrangian submanifolds, together with their common names.

| | Special $\frac{1}{2}\mathbb{A}$-Lagr. submfd. (type I) | Special $\frac{1}{2}\mathbb{A}$-Lagr. submfd. (type II) |
|---|---|---|
| $\mathbb{C}$ | $\omega\|_C = \operatorname{Im}\Omega\|_C = 0$ <br> (special Lagr. submfd. with phase 0) | $\omega\|_C = \operatorname{Re}\Omega\|_C = 0$ <br> (special Lagr. submfd. with phase $\pi/2$) |
| $\mathbb{H}$ | $\Omega\|_C = 0$ <br> (Complex Lagrangian submanifolds) | $\Omega\|_C = 0$ <br> (Complex Lagrangian submanifolds) |
| $\mathbb{O}$ | $\times$ preserves $C$ (or $\chi\|_C = 0$) <br> (Associative submanifolds) | $\Omega\|_C = 0$ <br> (Coassociative submanifolds) |

## 4.4 Volume calibrations

This short subsection is included for completeness, readers should consult the paper [12] by Harvey and Lawson for a careful treatment.

**Definition 30** *(1) A differential form $\Phi \in \Omega^k(M)$ in an oriented Riemannian manifold $M$ is called a volume calibrating form if*

$$\begin{aligned} d\Phi &= 0 \\ \Phi|_P &\leq dv_P \quad \text{for all } P \in \widetilde{Gr}(k, T_M). \end{aligned}$$

*(2) A k-dimensional submanifold $C \subset M$ is calibrated by $\Phi$ if*

$$\Phi|_C = dv_C.$$

We have the following fundamental lemma.

**Lemma 31** *Any closed calibrated submanifold $C$ is homologically volume minimizing, i.e. $Vol(C) \leq Vol(C')$ provided $C$ and $C'$ represent the same homology class in $M$. Moreover if $Vol(C) = Vol(C')$ then $C'$ is also calibrated.*

For Riemannian $\mathbb{A}$-manifolds, there are natural calibrating forms $\Phi$ and their calibrating submanifolds are closely related to $\mathbb{A}$-submanifolds. We list them in the following table.

| | Holonomy (Riemannian $\mathbb{A}$-manifolds) | Calibrating form (Calibrated submanifolds) |
|---|---|---|
| $\mathbb{R}$ | $O(n)$ (Riemannian manifolds) | $\Phi = \exp(\nu ol_M)$ (Points and $M$) |
| $\mathbb{C}$ | $U(n)$ (Kahler manifolds) | $\Phi = \exp(\omega)$ (Complex submanifolds) |
| $\mathbb{H}$ | $Sp(n)\,Sp(1)$ (Quaternionic Kahler mfds) | $\Phi = \exp(\Theta)$ (Quaternionic submanifolds) |
| $\mathbb{O}$ | $Spin(7)$ ($Spin(7)$-manifolds) | $\Phi = \exp(\Theta)$ (Cayley submanifods) |

For special Riemannian $\mathbb{A}$-submanifolds, there are further calibrating forms, which are closely related to special $\frac{1}{2}\mathbb{A}$-Lagrangian submanifolds above. They are listed in the next table.

| | Holonomy (Special Riem. $\mathbb{A}$-manifolds) | Calibrating form (Calibrated submanifolds) |
|---|---|---|
| $\mathbb{R}$ | $SO(n)$ (Oriented manifolds) | $\Phi = v_M$ (Whole manifold $M$) |
| $\mathbb{C}$ | $SU(n)$ (Calabi-Yau manifolds) | Type I: $\Phi = \operatorname{Re}\Omega$ (SLag with phase 0) Type II: $\Phi = \operatorname{Im}\Omega$ (SLag with phase $\pi/2$) |
| $\mathbb{H}$ | $Sp(n)$ (Hyperkahler manifolds) | $\Phi_1 = \operatorname{Re}\Omega_I^n$ and $\Phi_2 = \operatorname{Re}\Omega_K^n$ (Complex Lagrangian submanifolds) |
| $\mathbb{O}$ | $G_2$ ($G_2$-manifolds) | Type I: $\Phi = \Omega$ (Associative submanifods) Type II: $\Phi = \Theta$ (Coassociative submanifolds) |

Remark: A middle dimension submanifold $C$ in a hyperkähler manifold $M$ is a complex Lagrangian if $\Omega_J|_C = 0$. Since $\Omega_J = \omega_I + i\omega_K$ we have $\omega_I = \omega_K = 0$. In this case $\Omega_I|_C = \omega_J$ (since $\omega_K = 0$), so $\operatorname{Im}\Omega_I^n|_C = 0$, i.e. $C$ is calibrated by $\operatorname{Re}\Omega_I^n$. Similar for $\operatorname{Re}\Omega_K^n$. The converse is also true. It is also calibrated by $\omega_J^n$.

# 5 Geometry and duality

On a $\mathbb{C}$-manifold $M$ (i.e. Kähler manifold), its $\mathbb{C}$-geometry studies cycles $(C, D_E)$ with $C$ a complex submanifold in $M$ and $E$ a holomorphic bundle over $C$. In algebraic geometry, one also allow $C$ and $E$ to be *singular* and consider $D^b(M)$ the derived category of coherent sheaves on $M$. When $M$ is special (i.e. Calabi-Yau manifold) we would also require $D_E$ to be a special $\mathbb{C}$-connection, i.e. a Hermitian Yang-Mills connection over $C$.

On the other hand, the $\mathbb{R}$-Lagrangian geometry of a Kähler (or symplectic) manifold $M$ studies cycles $(C, D_E)$ with $C$ a Lagrangian submanifold and $D_E$ a unitary flat connection over $C$. The space of morphisms between these cycles are the Floer homology groups. When $M$ is special, we also study special Lagrangians $C$. For instance the mirror Calabi-Yau manifold is conjectured to be the moduli space of certain special Lagrangian cycles, as in the SYZ mirror conjecture.

A novelty about geometry for manifolds with special holonomy is the *duality* transformation. For example, the mirror symmetry among Calabi-Yau manifolds (see e.g. [22]), motivated from physics, is still very mysterious to mathematicians. From the work of Strominger, Yau and Zaslow [32], we expect that it is a fiberwise Fourier transformation along a special Lagrangian torus fibration[6].

Fourier transformation on tori are well-studied in mathematics. We review it from our point of view: namely it should transform the $\mathbb{A}'$-geometry of one torus to the $\mathbb{A}'$-geometry of its dual torus, with $\mathbb{A}' = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. If we complexify a torus $T^n$ to $M = T^n \times i\mathbb{R}^n$, then it is a special $\mathbb{A}$-manifold with $\mathbb{A}' = \frac{1}{2}\mathbb{A}$. Moreover it has a natural fibration by special $\frac{1}{2}\mathbb{A}$-Lagrangian tori given by projection. The fiberwise Fourier transformation, or the SYZ transformation, should transform the $\mathbb{A}$-geometry on $M$ to the $\frac{1}{2}\mathbb{A}$-Lagrangian geometry on $W = T^{n*} \times i\mathbb{R}^{n*}$, and vice versa.

## 5.1 $\mathbb{A}$-geometry and $\frac{1}{2}\mathbb{A}$-Lagrangian geometry

Suppose $M$ is a Riemannian $\mathbb{A}$-manifold. We consider (i) the geometry of $\mathbb{A}$-cycles and (ii) the geometry of $\frac{1}{2}\mathbb{A}$-Lagrangian cycles on $M$.

**Definition 32** *Suppose $M$ is a Riemannian $\mathbb{A}$-manifold. A pair $(C, D_E)$ is called an (i) $\mathbb{A}$-cycle if $C$ is a $\mathbb{A}$-submanifold in $M$ and $D_E$ is a $\mathbb{A}$-connection over $C$ or (ii) $\frac{1}{2}\mathbb{A}$-Lagrangian cycle if $C$ is a $\frac{1}{2}\mathbb{A}$-Lagrangian submanifold in $M$ and $D_E$ is a special $\frac{1}{2}\mathbb{A}$-connection over $C$.*

[6] We also need a Legendre transformation along the base [21].

The following table gives the common names of these cycles.

| | $\mathbb{A}$-cycle on $M$ | $\frac{1}{2}\mathbb{A}$-Lagrangian cycle on $M$ |
|---|---|---|
| $\mathbb{C}$ | Complex submanifold + Holomorphic bundle | Lagrangian submanifold + Unitary flat bundle |
| $\mathbb{H}$ | Quaternionic submanifold + Bundle with $\mathbb{C}$-connection | $\mathbb{C}$-Lagrangian submanifold + Holomorphic bundle |
| $\mathbb{O}$ | The whole manifold $M$ + $Spin(7)$-Donaldson-Thomas bundle | Cayley submanifold + Anti-Self-Dual bundle |

**Definition 33** *Suppose $M$ is a special Riemannian $\mathbb{A}$-manifold. A pair $(C, D_E)$ is called an (i) special $\mathbb{A}$-cycle if $C$ is an $\mathbb{A}$-submanifold in $M$ and $D_E$ is a special $\mathbb{A}$-connection over $C$ or (ii) special $\frac{1}{2}\mathbb{A}$-Lagrangian cycle (or type I or II) if $C$ is a special $\frac{1}{2}\mathbb{A}$-Lagrangian submanifold (or type I or II) in $M$ and $D_E$ is a special $\frac{1}{2}\mathbb{A}$-connection over $C$.*

Notice that all special cycles are calibrated. For example a special $\mathbb{C}$-cycle $(C, D_E)$ in a Kähler manifold is calibrated by $\exp \omega$ because a complex submanifold $C$ of dimension $2k$ is volume calibrated by $\omega^k$ and a Hermitian Yang-Mills connection over $C$ is Yang-Mills calibrated by $\omega^{k-2}$. The following table gives the common names of special cycles and their calibrating forms.

| | Special $\mathbb{A}$-cycle (calibrating form) | Special $\frac{1}{2}\mathbb{A}$-Lagr. cycle, I (calibrating form) | Special $\frac{1}{2}\mathbb{A}$-Lagr. cycle, II (calibrating form) |
|---|---|---|---|
| $\mathbb{R}$ | Points or $M$ + Unitary flat bundle ($\exp \nu_M$) | n/a | n/a |
| $\mathbb{C}$ | Complex submanifold + Herm. Yang-Mills bundle ($\exp \omega$) | Lagr. submfd phase 0 + Unitary flat bundle ($\exp(\mathrm{Re}\,\Omega)$) | Lagr. submfd phase $\pi/2$ + Unitary flat bundle ($\exp(\mathrm{Im}\,\Omega)$) |
| $\mathbb{H}$ | Quaternionic submfd + ASD connection ($\exp \Theta$) | Complex Lagr. submfd + Herm. Yang-Mills bdl ($\exp(\mathrm{Re}\,\Omega_I^n)$ and $\exp \omega_J$) | Complex Lagr. submfd + Herm. Yang-Mills bdl ($\exp(\mathrm{Re}\,\Omega_I^n)$ and $\exp \omega_J$) |
| $\mathbb{O}$ | The manifold $M$ + $G_2$-Donaldson-Thomas bdl ($\exp \Theta_M$) | Associative submfd + Unitary flat bundle ($\exp \Omega_M$) | Coassociative submfd + ASD connection ($\exp \Theta_X$) |

(Special) $\mathbb{A}$-geometry studies (special) $\mathbb{A}$-cycles and (special) $\frac{1}{2}\mathbb{A}$-Lagrangian geometry studies (special) $\frac{1}{2}\mathbb{A}$-Lagrangian cycles on (special) Riemannian $\mathbb{A}$-manifolds.

Remark: (Special) $\mathbb{C}$-geometry and $\mathbb{R}$-Lagrangian geometry are basically the complex algebraic geometry and the symplectic geometry. Special $\mathbb{R}$-Lagrangian geometry is important in the SYZ mirror conjecture. Special $\mathbb{C}$-Lagrangian geometry is studied in [24] and is closely related to the classical Plucker formula. For (special) $\mathbb{O}$-manifolds, these geometries are discussed by Donaldson and Thomas [8], Hitchin, Gukov, Yau and Zaslow [11], Lee and the author [18], [20].

## 5.2 Fourier transformation of $\mathbb{A}$-geometry

First we review the Fourier transformation in geometry. Classically Fourier transformation is a duality between functions on a vector space $V \cong \mathbb{R}^n$ and on its dual vector space $V^*$. It is given by

$$f(x) \to \hat{f}(y) = \frac{1}{(2\pi)^n} \int_V f(x) e^{ix\cdot y} dx.$$

The Fourier transformation on the geometry of flat tori is similar. Suppose $T = V/\Lambda$ is any n dimensional torus, i.e. $\Lambda \cong \mathbb{Z}^n$ is a lattice in $V \cong \mathbb{R}^n$. The dual torus $T^*$ is defined as $V^*/\Lambda^*$ where $\Lambda^*$ consists of those $\phi \in V^*$ with $\phi(\Lambda) \subset \mathbb{Z}$. This relationship is reflexive, $(T^*)^* = T$. Moreover, $T^*$ can be naturally identified with the moduli space of flat $U(1)$-connections on $T$,

$$\mathcal{M}^{U(1)\text{-flat}}(T) \cong T^*.$$

On $T \times T^*$, there is *universal Poincaré line bundle* $\mathbf{L}$ with a universal connection, $\mathbf{D} = d + \pi i \sum_{j=1}^n \left(y^j dy_j - y_j dy^j\right)$, where $y_j$'s are coordinates on $T^*$ dual to the linear coordinates $y^j$'s on $T$. Its curvature, $\mathbf{F} = 2\pi i \Sigma dy^j \wedge dy_j$ plays the role of the kernel function in the classical Fourier transformation.

**Fourier transformation of $\mathbb{R}$-geometry on $T$**

On the topological level, the Fourier transformation is given by,

$$\mathcal{F} : H^k(T, \mathbb{Z}) \xrightarrow{\cong} H^{n-k}(T^*, \mathbb{Z})$$
$$\mathcal{F}(\phi) = \int_T \phi \wedge e^{\frac{i}{2\pi}\mathbf{F}},$$

and we also have a similar one for K-groups.

On the *flat* level, we consider flat bundles over $T$ or points in $T$, i.e. cycles $(C, D_E)$ that are calibrated by $\exp \nu_M$, where $\nu_M$ is the volume form on $T$. Any flat $U(r)$-bundle $E$ over $T$ is isomorphic to an direct sum of flat line bundles, unique up to permutations. This can be interpreted as an identification of moduli spaces of special $\mathbb{R}$-cycles on $T$ and on $T^*$, via the Fourier transformation.

**Fourier transform of $\mathbb{C}$-geometry on $T$**

When $T$ and $T^*$ are Abelian varieties, Mukai shows that the Fourier transformation,

$$\mathcal{F}(\cdot) = \mathbf{R}\pi_{1*}\left(\pi_2^*(\cdot) \otimes \mathcal{L}\right)$$

is an equivalence of derived categories of coherent sheaves, $D^b(T)$ and $D^b(T^*)$, with the inversion property. This has far reaching implications in the theory of Abelian varieties.

**Fourier transform of $\mathbb{H}$-geometry on $T$**

Unfortunately, we only know of a low dimension example in this case: A flat torus $T$ of dimension four is a special $\mathbb{H}$-manifold. Braam and Schenk (see e.g. [8]) show that the Fourier transformation of an ASD connection over $T$ without any flat factor is another ASD connection over $T^*$. Moreover this bijection between their moduli spaces is an isometry with respect to the Weil-Peterrson $L^2$-metrics.

In summary we expect that the Fourier transform on flat $\mathbb{A}$-tori gives a correspondence:

$$\mathbb{A}\text{-Geometry}(T) \longleftrightarrow \mathbb{A}\text{-Geometry}(T^*).$$

## 5.3 $\frac{1}{2}\mathbb{A}$-Lagrangian fibrations

On a Riemannian $\mathbb{A}$-manifold $M$, a $\frac{1}{2}\mathbb{A}$-*Lagrangian fibrations*

$$f : M \to B$$

is a smooth map $f$ such that its generic fibers are smooth $\frac{1}{2}\mathbb{A}$-Lagrangian submanifolds of $M$. We usually also assume that $f$ has a section which is also a $\frac{1}{2}\mathbb{A}$-Lagrangian submanifold.

$\frac{1}{2}\mathbb{A}$-Lagrangian fibrations have been playing important roles in various branches of mathematics and physics. In symplectic geometry, a Lagrangian fibration with a Lagrangian section is a very important structure. It is sometimes called a completely integrable system, or a real polarization when we try to quantize the symplectic manifold. Familiar examples include toric varieties.

In string theory, Strominger, Yau and Zaslow propose that mirror symmetry should be explained in terms of the fiberwise Fourier transformation along special Lagrangian fibrations on mirror Calabi-Yau manifolds [32].

Heuristically, one should view a $\frac{1}{2}\mathbb{A}$-Lagrangian fibration with a section on $M$ as a *global de-complexification* of $M$.

Remark: When $\mathbb{A}$ equals $\mathbb{C}$ or $\mathbb{H}$, a generic fiber of any $\frac{1}{2}\mathbb{A}$-Lagrangian fibration is a torus. However in a $Spin(7)$-manifold or a $G_2$-manifold, such a fiber is expected to be either a torus or a K3 surface, at least near a adiabatic limit (see [18]).

## 5.4 Mirror duality between different geometries

Following SYZ proposal, we would be interested in a fiberwise Fourier transformation on any Riemannian $\mathbb{A}$-manifold $M$ with a given $\frac{1}{2}\mathbb{A}$-Lagrangian fibration and a section. First we want to construct a dual torus fibration

$$g : W \to B.$$

First this would require each fiber of $f$ to have the same dimension. Typically, $M$ would then be a special $\mathbb{A}$-manifold, except in the octonionic case. Second, in order the perform the Fourier transformation on fibers, we require these fiber tori to be flat, at least in the limit. To be more precise, we assume that $M$ has an one parameter family of such metrics parametrized by $t \in [0, \infty)$, such that as $t \to \infty$ the second fundamental form of each smooth fiber goes to zero. This is related to the so-called *large structure limit* in the physics literature. Then one expects that (i) the total space of the dual torus fibration is also a special $\mathbb{A}$-manifold and (ii) the fiberwise Fourier transformation will give an equivalence of geometries:

$$\text{(Special) } \mathbb{A}\text{-Geometry}\,(M) \longleftrightarrow \text{(Special) } \frac{1}{2}\mathbb{A}\text{-Lagrangian Geometry}\,(W)\,.$$

Moreover the relationship between $M$ and $W$ should be reflexive.

The above picture has been over-simplified. There are many subtleties involved. Many of them are related to quantum corrections, an issue we have not addressed here. The most famous example is the mirror symmetry conjecture for special $\mathbb{C}$-manifolds (i.e. Calabi-Yau manifolds). The fiberwise Fourier transformation in this case has been studied by many people including Gross, Hitchin, Kontsevich, Ruan, Vafa, Witten, Yau, Zaslow, the author and many others.

We would indicate how the fiberwise Fourier transformation works in the simplest situation and show how the symplectic structure (determined by $\exp\omega$) is being transformed to the complex structure (determined by $\Omega$) on the mirror: Suppose $M = \mathbb{C}^n = \mathbb{R}^n \times \mathbb{R}^n$ with the standard complex structure $J$, i.e. $\Omega_M = dz^1 \wedge \cdots \wedge dz^n$, and with a symplectic structure $\omega_M = \Sigma \phi_{ij}(x)\, dx^i dy^j$ which is invariant under translations along $y$ directions. Then $W = \mathbb{R}^n \times \mathbb{R}^{n*}$ with coordinates $x^i$'s and $y_i$'s. By direct computations, under the fiberwise Fourier transformation on differential forms,

$$(\cdot) \to \int (\cdot)\, e^{\mathbf{F}} dy^1 dy^2 \cdots dy^n.$$

where $\mathbf{F} = i\Sigma dy^j \wedge dy_j$ is the universal curvature form, we have

$$\begin{aligned} \exp(\omega_M) \to \Omega_W &= \prod \left(\phi_{ij} dx^i + i dy_j\right) \\ \Omega_M \to \exp(\omega_W) &= \exp\left(\sum dx^i \wedge dy_i\right). \end{aligned}$$

Thus we see how variation of symplectic structures on $M$ corresponds to variation of complex structures on $W$ explicitly (see [21] for more details).

# 6 Remarks and questions

In this last section, we remark on some other aspects of geometry over $\mathbb{A}$ with $\mathbb{A} = \mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$, and mention a few interesting questions.

**Triality transformation**

As we discussed in section 5.4, mirror symmetry for Calabi-Yau manifolds and hyperkähler manifolds is a duality transformation for the geometry of these manifolds. In its simplest form, it can be viewed as the duality between a vector space and its dual vector space. The novelty about octonion in algebra is the *triality* (see e.g. [3]). It would be important to introduce the triality transformation to the geometry for $\mathbb{O}$-manifolds. The work of Gukov, Yau and Zaslow [11] might be related to this.

**Moduli space of connections**

In gauge theory, we consider the space of connections $\mathcal{A}$ on a bundle $E$ modulo the group of gauge transformations $\mathcal{G} = Aut(E)$, and call this the moduli space of connections on $E$. On a Kähler manifold $M$, the moduli space of special $\mathbb{C}$-connections (i.e. Hermitian Yang-Mills connections) is the symplectic reduction $\mu^{-1}(0)/\mathcal{G}$ of the space of $\mathbb{C}$-connections by $\mathcal{G}$, because

$$\begin{aligned} \mu : \mathcal{A} &\to \Omega^{2n}(M, ad(E)) \\ \mu(D_E) &= F_E \wedge \omega^{n-1} \end{aligned}$$

is the moment map for the action of $\mathcal{G}$ on $\mathcal{A}$. By the work of Donaldson, Uhlenbeck and Yau, we can also view this as the complex quotient of the space of $\mathbb{C}$-connections by $\mathcal{G}^{\mathbb{C}}$. Similarly on a hyperkähler manifold $M$, there is a hyperkähler moment map,

$$\begin{aligned} \mu : \mathcal{A} &\to \Omega^{4n}(M, ad(E)) \otimes \mathbb{R}^3 \\ \mu(D_E) &= \left(F_E \wedge \omega_I^{2n-1}, F_E \wedge \omega_J^{2n-1}, F_E \wedge \omega_K^{2n-1}\right), \end{aligned}$$

such that the moduli space of special $\mathbb{H}$-connections on $E$ is the hyperkähler quotient of $\mathcal{A}$ by $\mathcal{G}$, restricted to the space of $\mathbb{H}$-connections. On a $G_2$-manifold $M = X \times S^1$, one does not have the notion of an octonionic quotient. However we still can define $\mu$ in the same way and its zeros correspond to connections pullback from $X$. Therefore we have a similar picture as before, namely the space of $Spin(7)$-Donaldson-Thomas connections on $M$ with $\mu = 0$ quotienting by $\mathcal{G}$ is the moduli space of $G_2$-Donaldson-Thomas connections on $X$.

**$\mathbb{A}$-manifolds without metrics**

Real and complex manifolds are usually defined using coordinate charts and requiring their transition functions to be differentiable and holomorphic respectively. However such definitions can not be generalized to the quaternionic and octonionic cases. For example any smooth map which preserves quaternionic structures must be affine (e.g. [5]). The correct generalization is to use a torsion free connection on $M$ preserving an $\mathbb{A}$-structure on the frame bundle: We denote by $\mathcal{G}_{\mathbb{A}}(n)$ the group of twisted isomorphisms $\phi$ of $\mathbb{A}^n$, but do not require $\phi$ to be an isometry. Similarly, we have $\mathcal{H}_{\mathbb{A}}(n)$ for special twisted isomorphisms.

**Definition 34** *A smooth manifold $M$ is called a (special) $\mathbb{A}$-manifold if the structure group of its frame bundle has a reduction to $\mathcal{G}_{\mathbb{A}}(n)$ (resp. $\mathcal{H}_{\mathbb{A}}(n)$) together with a torsion free connection.*

The following table gives explicit descriptions of $\mathcal{G}_{\mathbb{A}}(n)$ and $\mathcal{H}_{\mathbb{A}}(n)$, together with the usual names for these manifolds.

| | $\mathcal{G}_{\mathbb{A}}(n)$<br>($\mathbb{A}$-manifolds) | $\mathcal{H}_{\mathbb{A}}(n)$<br>(Special $\mathbb{A}$-manifolds) |
|---|---|---|
| $\mathbb{R}$ | $GL(n,\mathbb{R})$<br>(Manifolds) | $GL^{+}(n,\mathbb{R})$<br>(Oriented manifolds) |
| $\mathbb{C}$ | $GL(n,\mathbb{C})$<br>(Complex manifolds) | $SL(n,\mathbb{C})$ |
| $\mathbb{H}$ | $GL(n,\mathbb{H})\,\mathbb{H}^{\times}$<br>(Quaternionic manifolds) | $GL(n,\mathbb{H})$<br>(Hypercomplex manifolds) |
| $\mathbb{O}$ | $Spin(7)$<br>($Spin(7)$-manifolds) | $G_2$<br>($G_2$-manifolds) |

**$\mathbb{A}$-torsions**

Ray and Singer define analytic torsions for real and complex manifolds, they are important non-local invariants for these manifolds and play important roles in the family index theory. The quaternionic generalization of the analytic torsion is discussed by Leung, Yi [26] and Koehler, Weingartmath [15]. It is natural to ask for their analog in the octonionic case and their common roles in the geometry over $\mathbb{A}$.

**$G_2$-symplectic manifolds**

A different generalization of Riemannian $\mathbb{A}$-manifolds is by not requiring the almost complex structures on $M$ to be integrable, but only require the *closedness* of the calibrating differential forms like the Kähler form or the holomorphic volume form. It turns out that integrability is automatic in the Calabi-Yau case, hyperkähler case and $Spin(7)$-case. And we only obtain two new classes of manifolds this way, namely (1) symplectic manifolds and (2) almost $G_2$-manifolds. In [25] the author discuss TQFT on almost $G_2$-manifolds.

Notice that $\frac{1}{2}\mathbb{A}$-Lagrangian submanifolds can be defined using vanishing of certain two forms, thus they are well-defined for such manifolds. In the symplectic case, they are simply Lagrangian submanifolds in the usual sense. Indeed, if $M$ is an almost $G_2$-manifold, i.e. there exists a closed non-degenerate three form on $M$, then the space $\mathcal{L}M$ of all unparametrized loop in $M$ has a natural symplectic structure, as observed by Movshev.

**Twistor theory for octonionic manifolds**

On any oriented Riemannian four manifold $M$, we can define a *twistor space* $Z$ with a fibration

$$f : Z \to M$$

whose fiber over $x$ is the $S^2$-family of Hermitian complex structures on $T_x M$. The twistor space $Z$ has a natural almost complex structure which is integrable if and only if the Weyl curvature tensor of $M$ is self-dual. Penrose's twistor transformation gives a correspondence between the conformal geometry of $M$ and the complex geometry of $Z$ (see e.g. [5]). There is a natural generalization of the twistor transform for any quaternionic Kähler manifold, which is a transformation between the quaternionic geometry of $M$ and the complex geometry of its twistor space $Z$.

In certain sense, the twistor theory is another form of a *de-complexification* of $M$. Therefore it is natural to ask whether there is an analog theory for $G_2$-manifolds, or even $Spin(7)$-manifolds. One would expect that such twistor spaces would be quaternionic manifolds.

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