Hyperbolic Mean Curvature Flow

Chun-Lei He*, De-Xing Kong† and Kefeng Liu‡

Abstract

In this paper we introduce the hyperbolic mean curvature flow and prove that the corresponding system of partial differential equations are strictly hyperbolic, and based on this, we show that this flow admits a unique short-time smooth solution and possesses the nonlinear stability defined on the Euclidean space with dimension larger than 4. Nonlinear wave equations satisfied by geometric quantities have been obtained. Moreover, we also discuss the relation between the equations for hyperbolic mean curvature flow and the equations for extremal surfaces in the Minkowski space-time.

Key words and phrases: Hyperbolic mean curvature flow, extremal surface, short-time existence, nonlinear stability.

2000 Mathematics Subject Classification: 58J45, 58J47.

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1 Introduction

Classical differential geometry is on the study of curved spaces and shapes, in which the time in general does not paly a role. However, in the last few decades, mathematicians have made great strides in understanding shapes that evolve in time. There are many processes by which a curve or surface or manifold can evolve, among them two successful examples are the mean curvature flow and the Ricci flow. For the Ricci flow, there are many deep and outstanding works, for example, it can be used to successfully solve the Poincaré conjecture and geometrization conjectures. In this paper we will focus on the mean curvature flow.

It is well known that the mean curvature flow is related on the motion of surfaces or manifolds. Much more well-known motion of surfaces are those equating the velocity $\frac{dX}{dt}$ with some scalar multiple of the normal of the surface. The scalar can be the curvature, mean curvature or the inverse of the mean curvature with suitable sign attached. This is the traditional mean curvature flow. For the traditional mean curvature flow, a beautiful theory has been developed by Hamilton, Huisken and other researchers (e.g., [4], [6], [10]), and some important applications have been obtained, for example, Huisken and Ilmanen developed a theory of weak solutions of the inverse mean curvature flow and used it to prove successfully the Riemannian Penrose inequality (see [11]).

A natural problem is as follows: in the above argument if we replace the velocity $\frac{dX}{dt}$ by the acceleration $\frac{d^2X}{dt^2}$, what happens? In fact, Yau in [15] has suggested the following equation related to a vibrating membrane or the motion of a surface

$$\frac{d^2X}{dt^2} = H\vec{n}, \tag{1.1}$$

where H is the mean curvature and the $\vec{n}$ is the unit inner normal vector of the surface, and pointed out that very little about the global time behavior of the hypersurfaces (see page 242 in [15]). In deed, according to the authors' knowledge, up to now only a few of the results on this aspect have been known: a hyperbolic theory for the evolution of the plane curves has been developed by Gurtin and Podio-Guidugli [5], and some applications to the crystal interfaces have been obtained (see [14]).

Here we would like to point out that the traditional mean curvature flow equation is parabolic, however the equation (1.1) is hyperbolic (see Section 2 for the details). Therefore, in this sense, we name the equation (1.1) as the hyperbolic version of mean curvature flow, or *hyperbolic mean curvature flow*. Analogous to our recent work [13], in which we introduced and studied the hyperbolic version of the Ricci flow — the hyperbolic geometric flow, in this paper we will investigate the hyperbolic mean curvature flow.

The paper is organized as follows. In Section 2, we introduce the hyperbolic mean curvature flow and give the short-time existence theorem. In Section 3, we construct some exact solutions to the hyperbolic mean curvature flow, these solutions paly an important role in applied fields. Section 4 is devoted to the study on the nonlinear stability of the hyperbolic mean curvature flow defined on the Euclidean space with the dimension larger than 4. In Section 5, we derive the nonlinear wave equations satisfied by some geometric quantities of the hypersurface $X(\cdot,t)$, these equations show the wave character of the curvatures. In Section 6, we illustrate the relations between the hyperbolic mean curvature flow and the equations for extremal surfaces in the Minkowski space $\mathbb{R}^{1,n}$.

2 Hyperbolic mean curvature flow

Let $\mathscr{M}$ be an n-dimensional smooth manifold and

$$X(\cdot,t): \quad \mathscr{M} \to \mathbb{R}^{n+1}$$

be a one-parameter family of smooth hypersuface immersions in $\mathbb{R}^{n+1}$. We say that it is a solution of the hyperbolic mean curvature flow if

$$\frac{\partial^2}{\partial t^2}X(x,t) = H(x,t)\vec{n}(x,t), \qquad \forall\, x \in \mathscr{M}, \qquad \forall\, t > 0, \tag{2.1}$$

where $H(x,t)$ is the mean curvature of $X(x,t)$ and $\vec{n}(x,t)$ is the unit inner normal vector on $X(\cdot,t)$.

In a local coordinate system $\{x^i\}$ $(1 \leqslant i \leqslant n)$, the induced metric $g = \{g_{ij}\}$ and the second fundamental form $A = \{h_{ij}\}$ on $\mathscr{M}$ can be computed as follows

$$g_{ij}(x,t) = \Big(\frac{\partial X(x,t)}{\partial x^i}, \frac{\partial X(x,t)}{\partial x^j}\Big)\ ,$$
$$h_{ij}(x,t) = \Big(\vec{n}(x,t), \frac{\partial^2 X(x,t)}{\partial x^i \partial x^j}\Big)\ \ .$$

Thus, the mean curvature $H(x,t)$ reads

$$H = g^{ij}h_{ij}\ \ .$$

Recall that the Gauss-Weingarten relations

$$\frac{\partial^2 X}{\partial x^i \partial x^j} = \Gamma^k_{ij}\frac{\partial X}{\partial x^k} + h_{ij}\vec{n}\ , \qquad \frac{\partial \vec{n}}{\partial x^j} = -h_{jl}g^{lm}\frac{\partial X}{\partial x^m}\ \ .$$

Thus, we have

$$\triangle_g X = g^{ij}\nabla_i\nabla_j X = g^{ij}\Big(\frac{\partial^2 X}{\partial x^i \partial x^j} - \Gamma^k_{ij}\frac{\partial X}{\partial x^k}\Big) = g^{ij}h_{ij}\vec{n} = H\vec{n}.$$

So the hyperbolic mean curvature flow equation (2.1) can be equivalent by written as

$$\frac{\partial^2 X}{\partial t^2} = \triangle_g X = g^{ij}\Big(\frac{\partial^2 X}{\partial x^i \partial x^j} - \Gamma^k_{ij}\frac{\partial X}{\partial x^k}\Big). \tag{2.2}$$

It is easy to see that the equation (2.2) is not strictly hyperbolic. Therefore, instead of considering the equation (2.2) we will follow a trick of DeTurck [3] by using harmonic coordinates under which (2.2) turns out to be strictly hyperbolic, so that we can apply the standard theory of hyperbolic equations.

A coordinate chart $(x^1, \cdots, x^n)$ on a Riemannian manifold $(\mathscr{M}, g)$ is called *harmonic* if

$$\triangle x^j = 0 \qquad j = 1, \cdots, n.$$

DeTurck [3] pointed out that a coordinate function x^k is *harmonic* if and only if

$$\Gamma^k = \Gamma^k_{ij} g^{ij} = - \triangle x^k = 0.$$

He also proved the following theorem on the existence of harmonic coordinates.

Lemma 2.1 *Let the metric g on a Riemannian manifold $(\mathscr{M}, g)$ be of class $C^{k,\alpha}$ (for $k \geqslant 1$) (resp. C^ω) in a local coordinate chart about some point p. Then there is a neighborhood of p in which harmonic coordinates exist, these new coordinates being $C^{k+1,\alpha}$ (resp. C^ω) functions of the original coordinates. Moreover, all harmonic coordinate charts defined near p have this regularity.*

By Lemma 2.1, we can choose the harmonic coordinates around a fixed point $p \in \mathscr{M}$ and for a fixed time $t \in \mathbb{R}^+$. Then the hyperbolic mean curvature flow (2.2) can be equivalent by written as

$$\frac{\partial^2 X}{\partial t^2} = g^{ij}\frac{\partial^2 X}{\partial x^i \partial X^j}\ . \tag{2.3}$$

By the standard theory of hyperbolic equations, we have the following result.

Theorem 2.1 (Local existences and uniqueness) *Let $\mathscr{M}$ be an n-dimensional smooth compact manifold, and X_0 be a smooth hypersuface immersion of $\mathscr{M}$ into $\mathbb{R}^{n+1}$. Then there exists a constant $T > 0$ such that the initial value problem*

$$\begin{cases} \dfrac{\partial^2}{\partial t^2} X(x,t) = H(x,t)\vec{n}(x,t), \\ X\Big|_{t=0} = X_0(x), \quad \dfrac{\partial X}{\partial t}(x,t)\Big|_{t=0} = X_1(x) \end{cases} \tag{2.4}$$

has a unique smooth solution $X(x,t)$ on $\mathscr{M} \times [0,T)$, where $X_1(x)$ is a smooth vector-valued function on $\mathscr{M}$.

3 Exact solutions

In order to understand further the hyperbolic mean curvature flow, in this section we investigate some exact solutions. These exact solutions play an important role in applied fields. To do so, we first consider the following initial value problem for an ordinary differential equation

$$\begin{cases} r_{tt} = -\dfrac{1}{r} \ , \\ r(0) = r_0 > 0 \ , \quad r_t(0) = r_1 \ . \end{cases} \tag{3.1}$$

For this initial value problem, we have the following lemma.

Lemma 3.1 *For arbitrary initial data $r_0 > 0$, if the initial velocity $r_1 \leqslant 0$, then the solution $r = r(t)$ decreases and attains its zero point at time t_0 (in particular, when $r_1 = 0$, we have $t_0 = \sqrt{\frac{\pi}{2}} r_0$); if the initial velocity is positive, then the solution r increases first and then decreases and attains its zero point in a finite time t_0.*

Proof. The proof is similar to the arguments in [14]. The following discussion is divided into two cases.

Case I The initial velocity is nonpositive, i.e., $r_1 \leqslant 0$.

We argue by contradiction. Let us assume that $r(t) > 0$ for all time $t > 0$. Then $r_{tt} < 0$ and $r_t(t) < r_t(0) = r_1 \leqslant 0$ for $t > 0$. Hence there exists a time t_0 such that $r(t_0) = 0$ (see Fig.1). This is a contradiction.

Moreover, for the case $r_1 = 0$, we can derive the explicit expression for t_0 according to the equation (3.1). Multiplying both side of $r_{tt} = -\frac{1}{r}$ by r_t, integrating, applying the initial condition $r_t(0) \leqslant 0$ and $r(t_0) = 0$, integrating once again yields

$$\int_0^{t_0} \frac{\sqrt{2}}{2r_0} dt = \int_0^{\infty} e^{-u^2} du = \frac{\sqrt{\pi}}{2},$$

where $u = \sqrt{\ln \frac{r_0}{r}}$. Thus we obtain

$$t_0 = \sqrt{\frac{\pi}{2}} r_0.$$

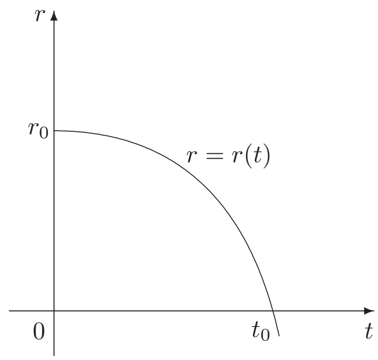

Fig. 1: $r_1 \leqslant 0$

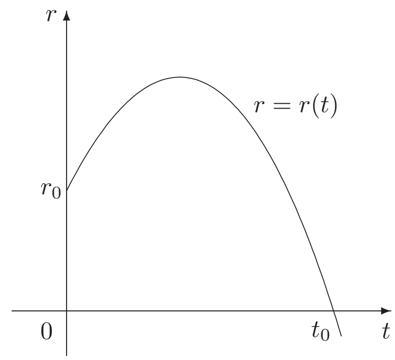

Fig. 2: $r_1 > 0$

Case II The initial velocity is positive, i.e., $r_1 > 0$.

By (3.1), we obtain

$$r_t^2 = -2\ln r + 2\ln r_0 + r_1^2 \ .$$

Then we have

$$r \leqslant e^{\frac{r_1^2}{2}} r_0 \ .$$

If r increases for all time, i.e., $r_t > 0$ for all time t, we have $r_0 < r \leqslant e^{\frac{r_1^2}{2}} r_0$ and $-\frac{1}{r_0} < r_{tt} \leqslant -e^{-\frac{r_1^2}{2}} \frac{1}{r_0}$. Thus, the curve r_t can be bounded by two straight lines $r_t = -\dfrac{1}{r_0} t + r_1$ and $r_t = -\frac{1}{r_0} e^{-\frac{r_1^2}{2}} t + r_1$. On the other hand, r_t is a convex function since

$$(r_t)_{tt} = \frac{r_t}{r^2} > 0.$$

Therefore, r_t will change sign and becomes negative at certain finite time, this contradicts to the hypothesis that r is always increasing. Thus, in this case, r increases first and then decreases and attains its zero point in a finite time (see Fig. 2). The proof is finished. ■

In what follows, we are interested in some exact solutions of hyperbolic mean curvature flow (2.1).

Example 1: Consider a family of spheres

$$X(x,t) = r(t)(\cos\alpha\cos\beta, \cos\alpha\sin\beta, \sin\alpha), \tag{3.2}$$

where $\alpha \in [-\dfrac{\pi}{2}, \dfrac{\pi}{2}]$, $\beta \in [0, 2\pi]$.

Clearly, the induced metric and the second fundamental form are, respectively,

$$g_{11} = r^2, \quad g_{22} = r^2\cos^2\alpha, \quad g_{12} = g_{21} = 0$$

and

$$h_{11} = r, \quad h_{22} = r\cos^2\alpha, \quad h_{12} = h_{21} = 0.$$

The mean curvature is

$$H = \frac{2}{r}.$$

On the other hand, the Christoffel symbols read

$$\Gamma^1_{11} = \Gamma^1_{12} = 0, \qquad \Gamma^1_{22} = \cos\alpha\sin\alpha,$$
$$\Gamma^2_{11} = \Gamma^2_{22} = 0, \qquad \Gamma^2_{12} = -\frac{\sin\alpha}{\cos\alpha}.$$

Thus, we obtain from (2.1) or (2.2) that

$$r_{tt} = -\frac{2}{r}. \tag{3.3}$$

By Lemma 3.1, it can be easily observed that, for arbitrary $r(0) > 0$, if $r_t(0) \leqslant 0$, the evolving sphere will shrink to a point; if $r_t(0) > 0$, the evolving sphere will expand first and then shrink to a point. ■

In fact, this phenomena can also be interpreted by physical principles. From (3.2), we have

$$X_t(x,t) = r_t(t)(\cos\alpha\cos\beta, \cos\alpha\sin\beta, \sin\alpha) \tag{3.4}$$

and

$$X_{tt}(x,t) = r_{tt}(t)(\cos\alpha\cos\beta, \cos\alpha\sin\beta, \sin\alpha). \tag{3.5}$$

By (3.3) and (3.5), the direction of acceleration is always the same as the inner normal vector. Thus, due to (3.4), if $r_t(0) \leqslant 0$, i.e., the initial velocity direction is the same as inner normal vector, then evolving sphere will shrink to a point; if $r_t(0) > 0$, i.e., the initial velocity direction is opposite to inner normal vector, then the evolving sphere will expand first and then shrink to a point.

Example 2: We now consider an exact solution with axial symmetry. In other words, we focus on the cylinder solution for the hyperbolic mean curvature flow which takes the following form

$$X(x,t) = (r(t)\cos\alpha, \; r(t)\sin\alpha, \; \rho),$$

where $\alpha \in [0, 2\pi], \; \rho \in [0, \rho_0]$.

Obviously, the induced metric and the second fundamental form read, respectively,

$$g_{11} = r^2, \quad g_{22} = 1, \quad g_{12} = g_{21} = 0$$

and

$$h_{11} = r, \quad h_{22} = 0, \quad h_{12} = h_{21} = 0.$$

The mean curvature is

$$H = \frac{1}{r}.$$

Moreover, the Christoffel symbols are

$$\Gamma_{ij}^{k} = 0, \qquad \forall\ i, j, k = 1, 2.$$

Then, we obtain from (2.1) or (2.2) that

$$r_{tt} = -\frac{1}{r}.$$

By Lemma 3.1, it can be easily found that the evolving cylinder will always shrink to a straight line for arbitrary $\rho_0 > 0, r(0) > 0$ and $r_t(0)$. ■

4 Nonlinear stability

In this section, we consider the nonlinear stability of the hyperbolic mean curvature flow defined on the Euclidean space with the dimension larger than 4.

Let $\mathscr{M}$ be an n-dimensional ($n > 4$) complete Riemannian manifold. Given the hypersurfaces $X_1(x)$ and $X_2(x)$ on $\mathscr{M}$, we consider the following initial value problem

$$\begin{cases} \dfrac{\partial^2}{\partial t^2} X(x,t) = H(x,t)\vec{n}(x,t)\ , \\ X(x,0) = X_0(x) + \varepsilon X_1(x), \quad \dfrac{\partial X}{\partial t}(x,0) = \varepsilon X_2(x), \end{cases} \tag{4.1}$$

where $\varepsilon > 0$ is a small parameter.

Definition 4.1 *$X_0(x)$ possesses the (locally) nonlinear stability with respect to $(X_1(x), X_2(x))$, if there exists a positive constant $\varepsilon_0 = \varepsilon_0(X_1(x), X_2(x))$ such that, for any $\varepsilon \in (0, \varepsilon_0]$, the initial value problem (4.1) has a unique (local) smooth solution $X(x,t)$;*

$X_0(x)$ is said to be (locally) nonlinear stable if it possesses the (locally) nonlinear stability with respect to arbitrary $X_1(x)$ and $X_2(x)$.

Theorem 4.1 *$X_0(x) = (x^1, x^2, \cdots, x^n, 0)$ ($n > 4$) is nonlinearly stable.*

Proof. Choose the harmonic coordinates around a fixed point $p \in \mathscr{M}$ and for a fixed time $t \in \mathbb{R}^+$. Then the equation (4.1) can be written as

$$\begin{cases} \dfrac{\partial^2}{\partial t^2} X(x,t) = g^{ij} \dfrac{\partial^2 X}{\partial x^i \partial x^j}\ , \\ X(x,0) = X_0(x) + \varepsilon X_1(x), \quad \dfrac{\partial X}{\partial t}(x,0) = \varepsilon X_2(x)\ . \end{cases} \tag{4.2}$$

Define $Y(x,t)=(y^1,\cdots,y^n,y^{n+1})$ in the following way

$$X(x,t)=X_0(x)+Y(x,t)\ .$$

Then for small $Y(x,t)$, we have

$$g_{ij}=\Big(\frac{\partial X}{\partial x^i},\frac{\partial X}{\partial x^j}\Big)=\delta_{ij}+\frac{\partial y^j}{\partial x^i}+\frac{\partial y^i}{\partial x^j}+y_{ij} \tag{4.3}$$

and

$$g^{ij}=\delta^{ij}-\frac{\partial y^j}{\partial x^i}-\frac{\partial y^i}{\partial x^j}-y_{ij}+O(||\lambda||)^2\ , \tag{4.4}$$

where

$$\delta_{ij}=\begin{cases}1, & i=j\ (i,j=1,\cdots,n),\\ 0, & i\neq j\ (i,j=1,\cdots,n)\ ,\end{cases}$$

$$y_{ij}=\Big(\frac{\partial Y}{\partial x^i},\frac{\partial Y}{\partial x^j}\Big)=\sum_{p=1}^{n+1}\frac{\partial y^p}{\partial x^i}\frac{\partial y^p}{\partial x^j}\ ,$$

$$\lambda=\left(\frac{\partial y^p}{\partial x^q}\right)\quad (p=1,2,\cdots,n+1;\quad q=1,2,\cdots,n).$$

The equation (4.2) can be rewritten as

$$\begin{cases}\dfrac{\partial^2}{\partial t^2}Y(x,t)=\Big(\delta^{ij}-\dfrac{\partial y^j}{\partial x^i}-\dfrac{\partial y^i}{\partial x^j}-y_{ij}+O(||\lambda||)^2\Big)\dfrac{\partial^2 Y}{\partial x^i\partial x^j}\ ,\\ Y(x,0)=\varepsilon X_1(x),\quad \dfrac{\partial Y}{\partial t}(x,0)=\varepsilon X_2(x)\ .\end{cases} \tag{4.5}$$

Define

$$\hat{\lambda}=\Big(\frac{\partial y^m}{\partial x^k},\frac{\partial^2 y^m}{\partial x^k\partial x^l}\Big)\ (m=1,\cdots,n+1;\ k,l=1,\cdots,n),$$

then for all p we have

$$\begin{aligned}\frac{\partial^2 y^p}{\partial t^2}&=\frac{\partial^2 y^p}{\partial x^i\partial x^i}+\Big(-\frac{\partial y^j}{\partial x^i}-\frac{\partial y^i}{\partial x^j}-y_{ij}+O(||\lambda||)^2\Big)\frac{\partial^2 y^p}{\partial x^i\partial x^j}\\ &=\frac{\partial^2 y^p}{\partial x^i\partial x^i}+O(||\hat{\lambda}||)^2.\end{aligned}$$

By the well-known result on the global existence for nonlinear wave equations (e.g., see [2], [7], [12]), there exists a unique global smooth solution $Y=Y(x,t)$ for the Cauchy problem (4.5). Thus, the proof is completed. ■

5 Evolution of metric and curvatures

From the evolution equation (2.1) for the hyperbolic mean curvature flow, we can obtain the evolution equations for some geometric quantities of the hypersurface $X(\cdot,t)$.

Lemma 5.1 *Under the hyperbolic mean curvature flow, the following identities hold*

$$\triangle h_{ij} = \nabla_i \nabla_j H + H h_{il} g^{lm} h_{mj} - |A|^2 h_{ij}, \tag{5.1}$$

$$\triangle |A|^2 = 2 g^{ik} g^{jl} h_{kl} \nabla_i \nabla_j H + 2|\nabla A|^2 + 2H\mathbf{tr}(A^3) - 2|A|^4, \tag{5.2}$$

where

$$|A|^2 = g^{ij} g^{kl} h_{ik} h_{jl}\ , \quad \mathbf{tr}(A^3) = g^{ij} g^{kl} g^{mn} h_{ik} h_{lm} h_{nj}\ .$$

Lemma 5.1 can be found in Zhu [16].

Theorem 5.1 *Under the hyperbolic mean curvature flow, it holds that*

$$\frac{\partial^2 g_{ij}}{\partial t^2} = -2H h_{ij} + 2\Big(\frac{\partial^2 X}{\partial t \partial x^i}, \frac{\partial^2 X}{\partial t \partial x^j} \Big), \tag{5.3}$$

$$\frac{\partial^2 \vec{n}}{\partial t^2} = -g^{ij} \frac{\partial H}{\partial x^i} \frac{\partial X}{\partial x^j} + g^{ij} \left(\vec{n}, \frac{\partial^2 X}{\partial t \partial x^i} \right) \Big[2 g^{kl} \Big(\frac{\partial X}{\partial x^j}, \frac{\partial^2 X}{\partial t \partial x^l} \Big) \frac{\partial X}{\partial x^k} + g^{kl} \Big(\frac{\partial X}{\partial x^l}, \frac{\partial^2 X}{\partial t \partial x^j} \Big) \frac{\partial X}{\partial x^k} - \frac{\partial^2 X}{\partial t \partial x^j} \Big] \tag{5.4}$$

and

$$\frac{\partial^2 h_{ij}}{\partial t^2} = \triangle h_{ij} - 2H h_{il} h_{mj} g^{lm} + |A|^2 h_{ij} + g^{kl} h_{ij} \left(\vec{n}, \frac{\partial^2 X}{\partial t \partial x^k} \right) \left(\vec{n}, \frac{\partial^2 X}{\partial t \partial x^l} \right) - 2 \frac{\partial \Gamma^k_{ij}}{\partial t} \left(\vec{n}, \frac{\partial^2 X}{\partial t \partial x^k} \right). \tag{5.5}$$

Proof. With the aids of the definitions of the metric, the second fundamental form and the Gauss-Weingarten relations, we can give a complete proof of Theorem 5.1. In fact, by the definition of the metric, we have

$$\begin{aligned}
\frac{\partial^2 g_{ij}}{\partial t^2} &= \frac{\partial^2}{\partial t^2} \Big(\frac{\partial X}{\partial x^i}, \frac{\partial X}{\partial x^j} \Big) = \Big(\frac{\partial^3 X}{\partial t^2 \partial x^i}, \frac{\partial X}{\partial x^j} \Big) + 2 \Big(\frac{\partial^2 X}{\partial t \partial x^i}, \frac{\partial^2 X}{\partial t \partial x^j} \Big) + (\frac{\partial X}{\partial x^i}, \frac{\partial^3 X}{\partial t^2 \partial x^j}) \\
&= \Big(\frac{\partial}{\partial x^i} (H\vec{n}), \frac{\partial X}{\partial x^j} \Big) + 2 \Big(\frac{\partial^2 X}{\partial t \partial x^i}, \frac{\partial^2 X}{\partial t \partial x^j} \Big) + \Big(\frac{\partial X}{\partial x^i}, \frac{\partial}{\partial x^j} (H\vec{n}) \Big) \\
&= H \Big(-h_{ik} g^{kl} \frac{\partial X}{\partial x^l}, \frac{\partial X}{\partial x^j} \Big) + 2 \Big(\frac{\partial^2 X}{\partial t \partial x^i}, \frac{\partial^2 X}{\partial t \partial x^j} \Big) + H \Big(\frac{\partial X}{\partial x^i}, -h_{jk} g^{kl} \frac{\partial X}{\partial x^l} \Big) \\
&= -2H h_{ij} + 2 \Big(\frac{\partial^2 X}{\partial t \partial x^i}, \frac{\partial^2 X}{\partial t \partial x^j} \Big).
\end{aligned}$$

This gives the proof of (5.3).

On the other hand,

$$\frac{\partial \vec{n}}{\partial t} = \left(\frac{\partial \vec{n}}{\partial t}, \frac{\partial X}{\partial x^i} \right) g^{ij} \frac{\partial X}{\partial x^j} = -\left(\vec{n}, \frac{\partial^2 X}{\partial t \partial x^i} \right) g^{ij} \frac{\partial X}{\partial x^j},$$

then

$$\begin{aligned}\frac{\partial^2\vec{n}}{\partial t^2} &= -\Big(\frac{\partial\vec{n}}{\partial t},\frac{\partial^2 X}{\partial t\partial x^i}\Big)g^{ij}\frac{\partial X}{\partial x^j} - \Big(\vec{n},\frac{\partial^3 X}{\partial t^2\partial x^i}\Big)g^{ij}\frac{\partial X}{\partial x^j} + \Big(\vec{n},\frac{\partial^2 X}{\partial t\partial x^i}\Big)g^{ik}g^{jl}\frac{\partial g_{kl}}{\partial t}\frac{\partial X}{\partial x^j} - g^{ij}\Big(\vec{n},\frac{\partial^2 X}{\partial t\partial x^i}\Big)\frac{\partial^2 X}{\partial t\partial x^j}\\
&= g^{ij}g^{kl}\Big(\vec{n},\frac{\partial^2 X}{\partial t\partial x^k}\Big)\Big(\frac{\partial X}{\partial x^l},\frac{\partial^2 X}{\partial t\partial x^i}\Big)\frac{\partial X}{\partial x^j} - \Big(\vec{n},\frac{\partial}{\partial x^i}(H\vec{n})\Big)g^{ij}\frac{\partial X}{\partial x^j}\\
&\quad + g^{ik}g^{jl}\Big(\vec{n},\frac{\partial^2 X}{\partial t\partial x^i}\Big)\Big[\Big(\frac{\partial^2 X}{\partial t\partial x^k},\frac{\partial X}{\partial x^l}\Big) + \Big(\frac{\partial X}{\partial x^k},\frac{\partial^2 X}{\partial t\partial x^l}\Big)\Big]\frac{\partial X}{\partial x^j} - \Big(\vec{n},\frac{\partial^2 X}{\partial t\partial x^i}\Big)g^{ij}\frac{\partial^2 X}{\partial t\partial x^j}\\
&= -g^{ij}\frac{\partial H}{\partial x^i}\frac{\partial X}{\partial x^j} - g^{ij}\Big(\vec{n},\frac{\partial^2 X}{\partial t\partial x^i}\Big)\frac{\partial^2 X}{\partial t\partial x^j}\\
&\quad + g^{ij}g^{kl}\Big(\vec{n},\frac{\partial^2 X}{\partial t\partial x^i}\Big)\Big[\Big(\frac{\partial^2 X}{\partial t\partial x^j},\frac{\partial X}{\partial x^l}\Big) + 2\Big(\frac{\partial X}{\partial x^j},\frac{\partial^2 X}{\partial t\partial x^l}\Big)\Big]\frac{\partial X}{\partial x^k}.\end{aligned}$$

This proves (5.4).

By virtue of

$$\frac{\partial h_{ij}}{\partial t} = \frac{\partial}{\partial t}\Big(\vec{n},\frac{\partial^2 X}{\partial x^i\partial x^j}\Big) = \Big(\frac{\partial\vec{n}}{\partial t},\frac{\partial^2 X}{\partial x^i\partial x^j}\Big) + \Big(\vec{n},\frac{\partial^3 X}{\partial t\partial x^i\partial x^j}\Big),$$

we have

$$\begin{aligned}\frac{\partial^2 h_{ij}}{\partial t^2} &= \Big(\frac{\partial^2\vec{n}}{\partial t^2},\frac{\partial^2 X}{\partial x^i\partial x^j}\Big) + 2\Big(\frac{\partial\vec{n}}{\partial t},\frac{\partial^3 X}{\partial t\partial x^i\partial x^j}\Big) + \Big(\vec{n},\frac{\partial^4 X}{\partial t^2\partial x^i\partial x^j}\Big)\\
&= -g^{kl}\Big(\frac{\partial H}{\partial x^k}\frac{\partial X}{\partial x^l},\frac{\partial^2 X}{\partial x^i\partial x^j}\Big) - g^{kl}\Big(\vec{n},\frac{\partial^2 X}{\partial t\partial x^k}\Big)\Big(\frac{\partial^2 X}{\partial t\partial x^l},\frac{\partial^2 X}{\partial x^i\partial x^j}\Big)\\
&\quad + g^{pq}g^{kl}\Big(\vec{n},\frac{\partial^2 X}{\partial t\partial x^p}\Big)\Big[\Big(\frac{\partial X}{\partial x^l},\frac{\partial^2 X}{\partial t\partial x^q}\Big) + 2\Big(\frac{\partial X}{\partial x^q},\frac{\partial^2 X}{\partial t\partial x^l}\Big)\Big](\frac{\partial X}{\partial x^k},\frac{\partial^2 X}{\partial x^i\partial x^j})\\
&\quad - 2g^{kl}\Big(\vec{n},\frac{\partial^2 X}{\partial t\partial x^k}\Big)\Big(\frac{\partial X}{\partial x^l},\frac{\partial^3 X}{\partial t\partial x^i\partial x^j}\Big) + \Big(\vec{n},\frac{\partial^2}{\partial x^i\partial x^j}(H\vec{n})\Big)\\
&= -\frac{\partial H}{\partial x^k}\Gamma^k_{ij} - g^{kl}\Gamma^m_{ij}\Big(\vec{n},\frac{\partial^2 X}{\partial t\partial x^k}\Big)\Big(\frac{\partial X}{\partial x^m},\frac{\partial^2 X}{\partial t\partial x^l}\Big) - g^{kl}h_{ij}\Big(\vec{n},\frac{\partial^2 X}{\partial t\partial x^k}\Big)\Big(\vec{n},\frac{\partial^2 X}{\partial t\partial x^l}\Big)\\
&\quad + g^{pq}\Gamma^l_{ij}\Big(\vec{n},\frac{\partial^2 X}{\partial t\partial x^p}\Big)\Big[\Big(\frac{\partial X}{\partial x^l},\frac{\partial^2 X}{\partial t\partial x^q}\Big) + 2\Big(\frac{\partial X}{\partial x^q},\frac{\partial^2 X}{\partial t\partial x^l}\Big)\Big]\\
&\quad - 2g^{kl}\Gamma^m_{ij}\Big(\vec{n},\frac{\partial^2 X}{\partial t\partial x^k}\Big)\Big(\frac{\partial X}{\partial x^l},\frac{\partial^2 X}{\partial t\partial x^m}\Big) - 2\frac{\partial\Gamma^k_{ij}}{\partial t}\Big(\vec{n},\frac{\partial^2 X}{\partial t\partial x^k}\Big)\\
&\quad + 2g^{kl}h_{ij}\Big(\vec{n},\frac{\partial^2 X}{\partial t\partial x^k}\Big)\Big(\vec{n},\frac{\partial^2 X}{\partial t\partial x^l}\Big) + \Big(\vec{n},\frac{\partial}{\partial x^i}\ \Big(\frac{\partial H}{\partial x^j}\vec{n} - Hh_{jk}g^{kl}\frac{\partial X}{\partial x^l}\Big)\ \Big)\\
&= \nabla_i\nabla_j H - Hh_{jk}g^{kl}h_{il} + g^{kl}h_{ij}\Big(\vec{n},\frac{\partial^2 X}{\partial t\partial x^k}\Big)\Big(\vec{n},\frac{\partial^2 X}{\partial t\partial x^l}\Big) - 2\frac{\partial\Gamma^k_{ij}}{\partial t}\Big(\vec{n},\frac{\partial^2 X}{\partial t\partial x^k}\Big)\ .\end{aligned}$$

Using (5.1), we obtain

$$\frac{\partial^2 h_{ij}}{\partial t^2} = \triangle h_{ij} - 2Hg^{kl}h_{il}h_{jk} + |A|^2 h_{ij} + g^{kl}h_{ij}\Big(\vec{n},\frac{\partial^2 X}{\partial t\partial x^k}\Big)\Big(\vec{n},\frac{\partial^2 X}{\partial t\partial x^l}\Big) - 2\frac{\partial\Gamma^k_{ij}}{\partial t}\Big(\vec{n},\frac{\partial^2 X}{\partial t\partial x^k}\Big).$$

This proves (5.5). Thus, the proof of Theorem 5.1 is completed . ■

Theorem 5.2 *Under the hyperbolic mean curvature flow,*

$$\begin{aligned}\frac{\partial^2 H}{\partial t^2} &= \triangle H + H|A|^2 - 2g^{ik}g^{jl}h_{ij}\Big(\frac{\partial^2 X}{\partial t\partial x^k},\frac{\partial^2 X}{\partial t\partial x^l}\Big) + Hg^{kl}\Big(\vec{n},\frac{\partial^2 X}{\partial t\partial x^k}\Big)\Big(\vec{n},\frac{\partial^2 X}{\partial t\partial x^l}\Big)\\
&\quad - 2g^{ij}\frac{\partial\Gamma^k_{ij}}{\partial t}\Big(\vec{n},\frac{\partial^2 X}{\partial t\partial x^k}\Big) + 2g^{ik}g^{jp}g^{lq}h_{ij}\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{kl}}{\partial t} - 2g^{ik}g^{jl}\frac{\partial g_{kl}}{\partial t}\frac{\partial h_{ij}}{\partial t}\ ,\end{aligned} \tag{5.6}$$

$$
\begin{aligned}
\frac{\partial^2}{\partial t^2}|A|^2 = & \triangle(|A|^2) - 2|\nabla A|^2 + 2|A|^4 + 2|A|^2 g^{pq}\Big(\vec{n}, \frac{\partial^2 X}{\partial t \partial x^p}\Big)\Big(\vec{n}, \frac{\partial^2 X}{\partial t \partial x^q}\Big) \\
& + 2g^{ij}g^{kl}\frac{\partial h_{ik}}{\partial t}\frac{\partial h_{jl}}{\partial t} - 8g^{im}g^{jn}g^{kl}h_{jl}\frac{\partial g_{mn}}{\partial t}\frac{\partial h_{ik}}{\partial t} - 4g^{im}g^{jn}g^{kl}h_{ik}h_{jl}\Big(\frac{\partial^2 X}{\partial t \partial x^m}, \frac{\partial^2 X}{\partial t \partial x^n}\Big) \\
& + 2g^{im}\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{mn}}{\partial t}h_{ik}h_{jl}\Big(2g^{jp}g^{nq}g^{kl} + g^{jn}g^{kp}g^{lq}\Big) - 4g^{ij}g^{kl}h_{jl}\frac{\partial \Gamma^p_{ik}}{\partial t}\Big(\vec{n}, \frac{\partial^2 X}{\partial t \partial x^p}\Big) \ .
\end{aligned} \tag{5.7}
$$

Proof. Noting

$$
g^{hm}g_{ml} = \delta^h_l,
$$

we get

$$
\begin{aligned}
\frac{\partial g^{ij}}{\partial t} &= -g^{ik}g^{jl}\frac{\partial g_{kl}}{\partial t} \ , \\
\frac{\partial^2 g^{ij}}{\partial t^2} &= 2g^{ik}g^{jp}g^{lq}\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{kl}}{\partial t} - g^{ik}g^{jl}\frac{\partial^2 g_{kl}}{\partial t^2} \ .
\end{aligned}
$$

By a direct calculation, we have

$$
\begin{aligned}
\frac{\partial^2 H}{\partial t^2} = & \frac{\partial^2 g^{ij}}{\partial t^2}h_{ij} + 2\frac{\partial g^{ij}}{\partial t}\frac{\partial h_{ij}}{\partial t} + g^{ij}\frac{\partial^2 h_{ij}}{\partial t^2} \\
= & \Big(2g^{ik}g^{jp}g^{lq}\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{kl}}{\partial t} - g^{ik}g^{jl}\frac{\partial^2 g_{kl}}{\partial t^2}\Big)h_{ij} - 2g^{ik}g^{jl}\frac{\partial g_{kl}}{\partial t}\frac{\partial h_{ij}}{\partial t} + g^{ij}\frac{\partial^2 h_{ij}}{\partial t^2} \\
= & 2g^{ik}g^{jp}g^{lq}h_{ij}\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{kl}}{\partial t} - 2g^{ik}g^{jl}\frac{\partial g_{kl}}{\partial t}\frac{\partial h_{ij}}{\partial t} - g^{ik}g^{jl}h_{ij}\Big[-2Hh_{kl} + 2\Big(\frac{\partial^2 X}{\partial t \partial x^k}, \frac{\partial^2 X}{\partial t \partial x^l}\Big)\Big] \\
& + g^{ij}\Big[\nabla_i\nabla_j H - Hh_{il}h_{jk}g^{lk} + g^{kl}h_{ij}\Big(\vec{n}, \frac{\partial^2 X}{\partial t \partial x^k}\Big)\Big(\vec{n}, \frac{\partial^2 X}{\partial t \partial x^l}\Big) - 2\frac{\partial \Gamma^k_{ij}}{\partial t}\Big(\vec{n}, \frac{\partial^2 X}{\partial t \partial x^k}\Big)\Big] \\
= & \triangle H + H|A|^2 - 2g^{ik}g^{jl}h_{ij}\Big(\frac{\partial^2 X}{\partial t \partial x^k}, \frac{\partial^2 X}{\partial t \partial x^l}\Big) + Hg^{kl}\Big(\vec{n}, \frac{\partial^2 X}{\partial t \partial x^k}\Big)\Big(\vec{n}, \frac{\partial^2 X}{\partial t \partial x^l}\Big) \\
& - 2g^{ij}\frac{\partial \Gamma^k_{ij}}{\partial t}\Big(\vec{n}, \frac{\partial^2 X}{\partial t \partial x^k}\Big) + 2g^{ik}g^{jp}g^{lq}h_{ij}\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{kl}}{\partial t} - 2g^{ik}g^{jl}\frac{\partial g_{kl}}{\partial t}\frac{\partial h_{ij}}{\partial t} \ .
\end{aligned}
$$

This is nothing but the desired (5.6).

On the other hand, by the definition of $|A|^2$ and the formula (5.2), a direct calculation gives

$$
\begin{aligned}
\frac{\partial^2}{\partial t^2}|A|^2 =& 2\frac{\partial^2 g^{ij}}{\partial t^2}g^{kl}h_{ik}h_{jl} + 2\frac{\partial g^{ij}}{\partial t}\frac{\partial g^{kl}}{\partial t}h_{ik}h_{jl} + 8\frac{\partial g^{ij}}{\partial t}g^{kl}\frac{\partial h_{ik}}{\partial t}h_{jl}\\
&+ 2g^{ij}g^{kl}\frac{\partial^2 h_{ik}}{\partial t^2}h_{jl} + 2g^{ij}g^{kl}\frac{\partial h_{ik}}{\partial t}\frac{\partial h_{jl}}{\partial t}\\
=& 2\Big(2g^{im}g^{jp}g^{nq}\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{mn}}{\partial t} - g^{im}g^{jn}\frac{\partial^2 g_{mn}}{\partial t^2}\Big)g^{kl}h_{ik}h_{jl}\\
&+ 2g^{im}g^{jn}g^{kp}g^{lq}h_{ik}h_{jl}\frac{\partial g_{mn}}{\partial t}\frac{\partial g_{pq}}{\partial t} - 8g^{im}g^{jn}\frac{\partial g_{mn}}{\partial t}g^{kl}\frac{\partial h_{ik}}{\partial t}h_{jl} + 2g^{ij}g^{kl}\frac{\partial h_{ik}}{\partial t}\frac{\partial h_{jl}}{\partial t}\\
&+ 2g^{ij}g^{kl}h_{jl}\Big[\nabla_i\nabla_k H - Hh_{ip}h_{kq}g^{pq} + g^{pq}h_{ik}\Big(\vec{n}, \frac{\partial^2 X}{\partial t\partial x^p}\Big)\Big(\vec{n}, \frac{\partial^2 X}{\partial t\partial x^q}\Big) - 2\frac{\partial \Gamma^p_{ik}}{\partial t}\Big(\vec{n}, \frac{\partial^2 X}{\partial t\partial x^p}\Big)\Big]\\
=& 4g^{im}g^{jp}g^{nq}g^{kl}\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{mn}}{\partial t}h_{ik}h_{jl} - 2g^{im}g^{jn}\Big[-2Hh_{mn} + 2\Big(\frac{\partial^2 X}{\partial t\partial x^m}, \frac{\partial^2 X}{\partial t\partial x^n}\Big)\Big]g^{kl}h_{ik}h_{jl}\\
&+ 2g^{im}g^{jn}g^{kp}g^{lq}h_{ik}h_{jl}\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{mn}}{\partial t} - 8g^{im}g^{jn}g^{kl}\frac{\partial g_{mn}}{\partial t}\frac{\partial h_{ik}}{\partial t}h_{jl}\\
&+ 2g^{ij}g^{kl}h_{jl}\nabla_i\nabla_k H - 2H\mathbf{tr}(A^3) + 2g^{pq}|A|^2\Big(\vec{n}, \frac{\partial^2 X}{\partial t\partial x^p}\Big)\Big(\vec{n}, \frac{\partial^2 X}{\partial t\partial x^q}\Big)\\
&- 4g^{ij}g^{kl}h_{jl}\frac{\partial \Gamma^p_{ik}}{\partial t}\Big(\vec{n}, \frac{\partial^2 X}{\partial t\partial x^p}\Big) + 2g^{ij}g^{kl}\frac{\partial h_{ik}}{\partial t}\frac{\partial h_{jl}}{\partial t}\\
=& \triangle(|A|^2) - 2|\nabla A|^2 + 2|A|^4 + 2|A|^2 g^{pq}\Big(\vec{n}, \frac{\partial^2 X}{\partial t\partial x^p}\Big)\Big(\vec{n}, \frac{\partial^2 X}{\partial t\partial x^q}\Big)\\
&+ 2g^{ij}g^{kl}\frac{\partial h_{ik}}{\partial t}\frac{\partial h_{jl}}{\partial t} - 8g^{im}g^{jn}g^{kl}h_{jl}\frac{\partial g_{mn}}{\partial t}\frac{\partial h_{ik}}{\partial t} - 4g^{im}g^{jn}g^{kl}h_{ik}h_{jl}\Big(\frac{\partial^2 X}{\partial t\partial x^m}, \frac{\partial^2 X}{\partial t\partial x^n}\Big)\\
&+ 2g^{im}\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{mn}}{\partial t}h_{ik}h_{jl}\Big(2g^{jp}g^{nq}g^{kl} + g^{jn}g^{kp}g^{lq}\Big) - 4g^{ij}g^{kl}h_{jl}\frac{\partial \Gamma^p_{ik}}{\partial t}\Big(\vec{n}, \frac{\partial^2 X}{\partial t\partial x^p}\Big) .
\end{aligned}
$$

This proves (5.7). ■

6 Relations between hyperbolic mean curvature flow and the equations for extremal surfaces in the Minkowski space $\mathbb{R}^{1,n}$

In this section, we study the relations between the hyperbolic mean curvature flow and the equations for extremal surfaces in the Minkowski space $\mathbb{R}^{1,n}$.

Let $v = (v_0, v_1, \cdots, v_n)$ be a position vector of a point in the $(1+n)$-dimensional Minkowski space $\mathbb{R}^{1,n}$. The scalar product of two vectors v and $w = (w_0, w_1, \cdots, w_n)$ is

$$\langle v, w\rangle = -v_0 w_0 + \sum_{i=1}^{n} v_i w_i .$$

The Lorentz metric of $\mathbb{R}^{1,n}$ reads

$$ds^2 = -dt^2 + \sum_{i=1}^{n}(dx^i)^2 .$$

A massless n-dimensional surface moving in $(1+n)$-dimensional Minkowski space can be defined by letting its action be proportional to the $(1+n)$-dimensional volume swept out in the Minkowski space. It is a natural generalization of the massless string theory, and it is interesting in its own right, as an example in which geometry, classical relativity and quantum mechanics are deeply connected. Hoppe et al [1], [8] and Huang and Kong [9] have obtained some interesting results about it.

We are interested in the following motion of an n-dimensional Riemannian manifold in $\mathbb{R}^{1,n+1}$ with the following parameter

$$(t, x^1, \cdots, x^n) \rightarrow \hat{X} = (t,\ X(t, x^1, \cdots, x^n)), \tag{6.1}$$

where $(x^1, \cdots, x^n) \in \mathscr{M}$ and $\hat{X}(\cdot, t)$ be a positive vector of a point in the Minkowski space $\mathbb{R}^{1,n+1}$. The induced Lorentz metric reads

$$\begin{cases} \hat{g}_{00} = -1 + \Big(\dfrac{\partial X}{\partial t}, \dfrac{\partial X}{\partial t}\Big), \\ \hat{g}_{0i} = \hat{g}_{i0} = \Big(\dfrac{\partial X}{\partial t}, \dfrac{\partial X}{\partial x^i}\Big), \qquad i, j = 1, \cdots, n. \\ \hat{g}_{ij} = g_{ij} = \Big(\dfrac{\partial X}{\partial x^i}, \dfrac{\partial X}{\partial x^j}\Big), \end{cases} \tag{6.2}$$

By the variational method or by vanishing mean curvature of the sub-manifold $\mathscr{M}$, we can obtain the following equation for the motion of $\mathscr{M}$ in the Minkowski space $\mathbb{R}^{1,n+1}$

$$\hat{g}^{\alpha\beta}\nabla_\alpha\nabla_\beta\hat{X} = \hat{g}^{\alpha\beta}\Big(\frac{\partial^2\hat{X}}{\partial x^\alpha\partial x^\beta} - \hat{\Gamma}^\gamma_{\alpha\beta}\frac{\partial\hat{X}}{\partial x^\gamma}\Big) = 0, \tag{6.3}$$

where $\alpha, \beta = 0, 1, \cdots, n$. It is convenient to fix the parametrization partially (see Bordemann and Hoppe [1]) by requiring

$$\hat{g}_{oi} = \hat{g}_{i0} = \Big(\frac{\partial X}{\partial t}, \frac{\partial X}{\partial x^i}\Big) = 0. \tag{6.4}$$

It is easy to see that the equation (6.3) is equivalent to the following system

$$\Big(\frac{\partial^2 X}{\partial t^2}, \frac{\partial X}{\partial t}\Big) - g^{ij}\Big(|X_t|^2 - 1\Big)\Big(\frac{\partial^2 X}{\partial t\partial x^i}, \frac{\partial X}{\partial x^j}\Big) = 0, \tag{6.5}$$

$$\begin{aligned} \frac{\partial^2 X}{\partial t^2} + g^{ij}\Big(\frac{\partial^2 X}{\partial x^i\partial x^j} - \Gamma^k_{ij}\frac{\partial X}{\partial x^k}\Big)\Big(|X_t|^2 - 1\Big) - \frac{1}{|X_t|^2 - 1}\left(\frac{\partial^2 X}{\partial t^2}, \frac{\partial X}{\partial t}\right)\frac{\partial X}{\partial t} \\ + g^{kl}\left(\frac{\partial^2 X}{\partial t\partial x^l}, \frac{\partial X}{\partial t}\right)\frac{\partial X}{\partial x^k} + g^{ij}\left(\frac{\partial^2 X}{\partial t\partial x^i}, \frac{\partial X}{\partial x^j}\right)\frac{\partial X}{\partial t} = 0, \end{aligned} \tag{6.6}$$

where

$$|X_t|^2 = \left(\frac{\partial X}{\partial t}, \frac{\partial X}{\partial t}\right).$$

We observe that, when $\frac{\partial X}{\partial t} \rightarrow 0$, the limit of the equation (6.5) reads

$$g^{ij}\Big(\frac{\partial^2 X}{\partial t\partial x^i}, \frac{\partial X}{\partial x^j}\Big) = 0\ , \tag{6.7}$$

i.e.,

$$\frac{\partial}{\partial t}\det(g_{ij}) = 0\ . \tag{6.8}$$

Moreover, the equation (2.2) is nothing but the limit of the equation (6.6) as $\frac{\partial X}{\partial t}$ approaches to zero.

Acknowledgements. C.-L. He would like to thank the Center of Mathematical Sciences at Zhejiang University for the great support and hospitality. The work of He and Kong was supported in part by the NNSF of China (Grant No. 10671124) and the NCET of China (Grant No. NCET-05-0390); the work of Liu was supported in part by the NSF and NSF of China.

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