Hyperbolic Geometric Flow

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Outline: Joint works with D. Kong and W. Dai

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1. Motivation

- Ricci flow:

  Geometric structure of manifolds

- Einstein equations and Penrose conjecture:

  Singularities in manifold and space-time

- Applications of hyperbolic PDEs to differential geometry:

  Wave character of metrics and curvatures

D. Christodoulou, S. Klainerman, M. Dafermos,
I. Rodnianski, H. Lindblad, N. Ziper $\cdots\cdots$

J. Hong (Nonlinear Analysis, 1995)

Kong et al (Comm. Math. Phys.; J. Math. Phys. 2006)

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2. Hyperbolic Geometric Flow

Let $(\mathscr{M}, g_{ij})$ be n-dimensional complete Riemannian manifold.

The Levi-Civita connection

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left\{ \frac{\partial g_{il}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l} \right\}$$

The Riemannian curvature tensors

$$R^k_{ijl} = \frac{\partial \Gamma^k_{jl}}{\partial x^i} - \frac{\partial \Gamma^k_{il}}{\partial x^j} + \Gamma^k_{ip}\Gamma^p_{jl} - \Gamma^k_{jp}\Gamma^p_{il}, \quad R_{ijkl} = g_{kp} R^p_{ijl}$$

The Ricci tensor

$$R_{ik} = g^{jl} R_{ijkl}$$

The scalar curvature

$$R = g^{ij} R_{ij}$$

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Hyperbolic geometric flow (HGF)

$$\frac{\partial^2 g_{ij}}{\partial t^2} = -2R_{ij} \tag{1}$$

for a family of Riemannian metrics $g_{ij}(t)$ on $\mathscr{M}$.

General version of HGF

$$\frac{\partial^2 g_{ij}}{\partial t^2} + 2R_{ij} + \mathscr{F}_{ij}\left(g, \frac{\partial g}{\partial t}\right) = 0 \tag{2}$$

- **Kong and Liu:**

  Wave Character of Metrics and Hyperbolic Geometric Flow, 2006

Physical background

- **Relation between Einstein equations and HGF**

Consider the Lorentzian metric

$$ds^2 = -dt^2 + g_{ij}(x,t)dx^i dx^j$$

Einstein equations in vacuum, i.e., $G_{ij} = 0$ become

$$\frac{\partial^2 g_{ij}}{\partial t^2} + 2R_{ij} + \frac{1}{2} g^{pq} \frac{\partial g_{ij}}{\partial t} \frac{\partial g_{pq}}{\partial t} - g^{pq} \frac{\partial g_{ip}}{\partial t} \frac{\partial g_{jq}}{\partial t} = 0 \qquad (3)$$

This is a special example of general version (2) of HGF. Neglecting the terms of first order gives the HGF (1).

(3) is named as **Einstein's hyperbolic geometric flow**

Three important types

- Hyperbolic geometric flow

$$\frac{\partial^2 g_{ij}}{\partial t^2} = -2R_{ij}$$

Wave type equation

- Einstein's hyperbolic geometric flow

$$\frac{\partial^2 g_{ij}}{\partial t^2} + 2R_{ij} + \frac{1}{2} g^{pq} \frac{\partial g_{ij}}{\partial t} \frac{\partial g_{pq}}{\partial t} - g^{pq} \frac{\partial g_{ip}}{\partial t} \frac{\partial g_{kq}}{\partial t} = 0$$

Wave type equation satisfying null condition

- Dissipative hyperbolic geometric flow

Wave type equation with dissipative terms

Laplace equation, heat equation and wave equation

- Laplace equation (elliptic equations)

$$\triangle u = 0$$

- Heat equation (parabolic equations)

$$u_t - \triangle u = 0$$

- Wave equation (hyperbolic equations)

$$u_{tt} - \triangle u = 0$$

Einstein manifold, Ricci flow, hyperbolic geometric flow

- Einstein manifold (elliptic equations)

$$R_{ij} = \lambda g_{ij}$$

- Ricci flow (parabolic equations)

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij}$$

- Hyperbolic geometric flow (hyperbolic equations)

$$\frac{\partial^2 g_{ij}}{\partial t^2} = -2R_{ij}$$

Laplace equation, heat equation and wave equation on manifolds in the Ricci sense

Geometric flows

$$\alpha_{ij}\frac{\partial^2 g_{ij}}{\partial t^2} + \beta_{ij}\frac{\partial g_{ij}}{\partial t} + \gamma_{ij} g_{ij} + 2R_{ij} = 0,$$

where $\alpha_{ij}, \beta_{ij}, \gamma_{ij}$ are certain smooth functions on $\mathscr{M}$ which may depend on t.

In particular,

$\alpha_{ij} = 1, \beta_{ij} = \gamma_{ij} = 0$: **hyperbolic geometric flow**

$\alpha_{ij} = 0, \beta_{ij} = 1, \gamma_{ij} = 0$: **Ricci flow**

$\alpha_{ij} = 0, \beta_{ij} = 0, \gamma_{ij} = const.$: **Einstein manifold**

Birkhoff Theorem holds for geometric flows

- Fu-Wen Shu and You-Gen Shen:
Geometric flows and black holes, arXiv: gr-qc/0610030

(Locally) any solutions of the Einstein equation are also the solutions of the Einstein hyperbolic geometric flow, which is relatively easier to deal with.

Complex geometric flows

If the underlying manifold $\mathscr{M}$ is a complex manifold and the metric is Kähler,

$$a_{ij}\frac{\partial^2 g_{i\bar{j}}}{\partial t^2} + b_{ij}\frac{\partial g_{i\bar{j}}}{\partial t} + c_{ij} g_{i\bar{j}} + 2R_{i\bar{j}} = 0,$$

where a_{ij}, b_{ij}, c_{ij} are certain smooth functions on $\mathscr{M}$ which may also depend on t.

3. Local Existence and Nonlinear Stability

Local existence theorem (Dai, Kong and Liu, 2006)

Let $(\mathscr{M}, g^0_{ij}(x))$ be a compact Riemannian manifold. Then there exists a constant $h > 0$ such that the initial value problem

$$\begin{cases} \dfrac{\partial^2 g_{ij}}{\partial t^2}(x,t) = -2R_{ij}(x,t), \\ g_{ij}(x,0) = g^0_{ij}(x), \ \dfrac{\partial g_{ij}}{\partial t}(x,0) = k^0_{ij}(x), \end{cases}$$

has a unique smooth solution $g_{ij}(x,t)$ on $\mathscr{M} \times [0,h]$, where $k^0_{ij}(x)$ is a symmetric tensor on $\mathscr{M}$.

Dai, Kong and Liu: Hyperbolic geometric flow (I): short-time existence and nonlinear stability

Method of proof: Two proofs.

- **Strict hyperbolicity**

Suppose $\hat{g}_{ij}(x,t)$ is a solution of the hyperbolic geometric flow (1), and $\psi_t : \mathscr{M} \to \mathscr{M}$ is a family of diffeomorphisms of $\mathscr{M}$. Let

$$g_{ij}(x,t) = \psi_t^* \hat{g}_{ij}(x,t)$$

be the pull-back metrics. The evolution equations for the metrics $g_{ij}(x,t)$ are strictly hyperbolic.

- **Symmetrization of hyperbolic geometric flow**

Introducing the new unknowns

$$g_{ij},\ h_{ij} = \frac{\partial g_{ij}}{\partial t},\ g_{ij,k} = \frac{\partial g_{ij}}{\partial x^k},$$

we have

$$\begin{cases} \dfrac{\partial g_{ij}}{\partial t} = h_{ij}, \\ g^{kl}\dfrac{\partial g_{ij,k}}{\partial t} = g^{kl}\dfrac{\partial h_{ij}}{\partial x^k}, \\ \dfrac{\partial h_{ij}}{\partial t} = g^{kl}\dfrac{\partial g_{ij,k}}{\partial x^l} + \widetilde{H}_{ij}. \end{cases}$$

Rewrite it as

$$A^0(u)\frac{\partial u}{\partial t} = A^j(u)\frac{\partial u}{\partial x^j} + B(u),$$

where the matrices A^0, A^j are symmetric.

Nonlinear stability

Let $\mathscr{M}$ be a n-dimensional complete Riemannian manifold. Given symmetric tensors g^0_{ij} and g^1_{ij} on $\mathscr{M}$, we consider

$$\begin{cases} \dfrac{\partial^2 g_{ij}}{\partial t^2}(t,x) = -2R_{ij}(t,x) \\ g_{ij}(x,0) = \bar{g}_{ij}(x) + \varepsilon g^0_{ij}(x), \quad \dfrac{\partial g_{ij}}{\partial t}(x,0) = \varepsilon g^1_{ij}(x), \end{cases}$$

where $\varepsilon > 0$ is a small parameter.

Definition: **The Ricci flat Riemannian metric $\bar{g}_{ij}(x)$ possesses the (locally) nonlinear stability with respect to (g^0_{ij}, g^1_{ij}), if there exists a positive constant $\varepsilon_0 = \varepsilon_0(g^0_{ij}, g^1_{ij})$ such that, for any $\varepsilon \in (0, \varepsilon_0]$, the above initial value problem has a unique (local) smooth solution $g_{ij}(t,x)$;**

$\overline{g}_{ij}(x)$ **is said to be (locally) nonlinear stable, if it possesses the (locally) nonlinear stability with respect to arbitrary symmetric tensors $g^0_{ij}(x)$ and $g^1_{ij}(x)$ with compact support.**

Nonlinear stability theorem (Dai, Kong and Liu, 2006)

The flat metric $g_{ij} = \delta_{ij}$ of the Euclidean space $\mathbb{R}^n$ with $n \geq 5$ is nonlinearly stable.

Remark: **The proof uses general theory of nonlinear wave equations.**

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Method of proof

Define a 2-tensor h

$$g_{ij}(x,t) = \delta_{ij} + h_{ij}(x,t).$$

Choose the elliptic coordinates $\{x^i\}$ **around the origin in** $\mathbb{R}^n$**. It suffices to prove that the following Cauchy problem has a unique global smooth solution**

$$\begin{cases} \dfrac{\partial^2 h_{ij}}{\partial t^2}(x,t) = \displaystyle\sum_{k=1}^{n} \dfrac{\partial^2 h_{ij}}{\partial x^k \partial x^k} + \bar{H}_{ij}\left(h_{kl}, \dfrac{\partial h_{kl}}{\partial x^p}, \dfrac{\partial^2 h_{kl}}{\partial x^p \partial x^q}\right), \\ h_{ij}(x,0) = \varepsilon g^0_{ij}(x), \quad \dfrac{\partial h_{ij}}{\partial t}(x,0) = \varepsilon g^1_{ij}(x). \end{cases}$$

Einstein's hyperbolic geometric flow

$$\frac{\partial^2 g_{ij}}{\partial t^2} + 2R_{ij} + \frac{1}{2} g^{pq} \frac{\partial g_{ij}}{\partial t} \frac{\partial g_{pq}}{\partial t} - g^{pq} \frac{\partial g_{ip}}{\partial t} \frac{\partial g_{kq}}{\partial t} = 0$$

satisfy the null condition. Existence and uniqueness for small initial data hold.

Einstein hyperbolic flow has nonlinear stability for all dimensions.

Global existence and nonlinear stability for small initial data (Dai, Kong and Liu)

4. Wave Nature of Curvatures

Under the hyperbolic geometric flow (1), the curvature tensors satisfy the following nonlinear wave equations

$$\frac{\partial^2 R_{ijkl}}{\partial t^2} = \Delta R_{ijkl} + \text{(lower order terms)},$$

$$\frac{\partial^2 R_{ij}}{\partial t^2} = \Delta R_{ij} + \text{(lower order terms)},$$

$$\frac{\partial^2 R}{\partial t^2} = \Delta R + \text{(lower order terms)},$$

where Δ is the Laplacian with respect to the evolving metric, the lower order terms only contain lower order derivatives of the curvatures.

Evolution equation for Riemannian curvature tensor

Under the hyperbolic geometric flow (1), the Riemannian curvature tensor R_{ijkl} satisfies the evolution equation

$$\begin{aligned}\frac{\partial^2}{\partial t^2}R_{ijkl} = \; & \triangle R_{ijkl} + 2\left(B_{ijkl} - B_{ijlk} - B_{iljk} + B_{ikjl}\right) \\ & - g^{pq}\left(R_{pjkl}R_{qi} + R_{ipkl}R_{qj} + R_{ijpl}R_{qk} + R_{ijkp}R_{ql}\right) \\ & + 2g_{pq}\left(\frac{\partial}{\partial t}\Gamma^p_{il}\frac{\partial}{\partial t}\Gamma^q_{jk} - \frac{\partial}{\partial t}\Gamma^p_{jl}\frac{\partial}{\partial t}\Gamma^q_{ik}\right),\end{aligned}$$

where $B_{ijkl} = g^{pr}g^{qs}R_{piqj}R_{rksl}$ and $\triangle$ is the Laplacian with respect to the evolving metric.

Evolution equation for Ricci curvature tensor

Under the hyperbolic geometric flow (1), the Ricci curvature tensor satisfies

$$\begin{aligned}\frac{\partial^2}{\partial t^2}R_{ik} &= \triangle R_{ik} + 2g^{pr}g^{qs}R_{piqk}R_{rs} - 2g^{pq}R_{pi}R_{qk} \\ &\quad + 2g^{jl}g_{pq}\left(\frac{\partial}{\partial t}\Gamma^p_{il}\frac{\partial}{\partial t}\Gamma^q_{jk} - \frac{\partial}{\partial t}\Gamma^p_{jl}\frac{\partial}{\partial t}\Gamma^q_{ik}\right) \\ &\quad - 2g^{jp}g^{lq}\frac{\partial g_{pq}}{\partial t}\frac{\partial}{\partial t}R_{ijkl} + 2g^{jp}g^{rq}g^{sl}\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rs}}{\partial t}R_{ijkl}\end{aligned}$$

Evolution equation for scalar curvature

Under the hyperbolic geometric flow (1), the scalar curvature satisfies

$$
\begin{aligned}
\frac{\partial^2}{\partial t^2}R =\ & \triangle R + 2|\mathrm{Ric}|^2 \\
& + 2g^{ik}g^{jl}g_{pq}\left(\frac{\partial}{\partial t}\Gamma^p_{il}\frac{\partial}{\partial t}\Gamma^q_{jk} - \frac{\partial}{\partial t}\Gamma^p_{jl}\frac{\partial}{\partial t}\Gamma^q_{ik}\right) \\
& - 2g^{ik}g^{jp}g^{lq}\frac{\partial g_{pq}}{\partial t}\frac{\partial}{\partial t}R_{ijkl} \\
& - 2g^{ip}g^{kq}\frac{\partial g_{pq}}{\partial t}\frac{\partial R_{ik}}{\partial t} + 4R_{ik}g^{ip}g^{rq}g^{sk}\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rs}}{\partial t}
\end{aligned}
$$

5. Exact Solutions and Birkhoff theorem

5.1. Exact solutions with the Einstein initial metrics

Definition: (Einstein metric on manifold) A Riemannian metric g_{ij} is called Einstein if $R_{ij} = \lambda g_{ij}$ for some constant λ. A smooth manifold $\mathscr{M}$ with an Einstein metric is called an Einstein manifold.

If the initial metric $g_{ij}(0, x)$ is Ricci flat, i.e., $R_{ij}(0, x) = 0$, then $g_{ij}(t, x) = g_{ij}(0, x)$ is obviously a solution to the evolution equation (1). Therefore, any Ricci flat metric is a steady solution of the hyperbolic geometric flow (1).

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If the initial metric is Einstein, that is, for some constant λ it holds

$$R_{ij}(0,x) = \lambda g_{ij}(0,x), \quad \forall\, x \in \mathscr{M},$$

then the evolving metric under the hyperbolic geometric flow (1) will be steady state, or will expand homothetically for all time, or will shrink in a finite time.

Note that we can choose suitable velocity to avoid singularity.

Let

$$g_{ij}(t,x) = \rho(t) g_{ij}(0,x)$$

By the definition of the Ricci tensor, one obtains

$$R_{ij}(t,x) = R_{ij}(0,x) = \lambda g_{ij}(0,x)$$

Equation (1) becomes

$$\frac{\partial^2(\rho(t) g_{ij}(0,x))}{\partial t^2} = -2\lambda g_{ij}(0,x)$$

This gives an ODE of second order

$$\frac{d^2 \rho(t)}{dt^2} = -2\lambda$$

One of the initial conditions is $\rho(0) = 1$, another one is assumed as $\rho'(0) = v$. The solution is given by

$$\rho(t) = -\lambda t^2 + vt + 1$$

General solution formula is

$$g_{ij}(t,x) = (-\lambda t^2 + vt + c)g_{ij}(0,x)$$

Remark: This is different from the Ricci flow!

Case I: The initial metric is Ricci flat, i.e., $\lambda = 0$.
In this case,

$$\rho(t) = vt + 1. \tag{4}$$

If $v = 0$, then $g_{ij}(t,x) = g_{ij}(0,x)$. This shows that $g_{ij}(t,x) = g_{ij}(0,x)$ is stationary.

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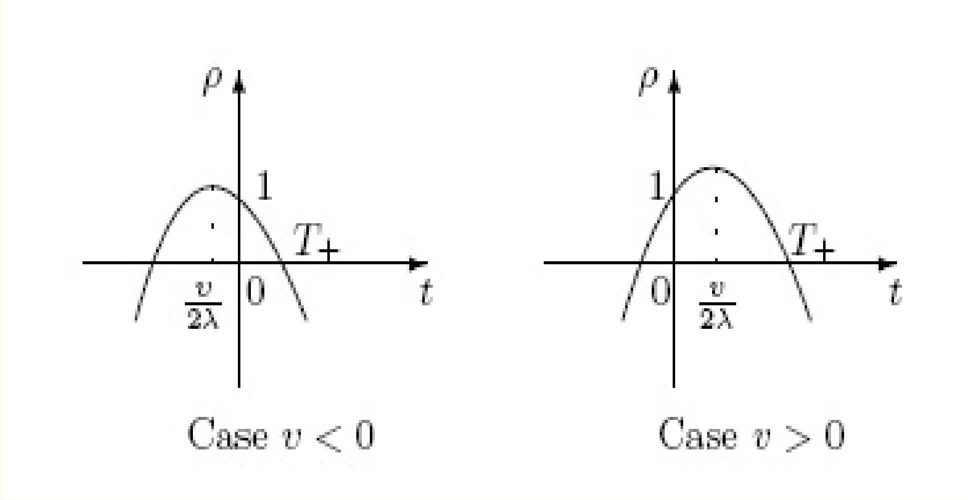
If $v > 0$, then $g_{ij}(t,x) = (1+vt)g_{ij}(0,x)$. This means that the evolving metric $g_{ij}(t,x) = \rho(t)g_{ij}(0,x)$ exists and expands homothetically for all time, and the curvature will fall back to zero like $-\frac{1}{t}$.

Notice that the evolving metric $g_{ij}(t,x)$ only goes back in time to $-v^{-1}$, when the metric explodes out of a single point in a "big bang".

If $v < 0$, then $g_{ij}(t,x) = (1+vt)g_{ij}(0,x)$. Thus, the evolving metric $g_{ij}(t,x)$ shrinks homothetically to a point as $t \nearrow T_0 = -\frac{1}{v}$. Note that, when $t \nearrow T_0$, the scalar curvature is asymptotic to $\frac{1}{T_0 - t}$. This phenomenon corresponds to the "black hole" in physics.

Case II: The initial metric has positive scalar curvature, i.e., $\lambda > 0$.

In this case, the evolving metric will shrink (if $v < 0$) or first expands then shrink (if $v > 0$) under the hyperbolic flow by a time-dependent factor.



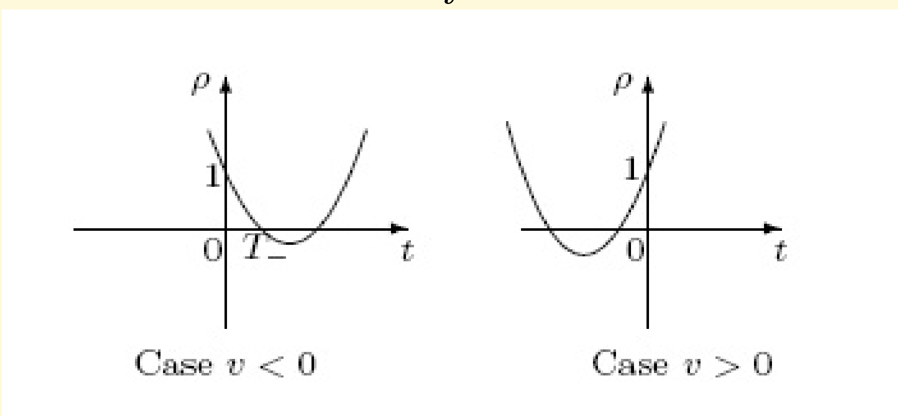
Case III: The initial metric has a negative scalar curvature, i.e., $\lambda < 0$.

In this case, we divide into three cases to discuss:

Case 1: $v^2 + 4\lambda > 0$.

(a) $v < 0$: the evolving metric will shrink in a finite time under the hyperbolic flow by a time-dependent factor;

(b) $v > 0$: the evolving metric $g_{ij}(t, x) = \rho(t) g_{ij}(0, x)$ exists and expands homothetically for all time, and the curvature will fall back to zero like $-\frac{1}{t^2}$.



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Case 2: $v^2 + 4\lambda < 0$.
In this case, the evolving metric $g_{ij}(t,x) = \rho(t) g_{ij}(0,x)$ exists and expands homothetically (if $v > 0$) or first shrinks then expands homothetically (if $v < 0$) for all time.
The scalar curvature will fall back to zero like $-\frac{1}{t^2}$.

Case 3: $v^2 + 4\lambda = 0$.
If $v > 0$, then evolving metric $g_{ij}(t,x) = \rho(t) g_{ij}(0,x)$ exists and expands homothetically for all time. In this case the scalar curvature will fall back to zero like $\frac{1}{t^2}$. If $v < 0$, then the evolving metric $g_{ij}(t,x)$ shrinks homothetically to a point as $t \nearrow T_* = \frac{v}{2\lambda} > 0$ and the scalar curvature is asymptotic to $\frac{1}{T_* - t}$.

Remark: A typical example of the Einstein metric is

$$ds^2 = \frac{1}{1-\kappa r^2}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2,$$

where κ is a constant taking its value -1, 0 or 1. It is easy to see that

$$ds^2 = R^2(t)\left\{\frac{1}{1-\kappa r^2}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2\right\}$$

is a solution of the hyperbolic geometric flow (1), where

$$R^2(t) = -2\kappa t^2 + c_1 t + c_2$$

in which c_1 and c_2 are two constants. This metric has interesting meaning in cosmology.

5.2. Exact solutions with axial symmetry

Consider

$$ds^2 = f(t,z)dz^2 - \frac{t}{g(t,z)}\left[(dx - \mu(t,z)dy)^2 + g^2(t,z))dy^2\right],$$

where f, g are smooth functions with respect to variables.

Since the coordinates x and y do not appear in the preceding metric formula, the coordinate vector fields ∂_x and ∂_y are Killing vector fields. The flow ∂_x (resp. ∂_y) consists of the coordinate translations that send x to $x + \Delta x$ (resp. y to $y + \Delta y$), leaving the other coordinates fixed. Roughly speaking, these isometries express the x-invariance (resp. y-invariance) of the model. The x-invariance and y-invariance show that the model possesses the z-axial symmetry.

Hyperbolic geometric flow gives

$$g_t = \mu_t = 0$$

$$f = \frac{1}{2g^2}\left[g_z^2 + \mu_z^2\right] + \frac{1}{g^4}\mu_z^2(c_1 t + c_2),$$

where g_z and μ_z satisfy

$$g g_z^2 - g g_z \mu_{zz} \mu_z^{-1} + g_z^2 + \mu_z^2 = 0$$

Birkhoff Theorem holds for axial-symmetric solutions!

Angle speed μ is independent of t!

6. Dissipative hyperbolic geometric flow

Let $\mathscr{M}$ be an n-dimensional complete Riemannian manifold with Riemannian metric g_{ij}. Consider the hyperbolic geometric flow

$$\frac{\partial^2 g_{ij}}{\partial t^2} = -2R_{ij} + 2g^{pq}\frac{\partial g_{ip}}{\partial t}\frac{\partial g_{jq}}{\partial t} + \left(d - 2g^{pq}\frac{\partial g_{pq}}{\partial t}\right)\frac{\partial g_{ij}}{\partial t} + \left(c + \frac{1}{n-1}\left(g^{pq}\frac{\partial g_{pq}}{\partial t}\right)^2 + \frac{1}{n-1}\frac{\partial g^{pq}}{\partial t}\frac{\partial g_{pq}}{\partial t}\right)g_{ij}$$

for a family of Riemannian metrics $g_{ij}(t)$ on $\mathscr{M}$, where c and d are arbitrary constants.

This equation also has strong feature of Ricci flow.

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By calculations, we obtain the following evolution equation of the scalar curvature R with respect to the metric $g_{ij}(x,t)$,

$$
\begin{aligned}
\frac{\partial^2 R}{\partial t^2} = \; & \triangle R + 2|Ric|^2 + \left(d - 2g^{pq}\frac{\partial g_{pq}}{\partial t}\right)\frac{\partial R}{\partial t} - \\
& \left(c + \frac{1}{n-1}\left(g^{pq}\frac{\partial g_{pq}}{\partial t}\right)^2 + \frac{1}{n-1}\frac{\partial g^{pq}}{\partial t}\frac{\partial g_{pq}}{\partial t}\right)R + \\
& 2g^{ik}g^{jl}g_{pq}\frac{\partial \Gamma^p_{ij}}{\partial t}\frac{\partial \Gamma^q_{kl}}{\partial t} - 2g^{ik}g^{jl}g_{pq}\frac{\partial \Gamma^p_{ik}}{\partial t}\frac{\partial \Gamma^q_{jl}}{\partial t} + \\
& 8g^{ik}\frac{\partial \Gamma^q_{ip}}{\partial t}\frac{\partial \Gamma^p_{kq}}{\partial t} - 8g^{ik}\frac{\partial \Gamma^p_{ip}}{\partial t}\frac{\partial \Gamma^q_{kq}}{\partial t} - 8g^{ik}\frac{\partial \Gamma^q_{pq}}{\partial t}\frac{\partial \Gamma^p_{ik}}{\partial t}
\end{aligned}
$$

Introduce

$$y \triangleq g^{pq}\frac{\partial g_{pq}}{\partial t} = Tr_g\left\{\frac{\partial g_{pq}}{\partial t}\right\}$$

and

$$z \triangleq g^{pq}g^{rs}\frac{\partial g_{pr}}{\partial t}\frac{\partial g_{qs}}{\partial t} = \left|\frac{\partial g_{pq}}{\partial t}\right|^2.$$

Then we have proved the following

$$\frac{\partial y}{\partial t} = -2R - \frac{n-2}{n-1}y^2 + dy - \frac{1}{n-1}z + cn$$

Dissipative Hyperbolic Geometric Flow

- Dai, Kong and Liu, 2006

7. Riemann surfaces

7.1. Global existence

Consider the evolution of a Riemannian metric g_{ij} on a complete non-compact surface M under the hyperbolic geometric flow equation

$$\frac{\partial^2 g_{ij}}{\partial t^2} = -2R_{ij}.$$

Let us consider the case $\mathbb{R}^2$ with the following initial metric

$$t = 0: \quad ds^2 = u_0(x)(dx^2 + dy^2),$$

where $u_0(x)$ is a smooth function with bounded $\mathcal{C}^2$ norm and satisfies

$$0 < m \leqslant u_0(x) \leqslant M < \infty,$$

in which m, M are two positive constants.

Theorem 1: (Kong and Liu, 2007) Given the above initial metric, for any smooth function $u_1(x)$ satisfying

(1) $u_1(x)$ has bounded $\mathcal{C}^1$ norm;

(2) it holds that

$$u_1(x) \geqslant \frac{|u_0'(x)|}{\sqrt{u_0(x)}}, \quad \forall x \in \mathbb{R},$$

the Cauchy problem

$$\begin{cases} \dfrac{\partial^2 g_{ij}}{\partial t^2} = -2R_{ij} \quad (i,j=1,2), \\[2ex] t=0: \quad g_{ij} = u_0(x)\delta_{ij}, \quad \dfrac{\partial g_{ij}}{\partial t} = u_1(x)\delta_{ij} \quad (i,j=1,2) \end{cases}$$

has a unique smooth solution for all time, and the solution metric g_{ij} possesses the form $g_{ij} = u(t,x)\delta_{ij}$. $\square$

Theorem 2: (Kong and Liu, 2007) Under the assumptions mentioned in Theorem 1, suppose that there exists a small positive constant ε such that

$$u_1(x) \geqslant \frac{|u_0'(x)|}{\sqrt{u_0(x)}} + \varepsilon, \quad \forall\, x \in \mathbb{R},$$

then the Cauchy problem has a unique smooth solution with the above form for all time, moreover the solution metric g_{ij} converges to flat metric at an algebraic rate $\dfrac{1}{(1+t)^k}$, where $k \leqslant 2$ is a positive constant depending on ε, M, the C^2 norm of u_0 and C^1 norm of u_1. □

Method of proof.

(1) **Cauchy problem for a kind of nonlinear wave equation**

$$\begin{cases} u_{tt} - \triangle \ln u = 0, \\ t = 0: \quad u = u_0(x), \; u_t = u_1(x) \end{cases}$$

In the present situation,

$$\begin{cases} u_{tt} - (\ln u)_{xx} = 0, \\ t = 0: \quad u = u_0(x), \; u_t = u_1(x) \end{cases}$$

Let $\phi = \ln u$, then the equation is reduced to

$$\phi_{tt} - e^{-\phi}\phi_{xx} = -\phi_t^2.$$

(2) Quasilinear hyperbolic system

Let

$$v = \phi_t, \;\; w = \phi_x.$$

Then

$$\begin{cases} \phi_t = v, \\ w_t - v_x = 0, \\ v_t - e^{-\phi} w_x = -v^2 \end{cases}$$

Introduce

$$p = v + e^{-\frac{\phi}{2}} w, \;\; q = v - e^{-\frac{\phi}{2}} w.$$

Lemma: $p, \; q$ **satisfy**

$$\begin{cases} p_t - \lambda p_x = -\frac{1}{4}\{p^2 + 3pq\}, \\ q_t + \lambda q_x = -\frac{1}{4}\{q^2 + 3pq\}, \end{cases}$$

where

$$\lambda = e^{-\frac{\phi}{2}}.$$

Lemma: It holds that

$$\begin{cases} p_t - (\lambda p)_x = pq, \\ q_t + (\lambda q)_x = pq. \end{cases} \qquad \square$$

Let

$$r = p_x, \quad s = q_x.$$

Lemma: r and s satisfy

$$\begin{cases} r_t - \lambda r_x = -\frac{1}{4}\left[(2q+3p)r + 3ps\right], \\ s_t + \lambda s_x = -\frac{1}{4}\left[(2p+3q)s + 3qr\right]. \end{cases} \qquad \square$$

(3) **Uniform estimates on p, q and r, s**

“Maximum principle” for hyperbolic systems

(4) **Global existence of HGF.**

(5) **Estimate on R**

$$\boldsymbol{R = \frac{(\ln u)_{xx}}{u} = \frac{1}{2}\left\{(r-s)e^{-\frac{\phi}{2}} + \left(\frac{p-q}{2}\right)^2\right\}.}$$

$$\Downarrow$$

$$\boldsymbol{|R(t,x)| \leqslant C\frac{1}{(1+t)^{\min\{2,C\}}}, \quad \forall (t,x) \in \mathbb{R}^+ \times \mathbb{R}.}$$

7.2. Formation of singularities

Consider the metric

$$t = 0 : \quad ds^2 = u_0(x)(dx^2 + dy^2).$$

Without loss of generality, we assume that there exists a point $x_0 \in \mathbb{R}$ such that

$$u_0'(x_0) < 0.$$

We choose

$$u_1(x) \equiv \frac{u_0'(x)}{\sqrt{u_0(x)}}, \quad \forall\, x \in \mathbb{R}.$$

Theorem 3: For the initial data $u_0(x)$ and $u_1(x)$ mentioned above, the Cauchy problem has a unique smooth solution only in $[0, \tilde{T}_{\max}) \times \mathbb{R}$, where

$$\tilde{T}_{\max} = -\frac{4}{\inf_{x \in \mathbb{R}} \{p_0(x)\}}$$

where

$$p_0(x) = 2u_0'(x) u_0^{-\frac{3}{2}}(x).$$

Moreover, there exists some point $(\tilde{T}_{\max}, x_*)$ such that the scalar curvature $R(t, x)$ satisfies

$$R(t, x) \to \infty \qquad \text{as} \quad (t, x) \to (\tilde{T}_{\max}, x_*). \qquad \square$$

By choosing suitable velocity, we may avoid doing surgery!

We have precise control on

- ♣ Blowup set.
- ♣ Solution character close to blowup point.

8. Some results related to HGF

8.1. Time-periodic solutions of Einstein equations

Einstein equation:

$$G_{\mu\nu} \triangleq R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$$

♣ **A new solution: Breather**

(1) time periodic space-time,

(2) asymptotically flat in space.

♣ **Physical characters: Time periodic solution, asymptotical flatness of t-sections, naked singularity ($r = 0$).**

8.2. Hyperbolic mean curvature flow for hypersurfaces

$$\frac{\partial^2 X}{\partial t^2} = -H\vec{n}$$

where $X = X(t, x_1, \cdots, x_n)$, H — mean curvature, $\vec{n}$ out normal vector.

Our recent results: (He-Kong-Liu)

♣ Extrinsic flow: short time existence, nonlinear stability, relations between the HGF and the HMCF flow equations for extremal hypersurfaces in space-time.

♣ Intrinsic HGF flow and extrinsic HMCF flow, both have interesting physical meaning.

9. Open Problems

◇ Yau has several conjectures about complete noncompact manifolds with nonnegative curvature. HGF may provide a promising way to approach these conjectures. Hyperbolic PDE has advantage in dealing with noncompact manifolds.

◇ Given initial metric g_{ij}^0 and symmetric tensor k_{ij}, study the singularity of the Einstein HGF with these initial data. Singularities of Einstein equation belong to these singularities. This should be related to the Penrose cosmic censorship conjecture.

◇ HGF on general manifolds, global existence and singularity. HGF has global solution for small initial data. Choosing velocity to avoid surgery to get geometric structure.

◇ There was an approach of geometrization by using Einstein equation which is too complicated to use. May HGF shed some light on this approach, by adjusting speed to avoid surgery?

◇ Hyperbolic Yang-Mills flow?

Thank You All!