# THE N-POINT FUNCTIONS FOR INTERSECTION NUMBERS ON MODULI SPACES OF CURVES

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Abstract. We derive from Witten's KdV equation a simple formula of the $n$-point functions for intersection numbers on moduli spaces of curves, generalizing Dijkgraaf's two-point function and Zagier's three-point function. This formula uncovers many new identities about integrals of $\psi$ classes and provides an elementary and more efficient algorithm to compute intersection numbers other than the celebrated Witten-Kontsevich theorem.

## 1. Introduction

We denote by $\overline{\mathcal{M}}_{g,n}$ the moduli space of stable $n$-pointed genus $g$ complex algebraic curves. Let $\psi_i$ be the first Chern class of the line bundle whose fiber over each pointed stable curve is the cotangent line at the $i$-th marked point.

We adopt Witten's notation in this paper,

$$\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n}.$$

These intersection numbers are the correlation functions of two dimensional topological quantum gravity. In the famous paper [7], Witten made the remarkable conjecture (proved by Kontsevich [4]) that the generating function of above intersection numbers are governed by KdV hierarchy, which provides a recursive way to compute all these intersection numbers. Witten's conjecture was reformulated by Dijkgraaf, Verlinde, and Verlinde [DVV] in terms of the Virasoro algebra.

**Definition 1.1.** We call the following generating function

$$F(x_1, \dots, x_n) = \sum_{g=0}^{\infty} \sum_{\sum d_j = 3g-3+n} \langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g \prod_{j=1}^{n} x_j^{d_j}$$

the $n$-point function.

The $n$-point function encodes all information of the correlation functions of two dimensional topological quantum gravity. Okounkov [6] obtained an analytic expression of the $n$-point functions using $n$-dimensional error-function-type integrals. Brézin and Hikami [1] apply correlation functions of GUE ensemble to find explicit formulae of $n$-point functions.

The first key point is to consider the following "normalized" $n$-point function

$$G(x_1, \dots, x_n) = \exp\left( \frac{-\sum_{j=1}^{n} x_j^3}{24} \right) \cdot F(x_1, \dots, x_n).$$

In particular, we have 1-point function $G(x)=\frac{1}{x^2}$, Dijkgraaf's 2-point function

$$G(x,y)=\frac{1}{x+y}\sum_{k\geq 0}\frac{k!}{(2k+1)!}\left(\frac{1}{2}xy(x+y)\right)^k$$

and Zagier's 3-point function [8] which we learned from Faber,

$$G(x,y,z)=\sum_{r,s\geq 0}\frac{r!S_r(x,y,z)}{4^r(2r+1)!!\cdot 2}\cdot\frac{\Delta^s}{8^s(r+s+1)!},$$

where $S_r(x,y,z)$ and $\Delta$ are the homogeneous symmetric polynomials defined by

$$S_r(x,y,z)=\frac{(xy)^r(x+y)^{r+1}+(yz)^r(y+z)^{r+1}+(zx)^r(z+x)^{r+1}}{x+y+z}\in\mathbb{Z}[x,y,z],$$

$$\Delta(x,y,z)=(x+y)(y+z)(z+x)=\frac{(x+y+z)^3}{3}-\frac{x^3+y^3+z^3}{3}.$$

Although two and three point functions are found in the early 1990's, it's not obvious at all that clean explicit formulae of general $n$-point functions should exist. Recall that we only have closed formula of intersection numbers in genus zero and one. Now we state the main theorem of this note.

**Theorem 1.2.** *For $n\geq 2$,*

$$G(x_1,\dots,x_n)=\sum_{r,s\geq 0}\frac{(2r+n-3)!!}{4^s(2r+2s+n-1)!!}P_r(x_1,\dots,x_n)\cdot\Delta(x_1,\dots,x_n)^s, \tag{1}$$

*where $P_r$ and $\Delta$ are homogeneous symmetric polynomials defined by*

$$\Delta(x_1,\dots,x_n)=\frac{(\sum_{j=1}^n x_j)^3-\sum_{j=1}^n x_j^3}{3},$$

$$P_r(x_1,\dots,x_n)=\left(\frac{1}{2\cdot\sum_{j=1}^n x_j}\sum_{\underline{n}=I\coprod J}(\sum_{i\in I}x_i)^2\cdot(\sum_{i\in J}x_i)^2\cdot G(x_I)\cdot G(x_J)\right)_{3r+n-3}$$

$$=\frac{1}{2\cdot\sum_{j=1}^n x_j}\sum_{\underline{n}=I\coprod J}(\sum_{i\in I}x_i)^2\cdot(\sum_{i\in J}x_i)^2\cdot\sum_{r'=0}^{r}G_{r'}(x_I)G_{r-r'}(x_J), \tag{2}$$

*where $I,J\neq\emptyset$, $\underline{n}=\{1,2,\dots,n\}$ and $G_g(x_I)$ denotes the degree $3g+|I|-3$ homogeneous component of the normalized $|I|$-point function $G(x_{k_1},\dots,x_{k_{|I|}})$, where $k_j\in I$.*

Note that the degree $3r+n-3$ polynomial $P_r(x_1,\dots,x_n)\in\mathbb{Q}[x_1,\dots,x_n]$ is expressed by normalized $|I|$-point functions $G(x_I)$ with $|I|<n$. So we can recursively obtain an explicit formula of the $n$-point function

$$F(x_1,\dots,x_n)=\exp\left(\frac{\sum_{j=1}^n x_j^3}{24}\right)\cdot G(x_1,\dots,x_n),$$

thus we have an elementary algorithm to calculate all intersection numbers of $\psi$ classes other than the celebrated Witten-Kontsevich's theorem [4, 7], which is the only feasible way known before to calculate all intersection numbers of $\psi$ classes.

Since $P_0(x,y)=\frac{1}{x+y}$, $P_r(x,y)=0$ for $r>0$, we get Dijkgraaf's 2-point function. From

$$P_r(x,y,z)=\frac{r!}{2^r(2r+1)!}\cdot\frac{(xy)^r(x+y)^{r+1}+(yz)^r(y+z)^{r+1}+(zx)^r(z+x)^{r+1}}{x+y+z},$$

we also easily recover Zagier's 3-point function obtained more than ten years ago.

There is another slightly different formula of $n$-point functions. When $n=3$, this has also been obtained by Zagier [8].

**Theorem 1.3.** *For $n\geq 2$,*

$$F(x_1,\dots,x_n)=\exp\frac{(\sum_{j=1}^n x_j)^3}{24}\sum_{r,s\geq 0}\frac{(-1)^s P_r(x_1,\dots,x_n)\cdot\Delta(x_1,\dots,x_n)^s}{8^s(2r+2s+n-1)s!}$$

*where $P_r$ and $\Delta$ are the same polynomials as defined in theorem 1.2.*

Theorem 1.3 follows from Theorem 1.2 and the following lemma.

**Lemma 1.4.** *Let $n\geq 2$ and $r,s\geq 0$. Then the following identity holds,*

$$\frac{(-1)^s}{8^s(2r+2s+n-1)s!}=\sum_{k=0}^{s}\frac{(-1)^k}{8^k k!}\cdot\frac{(2r+n-3)!!}{4^{s-k}(2r+2s-2k+n-1)!!} \tag{3}$$

*Proof.* Let $p=2r+n\geq 2$ and

$$f(p,s)=\sum_{k=0}^{s}\frac{(-1)^k}{2^k k!(p+2s-2k-1)!!}.$$

We have

$$\begin{aligned}f(p,s)&=\sum_{k=0}^{s}\frac{(-1)^k(p+2s+1)}{2^k k!(p+2s-2k+1)!!}+\sum_{k=0}^{s}\frac{2k(-1)^{k-1}}{2^k k!(p+2s-2k+1)!!}\\&=(p+2s+1)\left(f(p,s+1)-\frac{(-1)^{s+1}}{2^{s+1}(s+1)!(p-1)!!}\right)+f(p,s)-\frac{(-1)^s}{2^s s!(p-1)!!}.\end{aligned}$$

So we have the following identity

$$f(p,s+1)=\frac{(-1)^{s+1}}{2^{s+1}(p+2s+1)(s+1)!(p-3)!!},$$

which is just the identity (3) if $s+1$ is replaced by $s$. □

In Section 2 we give a proof of the main theorem. Section 3 contains many new identities of the intersection numbers of the $\psi$ classes derived from our formula of the $n$-point functions. In Section 4 we briefly discuss other applications of the $n$-point functions.

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## 2. Proof of the main theorem

We can derive from Witten's KdV equation the following coefficient equation (see [3, 7]),

$$(2d_1+1)\langle\tau_{d_1}\tau_0^2\prod_{j=2}^n\tau_{d_j}\rangle=\frac{1}{4}\langle\tau_{d_1-1}\tau_0^4\prod_{j=2}^n\tau_{d_j}\rangle$$
$$+\sum_{\{2,\ldots,n\}=I\coprod J}\left(\langle\tau_{d_1-1}\tau_0\prod_{i\in I}\tau_{d_i}\rangle\langle\tau_0^3\prod_{i\in J}\tau_{d_i}\rangle+2\langle\tau_{d_1-1}\tau_0^2\prod_{i\in I}\tau_{d_i}\rangle\langle\tau_0^2\prod_{i\in J}\tau_{d_i}\rangle\right),$$

which is equivalent to the following differential equation of $n$-point functions $F(x_1,\ldots,x_n)$,

$$\left(2x_1\frac{\partial}{\partial x_1}+1\right)\left((\sum_{j=1}^n x_j)^2\cdot F(x_1,\ldots,x_n)\right)=\left(\frac{x_1}{4}(\sum_{j=1}^n x_j)^4+x_1\sum_{j=1}^n x_j\right)\cdot F(x_1,\ldots,x_n)$$
$$+\frac{x_1}{2}\sum_{\underline{n}=I\coprod J}\left((\sum_{i\in I}^n x_i)(\sum_{i\in J}^n x_i)^3+2(\sum_{i\in I}^n x_i)^2(\sum_{i\in J}^n x_i)^2\right)F(x_I)F(x_J).$$

So in order to prove Theorem 1.2, we need to check that

$$E(x_1,\ldots,x_n):=(\sum_{j=1}^n x_j)\cdot G(x_1,\ldots,x_n)$$

satisfies the following differential equation,

$$\left(2x_1\sum_{j=1}^n x_j\right)\frac{\partial}{\partial x_1}E(x_1,\ldots,x_n)+\left(x_1+\frac{x_1^3}{4}\sum_{j=1}^n x_j+\sum_{j=1}^n x_j-\frac{x_1}{4}(\sum_{j=1}^n x_j)^3\right)E(x_1,\ldots,x_n)$$
$$=\frac{x_1}{2}\sum_{\underline{n}=I\coprod J}\left((\sum_{i\in J}x_i)^2+2(\sum_{i\in I}x_i)\cdot(\sum_{i\in J}x_i)\right)E(x_I)E(x_J). \tag{4}$$

The verification is straightforward from the definition of $G(x_1,\ldots,x_n)$ in Theorem 1.2.

We now prove the following initial value condition of $G(x_1,\ldots,x_n)$, thus conclude the proof of Theorem 1.2.

$$G(x_1,\ldots,x_n,0)=(\sum_{j=1}^n x_j)\cdot G(x_1,\ldots,x_n).$$

Let

$$M_r(x_1,\ldots,x_n):=\sum_{\underline{n}=I\coprod J}(\sum_{i\in I}x_i)^3\cdot(\sum_{i\in J}x_i)^2\cdot\sum_{r'=0}^r G_{r'}(x_I)G_{r-r'}(x_J).$$

For the left hand side, we have

$$
\begin{aligned}
&(\sum_{j=1}^{n} x_j)\cdot \mathrm{LHS}\\
&=\sum_{r,s\geq 0}\frac{(2r+n-2)!!}{4^s(2r+2s+n)!!}\cdot\left(M_r+(\sum_{j=1}^{n}x_j)^2G_r(x_1,\dots,x_n)\right)\cdot\Delta(x_1,\dots,x_n)^s\\
&=\sum_{r,s\geq 0}\frac{(2r+n-2)!!}{4^s(2r+2s+n)!!}M_r\Delta^s+\sum_{r,s\geq 0}\frac{(2r+n-2)!!}{4^s(2r+2s+n)!!}\sum_{p+q=r}\frac{(2p+n-3)!!}{4^q(2r+n-1)!!}M_p\Delta^{q+s}\\
&=\sum_{r,s\geq 0}\left(\frac{(2r+n-2)!!}{4^s(2r+2s+n)!!}+\sum_{k=r}^{r+s}\frac{(2k+n-2)!!(2r+n-3)!!}{4^s(2r+2s+n)!!(2k+n-1)!!}\right)M_r\Delta^s,
\end{aligned}
$$

where in the last equation we have used change of variables.

While for the right hand side,

$$(\sum_{j=1}^{n} x_j)\cdot \mathrm{RHS}=\sum_{r,s\geq 0}\frac{(2r+n-3)!!}{4^s(2r+2s+n-1)!!}\cdot M_r\cdot\Delta^s.$$

So we need only prove the following combinatorial identity

$$\frac{(2r+n-2)!!}{4^s(2r+2s+n)!!}+\sum_{k=r}^{r+s}\frac{(2k+n-2)!!(2r+n-3)!!}{4^s(2r+2s+n)!!(2k+n-1)!!}=\frac{(2r+n-3)!!}{4^s(2r+2s+n-1)!!}$$

$$\text{i.e.}\qquad \frac{(2r+n-2)!!}{(2r+n-3)!!}+\sum_{k=r}^{r+s}\frac{(2k+n-2)!!}{(2k+n-1)!!}=\frac{(2r+2s+n)!!}{(2r+2s+n-1)!!}$$

for all $n\geq 2$ and $r,s\geq 0$. It follows easily from the following identity

$$\frac{(p+1)!!}{p!!}+\frac{(p+1)!!}{(p+2)!!}=\frac{(p+3)!!}{(p+2)!!}.$$

It is typical that from the formula of $n$-point functions in Theorem 1.2, many assertions about intersection numbers will be reduced to combinatorial identities.

## 3. New properties of the n-point functions

In this section we derive various new identities about the intersection numbers of the $\psi$ classes by using our simple formula of the $n$-point functions.

**Lemma 3.1.** *Let $n\geq 2$.*

(1) *We have the following recursion relation for normalized $n$-point functions*

$$G_g(x_1,\dots,x_n)=\frac{1}{(2g+n-1)}P_g(x_1,\dots,x_n)+\frac{\Delta(x_1,\dots,x_n)}{4(2g+n-1)}G_{g-1}(x_1,\dots,x_n).$$

(2) *The following identity holds*

$$\Delta(x_1,\dots,x_n)=x_1^2(\sum_{j=2}^{n}x_j)+x_1(\sum_{j=2}^{n}x_j)^2+\Delta(x_2,\dots,x_n).$$

*Proof.* We have

$$G_g(x_1,\dots,x_n)=\sum_{r+s=g}\frac{(2r+n-3)!!}{4^s(2g+n-1)!!}P_r(x_1,\dots,x_n)\cdot\Delta(x_1,\dots,x_n)^s$$
$$=\frac{1}{2g+n-1}P_g(x_1,\dots,x_n)+\sum_{r+s=g-1}\frac{(2r+n-3)!!}{4^{s+1}(2g+n-1)!!}P_r(x_1,\dots,x_n)\Delta(x_1,\dots,x_n)^{s+1}$$
$$=\frac{1}{(2g+n-1)}P_g(x_1,\dots,x_n)+\frac{\Delta(x_1,\dots,x_n)}{4(2g+n-1)}G_{g-1}(x_1,\dots,x_n).$$

The proof of (2) is easy. □

Let $\mathcal{C}\left(\prod_{j=1}^n x_j^{d_j}, P(x_1,\dots,x_n)\right)$ denotes the coefficient of $\prod_{j=1}^n x_j^{d_j}$ in a polynomial or formal power series $P(x_1,\dots,x_n)$. From the inductive structure in the definition of $n$-point functions, we have the following basic properties of $n$-point functions, their proofs are purely combinatorial.

First consider the normalized $(n+1)$-point function $G(z,x_1,\dots,x_n)$. Here we use the variable $z$ to distinguish one point. We have the following theorem about the coefficients of $G(z,x_1,\dots,x_n)$.

**Theorem 3.2.** *Let $2g-2+n\ge 0$.*

(1) *If $k>2g-2+n$, $d_j\ge 0$ and $\sum_{j=1}^n d_j=3g-2+n-k$, then*

$$\mathcal{C}\left(z^k\prod_{j=1}^n x_j^{d_j}, G_g(z,x_1,\dots,x_n)\right)=0,$$
$$\mathcal{C}\left(z^k\prod_{j=1}^n x_j^{d_j}, P_g(z,x_1,\dots,x_n)\right)=0.$$

(2) *Let $d_j\ge 0$, $\sum_{j=1}^n d_j=g$ and $a=\#\{j\mid d_j=0\}$. Then*

$$\mathcal{C}\left(z^{2g-2+n}\prod_{j=1}^n x_j^{d_j}, G_g(z,x_1,\dots,x_n)\right)=\frac{1}{4^g\cdot\prod_{j=1}^n(2d_j+1)!!},$$
$$\mathcal{C}\left(z^{2g-2+n}\prod_{j=1}^n x_j^{d_j}, P_g(z,x_1,\dots,x_n)\right)=\frac{a}{4^g\cdot\prod_{j=1}^n(2d_j+1)!!}.$$

(3) *Let $d_j\ge 0$, $\sum_{j=1}^n d_j=g+1$, $a=\#\{j\mid d_j=0\}$ and $b=\#\{j\mid d_j=1\}$. Then*

$$\mathcal{C}\left(z^{2g-3+n}\prod_{j=1}^n x_j^{d_j}, G_g(z,x_1,\dots,x_n)\right)=\frac{2g^2+(2n-1)g+\frac{n^2-n}{2}-3+\frac{5a-a^2}{2}}{4^g\cdot\prod_{j=1}^n(2d_j+1)!!},$$
$$\mathcal{C}\left(z^{2g-3+n}\prod_{j=1}^n x_j^{d_j}, P_g(z,x_1,\dots,x_n)\right)=\frac{a(2g^2+2ng-g+\frac{n^2-n-a^2+5a}{2}+3b-3)-3b}{4^g\cdot\prod_{j=1}^n(2d_j+1)!!}.$$

*Proof.* (1) is obvious from theorem 1.2. We now prove (2) inductively.

$$\begin{aligned}\mathcal{C}\left(z^{2g-2+n}\prod_{j=1}^n x_j^{d_j}, P_g(z,x_1,\ldots,x_n)\right) &= \sum_{j=1}^n \mathcal{C}\left(z^{2g-2+n}\prod_{j=1}^n x_j^{d_j}, G_g(z,x_1,\ldots,\hat{x}_j,\ldots,x_n)\right)\\ &= \frac{a}{4^g\cdot\prod_{j=1}^n(2d_j+1)!!},\end{aligned}$$

where $a=\#\{j\mid d_j=0\}$.

$$\begin{aligned}&\mathcal{C}\left(z^{2g-2+n}, G_g(z,x_1,\ldots,x_n)\right)\\ &=\sum_{r+s=g}\frac{(2r+n-2)!!}{4^s(2g+n)!!}\sum_{\sum d_j=r}\frac{a\cdot\prod_{j=1}^n x_j^{d_j}}{4^r\prod_{j=1}^n(2d_j+1)!!}(\sum_{j=1}^n x_j)^s\\ &=\frac{1}{2g+n}\sum_{\sum d_j=g}\frac{a\cdot\prod_{j=1}^n x_j^{d_j}}{4^g\prod_{j=1}^n(2d_j+1)!!}+\frac{\sum_{j=1}^n x_j}{4(2g+n)}\sum_{\sum d_j=g-1}\frac{\prod_{j=1}^n x_j^{d_j}}{4^{g-1}\prod_{j=1}^n(2d_j+1)!!}\\ &=\frac{1}{2g+n}\left(\sum_{\sum d_j=g}\frac{a\cdot\prod_{j=1}^n x_j^{d_j}}{4^g\prod_{j=1}^n(2d_j+1)!!}+\sum_{\sum d_j=g}\frac{(2g+n-a)\prod_{j=1}^n x_j^{d_j}}{4^g\prod_{j=1}^n(2d_j+1)!!}\right)\\ &=\sum_{\sum d_j=g}\frac{\prod_{j=1}^n x_j^{d_j}}{4^g\prod_{j=1}^n(2d_j+1)!!}.\end{aligned}$$

The statement (3) can be proved similarly. □

Now consider the normalized special $(n+2)$-point function $G(y,-y,x_1,\ldots,x_n)$. We have the following theorem about the coefficients of $G(y,-y,x_1,\ldots,x_n)$.

**Theorem 3.3.** *Let $g\geq 0$ and $n\geq 1$.*

(1) *If $k>2g$, $d_j\geq 0$ and $\sum_{j=1}^n d_j=3g+n-k$, then*

$$\mathcal{C}\left(y^k\prod_{j=1}^n x_j^{d_j}, \sum_{\underline{n}=I\coprod J}(y+\sum_{i\in I}x_i)^2\cdot(-y+\sum_{i\in J}x_i)^2\cdot G(y,x_I)\cdot G(-y,x_J)\right)=0,$$

*or equivalently,*

$$\sum_{\underline{n}=I\coprod J}\sum_{j=0}^k(-1)^j\langle\tau_j\tau_0^2\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\tau_{k-j}\tau_0^2\prod_{i\in J}\tau_{d_i}\rangle_{g-g'}=0.$$

(2) *If $d_j\geq 1$ and $\sum_{j=1}^n d_j=g+n$, then*

$$\begin{aligned}&\mathcal{C}\left(y^{2g}\prod_{j=1}^n x_j^{d_j}, \sum_{\underline{n}=I\coprod J}(y+\sum_{i\in I}x_i)^2\cdot(-y+\sum_{i\in J}x_i)^2\cdot G(y,x_I)\cdot G(-y,x_J)\right)\\ &=\frac{(2g+n+1)!}{4^g(2g+1)!\cdot\prod_{j=1}^n(2d_j-1)!!}.\end{aligned}$$

*or equivalently,*

$$\sum_{\underline{n}=I\coprod J}\sum_{j=0}^{2g}(-1)^j\langle\tau_j\tau_0^2\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\tau_{2g-j}\tau_0^2\prod_{i\in J}\tau_{d_i}\rangle_{g-g'}=\frac{(2g+n+1)!}{4^g(2g+1)!\cdot\prod_{j=1}^n(2d_j-1)!!}.$$

(3) *If* $d_j\geq 1$ *and* $\sum_{j=1}^n d_j=g+n$*, then*

$$\mathcal{C}\left(y^{2g}\prod_{j=1}^n x_j^{d_j},\sum_{\underline{n}=I\coprod J}(\sum_{i\in I}x_i)^2\cdot(\sum_{i\in J}x_i)^2\cdot G(y,-y,x_I)\cdot G(x_J)\right)=0.$$

*or equivalently,*

$$\sum_{\underline{n}=I\coprod J}\sum_{j=0}^{2g}(-1)^j\langle\tau_j\tau_{2g-j}\tau_0^2\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\tau_0^2\prod_{i\in J}\tau_{d_i}\rangle_{g-g'}=0.$$

(4) *If* $d_j\geq 0$ *and* $\sum_{j=1}^n d_j=g+n$*, then*

$$\sum_{\underline{n}=I\coprod J}\sum_{j=0}^{2g}(-1)^j\left(\langle\tau_j\tau_0^2\prod_{i\in I}\tau_{d_i}\rangle\langle\tau_{2g-j}\tau_0^2\prod_{i\in J}\tau_{d_i}\rangle+\langle\tau_j\tau_{2g-j}\tau_0^2\prod_{i\in I}\tau_{d_i}\rangle\langle\tau_0^2\prod_{i\in J}\tau_{d_i}\rangle\right)$$
$$=(2g+n+1)\cdot\sum_{j=0}^{2g}(-1)^j\langle\tau_0\tau_j\tau_{2g-j}\tau_{d_1}\cdots\tau_{d_n}\rangle_g$$

(5) *If* $k>2g$*,* $d_j\geq 0$ *and* $\sum_{j=1}^n d_j=3g-1+n-k$*, then*

$$\mathcal{C}\left(y^k\prod_{j=1}^n x_j^{d_j},G_g(y,-y,x_1,\ldots,x_n)\right)=0,$$
$$\mathcal{C}\left(y^k\prod_{j=1}^n x_j^{d_j},P_g(y,-y,x_1,\ldots,x_n)\right)=0.$$

(6) *If* $d_j\geq 0$ *and* $\sum_{j=1}^n d_j=g-1+n$*, then*

$$\mathcal{C}\left(y^{2g}\prod_{j=1}^n x_j^{d_j},P_g(y,-y,x_1,\ldots,x_n)\right)=(2g+n+1)\mathcal{C}\left(y^{2g}\prod_{j=1}^n x_j^{d_j},G_g(y,-y,x_1,\ldots,x_n)\right).$$

*If moreover we have* $d_j\geq 1$*, then*

$$\mathcal{C}\left(y^{2g}\prod_{j=1}^n x_j^{d_j},G_g(y,-y,x_1,\ldots,x_n)\right)=\frac{(2g+n-1)!}{4^g(2g+1)!\cdot\prod_{j=1}^n(2d_j-1)!!}.$$

*Proof.* We first show that (1) and (2) imply the statements (3)-(6).

(3) is obvious, since for $d_i\geq 1$, we have

$$\langle\tau_0^2\prod_{i\in J}\tau_{d_i}\rangle_0=0.$$

(4), (5) and the first identity of (6) follow easily from Theorem 1.2.

Let $d_1 \geq d_2 \geq \cdots \geq d_n \geq 1$. We prove the second identity of (6) by inducting on $d_1$, the maximum index.

$$\begin{aligned}
&\sum_{j=0}^{2g}(-1)^j\langle\tau_{2g-j}\tau_j\tau_{d_1}\cdots\tau_{d_n}\rangle_g\\
&=\sum_{j=0}^{2g}(-1)^j\langle\tau_0\tau_{2g-j}\tau_j\tau_{d_1+1}\tau_{d_2}\cdots\tau_{d_n}\rangle_g-\sum_{k=2}^{n}\sum_{j=0}^{2g}(-1)^j\langle\tau_{2g-j}\tau_j\tau_{d_1+1}\tau_{d_2}\cdots\tau_{d_k-1}\cdots\tau_{d_n}\rangle_g\\
&=\frac{(2g+n)!}{4^g(2g+1)!\cdot\prod_{j=1}^n(2d_j-1)!!(2d_1+1)}-\sum_{k=2}^{n}\frac{(2g+n-1)!(2d_k-1)}{4^g(2g+1)!\cdot\prod_{j=1}^n(2d_j-1)!!(2d_1+1)}\\
&=\frac{(2g+n-1)!}{4^g(2g+1)!\cdot\prod_{j=1}^n(2d_j-1)!!},
\end{aligned}$$

where we have used (4). In fact the above identity still holds if there is only one $d_j=0$.

By explicitly writing down the $n$-point functions, we give a proof of (1) and (2) in the case $n=2$, the general case can be proved similarly. Note also that it is easy to prove Theorem 3.3 for $g=0$ since $G_0(x_1,\dots,x_n)=(x_1+\cdots+x_n)^{n-3}$ (see Lemma 4.3 and Corollary 4.4).

It is easy to prove the following identity by inducting on $g$.

$$\begin{aligned}
0&=\mathcal{C}\left(y^{2g+2},\sum_{\underline{2}=I\coprod J}(y+\sum_{i\in I}x_i)^2\cdot(-y+\sum_{i\in J}x_i)^2\cdot\sum_{g'=0}^{g}G_{g'}(y,x_I)G_{g-g'}(-y,x_J)\right)\\
&=2\sum_{r+s=g}\left(\frac{(2r)!!}{4^s(2g+2)!!}\frac{1}{4^r(2r+1)!!}(x_1^r+x_2^r)(x_1+x_2)^s-\frac{1}{4^r(2r+1)!!}\frac{1}{4^s(2s+1)!!}x_1^rx_2^s\right).
\end{aligned}$$

Because we have

$$\begin{aligned}
&\mathcal{C}\left(y^{2g},\sum_{\underline{2}=I\coprod J}(y+\sum_{i\in I}x_i)^2\cdot(-y+\sum_{i\in J}x_i)^2\cdot\sum_{g'=0}^{g}G_{g'}(y,x_I)G_{g-g'}(-y,x_J)\right)\\
=&2\sum_{r+s=g}\left(\frac{(2r)!!}{4^s(2g+2)!!}\frac{1}{4^r(2r+1)!!}\left(s(x_1^r+x_2^r)(x_1+x_2)^sx_1x_2+\frac{s^2+s}{2}(x_1^r+x_2^r)(x_1+x_2)^{s+2}\right.\right.\\
&+(s+1)(r+1)(x_1^{r+1}+x_2^{r+1})(x_1+x_2)^{s+1}+\left.\frac{r^2+r}{2}(x_1^{r+2}+x_2^{r+2})(x_1+x_2)^s\right)\\
&+\left.\frac{1}{4^r(2r+1)!!}\frac{1}{4^s(2s+1)!!}\left((r+1)(s+1)x_1^{r+1}x_2^{s+1}-\frac{s^2+s}{2}x_1^rx_2^{s+2}-\frac{r^2+r}{2}x_1^{r+2}x_2^s\right)\right)\\
&+\frac{1}{4^g(2g)!!}(x_1+x_2)^{g+2},
\end{aligned}$$

the proof of (2) is also easy. ☐

It is easy to see that the statements (5) and (6) of Theorem 3.3 imply the following identities of intersection numbers which we have announced in [5]. They are related to Faber's intersection number conjecture.

**Corollary 3.4.**

(1) *Let* $d_j \ge 0$, $\#\{j \mid d_j = 0\} \le 1$ *and* $\sum_{j=1}^n d_j = g+n-1$. *Then*

$$\sum_{j=0}^{2g}(-1)^j\langle\tau_{2g-j}\tau_j\tau_{d_1}\cdots\tau_{d_n}\rangle_g = \frac{(2g+n-1)!}{4^g(2g+1)!\prod_{j=1}^n(2d_j-1)!!}.$$

*If* $\#\{j \mid d_j = 0\} = 2$ *and* $a = \#\{j \mid d_j = 1\}$, *then the right hand side becomes*

$$\frac{(2g+n-1)!}{4^g(2g+1)!\prod_{j=1}^n(2d_j-1)!!}\cdot\frac{2g+n-a}{2g+n-1-a}.$$

(2) *Let* $k > 2g$, $d_j \ge 0$ *and* $\sum_{j=1}^n d_j = 3g-1+n-k$. *Then*

$$\sum_{j=0}^{k}(-1)^j\langle\tau_{k-j}\tau_j\tau_{d_1}\cdots\tau_{d_n}\rangle_g = 0.$$

We also have the following generalization of statements (1) and (2) of Theorem 3.3. The proof is similar.

**Theorem 3.5.** *Let* $g \ge 0$, $n \ge 1$ *and* $r, s \ge 0$.

(1) *If* $k > 2g+r+s$, $d_j \ge 0$ *and* $\sum_{j=1}^n d_j = 3g+r+s+n-k$, *then*

$$\mathcal{C}\left(y^k\prod_{j=1}^n x_j^{d_j}, \sum_{\underline{n}=I\coprod J}(y+\sum_{i\in I}x_i)^{2+s}\cdot(-y+\sum_{i\in J}x_i)^{2+r}\cdot G(y,x_I)\cdot G(-y,x_J)\right) = 0,$$

*or equivalently,*

$$\sum_{\underline{n}=I\coprod J}\sum_{j=0}^{k}(-1)^j\langle\tau_j\tau_0^{2+r}\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\tau_{k-j}\tau_0^{2+s}\prod_{i\in J}\tau_{d_i}\rangle_{g-g'} = 0.$$

(2) *If* $d_j \ge 1$ *and* $\sum_{j=1}^n d_j = g+n$, *then*

$$\mathcal{C}\left(y^{2g+r+s}\prod_{j=1}^n x_j^{d_j}, \sum_{\underline{n}=I\coprod J}(y+\sum_{i\in I}x_i)^{2+s}\cdot(-y+\sum_{i\in J}x_i)^{2+r}\cdot G(y,x_I)\cdot G(-y,x_J)\right)$$
$$= \frac{(-1)^r(2g+n+r+s+1)!}{4^g(2g+r+s+1)!\cdot\prod_{j=1}^n(2d_j-1)!!}.$$

*or equivalently,*

$$\sum_{\underline{n}=I\coprod J}\sum_{j=0}^{2g+r+s}(-1)^j\langle\tau_j\tau_0^{2+r}\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\tau_{2g+r+s-j}\tau_0^{2+s}\prod_{i\in J}\tau_{d_i}\rangle_{g-g'}$$
$$= \frac{(-1)^r(2g+n+r+s+1)!}{4^g(2g+r+s+1)!\cdot\prod_{j=1}^n(2d_j-1)!!}.$$

## 4. Other applications of n-point functions

From the new identities in Section 3 and their derivations, we can see that the simple formula of $n$-point functions may be used to prove the following equivalent statement of the Faber's intersection number conjecture:

**Conjecture 4.1.** *Let $d_j \geq 0$ and $\sum_{j=1}^n d_j = g+n-2$. Then*

$$\begin{aligned}\langle \tau_{d_1}\cdots\tau_{d_n}\tau_{2g}\rangle_g &= \sum_{j=1}^n \langle \tau_{d_1}\cdots\tau_{d_{j-1}}\tau_{d_j+2g-1}\tau_{d_{j+1}}\cdots\tau_{d_n}\rangle_g \\ &\quad -\frac{1}{2}\sum_{\underline{n}=I\coprod J}\sum_{j=0}^{2g-2}(-1)^j\langle\tau_j\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\tau_{2g-2-j}\prod_{i\in J}\tau_{d_i}\rangle_{g-g'}.\end{aligned}$$

It is clear that our explicit formula of $n$-point functions should also shed light on the following conjectural identity as stated in [5] where the case of $n=1$ has been proved.

**Conjecture 4.2.** *Let $g \geq 2$, $d_j \geq 1$ and $\sum_{j=1}^n (d_j-1) = g$. Then*

$$\begin{aligned}&\frac{(2g-3+n)!}{2^{2g+1}(2g-3)!\prod_{j=1}^n(2d_j-1)!!} \\ &\qquad = \langle\tau_{d_1}\cdots\tau_{d_n}\tau_{2g-2}\rangle_g - \sum_{j=1}^n\langle\tau_{d_1}\cdots\tau_{d_{j-1}}\tau_{d_j+2g-3}\tau_{d_{j+1}}\cdots\tau_{d_n}\rangle_g \\ &\qquad\qquad + \frac{1}{2}\sum_{\underline{n}=I\coprod J}\sum_{j=0}^{2g-4}(-1)^j\langle\tau_j\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\tau_{2g-4-j}\prod_{i\in J}\tau_{d_i}\rangle_{g-g'}\end{aligned}$$

If we have $d_j \geq 0$, $\#\{j \mid d_j = 0\} = 1$ and $a = \#\{j \mid d_j = 1\}$ in the above conjecture, then the left hand side becomes

$$\frac{(2g-3+n)!}{2^{2g+1}(2g-3)!\prod_{j=1}^n(2d_j-1)!!}\cdot\frac{2g+n+1-a}{2g+n-3-a}.$$

We will discuss the relation of the above conjectures with our simple formula of the $n$-point functions in a forthcoming paper. Here as the first step we only prove two interesting combinatorial identities.

**Lemma 4.3.** *Let $n \geq 2$.*

(1) *Assume that if $I = \emptyset$, then $(\sum_{i\in I} x_i)^{|I|} = 1$. We have*

$$\sum_{\{2,\dots,n\}=I\coprod J}(x_1+\sum_{i\in I}x_i)^{|I|}(-x_1+\sum_{i\in J}x_i)^{|J|} = \sum_{\{2,\dots,n\}=I\coprod J}(\sum_{i\in I}x_i)^{|I|}(\sum_{i\in J}x_i)^{|J|}$$

(2) *We have*

$$\sum_{\substack{\underline{n}=I\coprod J\\ I,J\neq\emptyset}}(\sum_{i\in I}x_i)^{|I|-1}(\sum_{i\in J}x_i)^{|J|-1} = 2(n-1)(\sum_{j=1}^n x_j)^{n-2} \tag{5}$$

*Proof.* Let $\prod_{j=1}^n x_j^{d_j}$ be any monomial of

$$\sum_{\{2,\dots,n\}=I\coprod J}(x_1+\sum_{i\in I}x_i)^{|I|}(-x_1+\sum_{i\in J}x_i)^{|J|}. \tag{6}$$

Since $\sum_{j=1}^n d_j = n-1$, so if $d_1>0$, then their must exist some $j>1$ such that $d_j=0$.

The statement (1) means that the polynomial (6) does not contain $x_1$, so we need only prove that after substitute $x_n=0$ in (6), the resulting polynomial does not contain $x_1$.

$$\begin{aligned}&\sum_{\{2,\dots,n-1\}=I\coprod J}\left((x_1+\sum_{i\in I}x_i)^{|I|+1}(-x_1+\sum_{i\in J}x_i)^{|J|}+(x_1+\sum_{i\in I}x_i)^{|I|}(-x_1+\sum_{i\in J}x_i)^{|J|+1}\right)\\ &=(\sum_{j=2}^{n-1}x_j)\sum_{\{2,\dots,n-1\}=I\coprod J}(x_1+\sum_{i\in I}x_i)^{|I|}(-x_1+\sum_{i\in J}x_i)^{|J|}.\end{aligned}$$

So statement (1) follows by induction.

We prove statement (2) by induction. Regard the LHS and RHS of the identity (5) as polynomials in $x_n$ with degree $n-2$, we need to prove the equality of (5) when substitute $x_n=-x_i$ for $i=1\dots n-1$. It's sufficient to check the case $x_n=-x_{n-1}$.

$$\begin{aligned}LHS&=2\sum_{\underline{n-2}=I\coprod J}\left((x_{n-1}+\sum_{i\in I}x_i)^{|I|}(-x_{n-1}+\sum_{i\in J}x_i)^{|J|}+(\sum_{i\in I}x_i)^{|I|+1}(\sum_{i\in J}x_i)^{|J|-1}\right)\\ &=2\sum_{\underline{n-2}=I\coprod J}\left((\sum_{i\in I}x_i)^{|I|}(\sum_{i\in J}x_i)^{|J|}+(\sum_{i\in I}x_i)^{|I|+1}(\sum_{i\in J}x_i)^{|J|-1}\right)\\ &=4(\sum_{j=1}^{n-2}x_j)^{n-2}+(\sum_{j=1}^{n-2}x_j)^2\sum_{\substack{\underline{n-2}=I\coprod J\\ I,J\neq\emptyset}}(\sum_{i\in I}x_i)^{|I|-1}(\sum_{i\in J}x_i)^{|J|-1}\\ &=2(n-1)(\sum_{j=1}^{n-2}x_j)^{n-2}=RHS.\end{aligned}$$

Note that if a term has power $|J|-1$, then $J\neq\emptyset$ is assumed. □

Finally as an interesting exercise we give a proof of the following well-known formula by using our formula of the $n$-point functions.

**Corollary 4.4.** *Let $n\geq 3$, $d_j\geq 0$ and $\sum_{j=1}^n d_j=n-3$. Then*

$$\langle\tau_{d_1}\cdots\tau_{d_n}\rangle_0=\binom{n-3}{d_1,\dots,d_n}.$$

*Proof.* It's equivalent to prove that for $n \geq 3$

$$
\begin{aligned}
(\sum_{j=1}^{n} x_j)^{n-3} &= G_0(x_1, \ldots, x_n) \\
&= \frac{1}{2(n-1)\sum_{j=1}^{n} x_j} \sum_{\underline{n}=I\coprod J} (\sum_{i\in I} x_i)^2 (\sum_{i\in J} x_i)^2 G_0(x_I) G_0(x_J) \\
&= \frac{1}{2(n-1)\sum_{j=1}^{n} x_j} \sum_{\underline{n}=I\coprod J} (\sum_{i\in I} x_i)^{|I|-1} (\sum_{i\in J} x_i)^{|J|-1}.
\end{aligned}
$$

This is just the Lemma 4.3 (2). □

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