Wave Character of Metrics and Hyperbolic Geometric Flow

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In this letter, we illustrate the wave character of the metrics and curvatures of manifolds, and introduce a new understanding tool - the hyperbolic geometric flow. This kind of flow is new and very natural to understand certain wave phenomena in the nature as well as the geometry of manifolds. It possesses many interesting properties from both mathematics and physics. Several applications of this method have been found.

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1. Introduction. Let us observe the water in the beautiful Westlake in Hangzhou. If there is no wind, the water surface can be regarded as a plane with a flat Riemmannian metric $\delta_{ij}(i,j=1,2)$. When wind blows over, the water wave propagates from one side to another side. In this case, the metric of the water surface is not flat globally, and changes along the time. There exists a front, called *wave front*, such that the metric is not flat after the front and is still flat before the front. We would like to call this phenomenon *the wave character of the metric*. Motivated by such wave character of metric as well as the work of Ricci flow, we introduce and study the following evolution equation which we would like to call the *hyperbolic geometric flow*: let $\mathscr{M}$ be n-dimensional complete Riemannian manifold with Riemannian metric g_{ij}, the Levi-Civita connection is given by the Christoffel symbols

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left\{ \frac{\partial g_{il}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l} \right\},$$

where g^{ij} is the inverse of g_{ij}. The Riemannian curvature tensors read

$$R^k_{ijl} = \frac{\partial \Gamma^k_{jl}}{\partial x^i} - \frac{\partial \Gamma^k_{il}}{\partial x^j} + \Gamma^k_{ip}\Gamma^p_{jl} - \Gamma^k_{jp}\Gamma^p_{il}, \quad R_{ijkl} = g_{kp} R^p_{ijl}.$$

The Ricci tensor is the contraction

$$R_{ik} = g^{jl} R_{ijkl}$$

and the scalar curvature is

$$R = g^{ij} R_{ij}.$$

The hyperbolic geometric flow considered here is the evolution equation

$$\frac{\partial^2 g_{ij}}{\partial t^2} = -2R_{ij}. \tag{1}$$

for a family of Riemannian metrics $g_{ij}(t)$ on $\mathscr{M}$. (1) is a nonlinear system of second order partial differential equations on the metric g_{ij}. More generally, we can consider the following evolution equations

$$\frac{\partial^2 g_{ij}}{\partial t^2} + 2R_{ij} = \kappa T_{ij}, \tag{2}$$

where T_{ij} denotes the energy-momentum tensor of the matter and of the fields (other than gravitational) which may be present (for example, Maxwell tensor etc.), and κ is a positive physical constant. In the vacuum, i.e., in the regions with $T_{ij}=0$, the equation (2) reduces to (1).

The hyperbolic geometric flow is a very natural tool to understand the wave character of the metrics and wave phenomenon of the curvatures. We will prove that it has many surprisingly good properties, which have essential and fundamental differences from the Einstein field equations (see [1]) and the Ricci flow (see [6]). More applications of hyperbolic geometric flow to both mathematics and physics can be expected.

The elliptic and parabolic partial differential equations have been successfully applied to differential geometry and physics. Typical examples are Hamilton's Ricci flow and Schoen-Yau's solution of the positive mass conjecture (see [6], [7]). A natural and important question is if we can apply the well-developed theory of hyperbolic differential equations to solve problems in differential geometry and theoretical physics. This letter is an attempt to apply the hyperbolic equation techniques to study some geometrical problems and physical problems. We have already found interesting results in these directions (see [3]). The method may be more important than the results presented in this letter. Our results show that the hyperbolic geometric flow is a natural and powerful tool to study some problems arising form differential geometry such as singularities, existence and regularity.

2. Hyperbolic geometric flow. Hyperbolic geometric flow considered here is the evolution equation (1), it describes the wave character of the Riemannian metrics $g_{ij}(t)$ on an n-dimensional complete Riemannian manifold $\mathscr{M}$. The version (1) of the hyperbolic geometric flow is the unnormalized evolution equation. We next consider the normalized version of hyperbolic geometric flow (1), which preserves the volume of the flow.

The hyperbolic geometric flow and the normalized hyperbolic geometric flow differ only by a change of scale in space and a change of parametrization in time. We now derive the normalized version of (1). Assume that $g_{ij}(t)$ is a solution of the (unnormalized) hyperbolic geometric flow (1) and choose the normalization factor $\varphi = \varphi(t)$

such that

$$\widetilde{g}_{ij} = \varphi^2 g_{ij} \quad \text{and} \quad \int_M d\widetilde{V} = 1. \tag{3}$$

Next we choose a new time scale

$$\widetilde{t} = \int \varphi(t)dt. \tag{4}$$

Then, for the normalized metric $\widetilde{g}_{ij}$, we have

$$\widetilde{R}_{ij} = R_{ij}, \quad \widetilde{R} = \frac{1}{\varphi^2}R, \quad \widetilde{r} = \frac{1}{\varphi^2}r, \tag{5}$$

where $r = \int_M RdV / \int_N dV$ is the average scalar curvature. Noting the second equation in (3) gives

$$\int_M dV = \varphi^{-n}. \tag{6}$$

Then

$$\begin{aligned}
\frac{\partial \widetilde{g}_{ij}}{\partial \widetilde{t}} &= \varphi \frac{\partial g_{ij}}{\partial t} + 2\frac{d\varphi}{dt} g_{ij}, \\
\frac{\partial^2 \widetilde{g}_{ij}}{\partial \widetilde{t}^2} &= \frac{\partial^2 g_{ij}}{\partial t^2} + 3\left(\frac{d}{dt}\log\varphi\right)\frac{\partial g_{ij}}{\partial t} + \\
&\quad 2\left(\frac{d}{dt}\log\varphi\right)\left(\frac{d}{dt}\log\frac{d\varphi}{dt}\right) g_{ij} \\
&= -2\widetilde{R}_{ij} + 3\varphi^{-1}\left(\frac{d}{dt}\log\varphi\right)\frac{\partial \widetilde{g}_{ij}}{\partial \widetilde{t}} + \\
&\quad 2\varphi^{-2}\left(\frac{d}{dt}\log\varphi\right)\left\{\frac{d}{dt}\log\frac{d\varphi}{dt} - 3\frac{d}{dt}\log\varphi\right\}\widetilde{g}_{ij} \\
&\equiv -2\widetilde{R}_{ij} + a\frac{\partial \widetilde{g}_{ij}}{\partial \widetilde{t}} + b\widetilde{g}_{ij}.
\end{aligned}$$

By (6) and calculations, we observe that a and b are certain functions of t. The following evolution equation

$$\frac{\partial^2 \widetilde{g}_{ij}}{\partial \widetilde{t}^2} = -2\widetilde{R}_{ij} + a\frac{\partial \widetilde{g}_{ij}}{\partial \widetilde{t}} + b\widetilde{g}_{ij} \tag{7}$$

is called the *normalized version* of the hyperbolic geometric flow (1). Thus, studying the behavior of the hyperbolic geometric flow near the maximal existence time is equivalent to studying the long-time behavior of normalized hyperbolic geometric flow.

Motivated by (7), we may consider the following more general evolution equations

$$\alpha_{ij}\frac{\partial^2 g_{ij}}{\partial t^2} + \beta_{ij}\frac{\partial g_{ij}}{\partial t} + \gamma_{ij}g_{ij} + 2R_{ij} = 0, \tag{8}$$

where $\alpha_{ij}, \beta_{ij}, \gamma_{ij}$ are certain smooth functions on $\mathscr{M}$ which may depend on t. In particular, if $\alpha_{ij} = 1, \beta_{ij} = \gamma_{ij} = 0$, then (8) goes back to the hyperbolic geometric flow; if $\alpha_{ij} = 0, \beta_{ij} = 1, \gamma_{ij} = 0$, then (8) is nothing but the famous Ricci flow; if $\alpha_{ij} = 0, \beta_{ij} = 1, \gamma_{ij} = -\frac{2}{n}r$, then (8) is the normalized Ricci flow (see [6]). In this sense, we name the evolution equations (8) as *hyperbolic-parabolic geometric flow*. We will study its geometric and physical meanings later.

Similar to (2), we may also consider the following field equations with the energy-momentum tensor T_{ij}

$$\alpha_{ij}\frac{\partial^2 g_{ij}}{\partial t^2} + 2R_{ij} + \mathscr{F}_{ij}\left(g, \frac{\partial g}{\partial t}\right) = \kappa T_{ij}, \tag{9}$$

where α_{ij} are certain smooth functions on $\mathscr{M}$ which may also depend on t, $\mathscr{F}_{ij}$ are some given smooth functions of the Riemannian metric g and its first order derivative with respect to t. In the vacuum, a special case of the field equation (9) goes back to the equation (8).

At the end of this section, we remark that if the underlying manifold $\mathscr{M}$ is a complex manifold and the metric is Kähler, similar to (8) the following complex evolution equations are very natural to consider

$$a_{ij}\frac{\partial^2 g_{i\bar{j}}}{\partial t^2} + b_{ij}\frac{\partial g_{i\bar{j}}}{\partial t} + c_{ij}g_{i\bar{j}} + 2R_{i\bar{j}} = 0, \tag{10}$$

where a_{ij}, b_{ij}, c_{ij} are certain smooth functions on $\mathscr{M}$ which may also depend on t. The evolution equation (10) is called the *complex hyperbolic-parabolic geometric flow*.

3. Exact solutions with the Einstein initial metrics. This section is devoted to studying the exact solutions with the Einstein initial metrics for the hyperbolic geometric flow (1). These exact solutions are useful to understand the basic features of the hyperbolic geometric flow.

Definition 1. (Einstein metric and manifold) *A Riemannian metric g_{ij} is called Einstein if $R_{ij} = \lambda g_{ij}$ for some constant λ. A smooth manifold $\mathscr{M}$ with an Einstein metric is called an Einstein manifold.*

If the initial metric $g_{ij}(0, x)$ is Ricci flat, i.e., $R_{ij}(0, x) = 0$, then $g_{ij}(t, x) = g_{ij}(0, x)$ is obviously a solution to the evolution equation (1). Therefore, any Ricci flat metric is a stationary solution of the hyperbolic geometric flow (1).

If the initial metric is Einstein, that is, for some constant λ it holds

$$R_{ij}(0, x) = \lambda g_{ij}(0, x), \quad \forall\, x \in \mathscr{M}, \tag{11}$$

then the evolving metric under the hyperbolic geometric flow (1) will be steady state, or will expand homothetically for all time, or will shrink in a finite time.

Indeed, since the initial metric is Einstein, (11) holds for some constant λ. Let

$$g_{ij}(t, x) = \rho(t) g_{ij}(0, x). \tag{12}$$

By the defintion of the Ricci tensor, one obtains

$$R_{ij}(t, x) = R_{ij}(0, x) = \lambda g_{ij}(0, x). \tag{13}$$

In the present situation, the equation (1) becomes

$$\frac{\partial^2(\rho(t) g_{ij}(0, x))}{\partial t^2} = -2\lambda g_{ij}(0, x). \tag{14}$$

This gives an ODE of second order

$$\frac{d^2\rho(t)}{dt^2} = -2\lambda. \qquad (15)$$

Obviously, one of the initial conditions for (15) is

$$\rho(0) = 1. \qquad (16)$$

Another one is assumed as

$$\rho'(0) = v, \qquad (17)$$

where v is a real number standing for the initial velocity. The solution of the initial value problem (15)-(17) is given by

$$\rho(t) = -\lambda t^2 + vt + 1. \qquad (18)$$

Case I The initial metric is Ricci flat, i.e., $\lambda = 0$.

In this case,

$$\rho(t) = vt + 1. \qquad (19)$$

If $v = 0$, then $g_{ij}(t,x) = g_{ij}(0,x)$. This shows that $g_{ij}(t,x) = g_{ij}(0,x)$ is stationary.

If $v > 0$, then $g_{ij}(t,x) = (1+vt)g_{ij}(0,x)$. This means that the evolving metric $g_{ij}(t,x) = \rho(t)g_{ij}(0,x)$ exists and expands homothetically for all time, and the curvature will fall back to zero like $-\frac{1}{t}$.

Notice that the evolving metric $g_{ij}(t,x)$ only goes back in time to $-v^{-1}$, when the metric explodes out of a single point in a "big bang".

If $v < 0$, then $g_{ij}(t,x) = (1+vt)g_{ij}(0,x)$. Thus, the evolving metric $g_{ij}(t,x)$ shrinks homothetically to a point as $t \nearrow T_0 = -\frac{1}{v}$. Note that, when $t \nearrow T_0$, the scalar curvature is asymptotic to $\frac{1}{T_0 - t}$. This phenomenon corresponds to the "black hole" in physics.

Conclusion: *For the Ricci flat initial metric, if the initial velocity is zero, then the evolving metric g_{ij} is stationary; if the initial velocity is positive, then the evolving metric g_{ij} exists and expands homothetically for all time; if the intial velocity is negative, then the evolving metric g_{ij} shrinks homothetically to a point in a finite time.*

Remark 1. For the Ricci flat initial metric, the evolving metric under the Ricci flow is only stationary. From this point of view, the hyperbolic geometric flow is essentially different from the Ricci flow.

Case II The initial metric has positive scalar curvature, i.e., $\lambda > 0$.

In this case, the evolving metric will shrink (if $v < 0$) or first expands then shrink (if $v > 0$) under the hyperbolic flow by a time-dependent factor (see Figure 1). In fact, it follows from (18) that, when $t \nearrow T_+ = \frac{v+\sqrt{v^2+4\lambda}}{2\lambda}$, $\rho(t) \searrow 0$. This implies that the evolving metric $g_{ij}(t,x)$ shrinks homothetically to a point as $t \nearrow T_+$. Notice that as $t \nearrow T_+$, the scalar curvature is asymptotic to $\frac{1}{T_+ - t}$.

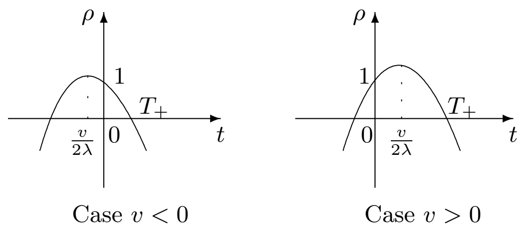

FIG. 1: The evolving metric shrinks (if $v < 0$) or first expands then shrinks (if $v > 0$)

Case III The initial metric has a negative scalar curvature, i.e., $\lambda < 0$.

In this case, we devide into three cases to discuss:

Case 1 $v^2 + 4\lambda > 0$.

(a) $v < 0$: the evolving metric will shrink in a finite time under the hyperbolic flow by a time-dependent factor (see Figure 2). In fact, if follows from (18) that, when $t \nearrow T_- = \frac{-v-\sqrt{v^2+4\lambda}}{-2\lambda} > 0$, $\rho(t) \searrow 0$. This shows that the evolving metric $g_{ij}(t,x)$ shrinks homothetically to a point as $t \nearrow T_-$ and the scalar curvature is asymptotic to $\frac{1}{T_- - t}$;

(b) $v > 0$: the evolving metric $g_{ij}(t,x) = \rho(t)g_{ij}(0,x)$ exists and expands homothetically for all time, and the curvature will fall back to zero like $-\frac{1}{t^2}$. See Figure 2.

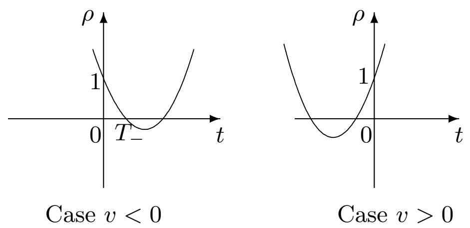

FIG. 2: The evolving metric shrinks (if $v < 0$) or expands (if $v > 0$)

Case 2 $v^2 + 4v < 0$.

In this case, the evolving metric $g_{ij}(t,x) = \rho(t)g_{ij}(0,x)$ exists and expands homothetically (if $v > 0$) or first shrinks then expands homothetically (if $v < 0$) for all time.

The scalar curvature will fall back to zero like $-\frac{1}{t^2}$.

Case 3 $v^2 + 4\lambda = 0$.

If $v > 0$, then evolving metric $g_{ij}(t,x) = \rho(t)g_{ij}(0,x)$ exists and expands homothetically for all time. In this case the scalar curvature will fall back to zero like $\frac{1}{t^2}$. If $v < 0$, then the evolving metric $g_{ij}(t,x)$ shrinks homothetically to a point as $t \nearrow T_* = \frac{v}{2\lambda} > 0$ and the scalar curvature is asymptotic to $\frac{1}{T_* - t}$.

4. Short-time existence and uniqueness. In this section we state the short-time existence and uniqueness result for the hyperbolic geometric flow (1) on a compact n-dimensional manifold $\mathscr{M}$. We can show that the hyperbolic geometric flow is a system of second order nonlinear weakly hyperbolic partial differential equations. The degeneracy of the system is caused by the diffeomorphism group of $\mathscr{M}$ which acts as the gauge group of the hyperbolic geometric flow. Because the hyperbolic geometric flow (1) is only weakly hyperbolic, the short-time existence and uniqueness result on a compact manifold does not come from the standard PDEs theory. However we can still prove the following short-time existence and uniqueness theorem.

Theorem 1. *Let $(\mathscr{M}, g^0_{ij}(x))$ be a compact Riemannian manifold. Then there exists a constant $\varepsilon > 0$ such that the initial value problem*

$$\begin{cases} \frac{\partial^2}{\partial t^2} g_{ij}(t,x) = -2R_{ij}(t,x), \\ g_{ij}(0,x) = g^0_{ij}(x), \ \frac{\partial g_{ij}}{\partial t}(0,x) = k^0_{ij}(x), \end{cases}$$

has a unique smooth solution $g_{ij}(t,x)$ on $M \times [0,\varepsilon)$, where $k^0_{ij}(x)$ is a symmetric tensor on $\mathscr{M}$.

The above short-time existence and uniqueness theorem can be proved by the following two ways: (a) using the gauge fixing idea as in Ricci flow, we can derive a system of second order nonlinear strictly hyperbolic partial differential equations, thus Theorem 1 comes from the standard PDEs theory; (b) we reduce the hyperbolic geometric flow (1) to a first-order quasilinear symmetric hyperbolic system, then using the Friedrich's theory [5] of symmetric hyperbolic system (more exactly, the quasilinear version [4]) we can also prove Theorem 1. See Dai, Kong and Liu [2] for the details.

5. Wave property of curvatures. The hyperbolic geometric flow is an evolution equation on the metric $g_{ij}(t,x)$. The evolution for the metric implies a nonlinear wave equation for the Riemannian curvature tensor R_{ijkl}, the Ricci curvature tensor R_{ij} and the scalar curvature R.

Theorem 2. *Under the hyperbolic geometric flow (1), the curvature tensors satisfy the evolution equations*

$$\frac{\partial^2 R_{ijkl}}{\partial t^2} = \Delta R_{ijkl} + \textit{(lower order terms)}, \qquad (20)$$

$$\frac{\partial^2 R_{ij}}{\partial t^2} = \Delta R_{ij} + \textit{(lower order terms)}, \qquad (21)$$

$$\frac{\partial^2 R}{\partial t^2} = \Delta R + \textit{(lower order terms)}, \qquad (22)$$

where Δ is the Laplacian with respect to the evolving metric, the lower order terms only contain lower order derivatives of the curvatures.

By (1), the equations (20)-(22) come from direct calculations. See Dai, Kong and Liu [3] for the details and geometric applications. The equations (20)-(22) show that the curvatures possess interesting wave property.

6. Summary and discussion. Hyperbolic partial differential equations can be used to describe the wave phenomena in the nature. In this letter, the hyperbolic geometric flow is introduced to illustrate the wave character of the metrics, which also implies the wave property of the curvature. Note that the hyperbolic geometric flow possesses very interesting geometric properties and dynamical behavior. A direct application of Theorem 2 gives the stability of solutions to the hyperbolic geometric flow equation on the Euclidean spaces under metric perturbations (see [3]). More applications of this flow to differential geometry and physics can be expected.

So far there have been many successes of elliptic and parabolic equations applied to mathematics and physics, but by now very few results on the applications of hyperbolic PDEs are known (see [4]). We believe that the hyperbolic geometric flow is a new and powerful tool to study geometric problems. Moreover, its physical application has been observed (see [3]). In the future we will study several fundamental problems, for examples, long-time existence, formation of singularities, as well as the physical and geometrical applications.

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