Gravity and the Noncommutative Residue for Manifolds with Boundary *

Yong Wang

†*Center of Mathematical Sciences Zhejiang University Hangzhou Zhejiang 310027, China ;*

E-mail: wangy581@nenu.edu.cn

Abstract We prove a Kastler-Kalau-Walze type theorem for the Dirac operator and the signature operator for manifolds with boundary. As a corollary, for the boundary flat case, we give two kinds of operator theoritic explainations of the gravitational action for boundary.

Subj. Class.: Noncommutative global analysis; Noncommutative differential geometry.

MSC: 58G20; 53A30; 46L87

Keywords: Noncommutative residue for manifolds with boundary; gravitational action for manifolds with boundary

1 Introduction

The noncommutative residue found in [Gu] and [Wo] plays a prominent role in noncommutative geometry. In [C1],Connes used the noncommutative residue to derive a conformal 4-dimensional Polyakov action analogy. In [C2], Connes proved that the noncommutative residue on a compact manifold M coincided with the Dixmier's trace on pseudodifferential operators of order $-\mathrm{dim}M$. Several years ago, Connes made a challenging observation that the noncommutative residue of the square of the inverse of the Dirac operator was proportional to the Einstein-Hilbert action, which was called Kastler-Kalau-Walze Theorem now. In [K], Kastler gave a brute-force proof of this theorem. In [KW], Kalau and Walze proved this theorem by the normal coordinates way simultaneously. In [A], Ackermann gave a note on a new proof of this theorem by the heat kernal expansion way.

On the other hand, Fedosov etc. defined a noncommutative residue on Boutet de Monvel's algebra and proved that it was a unique continues trace in [FGLS]. In [S], Schrohe gave the relation between the Dixmier trace and the noncommutative residue for manifolds with boundary. In [Wa1] and [Wa2], we generalized some results in [C1] and [U] to the case of manifolds with boundary . In [H], the gravitational action for manifolds with boundary was worked out (also see [B]). The motivation of this paper

*partially supported by NSF of NENU.

†also at School of Mathematics and Statistics, Northeast Normal University

is to give an operator theoritic explaination of the gravitational action for manifolds with boundary and prove a Kastler-Kalau-Walze type theorem for manifolds with boundary.

Let us recall Kastler-Kalau-Walze Theorem in [K], [KW],[A]. Let M be a 4-dimensional oriented spin manifold and D be the associated Dirac operator on the spinor bundle $S(TM)$. Let s be the scalar curvature and Wres denote the noncommutative residue (see [Wo],[FGV]). Then Kastler-Kalau-Walze Theorem gives a spectral explaination of the gravitational action and says that there exists a constant c_0, such that

$$\mathrm{Wres}(D^{-2}) = c_0 \int_M s\mathrm{dvol}_M. \tag{1.1}$$

For an oriented spin manifold M^4 with boundary ∂M, we use $\widetilde{\mathrm{Wres}}[(\pi^+D^{-1})^2]$ instead of $\mathrm{Wres}(D^{-2})$. Here $\widetilde{\mathrm{Wres}}$ denotes the noncommutative residue for manifolds with boundary in [FGLS] and π^+D^{-1} is an element in Boutet de Monvel's algebra (see [Wa1], Section 3).We hope to get the gravitational action for manifolds with boundary by computing $\widetilde{\mathrm{Wres}}[(\pi^+D^{-1})^2]$. For simplicity, we assume that the metric g^M on M has the following form near the boundary,

$$g^M = \frac{1}{h(x_n)} g^{\partial M} + dx_n^2, \tag{1.2}$$

where $g^{\partial M}$ is the metric on ∂M. $h(x_n) \in C^\infty([0,1)) = \{\widetilde{h}|_{[0,1)} | \widetilde{h} \in C^\infty((-\varepsilon,1))\}$ for some $\varepsilon > 0$ and satisfies $h(x_n) > 0$, $h(0) = 1$ where x_n denotes the normal directional coordinate. Through the computation, we find that the boundary extra term which we expect to get vanishes, so $\widetilde{\mathrm{Wres}}[(\pi^+D^{-1})^2]$ is also proportional to $\int_M s\mathrm{dvol}_M$. Fortunately, if we assume that ∂M is flat, then we can define $\int_{\partial M} \mathrm{res}_{1,1}[(\pi^+D^{-1})^2]$ and $\int_{\partial M} \mathrm{res}_{2,1}[(\pi^+D^{-1})^2]$ (see Section 2) and get the gravitational action for ∂M is proportional to $\int_{\partial M} \mathrm{res}_{1,1}[(\pi^+D^{-1})^2]$ and $\int_{\partial M} \mathrm{res}_{2,1}[(\pi^+D^{-1})^2]$, which gives two kinds of operator theoritic explainations of the gravitational action for boundary.

This paper is organized as follows: In Section 2, for the Dirac operator D, we compute $\widetilde{\mathrm{Wres}}[(\pi^+D^{-1})^2]$. In Section 3, we compute $\widetilde{\mathrm{Wres}}[(\pi^+D^{-1})^2]$ for the signature operator. Two kinds of operator theoritic explainations of the gravitational action for boundary will be given in Section 4. In Appendix, the proof of two facts in Scetion 2 will be given.

2 The Dirac operator case

In this section, we compute $\widetilde{\mathrm{Wres}}[(\pi^+D^{-1})^2]$ by the brute force way in [K] and the normal coordinates way in [KW].

Let M be a 4-dimensional compact oriented spin manifold with boundary ∂M and the metric g^M in (1.2). Let $U \subset M$ be the collar neighborhood of ∂M which is diffeomorphic to $\partial M \times [0,1)$. By the definition of $C^\infty([0,1))$ and $h > 0$, there exists $\widetilde{h} \in C^\infty((-\varepsilon,1))$ such that $\widetilde{h}|_{[0,1)} = h$ and $\widetilde{h} > 0$ for some sufficiently small $\varepsilon > 0$.

Then there exists a metric $\widehat{g}$ on $\widehat{M}$ which has the form on $U \cup_{\partial M} \partial M \times (-\varepsilon, 0]$

$$\widehat{g} = \frac{1}{\widetilde{h}(x_n)} g^{\partial M} + dx_n^2, \tag{2.2}$$

such that $\widehat{g}|_M = g$. We fix a metric $\widehat{g}$ on the $\widehat{M}$ such that $\widehat{g}|_M = g$. We can get the spin structure on $\widehat{M}$ by extending the spin structure on M. Let D be the Dirac operator associated to $\widehat{g}$ on the spinor bundle $S(T\widehat{M})$. We want to compute $\widetilde{\mathrm{Wres}}[(\pi^+ D^{-1})^2]$ (for the related definitions, see [Wa1], Section 2, 3). Let $\mathbf{S}$ ($\mathbf{S}'$) be the unit sphere about ξ (ξ') and $\sigma(\xi)$ ($\sigma(\xi')$) be the corresponding canonical $n-1$ ($n-2$) volume form. Denote by $\sigma_l(A)$ the l- order symbol of an operator A. By [Wa1], (2.6), (3.7), (3.14), (3.15) and p.10 line 2, we have,

$$\widetilde{\mathrm{Wres}}[(\pi^+ D^{-1})^2] = \int_M \int_{|\xi|=1} \mathrm{trace}_{S(TM)}[\sigma_{-4}(D^{-2})]\sigma(\xi)dx + \int_{\partial M} \Phi, \tag{2.1}$$

where

$$\Phi = \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{j,k=0}^{\infty} \sum \frac{(-i)^{|\alpha|+j+k+1}}{\alpha!(j+k+1)!}$$

$$\times \mathrm{trace}_{S(TM)}[\partial_{x_n}^j \partial_{\xi'}^\alpha \partial_{\xi_n}^k \sigma_r^+(D^{-1})(x', 0, \xi', \xi_n) \times \partial_{x'}^\alpha \partial_{\xi_n}^{j+1} \partial_{x_n}^k \sigma_l(D^{-1})(x', 0, \xi', \xi_n)] d\xi_n \sigma(\xi') dx', \tag{2.2}$$

where the sum is taken over $r - k - |\alpha| + l - j - 1 = -4, \;\; r, l \leq -1$ and $\sigma_r^+(D^{-1}) = \pi_{\xi_n}^+ \sigma_r(D^{-1})$ (for the definition of π^+, see [Wa1] (2.1)). By [K], [KW], [A], we have

$$\int_M \int_{|\xi|=1} \mathrm{tr}[\sigma_{-4}(D^{-2})]\sigma(\xi)dx = -\frac{\Omega_4}{3} \int_M s\mathrm{dvol}_M. \tag{2.3}$$

where $\Omega_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$. So we only to compute $\int_{\partial M} \Phi$.

Firstly, we compute the symbol $\sigma(D^{-1})$ of D^{-1}. Recall the definition of the Dirac operator D (see [BGV], [Y]). Let ∇^L denote the Levi-civita connection about g^M. In the local coordinates $\{x_i; 1 \leq i \leq n\}$ and the fixed orthonormal frame $\{\widetilde{e_1}, \cdots, \widetilde{e_n}\}$, the connection matrix $(\omega_{s,t})$ is defined by

$$\nabla^L(\widetilde{e_1}, \cdots, \widetilde{e_n}) = (\widetilde{e_1}, \cdots, \widetilde{e_n})(\omega_{s,t}). \tag{2.4}$$

$c(\widetilde{e_i})$ denotes the Clifford action. The Dirac operator

$$D = \sum_{i=1}^n c(\widetilde{e_i})[\widetilde{e_i} - \frac{1}{4} \sum_{s,t} \omega_{s,t}(\widetilde{e_i}) c(\widetilde{e_s}) c(\widetilde{e_t})]. \tag{2.5}$$

So we get,

$$\sigma_1(D) = \sqrt{-1} c(\xi); \sigma_0(D) = -\frac{1}{4} \sum_{i,s,t} \omega_{s,t}(\widetilde{e_i}) c(\widetilde{e_i}) c(\widetilde{e_s}) c(\widetilde{e_t}), \tag{2.6}$$

where $\xi=\sum_{i=1}^{n}\xi_i dx_i$ denotes the cotangent vector. Write

$$D_x^{\alpha}=(-\sqrt{-1})^{|\alpha|}\partial_x^{\alpha};\ \sigma(D)=p_1+p_0;\ \sigma(D^{-1})=\sum_{j=1}^{\infty}q_{-j}. \tag{2.7}$$

By the composition formula of psudodifferential operators, then we have

$$\begin{aligned}1=\sigma(D\circ D^{-1}) &= \sum_{\alpha}\frac{1}{\alpha!}\partial_{\xi}^{\alpha}[\sigma(D)]D_x^{\alpha}[\sigma(D^{-1})]\\ &= (p_1+p_0)(q_{-1}+q_{-2}+q_{-3}+\cdots)\\ &\quad +\sum_j(\partial_{\xi_j}p_1+\partial_{\xi_j}p_0)(D_{x_j}q_{-1}+D_{x_j}q_{-2}+D_{x_j}q_{-3}+\cdots)\\ &= p_1q_{-1}+(p_1q_{-2}+p_0q_{-1}+\sum_j\partial_{\xi_j}p_1D_{x_j}q_{-1})+\cdots,\end{aligned}$$

Thus, we get:

$$q_{-1}=p_1^{-1};\ q_{-2}=-p_1^{-1}[p_0p_1^{-1}+\sum_j\partial_{\xi_j}p_1D_{x_j}(p_1^{-1})]. \tag{2.8}$$

By (2.6), (2.8) and a direct computation, we have

Lemma 2.1

$$q_{-1}=\frac{\sqrt{-1}c(\xi)}{|\xi|^2};\quad q_{-2}=\frac{c(\xi)p_0c(\xi)}{|\xi|^4}+\frac{c(\xi)}{|\xi|^6}\sum_j c(dx_j)[\partial_{x_j}[c(\xi)]|\xi|^2-c(\xi)\partial_{x_j}(|\xi|^2)] \tag{2.9}$$

Since Φ is a global form on ∂M, so for any fixed point $x_0\in\partial M$, we can choose the normal coordinates U of x_0 in ∂M (not M) and compute $\Phi(x_0)$ in the coordinates $\widetilde{U}=U\times[0,1)\subset M$ and the metric $\frac{1}{h(x_n)}g^{\partial M}+dx_n^2$. The dual metric of g^M on $\widetilde{U}$ is $h(x_n)g^{\partial M}+dx_n^2$. Write $g_{ij}^M=g^M(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j});\ g_M^{ij}=g^M(dx_i,dx_j)$, then

$$[g_{i,j}^M]=\begin{bmatrix}\frac{1}{h(x_n)}[g_{i,j}^{\partial M}] & 0\\ 0 & 1\end{bmatrix};\quad [g_M^{i,j}]=\begin{bmatrix}h(x_n)[g_{\partial M}^{i,j}] & 0\\ 0 & 1\end{bmatrix}, \tag{2.10}$$

and

$$\partial_{x_s}g_{ij}^{\partial M}(x_0)=0,1\leq i,j\leq n-1;\quad g_{ij}^M(x_0)=\delta_{ij}. \tag{2.11}$$

Let $\{e_1,\cdots,e_{n-1}\}$ be the orthonormal frame field in U about $g^{\partial M}$ which is parallel along geodesics and $e_i(x_0)=\frac{\partial}{\partial x_i}(x_0)$, then $\{\widetilde{e_1}=\sqrt{h(x_n)}e_1,\cdots,\widetilde{e_{n-1}}=\sqrt{h(x_n)}e_{n-1},\widetilde{e_n}=dx_n\}$ is the orthonormal frame field in $\widetilde{U}$ about g^M. Locally $S(TM)|_{\widetilde{U}}\cong\widetilde{U}\times\wedge_{\mathbf{C}}^*(\frac{n}{2})$. Let $\{f_1,\cdots,f_4\}$ be the orthonormal basis of $\wedge_{\mathbf{C}}^*(\frac{n}{2})$. Take the spin frame field $\sigma:\ \widetilde{U}\to\mathrm{Spin}(M)$ such that $\pi\sigma=\{\widetilde{e_1},\cdots,\widetilde{e_n}\}$ where $\pi:\ \mathrm{Spin}(M)\to O(M)$ is a double covering, then $\{[(\sigma,f_i)],\ 1\leq i\leq 4\}$ is the orthonormal frame of $S(TM)|_{\widetilde{U}}$. In the following, since the global form Φ is independent of the choice of the local frame, so we can compute $\mathrm{tr}_{S(TM)}$ in the frame $\{[(\sigma,f_i)],\ 1\leq i\leq 4\}$. Let

$\{E_1, \cdots, E_n\}$ be the canonical basis of $\mathbf{R}^n$ and $c(E_i) \in \mathrm{cl}_{\mathbf{C}}(n) \cong \mathrm{Hom}(\wedge^*_{\mathbf{C}}(\frac{n}{2}), \wedge^*_{\mathbf{C}}(\frac{n}{2}))$ be the Clifford action. By [Y], then

$$c(\widetilde{e_i}) = [(\sigma, c(E_i))];\ c(\widetilde{e_i})[(\sigma, f_i)] = [(\sigma, c(E_i)f_i)];\ \frac{\partial}{\partial x_i} = [(\sigma, \frac{\partial}{\partial x_i})], \tag{2.12}$$

then we have in the above frame, $\frac{\partial}{\partial x_i} c(\widetilde{e_i}) = 0$

Lemma 2.2 $\partial_{x_j}(|\xi|^2_{g^M})(x_0) = 0,$ if $j < n;\ = h'(0)|\xi'|^2_{g^{\partial M}},$ if $j = n.$ (2.13)
$\partial_{x_j}[c(\xi)](x_0) = 0,$ if $j < n;\ = \partial_{x_n}[c(\xi')](x_0),$ if $j = n,$ (2.14)
where $\xi = \xi' + \xi_n dx_n$.
Proof. By the equality $\partial_{x_j}(|\xi|^2_{g^M})(x_0) = \partial_{x_j}(h(x_n)g^{l,m}_{\partial M}(x')\xi_l\xi_m + \xi_n^2)$ and (2.11), then (2.13) is correct. By Lemma A.1 in Appendix, (2.14) is correct. □

In order to compute $p_0(x_0)$, we need to compute $\omega_{s,t}(\widetilde{e_i})(x_0)$.

Lemma 2.3 *When* $i < n$, $\omega_{n,i}(\widetilde{e_i})(x_0) = \frac{1}{2}h'(0)$; *and* $\omega_{i,n}(\widetilde{e_i})(x_0) = -\frac{1}{2}h'(0)$, *In other cases,* $\omega_{s,t}(\widetilde{e_i})(x_0) = 0$
Proof. See Appendix. □

Lemma 2.4 $p_0(x_0) = c_0 c(dx_n)$, where $c_0 = -\frac{3}{4}h'(0)$.
Proof. This comes from (2.6), Lemma 2.3 and the relation $c(\widetilde{e_i})c(\widetilde{e_j}) + c(\widetilde{e_j})c(\widetilde{e_i}) = -2\delta_{i.j}$. □

Now we can compute Φ, since the sum is taken over $-r-l+k+j+|\alpha| = -3,\ r, l \leq -1$, then we have the following five cases:

case a) I) $r = -1,\ l = -1\ k = j = 0,\ |\alpha| = 1$

By (2.2), we get

$$\text{case a) I)} = -\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{|\alpha|=1}\mathrm{trace}[\partial^\alpha_{\xi'}\pi^+_{\xi_n}q_{-1} \times \partial^\alpha_{x'}\partial_{\xi_n}q_{-1}](x_0)d\xi_n\sigma(\xi')dx', \tag{2.14}$$

By Lemma 2.2, for $i < n$, then

$$\partial_{x_i}q_{-1}(x_0) = \partial_{x_i}\left(\frac{\sqrt{-1}c(\xi)}{|\xi|^2}\right)(x_0) = \frac{\sqrt{-1}\partial_{x_i}[c(\xi)](x_0)}{|\xi|^2} - \frac{\sqrt{-1}c(\xi)\partial_{x_i}(|\xi|^2)(x_0)}{|\xi|^4} = 0,$$

so case a) I) vanishes.

case a) II) $r = -1,\ l = -1\ k = |\alpha| = 0,\ j = 1$

By (2.2), we get

$$\mathrm{res}_{1,1}[(\pi^+D^{-1})^2] := -\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}[\partial_{x_n}\pi^+_{\xi_n}q_{-1} \times \partial^2_{\xi_n}q_{-1}](x_0)d\xi_n\sigma(\xi')dx', \tag{2.15}$$

By Lemma 2.1 and Lemma 2.2, we have

$$\partial^2_{\xi_n} q_{-1} = \sqrt{-1}\left(-\frac{6\xi_n c(dx_n) + 2c(\xi')}{|\xi|^4} + \frac{8\xi_n^2 c(\xi)}{|\xi|^6}\right); \tag{2.16}$$

$$\partial_{x_n} q_{-1}(x_0) = \frac{\sqrt{-1}\partial_{x_n} c(\xi')(x_0)}{|\xi|^2} - \frac{\sqrt{-1}c(\xi)|\xi'|^2 h'(0)}{|\xi|^4}. \tag{2.17}$$

By [Wa1] (2.1) and the Cauchy integral formula, then

$$\begin{aligned}
\pi^+_{\xi_n}\left[\frac{c(\xi)}{|\xi|^4}\right](x_0)|_{|\xi'|=1} &= \pi^+_{\xi_n}\left[\frac{c(\xi') + \xi_n c(dx_n)}{(1+\xi_n^2)^2}\right] \\
&= \frac{1}{2\pi i}\lim_{u\to 0^-}\int_{\Gamma^+}\frac{\frac{c(\xi')+\eta_n c(dx_n)}{(\eta_n+i)^2(\xi_n+iu-\eta_n)}}{(\eta_n - i)^2} d\eta_n \\
&= \left[\frac{c(\xi') + \eta_n c(dx_n)}{(\eta_n+i)^2(\xi_n-\eta_n)}\right]^{(1)}|_{\eta_n=i} \\
&= -\frac{ic(\xi')}{4(\xi_n - i)} - \frac{c(\xi') + ic(dx_n)}{4(\xi_n-i)^2}
\end{aligned} \tag{2.18}$$

Similarly,

$$\pi^+_{\xi_n}\left[\frac{\sqrt{-1}\partial_{x_n} c(\xi')}{|\xi|^2}\right](x_0)|_{|\xi'|=1} = \frac{\partial_{x_n}[c(\xi')](x_0)}{2(\xi_n - i)}. \tag{2.19}$$

By (2.17), (2.18), (2.19), then

$$\pi^+_{\xi_n}\partial_{x_n} q_{-1}(x_0)|_{|\xi'|=1} = \frac{\partial_{x_n}[c(\xi')](x_0)}{2(\xi_n - i)} + \sqrt{-1}h'(0)\left[\frac{ic(\xi')}{4(\xi_n - i)} + \frac{c(\xi') + ic(dx_n)}{4(\xi_n - i)^2}\right]. \tag{2.20}$$

By the relation of the Clifford action and $\mathrm{tr}AB = \mathrm{tr}BA$, then we have the equalities:

$$\begin{aligned}
&\mathrm{tr}[c(\xi')c(dx_n)] = 0;\quad \mathrm{tr}[c(dx_n)^2] = -4;\quad \mathrm{tr}[c(\xi')^2](x_0)|_{|\xi'|=1} = -4; \\
&\mathrm{tr}[\partial_{x_n} c(\xi')c(dx_n)] = 0;\quad \mathrm{tr}[\partial_{x_n} c(\xi')c(\xi')](x_0)|_{|\xi'|=1} = -2h'(0).
\end{aligned} \tag{2.21}$$

By (2.21) and a direct computation ,we have

$$\begin{aligned}
&h'(0)\mathrm{tr}\left\{\left[\frac{ic(\xi')}{4(\xi_n - i)} + \frac{c(\xi') + ic(dx_n)}{4(\xi_n - i)^2}\right] \times \left[\frac{6\xi_n c(dx_n) + 2c(\xi')}{(1+\xi_n^2)^2} - \frac{8\xi_n^2[c(\xi') + \xi_n c(dx_n)]}{(1+\xi_n^2)^3}\right]\right\}(x_0)|_{|\xi'|=1} \\
&= -4h'(0)\frac{-2i\xi_n^2 - \xi_n + i}{(\xi_n - i)^4(\xi_n + i)^3}.
\end{aligned} \tag{2.22}$$

Similarly, we have

$$-\sqrt{-1}\mathrm{tr}\left\{\left[\frac{\partial_{x_n}[c(\xi')](x_0)}{2(\xi_n - i)}\right] \times \left[\frac{6\xi_n c(dx_n) + 2c(\xi')}{(1+\xi_n^2)^2} - \frac{8\xi_n^2[c(\xi') + \xi_n c(dx_n)]}{(1+\xi_n^2)^3}\right]\right\}(x_0)|_{|\xi'|=1}$$

$$= -2\sqrt{-1}h'(0)\frac{3\xi_n^2-1}{(\xi_n-i)^4(\xi_n+i)^3}. \tag{2.23}$$

By (2.16), (2.20), (2.22), (2.23), then

$$\begin{aligned}
\mathrm{res}_{1,1}[(\pi^+D^{-1})^2] &= -\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\frac{ih'(0)(\xi_n-i)^2}{(\xi_n-i)^4(\xi_n+i)^3}d\xi_n\sigma(\xi')dx' \\
&= -ih'(0)\Omega_3\int_{\Gamma^+}\frac{1}{(\xi_n-i)^2(\xi_n+i)^3}d\xi_n dx' \\
&= -ih'(0)\Omega_3 2\pi i[\frac{1}{(\xi_n+i)^3}]^{(1)}|_{\xi_n=i}dx' \\
&= -\frac{3}{8}\pi h'(0)\Omega_3 dx'.
\end{aligned}$$

case a) III) $r=-1,\ l=-1\ j=|\alpha|=0,\ k=1$

By (2.2), we get

$$-\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}[\partial_{\xi_n}\pi^+_{\xi_n}q_{-1}\times\partial_{\xi_n}\partial_{x_n}q_{-1}](x_0)d\xi_n\sigma(\xi')dx', \tag{2.24}$$

By Lemma 2.2, we have

$$\partial_{\xi_n}\partial_{x_n}q_{-1}(x_0)|_{|\xi'|=1} = -\sqrt{-1}h'(0)\left[\frac{c(dx_n)}{|\xi|^4}-4\xi_n\frac{c(\xi')+\xi_n c(dx_n)}{|\xi|^6}\right]-\frac{2\xi_n\sqrt{-1}\partial_{x_n}c(\xi')(x_0)}{|\xi|^4}. \tag{2.25}$$

$$\partial_{\xi_n}\pi^+_{\xi_n}q_{-1}(x_0)|_{|\xi'|=1} = -\frac{c(\xi')+ic(dx_n)}{2(\xi_n-i)^2}. \tag{2.26}$$

Similar to (2.22), (2.23), we have

$$\mathrm{tr}\left\{\frac{c(\xi')+ic(dx_n)}{2(\xi_n-i)^2}\times\sqrt{-1}h'(0)\left[\frac{c(dx_n)}{|\xi|^4}-4\xi_n\frac{c(\xi')+\xi_n c(dx_n)}{|\xi|^6}\right]\right\}$$
$$= 2h'(0)\frac{i-3\xi_n}{(\xi_n-i)^4(\xi_n+i)^3}; \tag{2.27}$$

and

$$\mathrm{tr}\left[\frac{c(\xi')+ic(dx_n)}{2(\xi_n-i)^2}\times\frac{2\xi_n\sqrt{-1}\partial_{x_n}c(\xi')(x_0)}{|\xi|^4}\right]$$
$$= -2h'(0)\sqrt{-1}\frac{\xi_n}{(\xi_n-i)^4(\xi_n+i)^2}. \tag{2.28}$$

So we get case a) III)$=\frac{3}{8}\pi h'(0)\Omega_3 dx'$.

case b) $r=-2,\ l=-1,\ k=j=|\alpha|=0$

By (2.2), we get

$$\mathrm{res}_{2,1}[(\pi^+D^{-1})^2] := -i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}[\pi^+_{\xi_n}q_{-2}\times\partial_{\xi_n}q_{-1}](x_0)d\xi_n\sigma(\xi')dx', \quad (2.29)$$

By Lemma 2.1 and Lemma 2.2, we have

$$q_{-2}(x_0)=\frac{c(\xi)p_0(x_0)c(\xi)}{|\xi|^4}+\frac{c(\xi)}{|\xi|^6}c(dx_n)[\partial_{x_n}[c(\xi')](x_0)|\xi|^2-c(\xi)h'(0)|\xi|^2_{\partial M}]. \quad (2.30)$$

Then

$$\pi^+_{\xi_n}q_{-2}(x_0)|_{|\xi'|=1}=\pi^+_{\xi_n}\left[\frac{c(\xi)p_0(x_0)c(\xi)+c(\xi)c(dx_n)\partial_{x_n}[c(\xi')](x_0)}{(1+\xi_n^2)^2}\right]$$
$$-h'(0)\pi^+_{\xi_n}\left[\frac{c(\xi)c(dx_n)c(\xi)}{(1+\xi_n)^3}\right]:=B_1-B_2. \quad (2.31)$$

Similar to (2.18), we have

$$B_1=-\frac{A_1}{4(\xi_n-i)}-\frac{A_2}{4(\xi_n-i)^2}, \quad (2.32)$$

where

$$A_1=ic(\xi')p_0c(\xi')+ic(dx_n)p_0c(dx_n)+ic(\xi')c(dx_n)\partial_{x_n}[c(\xi')];$$
$$A_2=[c(\xi')+ic(dx_n)]p_0[c(\xi')+ic(dx_n)]+c(\xi')c(dx_n)\partial_{x_n}c(\xi')-i\partial_{x_n}[c(\xi')]. \quad (2.33)$$

$$\begin{aligned}B_2 &= h'(0)\pi^+_{\xi_n}\left[\frac{-\xi_n^2c(dx_n)^2-2\xi_nc(\xi')+c(dx_n)}{(1+\xi_n^2)^3}\right]\\ &= \frac{h'(0)}{2}\left[\frac{-\eta_n^2c(dx_n)-2\eta_nc(\xi')+c(dx_n)}{(\eta_n+i)^3(\xi_n-\eta_n)}\right]^{(2)}|_{\eta_n=i}\\ &= \frac{h'(0)}{2}\left[\frac{c(dx_n)}{4i(\xi_n-i)}+\frac{c(dx_n)-ic(\xi')}{8(\xi_n-i)^2}+\frac{3\xi_n-7i}{8(\xi_n-i)^3}[ic(\xi')-c(dx_n)]\right].\end{aligned} \quad (2.34)$$

$$\partial_{\xi_n}q_{-1}(x_0)|_{|\xi'|=1}=\sqrt{-1}\left[\frac{c(dx_n)}{1+\xi_n^2}-\frac{2\xi_nc(\xi')+2\xi_n^2c(dx_n)}{(1+\xi_n^2)^2}\right]. \quad (2.35)$$

By (2.34), (2.35), we have

$$\mathrm{tr}[B_2\times\partial_{\xi_n}q_{-1}(x_0)]|_{|\xi'|=1}=\frac{\sqrt{-1}}{2}h'(0)\mathrm{trace}$$
$$\left\{\left\{\left[\frac{1}{4i(\xi_n-i)}+\frac{1}{8(\xi_n-i)^2}-\frac{3\xi_n-7i}{8(\xi_n-i)^3}\right]c(dx_n)+\left[\frac{-1}{8(\xi_n-i)^2}+\frac{3\xi_n-7i}{8(\xi_n-i)^3}\right]ic(\xi')\right\}\right.$$
$$\left.\times\left\{\left[\frac{1}{1+\xi_n^2}-\frac{2\xi_n^2}{(1+\xi_n^2)^2}\right]c(dx_n)-\frac{2\xi_n}{(1+\xi_n^2)^2}c(\xi')\right\}\right\}$$

$$= \frac{\sqrt{-1}}{2} h'(0) \frac{-i\xi_n^2 - \xi_n + 4i}{4(\xi_n - i)^3(\xi_n + i)^2} \mathrm{tr}[\mathrm{id}]. \tag{2.36}$$

Note that

$$B_1 = \frac{-1}{4(\xi_n - i)^2} [(2 + i\xi_n)c(\xi')p_0 c(\xi') + i\xi_n c(dx_n)p_0 c(dx_n)$$

$$+(2 + i\xi_n)c(\xi')c(dx_n)\partial_{x_n} c(\xi') + ic(dx_n)p_0 c(\xi') + ic(\xi')p_0 c(dx_n) - i\partial_{x_n} c(\xi')]. \tag{2.37}$$

By (2.21), (2.35), (2.37), Lemma 2.4 and $\mathrm{tr}(AB) = \mathrm{tr}(BA)$, considering for $i < n$ $\int_{|\xi'|=1}\{\text{odd number product of } \xi_i\}\sigma(\xi') = 0$, then

$$\mathrm{tr}[B_1 \times \partial_{\xi_n} q_{-1}(x_0)]|_{|\xi'|=1} = \frac{-2ic_0}{(1 + \xi_n^2)^2} + h'(0) \frac{\xi_n^2 - i\xi_n - 2}{2(\xi_n - i)(1 + \xi_n^2)^2}. \tag{2.38}$$

By (2.31), (2.36) and (2.38), we have

$$\text{case b)} = -\Omega_3 \int_{\Gamma_+} \frac{2c_0(\xi_n - i) + ih'(0)}{(\xi_n - i)^3(\xi_n + i)^2} d\xi_n dx' = \frac{9}{8}\pi h'(0)\Omega_3 dx'. \tag{2.39}$$

case c) $r = -1,\ l = -2,\ k = j = |\alpha| = 0$

By (2.2), we get

$$= -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \mathrm{trace}[\pi^+_{\xi_n} q_{-1} \times \partial_{\xi_n} q_{-2}](x_0) d\xi_n \sigma(\xi') dx'. \tag{2.40}$$

By

$$\pi^+_{\xi_n} q_{-1}(x_0)|_{|\xi'|=1} = \frac{c(\xi') + ic(dx_n)}{2(\xi_n - i)}; \tag{2.41}$$

$$\partial_{\xi_n} q_{-2}(x_0)|_{|\xi'|=1} = \frac{1}{(1 + \xi_n^2)^3} [(2\xi_n - 2\xi_n^3)c(dx_n)p_0 c(dx_n) + (1 - 3\xi_n^2)c(dx_n)p_0 c(\xi')$$

$$+(1 - 3\xi_n^2)c(\xi')p_0 c(dx_n) - 4\xi_n c(\xi')p_0 c(\xi') + (3\xi_n^2 - 1)\partial_{x_n} c(\xi') - 4\xi_n c(\xi')c(dx_n)\partial_{x_n} c(\xi')$$

$$+2h'(0)c(\xi') + 2h'(0)\xi_n c(dx_n)] + 6\xi_n h'(0) \frac{c(\xi)c(dx_n)c(\xi)}{(1 + \xi_n^2)^4}, \tag{2.42}$$

then similar to the computation of the case b), we have

$$\mathrm{trace}[\pi^+_{\xi_n} q_{-1} \times \partial_{\xi_n} q_{-2}](x_0)|_{|\xi'|=1} = \frac{3h'(0)(i\xi_n^2 + \xi_n - 2i)}{(\xi_n - i)^3(\xi_n + i)^3} + \frac{12h'(0)i\xi_n}{(\xi_n - i)^3(\xi_n + i)^4}. \tag{2.43}$$

So case c) $= -\frac{9}{8}\pi h'(0)\Omega_3 dx'$. Now Φ is the sum of the case a), b) and c), so is zero. Then we get

Theorem 2.5 *Let M be a 4-dimensional compact spin manifold with the boundary ∂M and the metric g^M as above and D be the Dirac operator, then*

$$\widetilde{\mathrm{Wres}}[(\pi^+ D^{-1})^2] = -\frac{\Omega_4}{3} \int_M s\mathrm{dvol}_M. \tag{2.44}$$

3 The signature operator case

Let M be a 4-dimensional compact oriented Riemannian manifold with boundary ∂M and the metric in Section 2. $D = d + \delta : \wedge^*(T^*M) \to \wedge^*(T^*M)$ is the signature operator. Take the coordinates and the orthonormal frame as in Section 2. Let $\epsilon(\widetilde{e_j*})$, $\iota(\widetilde{e_j*})$ be the exterior and interior multiplications respectively. Write

$$c(\widetilde{e_j}) = \epsilon(\widetilde{e_j*}) - \iota(\widetilde{e_j*}); \quad \bar{c}(\widetilde{e_j}) = \epsilon(\widetilde{e_j*}) + \iota(\widetilde{e_j*}). \tag{3.1}$$

We'll compute $\mathrm{tr}_{\wedge^*(T^*M)}$ in the frame $\{dx_{i_1} \wedge \cdots \wedge dx_{i_k} |\ 1 \leq i_1 < \cdots < i_k \leq 4\}$. By [Y], we have

$$D = d + \delta = \sum_{i=1}^{n} c(\widetilde{e_i})[\widetilde{e_i} + \frac{1}{4}\sum_{s,t}\omega_{s,t}(\widetilde{e_i})[\bar{c}(\widetilde{e_s})\bar{c}(\widetilde{e_t}) - c(\widetilde{e_s})c(\widetilde{e_t})]]. \tag{3.2}$$

So

$$p_1 = \sigma_1(d+\delta) = \sqrt{-1}c(\xi);\ p_0 = \sigma_0(d+\delta) = \frac{1}{4}\sum_{i,s,t}\omega_{s,t}(\widetilde{e_i})c(\widetilde{e_i})[\bar{c}(\widetilde{e_s})\bar{c}(\widetilde{e_t}) - c(\widetilde{e_s})c(\widetilde{e_t})]. \tag{3.3}$$

Lemma 2.1-2.3 is also correct, by Lemma 2.3, then

$$p_0(x_0) = \widetilde{p_0}(x_0) - \frac{3}{4}h'(0)c(dx_n);\ \widetilde{p_0}(x_0) = \frac{1}{4}h'(0)\sum_{i=1}^{n-1}c(\widetilde{e_i})\bar{c}(\widetilde{e_n})\bar{c}(\widetilde{e_i})(x_0). \tag{3.4}$$

For the signature operator case,

$$\mathrm{tr}[\mathrm{id}] = 16; \quad \mathrm{tr}[c(\xi')\partial_{x_n}c(\xi')](x_0)|_{|\xi'|=1} = -8h'(0); \tag{3.5}$$

$$\mathrm{tr}[c(\xi')p_0c(\xi')c(dx_n)](x_0) = \mathrm{tr}[p_0c(\xi')c(dx_n)c(\xi')](x_0) = |\xi'|^2\mathrm{tr}[p_0c(dx_n)]. \tag{3.6}$$

$$\begin{aligned}
c(dx_n)\widetilde{p_0}(x_0) &= -\frac{1}{4}h'(0)\sum_{i=1}^{n-1}c(e_i)\bar{c}(e_i)c(e_n)\bar{c}(e_n) \\
&= -\frac{1}{4}h'(0)\sum_{i=1}^{n-1}[\epsilon(e_i*)\iota(e_i*) - \iota(e_i*)\epsilon(e_i*)][\epsilon(e_n*)\iota(e_n*) - \iota(e_n*)\epsilon(e_n*)]
\end{aligned}$$

By [U] Theorem 4.3,

$$\begin{aligned}
&\mathrm{tr}_{\wedge^m(T^*M)}\{[\epsilon(e_i*)\iota(e_i*) - \iota(e_i*)\epsilon(e_i*)][\epsilon(e_n*)\iota(e_n*) - \iota(e_n*)\epsilon(e_n*)]\} \\
&= a_{n,m} < e_i*, e_n* >^2 + b_{n,m}|e_i*|^2|e_n*|^2 = b_{n,m},
\end{aligned} \tag{3.7}$$

where $b_4^m = \begin{pmatrix} 2 \\ m-2 \end{pmatrix} + \begin{pmatrix} 2 \\ m \end{pmatrix} - 2\begin{pmatrix} 2 \\ m-1 \end{pmatrix}$. So

$$\mathrm{tr}_{\wedge^*(T^*M)}\{[\epsilon(e_i*)\iota(e_i*) - \iota(e_i*)\epsilon(e_i*)][\epsilon(e_n*)\iota(e_n*) - \iota(e_n*)\epsilon(e_n*)]\} = \sum_{m=0}^{4} b_{4,m} = 0.$$

Then

$$\mathrm{tr}_{\wedge^*(T^*M)}[c(dx_n)\widetilde{p_0}(x_0)] = 0. \tag{3.8}$$

By (3.4), (3.5), (3.6) and (3.8), then $\Phi_{\rm sig} = 4\Phi_{\rm Dirac} = 0$. So we get

Theorem 3.1 *Let M be a 4-dimensional compact oriented Riemaniann manifold with the boundary ∂M and the metric g^M as above and D be the signature operator, then*

$$\widetilde{\rm Wres}[(\pi^+D^{-1})^2] = \frac{8\Omega_4}{3}\int_M s{\rm dvol}_M. \tag{3.9}$$

4 The gravitational action for manifolds with boundary

Firstly, we recall the Einstein-Hilbert action for manifolds with boundary (see [H] or [B]),

$$I_{\rm Gr} = \frac{1}{16\pi}\int_M s{\rm dvol}_M + 2\int_{\partial M} K{\rm dvol}_{\partial_M} := I_{\rm Gr,i} + I_{\rm Gr,b}, \tag{4.1}$$

where

$$K = \sum_{1\leq i,j\leq n-1} K_{i,j}g^{i,j}_{\partial M};\quad K_{i,j} = -\Gamma^n_{i,j}, \tag{4.2}$$

and $K_{i,j}$ is the second fundamental form, or extrinsic curvature. Take the metric in Section 2, then by Lemma A.2, $K_{i,j}(x_0) = -\Gamma^n_{i,j}(x_0) = -\frac{1}{2}h'(0)$, when $i = j < n$, otherwise is zero. Then

$$K(x_0) = \sum_{i,j} K_{i.j}(x_0)g^{i,j}_{\partial M}(x_0) = \sum_{i=1}^{3} K_{i,i}(x_0) = -\frac{3}{2}h'(0),$$

So

$$I_{\rm Gr,b} = -3h'(0){\rm vol}_{\partial M}. \tag{4.3}$$

Now, we assume ∂M is flat , then $\{dx_i = e_i\}$, $g^{\partial M}_{i,j} = \delta_{i,j}$, $\partial_{x_s}g^{\partial M}_{i,j} = 0$. So ${\rm res}_{1,1}[(\pi^+D^{-1})^2]$ and ${\rm res}_{2,1}[(\pi^+D^{-1})^2]$ are two global forms locally defined by the aboved oriented orthonormal basis $\{dx_i\}$. By case a) II) and case b), then we have:

Theorem 4.1 *Let M be a 4-dimensional compact spin manifold with the boundary ∂M and the metric g^M as above and D be the Dirac operator. Assume ∂M is flat, then*

$$\int_{\partial M} {\rm res}_{1,1}[(\pi^+D^{-1})^2] = \frac{\pi}{8}\Omega_3 I_{\rm Gr,b}; \tag{4.4}$$

$$\int_{\partial M} {\rm res}_{2,1}[(\pi^+D^{-1})^2] = -\frac{3\pi}{8}\Omega_3 I_{\rm Gr,b}. \tag{4.5}$$

Theorem 4.2 *Let M be a 4-dimensional compact oriented Riemaniann manifold with the boundary ∂M and the metric g^M as above and D be the signature operator. Assume ∂M is flat, then*

$$\int_{\partial M} \mathrm{res}_{1,1}[(\pi^+ D^{-1})^2] = \frac{\pi}{2}\Omega_3 I_{\mathrm{Gr,b}}; \tag{4.6}$$

$$\int_{\partial M} \mathrm{res}_{2,1}[(\pi^+ D^{-1})^2] = -\frac{3\pi}{2}\Omega_3 I_{\mathrm{Gr,b}}. \tag{4.7}$$

Remark 1: We take N is a flat 3-dimensional oriented Riemannian manifold and $M = N \times [0,1]$, then $\partial M = N \oplus N$. Let $g^M = \frac{1}{h(x_n)} g^N + dx_n^2$, where $h(x_n) = 1 - x_n(x_n - 1) > 0$ for $x_n \in [0,1]$ and $h(0) = h(1) = 1$. The (M, g^M) satisfies the condition in Theorem 4.2. Similar construction is correct for Theorem 4.1. When ∂M is not connected, we still define the noncommutative residue with the loss of the unique property.

Remark 2: The reason that the extra term does not appear is perhaps that we ignore boundary conditions. We hope to compute the noncommutative residue $\widetilde{\mathrm{Wres}}[(\pi^+ D^{-1})^2]$ under certain boundary condition to get the extra term in the future. Grubb and Schrohe got the noncommutative residue for manifolds with boundary through asymptotic expansions in [GS]. Another problem is to compute $\widetilde{\mathrm{Wres}}[(\pi^+ D^{-1})^2]$ by asymptotic expansions.

Appendix

In this appendix, we will prove a fact used in Lemma 2.2 and Lemma 2.3.

Lemma A.1

$$\partial_{x_l} c(dx_j)(x_0) = 0, \textit{when } l < n;\ \partial_{x_l} c(dx_n) = 0$$

Proof. The fundamental setup is as in Section 2. Write $< \partial_{x_s}, e_i >_{g^{\partial M}} = H_{i,s}$, then by [Y] or [BGV], $\partial_{x_j} H_{i,s}(x_0) = 0$. Define $dx_j^* \in TM|_{\widetilde{U}}$ by $< dx_j^*, v >= (dx_j, v)$ for $v \in TM$. For $j < n$,

$$\begin{aligned} c(dx_j) &= c(dx_j^*) = c(\sum_i < dx_j^*, \widetilde{e_i} > \widetilde{e_i}) \\ &= \sum_{i,s} g^{s,j} < \partial_{x_s}, \widetilde{e_i} >_{g^M} c(\widetilde{e_i}) = \sum_{1 \le i,s < n} \frac{1}{\sqrt{h(x_n)}} g^{s,j} H_{s,i} c(\widetilde{e_i}) + \sum_{i=s=n} g^{n,j} c(\widetilde{e_n}). \end{aligned}$$

So for $i < n$, $\partial_{x_l} c(dx_j)(x_0) = 0$. □

The proof of Lemma 2.3:

Recall, let ∇^L be the Levi-Civita connection about g^M and

$$\nabla^L_{\partial_{x_i}}\partial_{x_j}=\sum_{k=1}^n\Gamma^k_{i,j}\partial_{x_k}, \tag{A.1}$$

then

$$\Gamma^k_{i,j}=\frac{1}{2}g^{kl}(\partial_{x_j}g_{li}+\partial_{x_i}g_{lj}-\partial_{x_l}g_{ij}). \tag{A.2}$$

Let

$$\partial_{x_i}=\sum_k h_{ik}\widetilde{e_k};\quad \widetilde{e_i}=\sum_k\widetilde{h_{ik}}\partial_{x_k}, \tag{A.3}$$

then the matrix $[h_{ik}]$ and $[\widetilde{h_{ik}}]$ are invertible, and $\widetilde{h_{ik}}(x_0)=\delta_{ik}$. By (A.1) and (A.3), then

$$\begin{aligned}\nabla^L_{\widetilde{e_i}}\widetilde{e_t}(x_0) &= \nabla^L_{\partial_{x_i}}(\sum_k\widetilde{h_{tk}}\partial_{x_k})\\ &= \sum_k\partial_{x_i}(\widetilde{h_{tk}})\partial_{x_k}+\sum_{k,l}\widetilde{h_{t,k}}\Gamma^l_{ik}\partial_{x_l}\\ &= \sum_s[\sum_k\partial_{x_i}(\widetilde{h_{tk}})h_{ks}+\sum_{k,l}\widetilde{h_{t,k}}\Gamma^l_{ik}h_{ls}]\widetilde{e_s}.\end{aligned}$$

By (2.4), then

$$\omega_{st}(\widetilde{e_i})(x_0)=\partial_{x_i}(\widetilde{h_{ts}})(x_0)+\Gamma^s_{it}(x_0)=-\partial_{x_i}h_{ts}(x_0)+\Gamma^s_{it}(x_0). \tag{A.4}$$

By (A.2) and the choices of g^M and the normal coordinate of x_0 in ∂M, then

Lemma A.2 *When $i<n$, then*

$$\Gamma^n_{ii}(x_0)=\frac{1}{2}h'(0);\ \Gamma^i_{ni}(x_0)=-\frac{1}{2}h'(0);\ \Gamma^i_{in}(x_0)=-\frac{1}{2}h'(0),$$

in other cases, $\Gamma^i_{st}(x_0)=0$.

By $h_{ts}=g^M(\partial_{x_t},\widetilde{e_s})=\frac{1}{\sqrt{h(x_n)}}H_{ts},\ (1\leq t,s<n)$, then we have

Lemma A.3 *When $i=n$, $t=s<n$, $-\partial_{x_i}h_{ts}(x_0)=\frac{1}{2}h'(0)$. In other cases,* $-\partial_{x_i}h_{ts}(x_0)=0$.

By Lemma A.2 and A.3 , (A.4), then we prove Lemma 2.3. □

Acknowledgement: The author is indebted to Professor Weiping Zhang for his encouragement and support. He also thanks Professors Xianzhe Dai and Siye Wu for their helpful discussions on the gravitational action for manifolds with boundary.

References

[A] T. Ackermann, *A note on the Wodzicki residue,* J. Geom. Phys., 20, 404-406, 1996.

[B] N. H. Barth, *The fourth-order gravitational action for manifolds with boundaries,* Class. Quantum Grav. 2, 497-513, 1985.
[BGV] N. Berline, E. Getzler, M. Vergne, *Heat Kernals and Dirac Operators,* Springer-Verlag, Berlin, 1992.
[C1] A. Connes, *Quantized calculus and applications,* XIth International Congress of Mathematical Physics (paris,1994), 15-36, Internat Press, Cambridge, MA, 1995.
[C2] A. Connes. *The action functinal in noncommutative geometry,* Comm. Math. Phys., 117:673-683, 1998.
[FGLS] B. V. Fedosov, F. Golse, E. Leichtnam, and E. Schrohe. *The noncommutative residue for manifolds with boundary,* J. Funct. Anal, 142:1-31,1996.
[FGV] H. Figueroa, J. Gracia-Bondía, and J. Várilly, *Elements of Noncommutative Geometry,* Birkhäuser Boston 2001.
[GS] G. Grubb, E. Schrohe, *Trace expansions and the noncommutative residue for manifolds with boundary,* J. Reine Angew. Math., 536:167-207, 2001.
[Gu] V.W. Guillemin, *A new proof of Weyl's formula on the asymptotic distribution of eigenvalues*, Adv. Math. 55 no.2, 131-160, 1985.
[H] S. W. Hawking, *in General Relativity: An Einstein Centenary Survey,* ed. S. Hawking and W.Israel, 1979.
[K] D. Kastler, *The Dirac operator and gravitiation,* Commun. Math. Phys, 166:633-643, 1995.
[KW] W. Kalau and M.Walze, *Gravity, non-commutative geometry, and the Wodzicki residue,* J. Geom. Phys., 16:327-344, 1995.
[S] E. Schrohe, *Noncommutative residue, Dixmier's trace, and heat trace expansions on manifolds with boundary,* Contemp. Math. 242, 161-186, 1999.
[U] W. J. Ugalde, *Differential forms and the Wodzicki residue,* arXiv: Math, DG/0211361.
[Wa1] Y. Wang, *Differential forms and the Wodzicki residue for manifolds with boundary,* to appear J. Geom. Phys., available online at *www.sciencedirect.com*
[Wa2] Y. Wang, *Differential forms and the noncommutative residue for manifolds with boundary in the non-product Case,* to appear Lett. math. Phys., available online at *www.cms.zju.edu.cn/frontindex.asp?version=english/priprint*
[Wo] M. Wodzicki, *Noncommutative residue, Chapter I. Fundamentals,* Lecture Notes in Math. 1289, 320-399, 1987.
[Y] Y. Yu, *The Index Theorem and The Heat Equation Method*, Nankai Tracts in Mathematics - Vol. 2, World Scientific Publishing, 2001.