# Global existence of the heat flow for $H$-systems in higher dimensions


Min-Chun Hong
Department of Mathematics
University of Queensland
Brisbane, QLD 4072, Australia

Deliang Hsu
Department of Mathematics
Shanghai JiaoTong University
Shanghai, China


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**Abstract**

In this paper we consider the heat flow for the $H$-system with variable function $H(\cdot)$ in higher dimensions, and establish global existence and uniqueness.


## 1 Introduction

Let $\Omega$ be a bounded smooth domain in $\mathbf{R}^n$ with $n \geq 2$. We call $u$ a (weak) solution of the $H$-system if $u \in \mathbf{W}^{1,n}\left(\Omega, \mathbf{R}^{n+1}\right)$ satisfies

$$\left\{ div\left(|\nabla u|^{n-2} \nabla u\right) = H(u)\, u_{x_1} \wedge \cdots \wedge u_{x_n}. \right. \tag{1}$$

The $H$-system (1) comes from the following constrained variational problem: for a given $\eta \in \mathbf{W}^{1,n}\left(\Omega, \mathbf{R}^{n+1}\right)$ and a constant $c$, we consider the minimization problem

$$\min_{\substack{u|_{\partial\Omega}=\eta \\ V(u)=c}} \int_\Omega |\nabla u|^n, \tag{2}$$

where $V(u) = \frac{1}{n+1}\int_\Omega u \cdot u_1 \wedge \cdots \wedge u_n$ is the well-defined (signed) volume for $u \in W^{1,n}$ and the boundary condition is to be interpreted in the usual trace sense. A minimizer of the problem (2) is a solution of (1), which has a natural geometric interpretation; namely, if $u$ fulfills certain additional conditions (conformality condition) then it represents a generalized hypersurface surface in $\mathbf{R}^{n+1}$ whose mean curvature at the point $u$ is $\frac{H(u)}{\sqrt{n^n}}$ see, e.g., [12], [14], [4] and [6]. Precisely, a map $u : \Omega \to \mathbf{R}^{n+1}$ is called conformal if

$$u_{x_i} \cdot u_{x_j} = \lambda^2 (x) \delta_{i,j}$$

holds on $\Omega$ for some real-valued function $\lambda(x)$. When $n = 2$, (1) becomes

$$\Delta u = H u_x \wedge u_y,$$

where $u$ is the parametrization of a surface whose mean curvature at the the point $u(x, y)$ is $H(u(x, y))$: such surfaces have studied by many authors (see e.g. [12] ).

For $n \geq 2$, Mou and Yang [14] proved existence of multiple solutions of the $H$-system (1) for constant $H$ for prescribed a given boundary data. Recently, Duzzer and Grotowski in [6] studied the existence of solutions of the $H$-system (1) for non-constant functions $H$ in higher dimensional compact Riemannian manifolds without boundary.

In the case of $n = 2$, the $H$-system flow was introduced to prove existence of the solution to the $H$-system by Struwe in [13] for constant function $H$ with a free boundary condition and studied by Rey in [16] for the case of non-constant function $H$.

In this paper, we establish existence of the heat flow for $H$-systems in the higher dimensional case. We assume the domain is a compact Riemannian manifold $M$ without boundary with $\dim M \geq 2$. For simplicity, we assume that $M$ is $\mathbf{R}^n$ or the torus $T^n$. More precisely, we study the following evolution problem on $M$ for prescribed initial data $u_0 \in W^{1,n}(M, \mathbf{R}^{n+1})$

$$\begin{cases} u_t - div\left(|\nabla u|^{n-2} \nabla u\right) = H(u) u_{x_1} \wedge \cdots \wedge u_{x_n} \\ u\,|_{t=0} = u_0 \,. \end{cases} \tag{3}$$

The main result of this paper is

**Theorem 1** *Assume that* $u_0 \in \mathbf{W}^{1,n}\left(M, \mathbf{R}^{n+1}\right) \cap \mathbf{L}^{\infty}\left(\Omega, \mathbf{R}^{n+1}\right)$, *and* $M$ *is* $\mathbf{R^n}$ *or the torus* $T^n$. *Assume that* $u_0$ *satisfies the conditions*

$$\|u_0\|_{L^{\infty}(M)} \|H\|_{L^{\infty}(\mathbf{R}^{n+1})} < \sqrt{n^n} \tag{$*$}$$

*and*

$$\|\nabla_y H(y)\|_{L^{\infty}} \leq C_0, \tag{$**$}$$

*where* $C_0$ *is a positive constant. Then we have that (3) has unique global weak solution* $u \in \mathbf{C}^{1,\alpha}\left((0, \infty) \times M, \mathbf{R}^{n+1}\right)$ *for some* $\alpha \in (0, 1)$.

In [9], Hungbühler established existence of a weak solution to the heat flow for $n$-harmonic maps from an $n$-dimensional Riemannian manifold into an arbitrary Riemannian manifold. The proof of Theorem 1 has similarities to the one for the $n$-harmonic map flow in [9]. Indeed, the nonlinear term which occurs in the right side of (3) is of the same order as the nonlinear term which occurs in the case of $n$-harmonic map flow. However, we have not any control on the $\mathbf{W}^{1,n}$-norm of the solution $u$ to the flow (3) with respect to $t$ unless we impose the initial data satisfying condition $(*)$. Moreover, it is difficult to handle the nonlinear term in (3) because the term involves a nonlinear function $H(u)$ and an exterior product term. In particular, there is no direct, simple an analogue of the energy inequality for $n$-harmonic map heat flow in the current setting. Fortunately, we can use a maximum principle argument for equations of parabolic type to control the enrgy and show that the condition $(*)$ is reserved along the flow and this gives us the desired bound for the $\mathbf{W}^{1,n}$-norm of the solution. We finally apply an interpolation type inequality to handle the nonlinear term $H(u)\, u_{x_1} \wedge \cdots \wedge u_{x_n}$ and get comparable estimates to the case of the $n$-harmonic map heat flow.

## 2 Some inequalities and priori estimates

In this section we establish some useful estimates for the solution of $H$-systems. First we need the following Nirenberg type inequality:

**Lemma 2** *Let $M$ be a compact manifold without boundary or $\mathbf{R^n}$. Then there exist a constant $C$ depending only on $\Omega$ such that for any measurable function $f : M \times [0,T]$ ($T > 0$ arbitrary), and $B_R(x_0) \subset \Omega$, and any function $\varphi \in L^\infty(B_R(x_0))$ depending only on the distance from $x_0$, i.e. $\varphi(y) = \varphi(|y - x_0|)$, $\varphi$ is non-increasing, the estimate*

$$\begin{aligned} &\int_0^T \int_M |f|^{2n} \varphi dx dt \qquad (4) \\ &\le C \sup_{0\le t\le T} \left( \int_{B_R(x_0)} |f|^n dx \right)^{\frac{2}{n}} \int_0^T \int_M \left| \nabla |f|^{n-1} \right|^2 \varphi dx dt \\ &+ \frac{C}{|B_R(x_0)|} \sup_{0\le t\le T} \int_{B_R(x_0)} |f|^n dx \cdot \int_0^T \int_M |f|^n \varphi dx dt \end{aligned}$$

*holds, provided $\varphi = 1$ on $B_{\frac{R}{2}}(x_0)$.*

For the proof we refer to [9] Lemma 3.1. The two-dimensional case was proved by Struwe in [11]. This inequality is also a key tool to get the higher order integrability of the gradient of the solution.

Now, we introduce two functionals on $W^{1,n}(M,\mathbf{R}^{n+1})$

$$\mathbf{E}(u)=\int_M |\nabla u|^n$$

and

$$\mathbf{J}(u)=\int_M\left(|\nabla u|^n+\frac{n}{n+1}Q(u)\cdot u_{x_1}\wedge\cdots\wedge u_{x_n}\right), \tag{5}$$

where

$$Q(u)=\left(\int_0^{u_1}H(s,u_2,\cdots,u_{n+1})\,ds,\cdots,\int_0^{u_{n+1}}H(u_1,u_2,\cdots,u_n,s)\,ds\right).$$

Note that a critical point of functional (5) is a solution of equation (1) (see Appendix at the end of paper).

Now we give the energy estimate of the solution of (1).

**Lemma 3** *Let $u\in\mathbf{C}^{1,\alpha}\left(M\times[0,T],\mathbf{R}^{n+1}\right)\cap\mathbf{W}^{1,2}\left([0,T],\mathbf{W}^{1,n}(M)\right)$ be a solution to (3) in the sense of distribution for some $\alpha$ with $0<\alpha<1$. Then the following energy equality holds for all $t_1,t_2$ with $0\le t_1\le t_2\le T$*

$$\int_{t_1}^{t_2}\int_M |u_t|^2\varphi\,dx\,dt+J(u(t_2))=J(u(t_1)).$$

**Proof.** Multiplying (3) by $u_t$ and integrating on $M\times[t_1,t_2]$, we get the results. Here we need to note in the calculation that

$$\frac{d}{dt}\int_M Q(u)\cdot u_{x_1}\wedge\cdots\wedge u_{x_n}=(n+1)\int_M H(u)\,u_t\cdot u_{x_1}\wedge\cdots\wedge u_{x_n}.$$

For convenience we give the proof of this formula in the Appendix. ■

The following lemma indicates that condition $(*)$ is preserved under the flow (3).

**Lemma 4** *Assume that $u_0\in\mathbf{W}^{1,n}\left(M,\mathbf{R}^{n+1}\right)\cap\mathbf{L}^{\infty}\left(M,\mathbf{R}^{n+1}\right)$ satisfies the condition $(*)$, and $u\in\mathbf{C}^{1,\alpha}\left(M\times[0,T],\mathbf{R}^{n+1}\right)\cap\mathbf{W}^{1,2}\left([0,T],\mathbf{W}^{1,n}(M)\right)$ is a solution of (3). Then*

*(i)* $\|u\|_{L^\infty(M\times[0,T])}\le\|u_0\|_{L^\infty(M)}$,

*(ii)* $\sup\limits_{[0,T]}\mathbf{E}(u)\le c\mathbf{J}(u_0)$.

**Proof.** Suppose $k$ is any number which satisfies

$$\|u_0\|_{L^\infty(M)} \|H\|_{L^\infty(M)} < k \|H\|_{L^\infty(M)} \leq \sqrt{n^n}.$$

For the map $u$, denote $f = |u|^2$ and the operator $\Delta_n g = div\left(|\nabla u|^{n-2} \nabla g\right)$ for any function $g$. Then we have the following

$$\begin{aligned} (\partial_t - \Delta_n) f &= 2u \cdot u_t - 2u \cdot div\left(|\nabla u|^{n-2} \nabla u\right) - 2|\nabla u|^n \\ &= 2\left(H(u)\, u \cdot u_{x_1} \wedge \cdots \wedge u_{x_n} - |\nabla u|^n\right) \\ &\leq 2|\nabla u|^n \left(\frac{\|H\|_{L^\infty} \max|u|}{\sqrt{n^n}} - 1\right) \end{aligned} \tag{6}$$

be satisfied in sense of distribution. By testing (6) with the function

$$\varphi = f - \min\left\{k^2, f\right\},$$

we have

$$\begin{aligned} &\frac{1}{2}\partial_t \int_{M\cap\{f\geq k^2\}} \left(f - k^2\right)^2 + \int_{M\cap\{f\geq k^2\}} |\nabla u|^{n-2} |\nabla f|^2 \\ &\leq 2\int_{M\cap\{f\geq k^2\}} |\nabla u|^n \left(\frac{\|H\|_{L^\infty} \max|u|}{\sqrt{n^n}} - 1\right)\left(f - k^2\right) \leq 0, \end{aligned}$$

for $t \in [0,T] \cap \left\{\frac{\|H\|_{L^\infty} \max|u|}{\sqrt{n^n}} - 1 \leq 0\right\}$. Since for $t = 0$, $f < k$, by a standard argument of closed and open interval for $t$ we get $f \leq k$ for all $t \geq 0$. This proves (i).

Concerning (ii), we remark that $|Q(u)| \leq \|H(u)\|_{L^\infty} \|u\|_{L^\infty}$, so that from (1) we have

$$|\nabla u|^n + \frac{n}{n+1} Q(u) \cdot u_{x_1} \wedge \cdots \wedge u_{x_n} \geq c(n)\, |\nabla u|^n.$$

Here $c(n) > 0$ depends only on $n$. This completes a proof of (ii). ■

The following we give the $\mathbf{L}^{2n}$-estimate for $|\nabla u|$, which is a crucial step to get higher regularity, and existence for the flow (3).

**Lemma 5** *There exists $\epsilon_1 > 0$ which only depends on $dist(x_0, \partial M)$ with the following property:*

*If $u \in C^2\left(B_{3R}(x_0) \times [0,T); \mathbf{R}^{n+1}\right)$ with $\mathbf{E}(u) \le c\mathbf{E}(u_0)$ is a solution of (3) on $B_{3R}(x_0) \times [0,T)$ for some $R \in \left(0, \frac{1}{3} dist(x_0, \partial M)\right)$ and if*

$$\sup\{\mathbf{E}(u(t)), B_R(x); 0 \le t \le T, x \in B_{2R}(x_0)\} < \epsilon_1$$

*then we have for every $x \in B_R(x_0)$*

$$\int_0^T \int_{B_R(x)} \left|\nabla^2 u\right|^2 |\nabla u|^{2n-4} < C\mathbf{E}(u_0)\left(1 + \frac{T}{R^n}\right) \tag{7}$$

*and*

$$\int_0^T \int_{B_R(x)} |\nabla u|^{2n} < C\mathbf{E}(u_0)\left(1 + \frac{T}{R^n}\right) \tag{8}$$

*for some constant $C$ which only depends on $dist(x_0, \partial M)$.*

**Proof.** The proof is close to the one in [9], but for completeness, we give details for the terms requiring a different treatment. here. Since we do not consider the estimate at boundary, for simplicity we consider the case of a flat torus $M = R^n/Z^n$. Let $\varphi \in C_0^\infty(B_{2R}(x_0))$ be a cutoff function satisfying $0 \le \varphi \le 1$, $\varphi \equiv 1$ on $B_R(x_0)$ and $|\nabla\varphi| \le \frac{2}{R}$. We now test (3) by the function $div\left(|\nabla u|^{n-2} \nabla u\right)\varphi^n$. Using condition $(*)$ and determinant inequality (see for example [14] P. 1195) we have

$$\left|u_t \Delta_n u \varphi^n - (\Delta_n u)^2 \varphi^n\right| \le c\,|\nabla u|^n\,|\Delta_n u|\,\varphi^n$$

with a constant $c$depending only on $\|H(u)\|_{L^\infty}$ and $n$. Integrating over $B_{2R}(x_0) \times [0,T)$ we obtain

$$\begin{aligned} &\int_0^T \int_{B_{2R}(x)} \left(\frac{1}{n}\frac{d}{dt}|\nabla u|^n \varphi^n + |\Delta_n u|^2 \varphi^n\right) \qquad (9)\\ &= \int_0^T \int_{B_{2R}(x)} \left[\left(-div\left(\varphi^n |\nabla u|^{n-2} \nabla u\right)\right) u_t + |\Delta_n u|^2 \varphi^n\right] \\ &= \int_0^T \int_{B_{2R}(x)} \left[-\varphi^n u_t \Delta_n u - n\varphi^{n-1} |\nabla u|^{n-2} u_t \nabla\varphi\nabla u + |\Delta_n u|^2 \varphi^n\right] \\ &\le \int_0^T \int_{B_{2R}(x)} \left[C\,|\nabla u|^n\,|\Delta_n u|\,\varphi^n + n\varphi^{n-1}\,|\Delta_n u|\,|\nabla u|^{n-1}\,|\nabla\varphi|\right] \\ &+ C\int_0^T \int_{B_{2R}(x)} n\varphi^{n-1}\,|\nabla u|^{2n-1}\,|\nabla\varphi|, \end{aligned}$$

where we use the condition $(*)$. By Young's inequality we estimate the last line in (9) by

$$\begin{aligned} &\int_0^T \int_{B_{2R}(x)} \left[\frac{1}{4}|\Delta_n u|^2 \varphi^n + C\varphi^n |\nabla u|^{2n} + C\varphi^{n-2}|\nabla\varphi|^2 |\nabla u|^{2n-2}\right] \\ &+ C\int_0^T \int_{B_{2R}(x)} n\varphi^{n-1}|\nabla u|^{2n-1}|\nabla\varphi| \\ &\le \int_0^T \int_{B_{2R}(x)} \left[\frac{1}{4}|\Delta_n u|^2 \varphi^n + C\varphi^n |\nabla u|^{2n} + C\varphi^{n-2}|\nabla\varphi|^2 |\nabla u|^{2n-2}\right]. \end{aligned} \tag{10}$$

On the other hand, integrating by parts twice we have

$$\begin{aligned} \int_0^T \int_{B_{2R}(x)} |\Delta_n u|^2 \varphi^n \ge \frac{1}{2}\int_0^T \int_{B_{2R}(x)} \left|\nabla^2 u\right|^2 |\nabla u|^{2n-2}\varphi^n \\ - c\int_0^T \int_{B_{2R}(x)} \varphi^{n-2}|\nabla\varphi|^2|\nabla u|^{2n-2} \end{aligned} \tag{11}$$

for some constant $c$ only depending on $n$. Plugging (10) and (11) into (9), we obtain

$$\begin{aligned} &\int_0^T \int_{B_{2R}(x)} \left|\nabla^2 u\right|^2 |\nabla u|^{2n-2}\varphi^n \\ &\le c\int_{B_{2R}(x_0)} |\nabla u_0|^n + c\int_0^T \int_{B_{2R}(x)} \left(\varphi^n|\nabla u|^{2n} + \varphi^{n-2}|\nabla\varphi|^2|\nabla u|^{2n-2}\right). \end{aligned} \tag{12}$$

The second term on the right hand of (12) may be estimated separately by Hőlder's and Young's inequality:

$$\int_0^T \int_{B_{2R}(x)} \varphi^{n-2}|\nabla\varphi|^2|\nabla u|^{2n-2} \le c\int_0^T \int_{B_{2R}(x)} \left(\varphi^n|\nabla u|^{2n} + |\nabla\varphi|^n|\nabla u|^n\right). \tag{13}$$

Combining (12) with (13) we have

$$\begin{aligned} &\int_0^T \int_{B_{2R}(x)} \left|\nabla^2 u\right|^2 |\nabla u|^{2n-2}\varphi^n \\ &\le c\int_{B_{2R}(x_0)} |\nabla u_0|^n + c\int_0^T \int_{B_{2R}(x)} \left(\varphi^n|\nabla u|^{2n} + |\nabla\varphi|^n|\nabla u|^n\right). \end{aligned} \tag{14}$$

It follows from Lemma (2.1) that

$$\int_0^T \int_{B_{2R}(x)} \varphi^n |\nabla u|^{2n} \tag{15}$$
$$\leq C \sup_{x\in B_{2R}(x_0)} \left( \int_{B_R(x)} |\nabla u|^n \, dx \right)^{\frac{2}{n}} \int_0^T \int_{B_{2R}(x_0)} \left|\nabla^2 u\right|^2 |\nabla u|^{2n-4} \varphi^n dxdt$$
$$+ \frac{C}{R^n} \sup_{0\leq t\leq T, x\in B_{2R}(x_0)} \int_{B_R(x)} |\nabla u|^n \, dx \int_0^T \int_{B_{2R}(x_0)} |\nabla u|^n \varphi^n dxdt.$$

Hence, for sufficiently small $\epsilon_1 > 0$, the assumption

$$\sup_{0\leq t\leq T, x\in B_{2R}(x_0)} \int_{B_R(x)} |\nabla u|^n < \epsilon_1$$

used in (15) and (14) implies the estimates (7) and (8). ■

# 3 Local Existence

In this section we prove local existence for the heat flow of the $H$-system (1). As in [9], we consider the regularized nonlinear parabolic problem

$$\begin{cases} u_t - div\left( \left( |\nabla u|^2 + \varepsilon \right)^{\frac{n-2}{2}} \nabla u \right) = H(u)\, u_{x_1} \wedge \cdots \wedge u_{x_n} \\ u\,|_{t=0} = u_0 \,. \end{cases} \tag{16}$$

By using semigroup theory in Banach space, Lunardi's Theorem (see [10]) yields:

**Lemma 6** *For any $\varepsilon > 0$, problem (16) has a unique solution $u_\varepsilon$ in the sense of Lunardi [10] on a time interval $[0, T_\varepsilon]$, and $u_\varepsilon \in \mathbf{C}^1\left([0,T_\varepsilon], \mathbf{L}^2(M)\right) \cap \mathbf{C}^0\left([0,T_\varepsilon], \mathbf{W}^{2,2}(M) \cap \mathbf{C}^{1,\alpha}(M)\right)$.*

For the proof of Lemma 6 we refer to Section 4 of [9].

Next, we need to prove that $T_\varepsilon \geq \mu > 0$, as $\varepsilon \to 0$. Before proving the local existence theorem, we give the following quantity which describe how much the energy is concentrated.

**Definition 7** *For a map $u : M\times[t_1,t_2] \to \mathbf{R}^{n+1}$, $u \in \mathbf{L}^\infty\left([t_1,t_2]; \mathbf{W}^{1,n}(M)\right)$, $\varepsilon > 0$ and $Q \subset M \times [t_1, t_2]$ let*

$$R^*(\varepsilon, u, Q) = \sup \left\{ R : \sup_{(x,t)\in Q} \left( E\left(u(t), B_R(x)\right)\right) < \epsilon \right\}.$$

**Theorem 8** *Let* $u \in \mathbf{C}^1\left([0,T_\varepsilon],\mathbf{L}^2(M)\right) \cap \mathbf{C}^0\left([0,T_\varepsilon],\mathbf{W}^{2,2}(M)\cap\mathbf{C}^{1,\alpha}(M)\right)$ *be a solution of (16) under the condition* $(*)$ *with initial value* $u_0$*. Assume that* $\varepsilon \le 1$*, then we have the following*

*(i)* $\sup_{[0,T]}\mathbf{E}_\varepsilon(u) \le c\mathbf{J}(u_0)$*, here we denotes* $\mathbf{E}_\varepsilon(u) = \int_M \left(\varepsilon + |\nabla u|^2\right)^{\frac{n}{2}}$.

*(ii) There exist constants* $c_1$*,* $c_2$*,* $\varepsilon > 0$*, depending only on* $n$*,* $M$ *(not on* $\varepsilon$ *and* $u_0$*) and a positive* $T_0 > 0$ *depending on additional* $\mathbf{E}_1(u_0)$ *and* $R^*(\varepsilon, u, M \times \{0\})$*, such that the condition*

$$\sup_{x\in M}\mathbf{E}_1(u_0, B_{2R}(x)) < \varepsilon$$

*implies*

$$\mathbf{E}_\varepsilon(u(t), B_R(x)) \le c_1\mathbf{E}_1(u_0, B_{2R}(x)) + c_2\mathbf{E}_1(u_0)^{1-\frac{1}{n}}\frac{t}{R^n},$$

*for all* $(x,t) \in M \times [0, \min\{T_\varepsilon, T_0\}]$.

*(3) There exists a constant* $\varepsilon_2 > 0$ *depending only on* $M$ *(but not on* $\varepsilon$ *and* $u_0$*) such that*

$$R^* = R^*(\varepsilon_2, u, M \times [0,T_\varepsilon]) > 0$$

*implies for every* $Q \subset M \times [0,T_\varepsilon]$ *with* $dist(Q, M\times\{0\}) > 0$

$$\|\nabla u\|_{L^\infty(Q)} \le C$$

*where constant* $C$ *depends only on* $n$*,* $\mathbf{E}_1(u_0)$*,* $M$*,* $R^*$ *and* $dist(Q, M\times\{0\})$*. Moreover if we assume that* $u_0 \in \mathbf{W}^{1,\infty}(M)$*, then we have*

$$\|\nabla u\|_{L^\infty(M\times[0,T_\varepsilon])} \le C$$

*where* $C$ *is a constant that depends on* $n$*,* $\mathbf{E}_1(u_0)$*,* $M$*,* $R^*$ *and* $\|\nabla u_0\|_{L^\infty(M)}$.

**Proof.** The proof of (i) is the same as that of Lemma 3 and Lemma 4 of [9].

Now we prove (ii). Let $\varphi \in \mathbf{C}_0^\infty(B_{2R}(x))$ be a cut off function which satisfies: $0 \le \varphi \le 1$, $\varphi \equiv 1$ on $B_R(x)$, and $|\nabla\varphi| \le \frac{2}{R}$, here $x \in M$ is an arbitrary point. Then we test (16) by the function $u_t\varphi^2$ and obtain

$$\begin{aligned}&\int_0^T\int_{B_{2R}(x)}\left(u_t\varphi^2 + \left(\varepsilon+|\nabla u|^2\right)^{\frac{n-2}{2}}\nabla u\left(\nabla u_t\varphi^2 + 2\varphi\nabla\varphi u_t\right)\right)\\ &= \int_0^T\int_{B_{2R}(x)}\varphi^2 u_t H(u)\, u_{x_1}\wedge\cdots\wedge u_{x_n}.\end{aligned}$$

Thus

$$\int_0^T \int_{B_{2R}(x)} \left( u_t^2 + \frac{1}{n} \frac{d}{dt} \left( \left| \varepsilon + |\nabla u|^2 \right|^{\frac{n}{2}} \right) \right) \varphi^2 + \int_0^T \int_{B_{2R}(x)} \frac{1}{n+1} \varphi^2 Q(u) \cdot u_{x_1} \wedge \cdots \wedge u_{x_n} = -2 \int_0^T \int_{B_{2R}(x)} \nabla u \, |\nabla u|^{n-2} \, \varphi \nabla \varphi u_t. \tag{17}$$

This implies

$$\begin{aligned} &\mathbf{J}_\varepsilon (u(T), B_R(x)) - \mathbf{J}_\varepsilon (u_0, B_{2R}(x)) \\ &\leq \int_0^T \int_{B_{2R}(x)} \frac{d}{dt} \left( \left| \varepsilon + |\nabla u|^2 \right|^{\frac{n}{2}} + \frac{1}{n+1} Q(u) \cdot u_{x_1} \wedge \cdots \wedge u_{x_n} \right) \varphi^2 \\ &= -n \int_0^T \int_{B_{2R}(x)} u_t^2 \varphi^2 - 2n \int_0^T \int_{B_{2R}(x)} \nabla u \, |\nabla u|^{n-2} \, \varphi \nabla \varphi u_t \\ &\leq c \int_0^T \int_{B_{2R}(x)} |\nabla u|^{2n-2} \, |\nabla \varphi|^2 , \end{aligned} \tag{18}$$

where we denote

$$\mathbf{J}_\varepsilon (u(T), B_R(x)) = \int_{B_R(x)} \left( \left( \varepsilon + |\nabla u|^2 \right)^{\frac{n}{2}} + \frac{1}{n+1} Q(u) \cdot u_{x_1} \wedge \cdots \wedge u_{x_n} \right).$$

So from (18) we get

$$\begin{aligned} &\mathbf{J}_\varepsilon (u(T), B_R(x)) - \mathbf{J}_\varepsilon (u_0, B_{2R}(x)) \\ &\leq \frac{c}{R^2} \int_0^T \int_{B_{2R}(x)} |\nabla u|^{2n-2} \\ &\leq c \frac{T^{\frac{1}{n}}}{R^2} \left( \int_0^T \int_{B_{2R}(x)} |\nabla u|^{2n} \right)^{\frac{n-2}{n}} , \end{aligned} \tag{19}$$

By the result in (i), it follows from (19) that

$$E_\varepsilon (u(T), B_R(x)) - c_1 E_\varepsilon (u_0, B_{2R}(x)) \leq c \frac{T^{\frac{1}{n}}}{R^2} \left( \int_0^T \int_{B_{2R}(x)} |\nabla u|^{2n} \right)^{\frac{n-2}{n}} . \tag{20}$$

By the well-known covering lemma, there exists a number $L$ only depending on the geometry of $M$ but not on $R$ such that every ball $B_{2R}(x)$ may be covered by at most $L$ balls $B_R(x_i)$. Now recall from Lemma 5 that there is a constant $\epsilon_1 > 0$ such that

$$\sup\left\{\mathbf{E}_\varepsilon\left(u\left(t\right)\right), B_{2R}\left(x\right); 0 \le t \le T, x \in B_{4R}\left(x_0\right)\right\} < \epsilon_1.$$

Then we have for every $x \in B_{4R}(x_0)$

$$\int_0^T \int_{B_{2R}(x)} \left(\varepsilon + |\nabla u|^2\right)^{\frac{n}{2}} < C\mathbf{E}_\varepsilon\left(u_0\right)\left(1 + \frac{T}{2^n R^n}\right). \tag{21}$$

Now we choose

$$\epsilon = \frac{\epsilon_1}{4Lc_1}$$

and suppose that $R_0$ is a number such that there holds

$$\sup_{x\in M} \mathbf{E}_\varepsilon\left(u_0, B_{2R_0}\left(x\right)\right) < \epsilon.$$

Then we can choose $T_0$ as

$$T_0 = \min\left\{1, \left(\frac{\epsilon_1 R_0}{4Lc\left(2cE_\varepsilon\left(u_0\right)\right)^{\frac{n-2}{n}}}\right)^n\right\}.$$

Now we claim: for all $(x,t) \in M \times [0, T_0]$ and all $R \le R_0$ there hold

$$\int_0^t \int_{B_{2R}(x)} \left(\varepsilon + |\nabla u|^2\right)^{\frac{n}{2}} \le C\mathbf{E}_\varepsilon\left(u_0\right)\left(1 + \frac{t}{2^n R^n}\right)$$

and

$$\sup_{0\le\tau\le t} \int_{B_{2R}(x)} \left(\varepsilon + |\nabla u|^2\right)^{\frac{n}{2}} \le \epsilon.$$

To see this let $T < T_0$ such that for all $t \in [0, T]$ the above two inequalities hold. Then it follows from (21), (20) and particular choice of $\varepsilon$, $T_0$ that

$$\begin{aligned}
&\sup_{(x,t)\in M\times[0,T]} \mathbf{E}_\varepsilon\left(u\left(t\right), B_{2R_0}\left(x\right)\right) \\
&\le L \sup_{(x,t)\in M\times[0,T]} \mathbf{E}_\varepsilon\left(u\left(t\right), B_{R_0}\left(x\right)\right) \\
&\le L\left(\sup_{x\in M} \mathbf{E}_\varepsilon\left(u_0, B_{2R_0}\left(x\right)\right) + c\frac{T^{\frac{1}{n}}}{R_0}\left(c\mathbf{E}_\varepsilon\left(u_0\right)\left(1 + \frac{T}{2^n R^n}\right)\right)^{\frac{n-2}{n}}\right) \\
&\le \frac{\epsilon}{2}.
\end{aligned}$$

Thus these two inequalities hold on some larger interval $[0, T+\delta]$. On the other hand the interval where the claimed inequalities hold is closed and nonempty. Hence we get the proof of the claim. This claim combined with (21) easily yields the result (2).

Once we have the $\mathbf{L}^{2n}$-estimate (Lemma 5) and localized energy monotonicity inequality (result (2)) in hand, we can get the higher regularity of the solution in the standard way as in [9] and [2]. However because of the particular nonlinear term in right-hand side of (16), the method in [9] cannot be directly applied in the current setting. Hence in the following calculation we outline the analysis of the right-hand side of (16). Now we give the proof of (3). Here we just sketch the proof for the case $\|\nabla u_0\|_{L^\infty(M)} < \infty$, the other case can be handled analogously by choosing a suitable time-dependent cutoff function. First let us fix some notation:

For $(x_0, t_0) \in M \times (0, T)$ let

$$\begin{aligned} P_R = P_R(x_0, t_0) &= B_R(x_0) \times (0, t_0) \\ P_R(\sigma) &= B_{R-\sigma R}(x_0) \times (0, t_0), \end{aligned}$$

here $\sigma \in (0,1)$. For $R$ small enough such that $P_R \subset M \times (0, T_\varepsilon)$, we take a cutoff function $\zeta$ with

$$\begin{aligned} \zeta &= 1, \quad \text{on } B_{R-\sigma R}(x_0) \\ \zeta &= 0, \quad \text{when } x \notin B_R(x_0) \end{aligned}$$

and

$$0 \le \zeta \le 1, |\nabla \zeta| \le \frac{2}{\sigma R}.$$

Using $-div\left(\left(\varepsilon + |\nabla u|^2\right)^l \zeta^2 \nabla u\right)$ as a test function in the equation (16), we have

$$\begin{aligned} &\int_{P_R} -u_t div\left(\left(\varepsilon + |\nabla u|^2\right)^l \zeta^2 \nabla u\right) \\ &+ \int_{P_R} div\left(\left(\varepsilon + |\nabla u|^2\right)^l \zeta^2 \nabla u\right) div\left(\left(\varepsilon + |\nabla u|^2\right)^{\frac{n-2}{2}} \nabla u\right) \\ &= -\int_{P_R} div\left(\left(\varepsilon + |\nabla u|^2\right)^l \zeta^2 \nabla u\right) H(u)\, u_{x_1} \wedge \cdots \wedge u_{x_n}. \end{aligned} \tag{22}$$

For simplicity, we set $v = \left(\varepsilon + |\nabla u|^2\right)$. A technical calculation (two terms in the left-hand side are treated by a similar argument as that found in [9]) yields

$$\frac{1}{2(l+1)}\int_{B_R(x_0)}\zeta^2 v^{l+1}(x,t_0)\,dx-\frac{1}{2(l+1)}\int_{B_R}\left(v^{l+1}\zeta^2\right)(x,0)\,dx \tag{23}$$
$$+\int_{P_R}\zeta^2\left(\frac{l(n-2)}{2}v^{\frac{n+2l-6}{2}}|\nabla v\nabla u|^2+\frac{l+n-2}{2}v^{\frac{n+2l-4}{2}}|\nabla v|^2\right)$$
$$+\int_{P_R}\zeta\nabla\zeta\left(v^{\frac{n+2l-2}{2}}\nabla v+(n-2)v^{\frac{n+2l-3}{2}}\nabla u\nabla v\right)+\int_{P_R}v^{\frac{n+2l-2}{2}}|\Delta u|^2\zeta^2$$
$$=\int_{P_R}v^l\zeta^2\nabla u\cdot\nabla\left(H(u)\,u_{x_1}\wedge\cdots\wedge u_{x_n}\right).$$

By Lemma 2.3 and the condition $(**)$, the term on the right hand is estimated by

$$\left|\int_{P_R}v^l\zeta^2\nabla u\cdot\nabla\left(H(u)\,u_{x_1}\wedge\cdots\wedge u_{x_n}\right)\right| \tag{24}$$
$$\le c\int_{P_R}v^l\zeta^2|\nabla u|^{n+2}+c\int_{P_R}v^l\zeta^2\left|\nabla^2 u\right||\nabla u|^n$$
$$\le c\int_{P_R}v^{\frac{n+2l+2}{2}}\zeta^2+\delta\int_{P_R}v^{\frac{n+2l-2}{2}}\zeta^2\left|\nabla^2 u\right|^2+C(\delta)\int_{P_R}v^{\frac{n+2l+2}{2}}\zeta^2,$$

where we use Young's inequality, and $C(\delta)$ is a positive constant depending only on $\delta$.

Plugging (24) into (23), interpolating the third term in the left-hand side, which involve $\nabla\zeta$, and using Young's inequality again, we have

$$\frac{1}{4(l+1)}ess\sup_{0<t<t_0}\int_{B_R(x_0)}\zeta^2 v^{l+1}(x,t)\,dx \tag{25}$$
$$+\left(\frac{n+l-2}{8}-\delta\right)\int_{P_R}v^{\frac{n+2l-4}{2}}|\nabla v|^2\,dxdt$$
$$\le\frac{1}{2(l+1)}\int_{B_R}\zeta^2 v^{l+1}(x,0)\,dx+C(n,\delta)\int_{P_R}v^{\frac{n+2l+2}{2}}\zeta^2$$
$$+2\left(\frac{1}{l+n-2}+\frac{n-2}{l}\right)\int_{P_R}v^{\frac{n+2l}{2}}|\nabla\zeta|^2\,dxdt.$$

Using Lemma 5, it follows from a standard argument as in [2] or [9] that

$$\begin{aligned}
&\frac{1}{4(l+1)} \sup_{0<t<t_0} \int_{B_R(x_0)} \zeta^2 v^{l+1}(x,t)\,dx \\
&+ c(n,\epsilon,\epsilon_1) \int_{t_0-R^n}^{t_0} \left\| v^{\frac{n+2l}{4}} \zeta \right\|_{L^{2^*}(B_R)} dt \\
&\leq \frac{1}{2(l+1)} \int_{B_R} \zeta^2 v^{l+1}(x,0)\,dx + c_1(n,l) \int_{P_R} v^{\frac{n+2l}{2}} |\nabla \zeta|^2 \,dxdt.
\end{aligned} \tag{26}$$

The standard iteration yields

$$\int_{P_R} v^q dxdt \leq C,$$

where $C$ depends on the initial datum and $q$, $n$ and $\epsilon$. To get the $L^\infty$ estimate, we use the Moser iteration method as in [9] and [1]. By the Hölder inequality and the Sobolev inequality, it follows from (26) that

$$\begin{aligned}
&\int_{P_R(\sigma)} v^{\left(l+\frac{n}{2}\right)\left(1+\frac{2}{n}\frac{2+2l}{n+2l}\right)} dxdt \\
&\leq c \left( \int_{B_R} \zeta^2 v^{l+1}(x,0)\,dx + \int_{P_R} v^{\frac{n+2l}{2}} |\nabla\zeta|^2 \,dxdt + l \int_{P_R} v^{\frac{n+2l+2}{2}} \zeta^2 dxdt \right)^{1+\frac{2}{n}}.
\end{aligned} \tag{27}$$

Iterating (27) implies

$$\|\nabla u\|_{L^\infty\left(P_R\left(\frac{1}{2}\right)\right)} \leq C,$$

where $C$ depends on the initial data $u_0$. This completes a proof of Theorem 8. ■

Once we have $\|\nabla u\|_{L^\infty}$ in hand, we can get high regularity estimate of solution of (16) in a standard way as in [9]. Note that these estimates do not depend on the parameter $\varepsilon$. In particular we use Theorem 8 to obtain the following theorem on an $\varepsilon$-independent existence interval.

**Theorem 9** *There exists a constant $\epsilon_0 > 0$ depending on $M$ with the following property:*

*For arbitrary $u_0 \in \mathbf{C}^{1,\alpha}\left(M, \mathbf{R}^{n+1}\right) \cap \mathbf{W}^{2,2}\left(M, \mathbf{R}^{n+1}\right)$ there exists a time $T_0 > 0$ only depending on $\mathbf{E}_0(u_0)$, $R^*(\varepsilon_0, u_0, M \times \{0\})$ and the geometry of $M$ such that for every $\varepsilon \in (0,1)$ there exists a solution $u \in$*

$\mathbf{C}^0\left([0,T_0],\mathbf{W}^{2,2}(M)\cap\mathbf{C}^{1,\alpha_1}(M)\right)\cap\mathbf{C}^1\left([0,T_0],\mathbf{L}^2(M)\right)$ *of (16) with initial value* $u_0$*. Moreover there hold the following* $\varepsilon$*-independent estimates:*

$$\begin{aligned}\|u_t\|_{L^2(M\times[0,T_0])}&\le C(u_0)\\ \|\nabla u\|_{L^\infty(M\times[0,T_0])}&\le C\left(\|\nabla u_0\|_{L^\infty}\right)\\ \|\nabla u\|_{C^{0,\alpha_1}(M\times[0,T_0])}&\le C\left(\|\nabla u_0\|_{C^{0,\alpha}(M)}\right).\end{aligned}$$

*where the constant also depend on* $n$ *and* $M$*, and the constant* $\alpha_1\le\alpha$ *depends on* $\alpha$ *and* $\|\nabla u_0\|_{\mathbf{L}^\infty}$*.*

The proof of Theorem 9 is almost parallel to the case of $n$-harmonic map flow as in [9]: in this case the right hand term do not cause any trouble in obtaing high regularity estimates. We refer to [9] Section 4. Combining Theorem 8 and Theorem 9, now we can prove local existence theorem of solution for the problem (3) by passing to the limit of $\varepsilon\to 0$.

**Theorem 10** *There exists a constant* $\epsilon_0>0$ *depending on* $M$ *with the following property:*

*For arbitrary* $u_0\in\mathbf{C}^{1,\alpha}\left(M,\mathbf{R}^{n+1}\right)\cap\mathbf{W}^{2,2}\left(M,\mathbf{R}^{n+1}\right)$ *there exists a time* $T_0>0$ *only depending on* $\mathbf{E}_0(u_0)$*,* $R^*(\epsilon_0,u_0,M\times\{0\})$ *and the geometry of* $M$ *such that there exists a weak solution* $u:M\times[0,T_0]\to\mathbf{R}^{n+1}$ *of (3). Furthermore* $u$ *satisfies the inequalities as in Theorem 9.*

**Proof.** Let $u_\varepsilon$ be the solution of (16) corresponding to a given $\varepsilon$ on the time interval $[0,T_0]$, where $T_0$ is constructed as in Theorem 8 and Theorem 9. The aim is to pass a limit on the distributional form of (16) on $[0,T_0]$. From the $\varepsilon$-independent estimates in Theorem 9 we know that at least $\{u_\varepsilon\}$ is bounded in $\mathbf{W}^{1,n}(M\times[0,T_0])$. Thus, we choose a sequence $\varepsilon_k\to 0$ such that

$$u_{\varepsilon_k}\rightharpoonup u\qquad\text{weakly in }\mathbf{W}^{1,2}(M\times[0,T_0]).$$

However from Theorem 9 we have $\|\nabla u_{\varepsilon_k}\|_{C^{0,\alpha_1}(M\times[0,T_0])}\le C\left(\|\nabla u_0\|_{C^{0,\alpha}(M)}\right)$, so we know that

$$\nabla u_{\varepsilon_k}\to\nabla u\qquad\text{strongly in }\mathbf{C}^{0,\frac{\alpha_1}{2}}(M\times[0,T_0]).$$

Now for any $\mathbf{C}_0^\infty$ test function $\varphi$ we pass the limit $\varepsilon_k\to 0$ in

$$\begin{aligned}&\int_0^{T_0}\int_M\left(\partial_t u_\varepsilon\varphi+\left(\varepsilon+|\nabla u_\varepsilon|^2\right)^{\frac{n-2}{2}}\nabla u_\varepsilon\nabla\varphi\right)dxdt\\ &=\int_0^{T_0}\int_M\varphi H(u_\varepsilon)(u_\varepsilon)_{x_1}\wedge\cdots\wedge(u_\varepsilon)_{x_n},\end{aligned}$$

which implies that $u$ is a weak solution of (3). To prove that $u$ satisfy the bounds as in Theorem 9, we notice that the estimates in Theorem 9 is independent on $\varepsilon$. ■

In fact we can prove the local existence result of (3) in a more general case of initial values $u_0 \in \mathbf{W}^{1,n}(M)$ by a approximation argument. We give the following as a corollary from Theorem 10.

**Corollary 11** *There exists a constant $\epsilon_0 > 0$ depending on $M$ with the following property: For given initial value $u_0 : M \to \mathbf{R}^{n+1}$, $u_0 \in \mathbf{W}^{1,n}\left(M, \mathbf{R}^{n+1}\right)$ there exists a time $T_0 > 0$ only depending on $\mathbf{E}_0(u_0)$, $R^*(\epsilon_0, u_0, M \times \{0\})$ and the geometry of $M$ such that there exists a weak solution $u : M \times [0, T_0] \to \mathbf{R}^{n+1}$ of (3). Moreover for any open set $Q$ with*

$$dist\left(Q, (M \times \{0\}) \cup \Sigma\right) = \mu > 0,$$

*there holds $\|\nabla u\|_{C^{0,\beta}(Q)} \leq C$ for some constant $C$ and $\beta \in (0,1)$ depending on $n$, $u_0$, $M$ and $\mu$.*

We refer the proof to [9] Section 4.

## 4 Global existence and proof of Theorem 1

In this section, we study the global existence of the solution of (3). Let us first give a result which characterizes finite time blow up phenomenon of the solution.

**Theorem 12** *For a given initial value $u_0 \in \mathbf{W}^{1,n}(M)$ with conditions $(*)$ and $(**)$, there exists a global weak solution $u$ of (3) which satisfies the following: There exists a set of finite points $\Sigma = \cup_{k=1}^{K} \Sigma_k \times \{T_k\}$, $0 < T_1 < \cdots < T_K \leq \infty$, such that on every open set $Q$ with $dist\left(Q, (M \times \{0\}) \cup \Sigma\right) = \mu > 0$ there holds $\|\nabla u\|_{\mathbf{C}^{0,\beta}(Q)} \leq C$ for some constant $C$ and $\beta \in (0,1)$ depending on $n$, $u_0$, $M$ and $\mu$. The number $K$ of singular times is bounded by*

$$K \leq \epsilon_0^{-1} c E(u_0)$$

*and the singular points $\left(x_i^k, T_k\right) \in \Sigma$ are characterized by the condition $\limsup_{t \nearrow T_k} \mathbf{E}\left(u(t), B_R\left(x_i^k\right)\right) \geq \epsilon_0$ for any $R > 0$. Furthermore at least a nonconstant, $\mathbf{C}^{1,\alpha}(\mathbf{R}^n)$ solution of (1) in entire space $\mathbf{R}^n$ separates in the sense that for sequence $R_m \to 0$, $t_m \nearrow T_k$, $x_m \to x_i^k$ as $m \to \infty$*

$$u_m(x) \equiv u\left(R_m(x - x_m), t_m\right) \to \overline{u} \quad \text{in } \mathbf{W}_{loc}^{1,n}\left(\mathbf{R}^n, \mathbf{R}^{n+1}\right).$$

*Here $\overline{u}$ is a solution of (1) in entire space $\mathbf{R}^n$.*

**Proof.** We extend the local solution of Theorem 10 or Corollary 11 by a standard way to a global weak solution to (3). This technique was used in the two dimensional case for harmonic map flow by Struwe [11] and $n$-dimensional case for $n$-harmonic map flow by Hungerbühler in [9]. In the follows, we describe the characterization of the blowing-up points $\Sigma$. Let $\left(x_i^k, T_k\right) \in \Sigma$ be any point of $\Sigma$. For simplicity, we assume $\left(x_i^k, T_k\right) = (x_0, T_1)$ and, when $t = T_1$, and assume that only one blowing-up point $x_0$ occurs. Denote

$$Q_T = [\delta, T], \text{ and}$$
$$\theta_T = \max_{Q_T} |\nabla u|,$$

where $\frac{T_1}{2} > \delta > 0$. It is obvious that $\theta_T$ is an increasing function. By the definition of $T_k$, we know that $\lim_{T \nearrow T_1} \theta_T = \infty$. Let $T^l$ be an increasing sequence with limit $T_1$ and $(a_l)$ a sequence in $\mathbf{M}$, where $l \to \infty$, such that

$$\theta_{T^l} = \left|\nabla u\left(a_l, T^l\right)\right|.$$

Up to a subsequence, we assume that $a_l \to x_0$. Let $U$ be a neighborhood of $x_0$ in $M$ and set

$$U_l = \left\{\zeta \in R^n \left| a_l + \frac{\zeta}{\theta_{T^l}} \in U \right.\right\},$$
$$I_l = \left[-\theta_{T^l}^n T^l, \theta_{T^l}^n \left(T_1 - T^l\right)\right]$$

and

$$v^l(\zeta, \tau) = u\left(a_l + \frac{\zeta}{\theta_{T^l}}, T^l + \frac{\tau}{\theta_{T^l}^n}\right) \quad \text{in } U_l \times I_l.$$

A simple calculation yields

$$\partial_\tau v^l - \Delta_n v^l = H\left(v^l\right) v_{\zeta_1}^l \wedge \cdots \wedge v_{\zeta_n}^l \tag{28}$$

and

$$\max_{U_l \times I_l} \left|\nabla v_\zeta^l(\zeta, \tau)\right| \le 1. \tag{29}$$

Let

$$h_l(\tau) = \int_{U_l} \left|\partial_\tau v^l\right|^2 d\zeta.$$

Then for any $\tau_0 > 0$

$$\begin{aligned}\int_{-\tau_0}^{0} h_l(\tau)\, d\tau &\leq \int_{T^l - \frac{\tau_0}{\theta^n_{T^l}}}^{T^l} \left( \int_M |u_t(x,t)|^2 \, dx \right) dt \\ &= \mathbf{J}\left( u\left( \cdot, T^l - \frac{\tau_0}{\theta^n_{T^l}} \right) \right) - \mathbf{J}\left( u\left( \cdot, T^l \right) \right) \\ &\to 0 \quad \text{as } l \to \infty.\end{aligned}$$

Thus, as $l \to \infty$, up to a subsequence, we assume that for a.e. $\tau \in [-\tau_0, 0]$

$$\int_{U_l} \left| \partial_\tau v^l(\zeta, \tau) \right|^2 d\zeta \to 0. \tag{30}$$

Because $M$ is without boundary, $U_l \to \mathbf{R}^n$ as $l \to \infty$ (in the following sense: for any $R > 0$, there exist $l_0 > 0$ such that for any $l \geq l_0$, we have $B(0, R) \subset U_l$). By the estimate in Theorem 3.4 and (29), we have

$$\sup_{\tau \in [-\tau_0, 0]} \|\nabla v(\tau, \cdot)\|_{\mathbf{C}^{0,\alpha_1}(B_R(0))} \leq C(R, \epsilon, \alpha_1), \tag{31}$$

for any $R > 0$ and some $\alpha_1 > 0$ in Section 3, where $\alpha_1$ could depend on $R$, but it does not cause trouble for us to pass to the limit. On the other hand, from (30) we have that

$$v^l(\zeta, \tau) \to 0 \quad \text{in } \mathbf{L}^2(B_R(0)) \tag{32}$$

for any $R > 0$ and a.e. $\tau \in [-\tau_0, 0]$. Combining (31) with (32) and using a diagonal argument we have that as $l \to \infty$, up to a subsequence

$$\begin{aligned} v^l(\zeta, \tau^*) &\to v^0(\zeta) \quad \text{in } \mathbf{C}^1(\mathbf{R}^n), \\ v^l(\zeta, \tau^*) &\to v^0(\zeta) \quad \text{in } \mathbf{C}^{1,\alpha_1}(B_R(0)), \end{aligned}$$

for any $R > 0$ and $\tau^*$ in some dense countable subset of $[-\tau_0, 0]$.

$v^0(\zeta)$ satisfies

$$\begin{aligned} \Delta_n v^0 &= H\left(v^0\right) v^0_{\zeta_1} \wedge \cdots \wedge v^0_{\zeta_n} \quad \text{in } \mathbf{R}^n, \\ \left\|v^0\right\|_{L^\infty(M)} \left\|H\left(v^0\right)\right\|_{L^\infty(M)} &\leq \sqrt{n^n}, \end{aligned} \tag{33}$$

in the weak sense. Let us show that $v^0$ is non-constant. We know that

$$\left| \nabla v^l(0,0) \right| = \frac{1}{\theta_{T^l}} \left| \nabla u\left(a_l, T^l\right) \right| = 1.$$

From (29), (28), and the estimate in Theorem 9 Theorem 10 and Corollary 11 that

$$\left\|\nabla_{\zeta} v^{l}(\tau, \varsigma)\right\|_{C^{0,\alpha_1}([-\tau_0,0]\times B_{\delta}(0))} \leq C$$

and therefore

$$\left|\nabla_{\zeta} v^{l}(0, \tau^{*}) - \nabla_{\zeta} v^{l}(0,0)\right| \leq C\left|\tau^{*}\right|^{\alpha_1}.$$

So we choose $|\tau^{*}|$ sufficiently small so that $\left|\nabla_{\zeta} v^{l}(0, \tau^{*})\right| \geq \frac{1}{2}$, for any $l > 0$, this prove that $v^0$ is not constant. Repeating the above argument to every point $\left(x_i^k, T_k\right)$ we can get the conclusion of theorem. This completes the proof. ■

To complete the proof of Theorem 1, we need the following lemma, which indicate that under the condition $(*)$, blow-up can not occur.

**Lemma 13** *If $v^0$ is a solution of*

$$\Delta_n v^0 = H\left(v^0\right) v^0_{\zeta_1} \wedge \cdots \wedge v^0_{\zeta_n} \quad \textit{in } \mathbf{R}^n$$

*and satisfies*

$$\left\|v^0\right\|_{L^{\infty}(M)} < \sqrt{n^n}, \tag{34}$$

*then $v_0$ must be a constant.*

**Proof.** Following [15] or [16], we argue it by contradiction.

Assume that $v_0$ is not a constant. Let $\varphi : \mathbf{R}^n \to \mathbf{R}$ be a smooth radial symmetric function such that

$$\varphi(x) = 1 \quad \text{for } |x| \leq 1,$$
$$\varphi(x) = 0 \quad \text{for } |x| \geq 2.$$

Denoting by

$$\varphi_k(x) = \varphi\left(\frac{x}{k}\right),$$

we have

$$\int_{R^n} \varphi_k v^0 \Delta_n v^0 dx = -\int_{R^n} \varphi_k \left|\nabla v^0\right|^n dx - \int_{R^n} \left|\nabla v^0\right|^{n-2} v^0 \nabla\varphi_k \nabla v^0 dx.$$

Note that $\left|\nabla v^0\right|^n \in \mathbf{L}^1(\mathbf{R}^n)$ and $\left|v^0 \Delta_n v^0\right| \leq \left\|v^0\right\|_{L^{\infty}} \|H\|_{L^{\infty}} \sqrt{n^n} \left|\nabla v^0\right|^n \in \mathbf{L}^1(\mathbf{R}^n)$. From Lebesgue's theorem we have

$$\lim_{k\to\infty} \int_{R^n} \varphi_k \left|\nabla v^0\right|^n dx = \int_{R^n} \left|\nabla v^0\right|^n dx,$$
$$\lim_{k\to\infty} \int_{R^n} \varphi_k v^0 \Delta_n v^0 dx = \int_{R^n} v^0 \Delta_n v^0 dx.$$

On the other hand

$$\left|\int_{R^n}\left|\nabla v^0\right|^{n-2} v^0\nabla\varphi_k\nabla v^0 dx\right| \leq \left\|v^0\right\|_{L^\infty}\left(\int_{R^n}\left|\nabla v^0\right|^n dx\right)^{\frac{n-1}{n}}\left(\int_{|x|\geq k}\left|\nabla\varphi_k\right|^n\right)^{\frac{1}{n}}$$

which goes to zero as $k$ goes to infinity. This implies

$$\int_{R^n} v^0\Delta_n v^0 dx = -\int_{R^n}\left|\nabla v^0\right|^n dx.$$

So by multiplying (33) by $v^0$ and integrating over $\mathbf{R}^n$ we obtain

$$\begin{aligned}\int_{R^n}\left|\nabla v^0\right|^n dx &= -\int_{R^n} H\left(v^0\right) v^0 v^0_{x_1}\wedge\cdots\wedge v^0_{x_n} \\ &\leq \sqrt{n^n}\left\|H\right\|_{L^\infty}\left\|v^0\right\|_{L^\infty}\int_{R^n}\left|\nabla v^0\right|^n dx,\end{aligned}$$

which gives

$$\left\|v^0\right\|_{L^\infty(M)}\left\|H\left(v^0\right)\right\|_{L^\infty(M)} \geq \sqrt{n^n}. \tag{35}$$

(35) contradicts with (34). ■

Now we complete the proof of Theorem 1.

**Proof of Theorem 1.** Combining Theorem 12, Lemma 4 and Lemma 13 we can easily get the global existence of solution of (3). Now we sketch the proof of uniqueness for the solution. By Lemma 13 we know that there is no blow-up point of the flow, which means there is no energy concentration. So by the estimate in Theorem 9 or Theorem 10 we know that $u \in \mathbf{L}^\infty\left((0,T); W^{1,\infty}(M)\right)$. By a similar argument to that of [9] Chapter 5, we know that the solution must be unique. ■

# 5 Appendix

In this appendix, we give the proof of following formula which is used in Lemma 3 and elsewhere in this paper.

$$\frac{d}{dt}\int_M Q(u)\cdot u_{x_1}\wedge\cdots\wedge u_{x_n} = (n+1)\int_M H(u)\, u_t\cdot u_{x_1}\wedge\cdots\wedge u_{x_n}.$$

For sake of clarity we denote $u_{x_i} = \frac{\partial u}{\partial x_i} = u_i$, $u_t = \frac{\partial u}{\partial t}$; $u = (u^\alpha)$, $Q(u) = (Q^\alpha)$, and $Q_{u^\alpha} = \frac{\partial Q}{\partial u^\alpha}$, $1 \le \alpha \le n+1$.

$$Q(u) \cdot u_{x_1} \wedge \cdots \wedge u_{x_n} = \det \begin{pmatrix} Q^1 & Q^2 & \cdot & \cdot & Q^{n+1} \\ u_1^1 & u_1^2 & \cdot & \cdot & u_1^{n+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ u_n^1 & u_n^2 & \cdot & \cdot & u_n^{n+1} \end{pmatrix}.$$

So we have

$$\begin{aligned}
&\frac{d}{dt}\int_M Q(u) \cdot u_{x_1} \wedge \cdots \wedge u_{x_n} dV \\
&= \int_M Q_{u^\alpha} u_t^\alpha \cdot u_{x_1} \wedge \cdots \wedge u_{x_n} dV + \sum_{i=1}^n \int_M Q(u) \cdot u_{x_1} \wedge \cdot \cdot (u_{x_i})_t \cdot \cdot \wedge u_{x_n} dV \\
&= \int_M Q_{u^\alpha} u_t^\alpha \cdot u_{x_1} \wedge \cdots \wedge u_{x_n} dV + \sum_{i=1}^n \int_M u_t \cdot \partial_{x_i} \left( u_{x_1} \wedge \cdot \cdot \underset{i}{Q(u)} \cdot \cdot \wedge u_{x_n} \right) dV \\
&= \int_M Q_{u^\alpha} u_t^\alpha \cdot u_{x_1} \wedge \cdots \wedge u_{x_n} dV + \sum_{i=1}^n \int_M u_t \cdot \left( u_{x_1} \wedge \cdot \cdot \partial_{x_i} \underset{i}{Q(u)} \cdot \cdot \wedge u_{x_n} \right) dV \\
&+ \sum_{i \ne k} \sum_{i=1}^n \int_M u_t \cdot \left( u_{x_1} \wedge \cdot \cdot \wedge u_{x_k x_i} \wedge \cdot \cdot \underset{i}{Q(u)} \cdot \cdot \wedge u_{x_n} \right) dV.
\end{aligned}$$

Use integration by parts, it is easy to see that

$$\sum_{i \ne k} \sum_{i=1}^n \int_M u_t \cdot \left( u_{x_1} \wedge \cdot \cdot \wedge u_{x_k x_i} \wedge \cdot \cdot \underset{i}{Q(u)} \cdot \cdot \wedge u_{x_n} \right) dV = 0.$$

Noting that

$$\partial_{x_i} Q(u) = Q_{u^\alpha} u_{x_i}^\alpha, \tag{36}$$

(here summation convention is used), we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_M Q(u) \cdot u_{x_1} \wedge \cdots \wedge u_{x_n} dV \\
&= \int_M Q_{u^\alpha} u_t^\alpha \cdot u_{x_1} \wedge \cdots \wedge u_{x_n} dV + \sum_{i=1}^n \int_M u_{x_i}^\alpha u_t \cdot \left( u_{x_1} \wedge \cdot \cdot \underset{i}{Q_{u^\alpha}} \cdot \cdot \wedge u_{x_n} \right) dV \\
&= \int_M Q_t \cdot u_{x_1} \wedge \cdots \wedge u_{x_n} dV + \sum_{i=1}^n \int_M u_t \cdot \left( u_{x_1} \wedge \cdot \cdot \underset{i}{Q_{x_i}} \cdot \cdot \wedge u_{x_n} \right) dV.
\end{aligned}$$

Now we claim that

$$Q_t \cdot u_{x_1} \wedge \cdots \wedge u_{x_n} + \sum_{i=1}^{n} u_t \cdot \left( u_{x_1} \wedge \cdot \cdot \underset{i}{Q_{x_i}} \cdot \cdot \wedge u_{x_n} \right) \tag{37}$$
$$= div\,(Q)\, u_t \cdot u_{x_1} \wedge \cdots \wedge u_{x_n}.$$

In fact, (37) is a result of linear algebra. Let $A$ be a $(n+1) \times (n+1)$ matrix. Let $a_1$,..., $a_{n+1}$ be arbitrary independent vectors in $\mathbf{R^{n+1}}$. Then we have

$$a_1 \wedge \cdots a_n \cdot A a_{n+1} + \sum_{l} \sum_{i=1}^{n} \left( a_1 \wedge \cdot \cdot \underset{i}{A a_i} \cdot \cdot \wedge a_n \right) \cdot a_{n+1}$$
$$= \text{trace}(A) a_1 \wedge \cdots a_n \cdot a_{n+1} \tag{38}$$

Note that both sides of (38) is linear in $a_1$, ..., $a_{n+1}$. It is easy to check (38) choosing $a_i = e_i$, where $\{e_i\}$ is the standard basis of $\mathbf{R^{n+1}}$.

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