Differential Forms and the Wodzicki Residue for Manifolds with Boundary II non-product metric case

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Abstract In this note, for a 4-dimensional compact manifold with boundary which has the non-product metric near the boundary, we compute $Ω_3(f_1, f_2)$.

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1 Introduction

Since the Wodzicki residue was found in [Wo], it was applied to many branches of mathematics. Especially, it was as the noncommutative counterpart of integral in NCG by [C1]. The Wodzicki residue also had been used to derive the gravitational action in the framework of NCG in [K], [KW]. In [C2], Connes used the Wodzicki residue to find a conformal 4-dimensional Polyakov action analogy. In [U], the Connes’ result was generalized to the higher dimensional case.

The Wodzicki type residue on Boutet de Monvel’s algebra for manifolds with boundary was found in [FGLS]. In [S], Schrohe gave the relation between Dixmier trace and the Wodzicki residue for manifolds with boundary. In [Wa1], the author proved the Kastler-Kaulu-Walze type theorem for manifolds with boundary and for the boundary flat case, he give two kinds of operator theoretic explanation of the gravitational action on boundary. In [Wa2], the author generalized the results in [C2] to the case of manifolds with boundary which have the product metric near the boundary. It is a natural question to compute the conformal invariant in [Wa2] in the non-product metric case. In this paper, for some special non-product metrics, we find that this conformal invariant vanishes which generalizes the result in [Wa2]. For the general metric, we point out the way of computations.

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This paper is organized as follows: In Section 2, we define a conformal invariant pair associated to the non-product metric case. In Section 3, for a 4-dimensional manifold, we compute this conformal invariant pair. Some remarks will be given in Section 4.

2 The conformal invariant pair \((\Omega_n(f_1, f_2), \Omega_{n-1}(f_1, f_2))\)

In this section, we construct a conformal invariant pair \((\Omega_n(f_1, f_2), \Omega_{n-1}(f_1, f_2))\) associated to a compact manifold with boundary which has the non-product metric. The fundamental setup is the same as Section 2 and Section 3 in [Wa2].

Let \((M, g)\) be an even dimensional, compact, oriented, Riemannian manifold with boundary \(\partial M\) and \(g\) need not have the product form near the boundary. Let \(\tilde{M} = M \cup \partial M\) be the double manifold. We fix a metric \(\tilde{g}\) on \(\tilde{M}\) such that \(\tilde{g}|_M = g\). Denote by \([(M, g)]\) a conformal manifold. Define a map \(\tilde{\phi}_g : [(M, g)] \mapsto [(\tilde{M}, \tilde{g})]\) by \(\tilde{\phi}_g((M, e^f g)) = (\tilde{M}, e^{\tilde{f}} \tilde{g})\) where \(\tilde{f}\) is an any extension to \(\tilde{M}\) of \(f\). Let \(\dim M = 2l\). As in [C2] or [U], we consider the operator for the manifold \((\tilde{M}, \tilde{g})\),

\[
F_{\tilde{g}} = \frac{d\delta - \delta d}{d\delta + \delta d} : \wedge^l(T^* \tilde{M}) \to \wedge^l(T^* \tilde{M}),
\]

then \(F_{\tilde{g}}\) does not depend on the choice of the metric in the conformal class \([(\tilde{M}, \tilde{g})]\). So is the operator \(\pi^+ f_0[\pi^+ F_{\tilde{g}}, \pi^+ f_1][\pi^+ F_{\tilde{g}}, \pi^+ f_2]\) (for the definitions, see [Wa2,p.6]). Similar to the discussions in [AM], \(\tilde{\text{Wres}}\) (for the definition, see [FGLS] or [Wa2]) does not depend on the metric. Now similar to (3.5) and (3.6) in [Wa2], we define the form pair \((\Omega_n(f_1, f_2)(\tilde{g}), \Omega_{n-1}(f_1, f_2)(\tilde{g}))\) through the following equality:

\[
\tilde{\text{Wres}}(\pi^+ f_0[\pi^+ F_{\tilde{g}}, \pi^+ f_1][\pi^+ F_{\tilde{g}}, \pi^+ f_2]) = \int f_0 \Omega_n(f_1, f_2)(\tilde{g}) + \int_{\partial M} f_0|_{\partial M} \Omega_{n-1}(f_1, f_2)(\tilde{g}). \tag{2.1}
\]

By [U], then \(\Omega_n(f_1, f_2)(\tilde{g}) = \int_{[\xi]=1} \text{tr} \left[ \sum_{\alpha! \alpha''! \beta! \delta!} \frac{1}{\alpha'! \alpha''! \beta! \delta!} D_x^2(f_1) D_x^{a'' + \delta}(f_2) \times \partial_x^{a' + a'' + \beta} (F_{\tilde{g}}) \partial_x^{\alpha} D_x^{a'} (F_{\tilde{g}}) \sigma(\xi)d\alpha \right] |_M , \tag{2.2}
\]

where \(\sigma_{\tilde{g}}\) denotes the order \(-j\) symbol of \(F_{\tilde{g}}\); \(D_x^\beta = (-i)^{|\beta|} \partial_x^\beta\) and the sum is taken over \(|\alpha'| + |\alpha''| + |\beta| + |\delta| + j + k = n; |\beta| \geq 1; |\delta| \geq 1; \alpha', \alpha'', \beta, \delta \in \mathbb{Z}_+^*; j, k \in \mathbb{Z}_+\). By [C2] or [U], \(\Omega_n(f_1, f_2)(\tilde{g})\) is a conformal invariant associated to \([(\tilde{M}, \tilde{g})]\). For a fixed \(\tilde{g}\), by the construction of \(\tilde{\phi}_g\), \(\Omega_n(f_1, f_2)(\tilde{g})\) is also a conformal invariant associated to \([(M, g)]\). Note that \(\Omega_n(f_1, f_2)(\tilde{g})\) does not depend on the choice of the extension \(\tilde{g}\) of \(g\) by (2.2), so we may write \(\Omega_n(f_1, f_2) = \Omega_n(f_1, f_2)(\tilde{g})\) which is a conformal invariant associated to \([(M, g)]\). Since \(\tilde{\text{Wres}}(\pi^+ f_0[\pi^+ F_{\tilde{g}}, \pi^+ f_1][\pi^+ F_{\tilde{g}}, \pi^+ f_2])\) and \(\Omega_n(f_1, f_2)(\tilde{g})\)
are conformal invariants, by (2.1), similar to the proof of the theorem 3.1 in [Wa2], so \( \Omega_{n-1}(f_1, f_2)(\tilde{g}) \) is also a conformal invariant. By (3.19) in [Wa2], then

\[
\Omega_{n-1}(f_1, f_2)(\tilde{g}) = \sum_{j,k=0}^{\infty} \sum_{|\beta|=1}^{-r} \sum_{|\delta|=1}^{-l} \frac{(-i)^{j+k+1+|\alpha|+|\beta|+|\delta|}(j!}{\alpha!|\beta|!|\delta|!(j+k+1)!} \times \int_{|\xi|_1=1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace} \left\{ \frac{\partial \xi^j}{\partial x_n} \left[ \frac{\partial^2 \xi^j}{\partial x_n^2}(f_1) \right] \frac{\partial^2 \xi^j}{\partial \xi_n \partial \xi_n} \frac{\partial^2 \xi^j}{\partial \xi_n \partial \xi_n} \left[ \frac{\partial \xi^j}{\partial x_n}(f_2) \right] \right\} \left| x_n = 0 \right| \left\{ \frac{d \xi_n \sigma(\xi)}{d \xi_n} \right\} dx, \tag{2.3}
\]

where the sum is taken over \( r - k - |\alpha| + l - j - 1 = -n \), \( r, l \leq -1 \), \( |\alpha| \geq 0 \). For the same reason, \( \Omega_{n-1}(f_1, f_2)(\tilde{g}) \) also does not depend on the choice of the extension \( \tilde{g} \) of \( g \) and we write \( \Omega_{n-1}(f_1, f_2) \). By Section 3 in [Wa2], then we have

**Theorem 2.1** The form pair \((\Omega_n(f_1, f_2), \Omega_{n-1}(f_1, f_2))\) is a uniquely determined conformal invariant by (2.1), (2.2), (2.3) and is symmetric in \( f_1 \) and \( f_2 \).

### 3 The computation of \((\Omega_4(f_1, f_2), \Omega_3(f_1, f_2))\)

Let \( M \) be a 4-dimensional compact oriented Riemannian manifold with boundary \( \partial M \) and the metric \( g^n \) which has the following form near the boundary,

\[
g^n = \frac{1}{h(x_n)} g^{\partial M} + dx_n^2, \tag{1.2}
\]

where \( g^{\partial M} \) is the metric on \( \partial M \), \( h(x_n) \) is a smooth function which satisfies \( h(x_n) > 0 \) and \( h(0) = 1 \) and \( x_n \) denotes the normal coordinate. We just care for the non-product metric, so for simplicity, we assume that

\(^{*}\) \( f_1, f_2 \) are independent of \( x_n \) near the boundary.

\( \Omega_4(f_1, f_2) \) is computed by the theorem 4.5 in [Wa2]. By (2.3) and assumption \(^{*}\), then

\[
\Omega_3(f_1, f_2) = \sum_{j,k=0}^{\infty} \sum_{|\beta|=1}^{-r} \sum_{|\delta|=1}^{-l} \frac{(-i)^{j+k+1+|\alpha|+|\beta|+|\delta|}(j!}{\alpha!|\beta|!|\delta|!(j+k+1)!} \times \int_{|\xi|_1=1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace}_{\lambda^2(T^*M)} \left\{ \frac{\partial \xi^j}{\partial x_n}(f_1) \times \frac{\partial^2 \xi^j}{\partial \xi_n \partial \xi_n}(f_1) \frac{\partial^2 \xi^j}{\partial \xi_n \partial \xi_n}(f_2) \right\} \left| x_n = 0 \right| \left\{ \frac{d \xi_n \sigma(\xi)}{d \xi_n} \right\} dx, \tag{3.1}
\]

where the sum is taken over \( -r + l + |\alpha| + k + j = 3 \), \( r, l \leq -1 \), \( \alpha, \beta, \delta \in \mathbb{Z}^\Delta_2 \). Since \( \Omega_3(f_1, f_2) \) is a global form on \( \partial M \), so for any fixed point \( x_0 \in \partial M \), we can choose the normal coordinates \( U \) of \( x_0 \) in \( \partial M \) (not \( M \)) and compute \( \Omega_3(f_1, f_2)(x_0) \) in
the coordinates \( \tilde{U} = U \times [0, 1] \subset M \) and the metric \( \frac{1}{\sqrt{x_n}} g^{\partial M} + dx_n^2 \). The dual metric of \( g^M \) on \( \tilde{U} \) is \( h(x_n)g^{\partial M} + dx_n^2 \). Write \( g^M = g^M(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}) \); \( g^{\partial M} = g^M(dx_i, dx_j) \), then

\[
[g^{\partial M}_{ij}] = \begin{bmatrix}
\frac{1}{h(x_n)} [g^{\partial M}] & 0 \\
0 & 1
\end{bmatrix}, \quad [g^M_{ij}] = \begin{bmatrix}
h(x_n) [g^{\partial M}] & 0 \\
0 & 1
\end{bmatrix},
\]

and

\[
\partial_x g^{\partial M}(x) = 0, 1 \leq i, j \leq n - 1; \quad g^{\partial M}_{ij}(x) = \delta_{ij}.
\] (3.2)

We’ll compute \( \mathrm{tr} \bigwedge^2(T^*M) \) in the frame \( \{dx_{i_1} \wedge dx_{i_2} \mid 1 \leq i_1 < i_2 \leq 4 \} \). Let \( \epsilon(\xi), \nu(\xi) \) be the exterior and interior multiplications respectively. Recall Lemma 2.2 in [Wa2]

\[
\partial_x(\|g^M_{ij}\|^2)(x) = 0, \text{ if } j < n; \quad = h'(0)\|\nu'(\xi)\|^2_{g^{\partial M}}, \text{ if } j = n.
\] (3.3)

By (3.2) and \( h(0) = 1 \), then \( \partial_x \nu(dx_j) = 0 \) and

\[
\partial_x \nu(dx_j)(x_0) = 0, \text{ if } l < n; \quad = h'(0)\nu(dx_j)(x_0), \text{ if } l = n.
\] (3.4)

So if \( i < n \), then

\[
\partial_x \nu(\xi)(x_0) = \partial_x \nu(\xi)(x_0) = 0; \quad \partial_x \nu(\xi)(x_0) = h'(0)\nu(\xi)(x_0).
\] (3.5)

**Theorem 3.1** Under the above conditions, then

\[
\Omega_3(f_1, f_2)(x_0) = h'(0) \sum_{1 \leq i, j \leq 3} a_{i,j} \partial_{x_i} f_1 \wedge \partial_{x_j} f_2 dx_1 \wedge dx_2 \wedge dx_3,
\] (3.6)

where \( a_{i,j} \) is a constant.

**Corollary 3.2** If \( h'(0) = 0 \) (for example \( h = 1 - x_2^2 \)), then \( \Omega_3(f_1, f_2) = 0 \). Especially, \( \Omega_3 \) has the product metric near the boundary, then \( \Omega_3(f_1, f_2) = 0 \) and

\[
\widetilde{\mathrm{Wres}}(\pi^+ f_0 | \pi^+ F; \pi^+ f_1 | \pi^+ F, \pi^+ f_2) = \int_M f_0 \Omega_3(f_1, f_2).
\] (3.7)

Now we prove Theorem 3.1. Since the sum is taken over \(- (r + l) + |\alpha| + k + j = 3, \quad r, l \leq -1 \), so \( \Omega_3(f_1, f_2) \) is the sum of the following five cases.

case a) 1) \( r = -1, \quad l = -1 \quad k = j = 0, \quad |\alpha| = 1 \)

By (3.1), we get

\[
\text{case a) 1) } \sum_{|\alpha|=1} \sum_{|\beta|=1} \sum_{|\delta|=1} \int_{\xi_1=1}^{+\infty} \int_{\xi_1=-\infty}^{+\infty} \mathrm{trace}_{\bigwedge^2(T^*M)} \left\{ \partial_{x^\prime}^\beta f_1 \times \partial_{x^\prime}^\alpha \pi^+ \sigma_1^\xi \right\} \\
\times \left\{ \partial_{x^\prime}^\alpha \pi^+ (f_2) \partial_{\xi_1} \partial_{x^\prime}^\delta \sigma_1^\xi + \partial_{x^\prime}^\beta f_1 \partial_{x^\prime}^\alpha \partial_{\xi_1} \partial_{x^\prime}^\delta \sigma_1^\xi \right\} (x_0) d\xi_0 \sigma(\xi') d^{n-1}x'.
\] (3.8)
It is necessary to compute trace_{\lambda_n}(T^*M)[\partial_{\xi_i}^{\alpha+\beta'}\pi^{\rho}_{\xi_\alpha}F_{\xi_\beta}^s \times \partial_{\xi_i}^{\alpha'}\partial_{\xi_\beta}^{\beta'}F_{\xi_\alpha}^s](x_0) and trace_{\lambda_n}(T^*M)[\partial_{\xi_i}^{\alpha+\beta'}\pi^{\rho}_{\xi_\alpha}F_{x}^s \times \partial_{\xi_i}^{\alpha'}\partial_{\xi_\beta}^{\beta'}F_{x}^s](x_0). Using the computations in [Wa2,p.17], for l, i, j < n, then

$$\partial_{\xi_i}\partial_{\xi_j}\partial_{\eta_j} \{ \text{trace} \left[ \pi^{\rho}_{\xi_\alpha}(F)(\xi', \xi_n) \times \partial_{\xi_n} \sigma_F(F(\eta', \xi_n)) \right] \} \big| \xi' = \eta' = l \sum \xi_i \xi_{i_2} \cdots \xi_{i_{2k+1}} f(\xi_n),$$

where 1 \leq i_1, \cdots, i_{2k+1} < n. Integration about |\xi'| = 1 is zero. By (3.3) and (3.5), then

$$\partial_{x_i}\sigma^s_F(x_0) = \partial_{x_i} \left[ \varepsilon(\xi)\iota(\xi) - \varepsilon(\xi)\iota(\xi) \right] (x_0) = 0,$$

so case a) I) is zero.

**case a) II)** $r = -1, l = -1 k = |\alpha| = 0, j = 1$

By (3.1), we get

$$\text{case a) II} = \frac{1}{2} \sum_{|\beta'| = 1} \sum_{|\beta'| = 1} \int_{|\xi'| = 1}^{+\infty} \hat{\partial}_{\beta'}^{\beta'}(f_1) \times \hat{\partial}_{\beta'}^{\beta'}(f_2)$$

$$\times \text{trace}_{\lambda_n}(T^*M)[\partial_{\xi_i}^{\alpha+\beta'}\pi^{\rho}_{\xi_\alpha}F_{x}^s \times \partial_{\xi_i}^{\alpha'}\partial_{\xi_\beta}^{\beta'}F_{x}^s](x_0) d\xi_n \sigma(\xi') d^n x'.$$

**case a) III)** $r = -1, l = -1 j = |\alpha| = 0, k = 1$

By (3.1), we get

$$\text{case a) III} = \frac{1}{2} \sum_{|\beta'| = 1} \sum_{|\beta'| = 1} \int_{|\xi'| = 1}^{+\infty} \hat{\partial}_{\beta'}^{\beta'}(f_1) \times \hat{\partial}_{\beta'}^{\beta'}(f_2)$$

$$\times \text{trace}_{\lambda_n}(T^*M)[\partial_{\xi_i}^{\alpha+\beta'}\partial_{\xi_n}^{\rho}F_{x}^s \times \partial_{\xi_i}^{\alpha'}\partial_{\xi_\beta}^{\beta'}F_{x}^s](x_0) d\xi_n \sigma(\xi') d^n x'.$$

Write $p(\xi) = \varepsilon(\xi)\iota(\xi) - \varepsilon(\xi)\iota(\xi)$. By (3.3),(3.4),(3.5), then

$$\partial_{x_n} p(\xi)(x_0) = h'(0)|\varepsilon(\xi)\iota(\xi') - \varepsilon(\xi')\iota(\xi)| (x_0);$$

$$\partial_{x_n} \sigma^s_F(x_0) = \frac{h'(0)|\varepsilon(\xi)\iota(\xi') - \varepsilon(\xi')\iota(\xi)| (x_0) - h'(0)|\xi|^{2}p(\xi)}{|\xi|^2}.\]

So case a) II+III) has the form in the theorem 3.1.

**case b)** $r = -2, l = -1, k = j = |\alpha| = 0$

By (3.1), we get

$$\text{case b} = \sum_{|\beta'| = 1} \sum_{|\beta'| = 1} \left[ (-1)^{2+|\beta'|} \frac{\beta'^!}{\beta'^1} \right] \times \int_{|\xi'| = 1}^{+\infty} \int_{-\infty}^{+\infty} \hat{\partial}_{\beta'}^{\beta'}(f_1) \times \hat{\partial}_{\beta'}^{\beta'}(f_2)$$

$$\times \text{trace}_{\lambda_n}(T^*M)[\partial_{\xi_i}^{\alpha+\beta'}\pi^{\rho}_{\xi_\alpha}F_{x}^s \times \partial_{\xi_i}^{\alpha'}\partial_{\xi_\beta}^{\beta'}F_{x}^s](x_0) d\xi_n \sigma(\xi') d^n x'.$$
In the following, we prove that

$$\begin{align*}
By (3.5), then \\
\text{similar to (3.12), then}
\end{align*}$$

will appear, then similar to case a) 1), it is zero after the integration. So \(|\beta'| = 1\). 
In the following, we prove that \(\sigma_{-1}(F)(x_0)\) has the coefficient \(h'(0)\). Write \(F = \frac{A}{h}\), where \(A = d\delta - \delta d, \Delta = d\delta + \delta d\), then by the composition formula of the symbol, we have

$$\begin{align*}
\sigma(F) &= \sum_{0\leq i \leq 2, j \geq 2} \frac{1}{\alpha_i} \partial_{\xi}^j (\sigma_i(A)) D_{\alpha_i}^j (\sigma_{-1}^{(1)}) \\
\sigma_{-1}(F) &= \sigma_1(A) \sigma_{-2}(\Delta^{-1}) + \sigma_2(A) \sigma_{-3}(\Delta^{-1}) + \sum_{|\alpha| = 1} \partial_{\xi}^j (\sigma_2(A)) D_{\alpha_i}^j (\sigma_{-2}(\Delta^{-1})). \quad (3.12)
\end{align*}$$

By (3.3), then

$$\sum_{|\alpha| = 1} \partial_{\xi}^j (\sigma_2(A)) D_{\alpha_i}^j (\sigma_{-2}(\Delta^{-1}))(x_0) = \frac{ih'(0)|\xi|^2 \partial_{\xi} \rho(\xi)}{|\xi|^4}. \quad (3.13)$$

Similar to (3.12), then

$$\begin{align*}
\sigma_1(d\delta) &= \sigma_1(d) \sigma_0(\delta) + \sigma_0(d) \sigma_1(\delta) - \sqrt{-1} \sum_i \partial_{\xi_i} (\sigma_1(d)) \partial_{\xi_i} (\sigma_1(\delta)); \\
\sigma_1(\delta d) &= \sigma_1(\delta) \sigma_0(\delta) + \sigma_0(\delta) \sigma_1(d) - \sqrt{-1} \sum_i \partial_{\xi_i} (\sigma_1(\delta)) \partial_{\xi_i} (\sigma_1(d)). \quad (3.14)
\end{align*}$$

Let \(\{e_1, \cdots, e_{n-1}\}\) be the orthonormal frame field in \(U\) about \(g^M\) which is parallel along geodesics and \(e_1(x_0) = \frac{\partial}{\partial x_1}(x_0)\), then \(\tilde{e}_1 = \sqrt{h(x_n)e_1}, \cdots, e_{n-1} = \sqrt{h(x_n)e_{n-1}}, e_n = dx_n\) is the orthonormal frame field in \(\tilde{U}\) about \(g^M\). By Lemma 2.3 and Section 3 in [Wa1], we have

$$\begin{align*}
\sigma_1(d) &= \sqrt{-1} \varepsilon(\xi), \quad \sigma_0(d)(x_0) = \frac{1}{4} h'(0) \sum_{i=1}^{n-1} \varepsilon(e_i^*) [\varepsilon(e_n) e_i - c(e_n) c(e_i)]; \quad (3.15) \\
\sigma_1(\delta) &= -\sqrt{-1} \varepsilon(\xi), \quad \sigma_0(\delta)(x_0) = -\frac{1}{4} h'(0) \sum_{i=1}^{n-1} \varepsilon(e_i^*) [\varepsilon(e_n) e_i - c(e_n) c(e_i)], \quad (3.16)
\end{align*}$$

where

$$c(e_j) = \varepsilon(e_j^*) - \varepsilon(e_j), \quad \varepsilon(e_j) = \varepsilon(e_j^*).$$

By (3.5), then

$$\begin{align*}
\sigma_1(d\delta)(x_0) &= \sqrt{-1} \varepsilon(\xi) \sigma_0(\delta)(x_0) - \sqrt{-1} \sigma_0(d)(x_0) \varepsilon(\xi) - \sqrt{-1} h'(0) \varepsilon(dx_n) \varepsilon(\xi')(x_0); \\
\sigma_1(\delta d)(x_0) &= -\sqrt{-1} \varepsilon(\xi) \sigma_0(d)(x_0) + \sqrt{-1} \sigma_0(\delta)(x_0) \varepsilon(\xi). \quad (3.17)
\end{align*}$$
By Lemma A.1 in [U] and (3.3), then
\[
\sigma_{-3}(\Delta^{-1})(x_0) = -\frac{1}{|\xi|^2} \left[ \sigma_1(\Delta) \frac{1}{|\xi|^2} \right] - \sqrt{-1} \sum_i \partial_{\xi_i}(1) \partial_{\xi_j}(\frac{1}{|\xi|^2})(x_0)
\]
\[
= -\frac{\sigma_1(\Delta)(x_0)}{|\xi|^2} - \frac{2\sqrt{-1}h'(0)|\xi|^2\xi_0}{|\xi|^0}.
\tag{3.19}
\]
By (3.12), (3.13), (3.15)-(3.19) and the definitions of \(A, \Delta\), we get \(\sigma_{-1}(F)(x_0) = h'(0)f(\xi)\). So case b) has the form in the theorem 3.1.

**Case c**) \(r = 1, l = -2, k = j = |\alpha| = 0\)

By (3.1), we get
\[
\text{case b) } = \sum_{|\beta'|=1} \sum_{|\delta'|=1} \frac{(-i)^2+|\delta'|}{\delta!} \times \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \partial_{\xi'}^{|\delta'|} (f_1) \times \partial_{\xi'}^{|\delta'|} (f_2)
\]
\[
\times \text{trace}_{\lambda^2(T^*M)} \partial_{\xi'}^{|\delta'|} \pi^+_\xi (\sigma^F_1(\xi_0) \times \partial_{\xi'}^{|\delta'|} \sigma^F_2(x_0)) d\xi_0 \sigma(\xi') d^{m-1}x'.
\tag{3.20}
\]
Similar to the discussions in case b), case c) also has the form in the theorem 3.1, so we proved the theorem 3.1.

## 4 Some remarks

In this section, since the computation of \(a_{ij}\) in the theorem 3.1 has a little tedious, so we just give some remarks on the computation way in this case.

**Remark 1** Since the computation of \(\pi^+_\xi \sigma^F_1(x_0)\) has a little tedious, so the computation of case c) is more direct than the computation of case b). So we try to use the computation of case c) and some simple computations instead of the computation of case b). By the Leibniz rule, trace property and “++” and “−−” vanishing after the integration about \(\xi_n\) (for details, see [FGLS]), then

\[
\int_{-\infty}^{+\infty} \text{trace}_{\lambda^2(T^*M)} [\partial_{\xi_0} \pi^+_\xi_0 (\sigma^F_1)(\xi', \xi_0) \times \sigma^F_1(\eta', \xi_0)](x_0) d\xi_0
\]
\[
= \int_{-\infty}^{+\infty} \text{trace}_{\lambda^2(T^*M)} [\partial_{\xi_0} (\sigma^F_1)(\xi', \xi_0) \times \sigma^F_1(\eta', \xi_0)](x_0) d\xi_0
\]
\[
- \int_{-\infty}^{+\infty} \text{trace}_{\lambda^2(T^*M)} [(\sigma^F_1)(\xi', \xi_0) \times \partial_{\xi_0} (\eta', \xi_0)](x_0) d\xi_0
\]
\[
= - \int_{-\infty}^{+\infty} \text{trace}_{\lambda^2(T^*M)} [(\sigma^F_1)(\xi', \xi_0) \times \partial_{\xi_0} (\eta', \xi_0)](x_0) d\xi_0
\]
then

As in [U,p.12-13], we write

By (3.5),

trace

the equality:

although we conjecture that it should vanish and \( \Omega \) may not get the sum of case b) and case c) is zero through the above computations although we conjecture that it should vanish and \( \Omega(\xi_1,\eta_1) \) is also zero.

**Remark 2** The computations of the trace of some operators will appear in this case. We just compute an example and the others is similar. In the following, we compute the equality:

\[
\text{trace}_{\lambda^2(T^*M)}\{[\partial_{x_n}p(\xi)p(\eta)](x_0) = h'(0)[a_{n,m}(\xi',\eta')^2 + b_{n,m}|\xi'|^2|\eta|^2](x_0) + 8h'(0)\xi_n\eta_n(\xi',\eta'),
\]

where \( C_n^m - a_{n,m} = b_{n,m} = C_{n-2}^m + C_n^m - 2C_{n-2}^m \) and \( C_n^m \) denotes the combinator number.

**Proof.** By (3.5),

\[
\partial_{x_n}p(\xi)(x_0) = h'(0)[\varepsilon(\xi)\varepsilon(\xi') - \varepsilon(\xi')\varepsilon(\xi)](x_0) = h'(0)p(\xi',0) + \xi_nB,
\]

where \( B = h'(0)[\varepsilon(dx_n)\varepsilon(\xi') - \varepsilon(\xi')\varepsilon(dx_n)](x_0) \). By the well-known equality

\[
\varepsilon_{m-1}(\xi)\varepsilon_m(\eta) + \varepsilon_{m+1}(\eta)\varepsilon_m(\xi) = \langle \xi,\eta \rangle I_m,
\]

then

\[
\varepsilon(dx_n)\varepsilon(\xi') - \varepsilon(dx_n)\varepsilon(\xi') = 2\varepsilon(dx_n)\varepsilon(\xi'); \ p(\eta) = 2\varepsilon(\eta)\varepsilon(\eta) - \langle \eta,\eta \rangle I_m.
\]

By (4.2) and (4.4) and the theorem 4.3 in [U],

\[
\text{trace}_{\lambda^2(T^*M)}\{[\partial_{x_n}p(\xi)p(\eta)](x_0) = h'(0)[a_{n,m}(\xi',\eta')^2 + b_{n,m}|\xi'|^2|\eta|^2](x_0)
\]

\[
+ 4h'(0)\xi_n\text{trace}_{\lambda^2(T^*M)}[\varepsilon(dx_n)\varepsilon(\xi')\varepsilon(\eta)\varepsilon(\eta)] - 2|\eta|^2h'(0)\xi_n\text{trace}_{\lambda^2(T^*M)}[\varepsilon(dx_n)\varepsilon(\xi')]
\]

By (4.3) and the trace property, we have

\[
\text{trace}_{\lambda^2(T^*M)}[\varepsilon(dx_n)\varepsilon(\xi')] = 0.
\]

As in [U,p.12-13], we write

\[
am(\xi_1,\xi_2,\eta_1,\eta_2) = \text{trace}_{\lambda^\infty(T^*M)}[\varepsilon_{m-1}(\xi_1)\varepsilon_m(\xi_2)\varepsilon_{m-1}(\eta_1)\varepsilon_m(\eta_2)].
\]

then

\[
am+1(\eta_1,\xi_2,\xi_1,\eta_2) = a_m(\xi_1,\xi_2,\eta_1,\eta_2) + \langle \xi_1,\xi_2 \rangle \langle \eta_1,\eta_2 \rangle [2A_{n,m} - C_n^m],
\]

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where $A_{n,m} = C_n^m - C_n^{m-1} + \cdots + (-1)^m C_n^0$. So $a_1(\xi_1,\xi_2,\eta_1,\eta_2) = \langle \eta_2, \xi_1 \rangle \langle \xi_2, \eta_1 \rangle$ and

$$a_2(\eta_1,\xi_2,\xi_1,\eta_2) = \langle \eta_2, \xi_1 \rangle \langle \xi_2, \eta_1 \rangle + \langle \xi_1, \xi_2 \rangle \langle \eta_1, \eta_2 \rangle [2A_{n,1} - C_n^1].$$

(4.8)

So by (4.8) and $n = 4$

$$\text{trace}_{\wedge_2((T^*M)\otimes E(dx_n)\otimes D(\xi'))}\epsilon(\eta) = a_2(dx^n, \xi', \eta, \eta) = 2\eta_2\langle \xi', \eta' \rangle.$$  

(4.9)

By (4.5), (4.6) and (4.9), we prove the equality (4.1).

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References