Geometric Analysis
(Talk for Chinese Mathematical Society)

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Let us start out with historical development:

- **Fermat’s principle of calculus of variation** (Shortest path in various media).

- **Calculus (Newton and Leibniz):** Path of bodies governed by law of nature.

- **Euler, Lagrange:** Foundation for variational principle and the study of partial differential equations. Derivations of equations for fluids and for minimal surfaces.

- **Fourier, Hilbert:** Decomposition of functions into eigenfunctions, spectral analysis.
• **Gauss, Riemann**: Concept of intrinsic geometry.

• **Riemann, Dirichlet, Hilbert**: Solving Dirichlet boundary value problem for harmonic function using variational method.

• **Christoffel, Levi-Civita, Bianchi, Ricci**: Calculus on manifolds.

• **Riemann, Poincarè, Koebe**: Riemann surface uniformation theory, conformal deformation.

• **Cartan**: Exterior differential system, connections on fiber bundle.
• **Einstein, Hilbert**: Einstein equation and Hilbert action.

• **Kähler, Hodge**: Kähler metric and Hodge theory.

• **Hilbert, Cohn-Vossen, Lewy Weyl, Hopf, Pogorelov, Effimov, Nirenberg**: Global surface theory in three space based on analysis.

• **Weistress, Riemann, Lebesgue, Courant, Douglas, Radó, Morrey**: Minimal surface theory.

• **Gauss, Green, Poincarè, Schauder, Morrey**: Potential theory, regularity theory for elliptic equations.
• Weyl, Hodge, Kodaira, de Rham, Bergman, Milgram-Rosenbloom, Atiyah-Singer: de Rham-Hodge theory, integral operators, kernel functions, index theory.

• Pontrjagin, Chern, Allendoerfer-Weil: Global topological invariants defined by curvature forms.

• Bochner-Kodaira: Vanishing of cohomology groups based on curvature consideration.

• De Giorgi-Nash-Morser: Regularity theory for higher dimensional elliptic equation and parabolic equation of divergence type.
• **Kodaira, Morrey, Hörmander, Kohn, Andreotti-Vesentini**: Embedding of complex manifolds, \( \bar{\partial} \)-Neumann problem.

• **Kodaira-Spencer, Newlander-Nirenberg**: Deformation of geometric structures.

• **Federer-Fleming, Almgren, Allard**: Varifolds and minimal varieties in higher dimension.

• **Eells-Sampson**: Existence of harmonic maps into manifolds with non-positive curvature.

• **Calabi**: Affine geometry and conjectures on Kähler Einstein metric.
Beginning of seventies, we began a more systematic study of interaction between geometry and analysis.

The following is the basic principle: Gel’fand-Naimark theorem says that the $C^*$-algebra of complex valued function defined on a Hausdorff space determines the topology of such space.

Algebraic geometer has defined Zariski topology of an algebraic variety using ring of rational functions.
In differential geometry, one should also be able to extract information of metric and topology of the manifolds by functions defined over it. Naturally, these functions should be defined either by geometric construction or by differential equations arised in geometry. (Integral equations have not be used extensively as the idea of linking local geometry to global geometry is more related to differential equation.) A natural generalization of function consist of following: differential forms, spinors, sections of vector bundles.

We shall now discuss various process to construct functions of interest to geometry.
I. Polynomials from ambient space.

If the manifold is isometrically embedded into Euclidean space, there are polynomial functions restricted from Euclidean space. However, isometric embedding is in general not rigid. Functions constructed in such a way are not too useful.
On the other hand, if a manifold is embedded into Euclidean space in an canonical manner and the geometry of this submanifold is defined by some group of linear transformations of the Euclidean space, the polynomials restricted to the submanifold do play important roles. S.Y.Cheng and I (1974,1975) did develop several important gradient estimates of these functions to control the geometry of such submanifolds.
The first important theorem is a spacelike hypersurface $M$ in the Minkowski space $\mathbb{R}^{n,1}$. A very important question: Since the metric on $\mathbb{R}^{n,1}$ is $\sum (dx^i)^2 - dt^2$, the restriction of this metric on $M$ need not be complete even though $M$ may be complete with respect to the Euclidean metric. In order to prove the equivalence of these two concepts for hypersurfaces where we can control their mean curvature, Cheng and I proved gradient estimate of the function

$$\langle X, X \rangle = \sum_i (x^i)^2 - t^2$$

restricted on the hypersurface.
By choosing coordinate system, the function $\langle X, X \rangle$ can be assumed to be positive and proper on $M$. For any positive proper function $f$ defined on $M$, if we prove the following gradient estimate

$$\frac{|\nabla f|}{f} \leq C$$

where $C$ is independent of $f$, then we can prove the metric on $M$ is complete. This is obtained by integrating the inequality. Once we know the metric is complete, we proved the Bernstein theorem which says that maximal spacelike hypersurface must be linear. Such work was then generalized by Treibergs, C.Gerhardt and R.Bartnik for hypersurfaces in more general spacetime. (It is still an important problem to treat this problem for the most general spacetime when we assume the Einstein equation and the initial data is nonsingular.)
Another important example is the study of affine hypersurfaces $M^n$ in an affine space $A^{n+1}$. These are the improper affine sphere

$$\det(u_{ij}) = 1$$

where $u$ is a convex function or the hyperbolic affine spheres

$$\det(u_{ij}) = \left(-\frac{1}{u}\right)^{n+2}$$

where $u$ is convex and zero on $\partial \Omega$ and $\Omega$ is a convex domain.
For affine geometry, there is an affine invariant metric defined on $M$ which is

$$(\det h_{ij})^{-\frac{1}{n+2}} \sum h_{ij} d x^i d x^j$$

where $h_{ij}$ is the second fundamental form of $M$. It is a fundamental question to know whether this metric is complete or not.

By choosing coordinate system in $A^{n+1}$, the height function is a proper positive function defined on $M$. The gradient estimate of the height function gives a way to prove completeness of the affine metric.
Once completeness of the affine metric is known, it is trivial to prove properties of affine spheres: some of these were conjectured by Calabi. For example we proved that improper affine sphere is a paraboloid and that every proper convex cone admits a foliation of hyperbolic affine spheres. The statement about improper affine sphere was first proved by Jörgens, Calabi and Pogorelov. Conversely, we also proved that every hyperbolic affine sphere is asymptotic to a convex cone.
This kind of argument to use gradient estimate of some naturally defined function was also used by me to prove that the Kähler Einstein metric constructed by Cheng and myself is complete for any bounded pseudoconvex domain.

**Basic Principle:** To control a metric, find a function that we know well and give a gradient or higher order estimate of this function. (It appeared in my paper with Mok.)
This principle was used by Cheng-Li-Yau in 1982 to give a comparison theorem of heat kernel for minimal submanifolds in Euclidean space, spheres and hyperbolic space. Since any complex submanifold in $\mathbb{C}P^n$ can be lifted to a minimal submanifold in $S^{2n+1}$, the much later work of Li-Tian on complex submanifold of $\mathbb{C}P^n$ is a corollary.
Another very important property of linear function is that when it restricts to a minimal hypersurface in a sphere $S^{n+1}$, it is automatically an eigenfunction. When the hypersurface is embedded, I conjectured that the first eigenvalue of the hypersurface is equal to $n$. While this result is not completely solved, the work of Choi-Wang gives a strong support that the first eigenvalue has a lower bound depending only on $n$. Such a result is good enough for Choi-Schoen to prove a compactness result for embedded minimal surfaces in $S^3$. 
II. Geometric construction of functions

When manifolds cannot be embedded into linear spaces, there are ways to construct functions adapted to metric structure. Obviously distance function is the first major function we use. Out of distance function, we can construct Busemann function in the following way:

Given a geodesic ray \( \gamma : [0, \infty) \to M \) so that
\[
\text{distance}(\gamma(t_1), \gamma(t_2)) = t_2 - t_1
\]
where \( \left\| \frac{d\gamma}{dt} \right\| = 1 \), one defines
\[
B_\gamma(x) = \lim_{t \to \infty} (d(x, \gamma(t)) - t).
\]
This function generalizes the concept of linear function. For hyperbolic spaceform, its level set defines horospheres. For manifolds with positive curvature, it is concave. Cohn-Vossen (for surface) and Gromoll-Meyer used it to prove complete non-compact manifold with positive curvature is diffeomorphic to $\mathbb{R}^n$. 
A very important property of the Busemann function is that it is superharmonic on complete manifolds with nonnegative Ricci curvature in the sense of distribution. This is the key to prove the splitting principle of Cheeger-Gromoll. Various versions of this splitting principle have been important for applications to structure of manifolds. When I proved the Calabi conjecture, the splitting principle was used by me and others to prove the structure theorem for Kähler manifolds with nonnegative first Chern Class.
In 1974, I was able to use the Busemann function to estimate volume of complete manifolds with nonnegative Ricci curvature. This work was generalized by Gromov and has been useful for the recent works of Perelman on Hamilton’s flow.
If we consider $\inf_\gamma B_\gamma$, where $\gamma$ ranges from all geodesic rays from a point on the manifold, we may be able to obtain a proper exhaustion of the manifold. When $M$ is a complete manifold with finite volume and curvature is pinched by two negative constants, Siu and I did prove that such function gives a concave exhaustion of the manifold. If the manifold is Kähler, we were able to prove that we can compactify such manifolds by adding a point to each end to form a compact complex variety.
Besides taking distance function to a point, we can also taking distance function to a submanifold. In such cases, its Laplacian will involve Ricci curvature of the ambient manifold and the mean curvature of the submanifold. Such functions can be used as barrier for construction of minimal submanifolds.
If we look at the space of loops in a manifold, we can take the length of each loop and define a natural function on the space of loops, this is a function where Morse theory found rich application. When the manifold has negative curvature, this function is related to the displacement function defined in the following way:
If $\gamma$ is an element of the fundamental group acting on the universal cover of a complete manifold with nonpositive curvature, we can consider the function $d(x, \gamma(x))$: The study of such a function gives rise to properties of compact manifolds with nonpositive curvature. For example, in my thesis, I generalized Preissman theorem to the effect that every solvable subgroup of the fundamental group must be a finite extension of an abelian group which is the fundamental group of a totally geodesic flat subtorus. Gromoll-Wolf and Lawson-Yau also proved that if the fundamental group of such manifold has no center and splits as a product, the manifold splits as a metric product.
Busemann function gives a way to detect the "angular structure" at infinity of the manifold. It can be used to construct the Poisson Kernel of hyperbolic space form. For simply connected complete manifold with bounded and strongly negative curvature, it is used as a barrier to solve Dirichlet problem for bounded harmonic functions, after modification at infinity.
III Functions defined by differential equations

The most important differential operator for a manifold is the Laplacian. Its spectral resolution gives rise to eigenfunctions. Harmonic functions are therefore the simplest function that play important roles in geometry.

If the manifold is compact, maximum principle shows that harmonic functions must be constant. However when we try to understand singularities of compact manifolds, we may create noncompact manifolds by scaling and blowing up process. Harmonic functions then play important roles.
The first important question for harmonic function on a complete manifold is the Liouville theorem. I started my research on analysis by understanding the right formulation of Liouville theorem. In 1971, I thought that it is natural to prove that for complete manifolds with non-negative Ricci curvature, there is no nontrivial harmonic functions. In fact, for complete Kähler manifolds with positive bisectional curvature, the manifold should be biholomorphic to complex Euclidean space. (This was partially motivated by some preliminary works of Greene-Wu on Cousin problem on these manifolds.) I also thought that on the opposite case, if a complete manifold has strongly negative curvature and is simply connected, one should be able to solve Dirichlet problem for bounded harmonic functions.
The gradient estimates that I derived for positive harmonic function come from interpretation of Schwarz lemma in complex analysis. In fact, I generalized the Ahlfors Schwarz lemma before I understood how to work out the gradient estimates for harmonic function. The generalized Schwarz lemma says that holomorphic maps mapping from a complete Kähler manifold with Ricci curvature bounded from below, to a Hermitian manifold with holomorphic bisectional curvature bounded from above by a negative constant, is distance decreasing where the constants depend only on the bound of the curvature.
The gradient estimate that I found can be generalized to cover eigenfunctions. And Peter Li was the first one to apply it to find estimate of eigenvalues for manifolds with positive Ricci curvature. Li-Yau then solved the problem of estimating eigenvalue of manifolds in terms of its diameter and the lower bound of the Ricci curvature. The precise upper bound of the eigenvalue was first obtained by S.Y. Cheng. Cheng’s theorem provide a very good demonstration of how analysis of functions provide information to geometry. As a corollary of his theorem, he proved that if a compact manifold $M^n$ has Ricci curvature $\geq n - 1$ and the diameter is equal to $\pi$, then the manifold id isometric to the sphere. He used a lower estimate for eigenvalue due to Lichnerowicz and Obata. Cheng’s argument’s is
flexible enough that Colding was able to use it to give a pinched version: If the diameter is close enough to $\pi$, it is diffeomorphic to sphere.

The classical Liouville theorem has a natural generalization: Polynomial growth harmonic functions are in fact polynomials. Motivated by this fact and several complex variable, I asked whether the space of polynomial growth harmonic functions with a fixed growth rate is finite dimension with upper bound of dimension depending only on the growth rate. This was proved by Colding-Minicozzi and generalized by Peter Li.
Most of the works can be generalized to those manifolds where Sobolev and Poincarè inequality hold. These inequalities are all related to isoperimetric inequalities. C.Croke was able to follow my work on Poincarè inequality to prove Sobolev inequality depending only on volume, diameter and the lower bound of Ricci curvature. Arguments of John Nash was then used by Cheng-Li-Yau to give estimates of heat kernel and its higher derivatives. In the course of estimate, an estimate of injectivity radius was derived and thus estimate turn out to play a role in Hamilton’s theory of Ricci flow.
The estimates of heat kernel was later generalized by Saloff-Coste and Grigor’yan to complete manifolds with polynomial value growth and volume doubling properties. The advantage of such works are that they are quasi-isometric invariants and can be generalized to cover analysis on graph or discrete groups.
On the other hand, the original gradient estimate that I derived is a pointwise inequality that is much more adaptive to nonlinear theory. Peter Li and I were able to find a parabolic version of it in 1984. We observed its significance on estimate of heat equation and its relation to variational principle for paths on spacetime. Such ideas turn out to provide fundamental estimates for Hamilton’s Ricci flow.
A very precise estimate of eigenvalue of Laplacian has been important in many areas of mathematics. For example, the idea of Szegö-Hersch on upper bound of first eigenvalue was generalized by me to higher genus in joint works with P.Yang and P.Li. I applied it to prove that a Riemann surface defined by Arithmetic group must have a relative high degree when it is branched over the sphere. There are also beautiful works by P.Sarnak on understanding eigenfunctions for such Riemann surfaces. He pointed out their relevance to number theory.
There are many important properties of eigenfunctions that were studied in the seventies. For example, Cheng was able to give a beautiful estimate of multiplications of eigenvalues based only on genus. The idea was used Colin de Verdière to study properties on graph theory.
There are several important questions related to nodal sets and the number of critical points of eigenfunctions. I made a conjecture on the area of the nodal set where Donnelly, Fefferman and Dong made some important contributions. The number of critical points of eigenfunction is difficult to deal with. I managed to prove existence of an critical point near the nodal set.
When there is potential, the eigenvalues of $-\triangle + V$ are also important. Efforts to study the gap $\lambda_2 - \lambda_1$ was made by me and coauthors. When $V$ is the scalar curvature, this was studied by Schoen and myself extensively. They are naturally related to conformal deformation, to stability of minimal surfaces, etc.