RECTIFIABILITY, REMOVABILITY, AND SINGULAR INTEGRALS

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In this lecture I discuss the interplay between the structure of removable singularities of bounded analytic functions and of Lipschitz harmonic functions, behaviour of singular integrals on m-dimensional subsets of $\mathbb{R}^n$, and rectifiability. For much more of recent developments and further references see the lecture notes of Pajot [P]. I shall mostly consider m-dimensional AD- (Ahlfors-David-) regular closed subsets of $\mathbb{R}^n$. This means that there is a constant C such that

$$r^m/C \leq H^m(E \cap B(x,r)) \leq Cr^m \text{ for all } x \in E, 0 < r < d(E).$$

Here H^m is the m-dimensional Hausdorff measure, $B(x,r)$ is the closed ball with centre x and radius r, and $d(E)$ is the diameter of E. I start with the following result from [MMV] for 1-dimensional sets:

1. **Theorem.** *Let $E \subset \mathbb{R}^n$ be a 1-dimensional AD-regular subset of $\mathbb{R}^n$. The following three conditions are equivalent.*

(1) $\int_E |\int_{E\setminus B(x,r)} \frac{x-y}{|x-y|^2} g(y) dH^1 y|^2 dH^1 x \leq C \int_E |g|^2 dH^1$ *for all $g \in L^2(E)$.*
(2) $\int_B \int_B \int_B c(x,y,z)^2 dH^1 x dH^1 y dH^1 z \leq C d(B)$ *for all $x \in \mathbb{R}^n, r > 0$.*
(3) $E \subset \Gamma$ *where Γ is a curve with $H^1(\Gamma \cap B(x,r)) \leq Cr$ for all $x \in \mathbb{R}^n, r > 0$.*

The key for the proof was the following identity found by Melnikov in [Me]:

$$c(x,y,z)^2 = \sum_\sigma \frac{1}{(z_{\sigma(1)} - z_{\sigma(3)})\overline{(z_{\sigma(2)} - z_{\sigma(3)})}}.$$

Here σ runs through all six permutations of 1,2 and 3, and $c(x,y,z)$ is the reciprocal of the radius of the circle passing through x, y and z. It is called the Menger curvature of this triple. It vanishes exactly when the three points lie on the same line. In general it measures how far they are from being collinear. Melnikov and Verdera used this identity to give a new proof for the boundedness of the Cauchy singular integral operator on Lipschitz graphs in [MV].

Based on earlier work of many people the above theorem gives the following corollary:

2**. Corollary.** *Let E be a compact 1-dimensional AD-regular subset of the complex plane. The following three conditions are equivalent.*

(1) *E is removable for bounded analytic functions.*
(2) *E is removable for Lipschitz harmonic functions.*
(3) *E is purely unrectifiable.*

Here the pure unrectifiablity of E means that E meets every rectifiable curve in zero length. The removability of E means that if E is contained in an open set U, any bounded analytic function in $U \setminus E$ can be extended analytically to U. Since Lipschitz functions can always be extended, (3) means that any Lipschitz function in U which is harmonic in $U \setminus E$ is harmonic in U.

David showed later in [D] that instead of AD-regularity it is enough to assume that E has finite H^1 measure. Still later Tolsa gave in [T] a characterization of removability for all compact subsets of the complex plane in terms of Menger curvature. A consequence of this is that (1) and (2) in the above corollary are equivalent for any compact set E. For these and other related results, see also [P].

The higher dimensional analogues of the above results are unknown. The natural questions are: is it true that

(a)

$$\int_E | \int_{E\setminus B(x,r)} \frac{x-y}{|x-y|^{m+1}} g(y) dH^m y|^2 dH^m x \leq C \int_E |g|^2 dH^m$$

for all $g \in L^2(E)$ if and only if E is uniformly rectifiable,

(b) when $m = n-1$, E is removable for Lipschitz harmonic functions if and only if E is purely unrectifiable?

The m-dimensional pure unrectifiability can be defined, for example, as the property that the set intersects every m-dimensional C^1 surface in a set of zero m-dimensional measure. The uniform rectifiability is a quantitative concept of rectifiability due to David and Semmes, [DS]. For 1-dimensional sets it means exactly the condition (3) of Theorem 1. It is known that the 'if'-part in (a) and the 'only if'-part in (b) are true. Some partial results for the converse can be found in [MP], [M] and [L].

One attempt to attack the higher dimensional case would be to try to use the following result from [MPr]:

3. **Theorem.** *Let E be an m-dimensional AD-regular subset of $\mathbb{R}^n$. If for H^m almost all $x \in E$ the limit*

$$\lim_{r\to 0}\int_{E\setminus B(x,r)} \frac{x-y}{|x-y^{m+1}} dH^m y$$

exists, then E is m-rectifiable, that is, there exist m-dimensional C^1 surfaces $S_1, S_2, \dots$ such that

$$H^m(E \setminus \bigcup_i S_i) = 0.$$

The main problem then would be to prove that the L^2-boundedness condition in (a) implies the almost everywhere convergence of principal values as in Theorem 3. Since I have not found any way to do this, I tried with another type of average convergence:

4. **Theorem.** *Let E be an m-dimensional AD-regular subset of $\mathbb{R}^n$. Suppose that the L^2-boundedness of (a) holds. Then for H^m almost all $z \in E$ the limit*

$$\lim_{r\to 0}\frac{1}{H^m(E\cap B(z,r))}\int_{E\cap B(z,r)}\int_{E\setminus B(z,r)} \frac{x-y}{|x-y|^{m+1}} dH^m y dH^m x$$

exists.

The problem now is that so far I have not been able to show that this convergence would imply rectifiability.

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