Chern's Work in Geometry

May 13, Harvard

Memorial Conference for Chern

Cartan is the grandfather of differential geometry.

Chern is the father of modern differential geometry.

Together they have created a beautiful and rich subject that have reached out to every branch of mathematics and physics.

Right before he died, Chern said that he is going to see the great Greek geometers.

There is no doubt that he had reached the same status as these great geometers.

Major events in geometry:

Pythagoras: (524 B.C.)	Pythagoras theorem.
Euclid: (350 B.C.)	Axioms for Euclidean geometry.
Archimedes:	Infinite process, study of conics.
Descartes: (1596-1650)	Introduction of coordinates, birth of analytic geometry, algebra and geometry are merged.
G.Desargues: (1591-1661)	Projective geometry.

Fermat: (1601-1665)	Variational principle.
Newton: (1642-1727) **Leibniz**: (1646-1716)	Calculus.
Euler: (1707-1783)	Combinational geometry, method of calculus of variation.
Gauss: (1777-1885)	Intrinsic geometry.
Riemann: (1826-1866)	Announced Riemannian geometry in 1854 in his Habitationschrift.

Sophus Lie: (1842-1899) Transformation group, contact geometry.

F. Klein: In 1872, announced the Erlanger programm which defines geometry as the study of a space with a group of transformation.

The group of projective collineations is the most encompassing group. The resulting geometry is projective geometry. Contributors include:

J.V. Poncelet (1788-1867)
A.F. Möbius (1790-1868)
M. Chasles (1793-1880)
J. Stenier (1796-1863)

Other geometries are affine geometry and conformal geometry.

A. Weil:

The psychological aspects of true geometric intuition will perhaps never be cleared up. ... Whatever the truth of the matter, mathematics in our century would not have made such impressive progress without the geometric sense of **Elie Cartan**, **Heinz Hopf**, **Chern** and a very few more. It seems safe to predict that such men will always be needed if mathematics is to go on as before.

Birth of modern differential geometry

Cartan completed the foundation works since Gauss-Riemman. Combining his development of Lie group theory and invariant theory of differential system, he introduced modern gauge theory.

Cartan defined generalized spaces which includes both Klein's homogeneous spaces and Riemann's local geometry. In morden terms, it is called "a connection in a fiber bundle". It generalizes the Levi-Civita parallelism.

In general, we have a fiber bundle $\pi : E \to M$, whose fibers $\pi^{-1}(x)$, $x \in M$, are homogeneous spaces acted on by a Lie group G. A connection is an infinitesimal transport of the fibers compatible with the group action by G.

While Grassmann introduced exterior forms, Cartan introduced the operation of exterior differentiation. His theory of Pfaffian system and theory of prolongation created invariants for solving equivalence problem in geometry.

Cartan's view of building invariants by moving frame had deep influence on Chern.

H. Hopf

H. Hopf initiated the study of differential topology, e.g., vector fields on manifold. His student Stiefel (1936) generalized Hopf's theorem to obtain Stiefel-Whitney Class.

H. Hopf did the hypersurface case of Gauss-Bonnet in 1925 in his thesis. In 1932, Hopf emphasis the integrand can be written as polynomial of components of Riemann curvature tensor.

These works have deep influence on Chern's later work.

Chern: Father of global intrinsic geometry

Chern: Riemannian geometry and its generalization in differential geometry are local in character. It seems a mystery to me that we do need a whole space to piece the neighborhood together. This is achieved by topology.

Both Cartan and Chern saw the importance of fiber bundle on problems in differential geometry.

It is certainly true that global differential geometry was studied by other great mathematics:

Cohn-Vossen
Minkowski
Hilbert
Weyl

But most of their works are focus on global surfaces in three dimensional Euclidean space.

Chern built bridge between intrinsic geometry and algebraic topology. (Besides his work on symmetric space, Cartan's work is more local in nature.)

Chern's education

(Tsing Hua University)

Undergraduate days. He studied:

1. Coolidge's non-Euclidean geometry: geometry of the circle and sphere.

2. Salmon's book: Conic sections and analytic geometry of three dimensions.

3. Castelnuovo's book: Analytic and projective geometry.

4. Otto Stande's book: Fadenkonstruktionen.

His teacher Professor Dan Sun studied projective differential geometry (found by E.J. Wilczynski in 1901 and followed by G. Fubini, E. Cěch).

Chern's master thesis was on projective line geometry which studies hypersurface in the space of all lines in three dimensional projective space. He studied line congruences: two dimensional submanifold of lines and their oscillation by quadratic line complex.

Chern's education

(Blaschke)

In 1932, Blaschke visited Peking. He lectured on "topological questions in differential geometry". He discussed pseudogroup of diffeomorphisim and their local invariants.

Chern started to think about global differential geometry and realized the importance of algebraic topology. He read Veblen's book "Analysis Situs" (1922).

In 1934, he studied in Hamburg under Blaschke. Both Artin, Hecke and Kähler were there. Blaschke worked on web geometry and integral geometry at that time. Chern started to read Seifert-Thrilfall (1934) and Alexandroff-Hopf (1935).

Chern's education

(Kähler, Cartan)

In Hamburg, Kähler lectured on Cartan-Kähler theory "Einführung in die Theorie der systeme von Differentialeichungen".

In 1936 to 1937, Chern went to Paries, to study with E. Cartan on moving frames, the method of equivalence and more on Cartan-Kähler theory. He spent ten months in Paries and met Cartan every two weeks.

Chern went back to China in the summer of 1937. He spent a few years to study Cartan's work. He said that Cartan wrote more than six thousand pages in his whole life. Chern has read at least seventy to eighty percent of these works. Some of the works he read it over and over again. During the War, it is great to spend full time to read and think in isolation.

Chern's comment on Cartan:

Undoubtedly one of the greatest mathematician of this century, his career was characterized by a rare harmony of genius and modesty.

Chern: In 1940, I was struggling in learning Elie Cartan. I realized the central role to be played by the notion of a connection and wrote several papers associating a connection to a given geometrical structures.

Weyl: Cartan is undoubtedly the greatest living master in differential geometry. . . I must admit that I found the book, like most of Cartan's papers, hard reading. . .

Most of the works of Chern are related to problem of equivalence:

Equivalence problem

1869:

E. Christoffel and R. Lipschitz solved the fundamental problem in Riemannian geometry. It was called the form problem:

> To decided when two ds^2's differ by a change of coordinate, Christoffel introduced the covariant differentiation now called Levi-Civita connection.

Cartan's equivalence problem:

> Given two sets of linear differential forms θ^i, θ^{*j} in the coordinates x^k, x^{*l} respectively, $1 \leq i, j, k, l \leq n$, both linearly independent.
>
> Given a Lie group $G \in GL(n, \mathbb{R})$, find the conditions that there are functions
>
> $$x^{*i} = x^{*i}(x^1, x^2, ..., x^n)$$
>
> such that θ^{*j}, after the substitution of these functions, differ from θ^i by a transformation of G.

The problem generally involves local invariants, and Cartan gave a procedure to generate such invariants.

Chern (1932-1943)

Projective differential geometry:

> Find a complete system of local invariants of a submanifold under the projective group and interpret them geometrically through osculation by simple geometrical figures.

Chern studied web geometry, projective line geometry, invariants of contact of pairs of submanifolds in projective space, transformations of surfaces (related to Bäcklund transform in soliton theory).

Chern (1932-1943)

Another typical problem in projective differential geometry:

Study the geometry of path structure by normal projective connections.

> Tresse (a student of Sophis Lie): Study pathes defined by integral curves of
>
> $$y'' = F(x, y, y')$$
>
> by normal projective connections in space (x, y, y').

Chern generalized this to n-dimension. Given $2(n-1)$ dimensional family of curves satisfying a differential system such that through any point and tangent to any direction at the point, there is exactly one such curve. Chern defines a normal projective connection. He then extended it to families of submanifolds.

Chern (1940-1942)

Integral geometry was developed by Crofton, Blaschke. Chern observed that such thery can be best understood in terms of two homogeneous spaces with the same Lie group G.

Hence there are two subgroups H and K

$$
\begin{tikzcd}
 & G \arrow[dl] \arrow[dr] & \\
G/H & & G/K
\end{tikzcd}
$$

Two cosets aH and bK is incident to each other if they intersect in G.

He generalized many important formula of Crofton. In 1952, He used it to generalize the kinematic formula of Poincare, Santalo and Blaschke.

Weil: It lifted the whole subject at one stroke to a higher plane than where Blaschke's school had lift it. I was impressed by the unusual talent and depth of understanding that shone through it.

Chern's visit of Princeton

(1943)

In 1943, Chern went from Kunming to Princeton, invited by Veblen and Weyl. Weyl was his hero. Fiber bundle theory was evolving starting from the works of Cartan and Whitney. Stiefel-Whitney classes were only defined mod two. Weil just published his work on Gauss-Bonnet formula and told Chern the works of Todd and Eger on "canonical classes" in algebraic geometry. (These works were done in the spirit of Italian geometry and rested on some unproved assumptions.)

Chern consider his best work is his intrinsic proof of Gauss-Bonnet formula.

F. Gauss did it for geodesic triangle (1827):

> *Disquistiones Circa superficies Curvas*.

He considered surface in $\mathbb{R}^3$ and used the Gauss map.

O. Bonnet (1948) generalized to any simply connected domain bounded by an arbitrary curve:

> *Mémoire sur la théorie générale des surfaces*, J. de l'Ecole Poly. Tome 19, Cahier **32**(1848)1-146.

W. Dyck generalized it to arbitrary genus:

> *Beiträge zur analysis situs*, Math Annalan **32**(1888)457-512.

H. Hopf generalized the formula to codimensional one hypersurface in $\mathbb{R}^n$.

C.B. Allendoerfer (1940) and W. Fenchel studied closed orientable Riemannian manifold which can be embedded in Euclidean space.

C.B. Allendoerfer and A. Weil (1843),

> *The Gauss-Bonnet theorem for Riemannian polyhedra*, Trans. Amer. Math. Soc., **53**(1943)101-129,

extended the formula to closed Riemannian polyhedron and hence to general closed Riemannian manifold.

But the proof still use embedding of the manifold in Eucliden space.

Weil: Following the footsteps of H. Weyl and other writers, the latter proof, resting on the consideration of ”tubes”, did depend (although this was not apparent at that time) on the construction of an sphere-bundle, but of a non-intrinsic one, viz. the transversal bundle for a given immersion.

Weil: Chern's proof operated explicitly for the first time with an intrinsic bundle, the bundle of tangent vectors of length one, thus clarifying the whole subject once and for all.

One century ago, Gauss established the concept of intrinsic geometry. Chern's proof of Gauss-Bonnet opened up a new horizon. Global topology is linked with intrinsic geometry through the concept of fiber bundle and transgression on the tangent sphere bundle. We see a new era of global intrinsic geometry.

In terms of moving frame, the structure equation for surface is

$$\begin{aligned} d\omega_1 &= \omega_{12} \wedge \omega_2 \\ d\omega_2 &= \omega_1 \wedge \omega_{12} \\ d\omega_{12} &= -K\omega_1 \wedge \omega_2 \end{aligned}$$

Here ω_{12} is the connection form and K is the Gauss curvature.

If the unit vector e_1 is given by a globally defined vector field V by

$$e_1 = \frac{V}{\| V \|}$$

at points where $V \neq 0$. Then we can apply Stoke's formula to obtain

$$-\int_M K\omega_1 \wedge \omega_2 = \sum_i \int_{\partial B(x_i)} \omega_{12}$$

where $B(x_i)$ is a small disk around x_i where $V(x_i) = 0$.

$\int_{\partial B(x_i)} \omega_{12}$ can be computed from the index of the vector field V at x_i.

According to the theorem of H. Hopf, summation of indices of a vector field is the Eucler number.

This proof of Chern is new even in two dimension.

In higher dimensional proof, the bundle is the unit tangent sphere bundle.

The curvature form Ω_{ij} is skew symmetric. The Pfaffian is

$$\mathrm{Pf} = \sum \epsilon_{i_1 \ldots i_{2n}} \Omega_{i_1 i_2} \wedge \ldots \wedge \Omega_{i_{2n-1} i_{2n}}.$$

The Gauss-Bonnet formula is

$$(-1)^n \frac{1}{2^{2n} \pi^n n!} \int \mathrm{Pf} = \chi(M).$$

Chern has to find a form Π on the unit sphere bundle so that $d\Pi$ is the lift of Pf. This is the birth of transgression.

Chern Class

Chern: My introduction to Characteristic class was through the Gauss-Bonnet formula, known to every student of surfaces theory. Long before 1943, when I gave an intrinsic proof of the n-dimensional Gauss-Bonnet formula, I know, by using orthonormal frames in surface theory, that the classical Gauss-Bonnet is but a global consequence of the Gauss formula which expresses the ”theorime egregium”. The algebraic aspect of this proof is the first instance of a construction later known as transgression, which is destined to play a fundamental role in the homology theory of fiber bundle, and in other problems.

Chern Class

Cartan's work on frame bundles and deRham's theorem have been always behind Chern's thinking.

Topology of fiber bundle:

E. Stiefel (1936)
Whitney (1937)

introduced Stiefel-Whitney Classes. It is only defined mod two.

J. Feldbau (1939),
C. Ehresmam (1941, 1942, 1943)
Chern (1944, 1945)
N. Steenord (1944)

studied topology of fiber bundles.

Pontrjagin (1942)

introduced Pontrjagin Class. He also associated topological invariants to curvature of Riemannian manifolds in 1944 (Doklady). It depends on embedding of manifolds and he did not know that these invariants are Pontrjagin Classes.

In the proof of Gauss-Bonnet formula, we can look for k vector fields $s_1, ..., s_k$ in general position. The points where they are linearly independent form a $(k-1)$ dimensional cycle independent of the choice of s_i. This was done by E. Stiefel in his thesis (1936).

H. Whitney (1937) considered sections for more general sphere bundle and looked at it from the point of view of obstruction theory.

Whitney noticed the importance of the universal bundle over the Grassmannian $G(q,n)$ of q planes in $\mathbb{R}^N$. He (1937) showed that any rank q bundle over the manifold can be induced by a map $f : M \to G(q,N)$ from this bundle.

When N is large, Pontrjagin (1942) and Steenrod (1944) observed that the map f is defined up to homotopy. The characteristic classes of the bundle is given by

$$f^*H^*(Gr(q,N)) \subset H^*(M)$$

The cohomology of $H^*(Gr(q,N))$ was studied by C. Ehresmann (1936) and they are generated by Schubert Cells.

Chern: It was a trivial observation and a stroke of luck, when I saw in 1944 that the situation for complex vector bundles is far simpler, because most of the classical complex spaces, such as the classical complex Grassmann manifolds, the complex Stiefel manifolds, etc. have no torsion.

For a complex vector bundle E, the Chern Classes $c_i(E) \in H^{2i}(M, \mathbb{Z})$.

Chern defined it in three different ways: by obstruction theory, by Schubert Cells and by curvature forms of a connection on the bundle. And he proved their equivalence.

The fundamental paper of Chern (1946)

In the paper,

> *Characteristic classes of Hermitian manifolds*,

Chern also laid the foundation of Hermitian geometry on complex manifolds. The concept of Hermitian connections was introduced, for example.

If Ω is the curvature form of the vector bundle, one defines

$$\det\left(I + \frac{\sqrt{-1}}{2\pi}\Omega\right) = 1 + c_1(\Omega) + \cdots + c_q(\Omega)$$

The advantage of defining Chern Classes by differential forms have tremendous importance in geometry and in modern physics.

An example is the concept of transgression created by Chern.

Transgression

Let φ be the connection form defined on the frame bundle associated to the vector bundle. Then the curvature form

$$\Omega = d\varphi - \varphi \wedge \varphi$$

Hence

$$\begin{aligned} c_1(\Omega) &= \frac{\sqrt{-1}}{2\pi}\mathrm{Tr}\Omega \\ &= \frac{\sqrt{-1}}{2\pi} d(\mathrm{Tr}\varphi) \end{aligned}$$

Similarly,

$$\begin{aligned} & \mathrm{Tr}(\Omega \wedge \Omega) \\ = \ & d(\mathrm{Tr}(\varphi \wedge \varphi) + \frac{1}{3}\mathrm{Tr}(\varphi \wedge \varphi \wedge \varphi)) \\ = \ & d(CS(\varphi)) \end{aligned}$$

This term $CS(\varphi)$ is called Chern-Simons form and has played fundamental role in three dimensional manifolds, in anomoly cancellation, in string theory and in solid state physics.

The idea of doing transgression on form level also gives rise to secondary operation on homology, eg. Massey product it appeared in K.T. Chen's work on iterated integral.

When the manifold is a complex manifold, we can write

$$d = \partial + \bar{\partial}$$

In a fundamental paper, Bott-Chern (1965) found:

> there is canonically constructed $(i-1, i-1)$-form $\tilde{T}c_i(\Omega)$ so that $c_i(\Omega) = \bar{\partial}\partial(\tilde{T}(c_i(\Omega)))$.

Chern made use of this theorem to generalize Nevanlinna theory of value distribution to holomorphic maps between higher dimensional complex manifolds.

The form $\tilde{T}c_i(\Omega)$ plays a fundamental role in Arekelov theory.

Donaldson used the case of $i = 2$ to prove the Donaldson-Uhlenbeck-Yau theorem for existence of Hermitian Yang Mills connection on algebraic surfaces.

For $i = 1$,

$$c_1 = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\det(h_{i\bar{j}})$$

where $h_{i\bar{j}}$ is the hermitian metric.

The right hand side is the Ricci tensor of the metric. The simplicity of the first Chern form motivates the Calabi conjecture.

The simplicity and beauty of geometry over complex number can not be exaggerated.

Chern (Chicago days)

After the fundamental paper on Chern Class in 1946, he explored more detail on the multiplicative structure of the characteristic classes.

In 1951, he had a paper with E. Spanier on the Gysin sequence on fiber bundle. They proved the Thom isomorphism independently of thom.

Splitting principle

In the paper (1953),

> *On the characteristic classes of complex sphere bundle and algebraic varieties*,

Chern showed that by considering an associated bundle with the flag manifold as fibers the characteristic classes can be defined in terms of line bundles. As a consequence the dual homology class of a characteristic class of an algebraic manifold contains a representative of algebraic cycle.

This paper provides the splitting principle in K-theory and coupled with Thom isomorphism allows one to give the definition of Chern classes on the associated bundle as was done by Grothendick later.

Chern's ability to create invariants for important geometric structure is unsurpassed by any mathematicians that I have ever known. His works on projective differential geometry, on affine geometry, on Chern-Moser invariants for pseudo-convex domains demonstrate his strength. The intrinsic norm on cohomology of complex manifold that he defined with Levine and Nirenberg has not been fully exploited yet. Before he died, a major program for him was to carry out Cartan-Kähler system for more general geometric situation.

Chern: The importance of complex numbers in geometry is a mystery to me. It is well-organized and complete.

Chern always regret that ancient Chinese mathematicians never discovered complex number. Chern's everlasting works in complex geometry make up the loss of Chinese mathematics for the last two thousand years.