MUMFORD-TATE GROUPS, FAMILIES OF CALABI-YAU VARIETIES AND ANALOGUE ANDRÉ-OORT PROBLEMS I.

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Abstract. We study Mumford-Tate groups of families of Calabi-Yau manifolds in this notes, for example, we have interest in the relations between the geometry of families of Calabi-Yau manifolds and their Mumford-Tate groups.

At first, we show some relations between the global monodromy groups and Mumford-Tate groups. We give an explicit proof that the identity component of monodromy group of a family of Calabi-Yau manifolds belongs to the Mumford-Tate group of the characteristic VHS of the family. Then, we show that a family of Calabi-Yau manifolds with big algebraic monodromy is rigid in meaning of Shafarevich's problem.

In the rest of this notes, analogue to Abelian varieties of CM-type, we introduce CM-type polarized rational Hodge structures and then Calabi-Yau manifolds of CM-type. Following Mumford and Shimura's work, we obtained that the subset of CM points is Zariski dense in the Shimura variety which is a quotient of the classifying space of the given polarized rational Hodge structures. Finally, we introduce the analogue André-Oort problems and present some questions which we are working on.

Contents

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Notations. For any rational vector space V, we denote $V_{\mathbb{R}} = V(\mathbb{R}) = V \otimes_{\mathbb{Q}} \mathbb{R}$ and

$$V_{\mathbb{C}} = V(\mathbb{C}) = V \otimes_{\mathbb{Q}} \mathbb{C} = V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}.$$

Respectively, we define similar notations for any real vector space. We also denote $V_{\mathbb{Q}}$ (resp. $V_{\mathbb{R}}, V_{\mathbb{C}}$) to be a $\mathbb{Q}$- (resp. $\mathbb{R}$-, $\mathbb{C}$-) vector space.

For any rational vector space V, using notion $\mathbf{GL}(V)$ and $\mathbf{SL}(V)$ to denote rational algebraic groups of automorphisms. The set of $\mathbb{Q}$-points of $\mathbf{GL}(V)$ (resp. $\mathbf{SL}(V)$) is denoted by $\mathrm{GL}(V_{\mathbb{Q}})$ (resp. $\mathrm{SL}(V_{\mathbb{Q}})$), also we denote $\mathrm{GL}(V_{\mathbb{R}})$ and $\mathrm{GL}(V_{\mathbb{C}})$ (resp. $\mathrm{SL}(V_{\mathbb{R}})$ and $\mathrm{SL}(V_{\mathbb{C}})$) for the set of $\mathbb{R}$-points and the set of $\mathbb{C}$-points of $\mathbf{GL}(V)$ (resp. $\mathbf{SL}(V)$). In that way, similar signs are denoted for real vector spaces and for complex vector spaces.

In this notes, we only consider vector spaces of finite rank and we work over base fields of characteristic zero, and all varieties in this notes are assumed to be defined over $\mathbb{C}$.

1. Hodge Structures on Finite Dimension Vector Spaces

The scalar group and Hodge structures. Let $\mathbb{G}_m$ be the multiplicative group. The scalar group

$$\mathbb{S} = R_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m := \prod_{\sigma\in\mathrm{Gal}(\mathbb{C}/\mathbb{R})} \mathbb{G}_m^{\sigma}$$

is invariant under the complex conjugate, thus $\mathbb{S}$ is a real algebraic group. $\mathbb{G}_m(\mathbb{C}) = \mathbb{C}^*$ is a complex Lie group and so $\mathbb{S}(\mathbb{R}) = \mathbb{C}^*$. Actually, $\mathbb{S} = \mathrm{Spec}\frac{\mathbb{R}[X,Y,Z]}{((X^2+Y^2)Z-1)}$ is a torus over $\mathbb{R}$.

Proposition 1.1 (cf. [7]). *Here are useful properties of the scalar group :*

1. *There is an isomorphism $\zeta : \mathbb{S}(\mathbb{C}) = (\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C})^* \xrightarrow{\sim} \mathbb{C}^* \times \mathbb{C}^*$ given by $\zeta(a \otimes b) = (ab, a\bar{b})$. Actually, ζ is determined by $\zeta : \mathbb{S}(\mathbb{R}) = \mathbb{C}^* \hookrightarrow \mathbb{C}^* \times \mathbb{C}^* = \mathbb{S}(\mathbb{C})$, $z \mapsto (z, \overline{z})$. Therefore, we always regard $\mathbb{S}_{\mathbb{C}} = \mathbb{G}_m \times \mathbb{G}_m$ under fixing the morphism $\zeta : \mathbb{S}_{\mathbb{C}} \xrightarrow{\cong} \mathbb{G}_m \times \mathbb{G}_m$.*
2. *This is a tautological inclusion $t : \mathbb{G}_m \longrightarrow \mathbb{S}$ determined by the natural homomorphism*
$$t_{\mathbb{R}} : \mathbb{R}^* = \mathbb{G}_m(\mathbb{R}) \xrightarrow{\hookrightarrow} \mathbb{S}(\mathbb{R}) = \mathbb{C}^*,$$
and $t_{\mathbb{R}}$ gives rise to the diagonal map $t_{\mathbb{C}} : \mathbb{C}^* = \mathbb{G}_m(\mathbb{C}) \to \mathbb{S}(\mathbb{C}) = \mathbb{C}^* \times \mathbb{C}^*$.
3. *There is a canonical norm homomorphism $\mathrm{N} : \mathbb{S} \to \mathbb{G}_m$ defined over $\mathbb{C}$, which on complex valued points is given by $\mathrm{N}_{\mathbb{C}} \circ \zeta^{-1}(a, b) = ab$, and on real valued points it is given by $\mathrm{N}_{\mathbb{R}} : \mathbb{C}^* \to \mathbb{R}^*$, $z \mapsto z\overline{z}$. Let $U^1 = \mathrm{Ker}(\mathbb{S} \xrightarrow{\mathrm{N}} \mathbb{G}_m)$, then $U^1(\mathbb{R}) = \{z \in \mathbb{C}^* \mid z\overline{z} = 1\}$.*
4. *Let G be a real algebraic group. For any homomorphism $h : \mathbb{S} \to G$ of real algebraic groups, there exists an associated homomorphism $\mu_h : \mathbb{G}_m \to G_{\mathbb{C}}$, which is determined by $z \mapsto h_{\mathbb{C}}(z, 1)$, $\forall z \in \mathbb{G}_m(\mathbb{C}) = \mathbb{C}^*$. Moreover,*
$$h_{\mathbb{C}}(z_1, z_2) = \mu_h(z_1)\overline{\mu_h}(z_2)\,\forall(z_1, z_2) \in \mathbb{C}^* \times \mathbb{C}^*,$$
and so $h(z) = \mu_h(z) \cdot \overline{\mu_h(z)}$. Let $w_h = h \circ t : \mathbb{G}_m \to G$ be a homomorphism. Then w_h is defined over $\mathbb{R}$, we called it weight homomorphism. We have $w_h(r) = h(r)\,\forall r \in \mathbb{G}_m(\mathbb{R})$, and $h(\sqrt{-1}) \equiv \mu_h(-1) \mod w_h(\mathbb{G}_m)$. $C := h(\sqrt{-1}) \in h(U^1(\mathbb{R}))$ is called Weil-operator.

Definition 1.2. A real (resp. rational) Hodge structure is a finite dimensional $\mathbb{R}$-vector space $V_{\mathbb{R}}$ (resp. $V_{\mathbb{Q}}$) together with a decomposition

$$V_{\mathbb{C}} = \bigoplus_{p,q\in\mathbb{Z}} V^{p,q}$$

such that $V_{p,q}$ are complex subspaces satisfying $\overline{V^{p,q}} = V^{q,p}$.

Let Y be a smooth compact Kähler manifold, then the Hodge decomposition

$$H^r(Y,\mathbb{C}) \simeq \bigoplus_{p+q=r} H^q(Y,\Omega^p_Y)$$

gives rise to a rational Hodge structure on $H^r(Y,\mathbb{Q})$. Actually, the Hodge decomposition of cohomology groups of any compact Kähler manifold is essentially an SL_2-representation.

In [7], the intrinsic meaning of Hodge structures is exposed, that is, a Hodge structure on V is equivalent to a representation of the scalar group $\mathbb{S}$:

Definition 1.3 (cf. [7])**.** a) A *real* Hodge structure is a pair (V,h) consisting of a finite dimensional $\mathbb{R}$-vector space V and a homomorphism of real algebraic groups $h : \mathbb{S} \to \mathbf{GL}(V)_{\mathbb{R}}$.

b) A *rational* Hodge structure is a $\mathbb{Q}$-vector space V with a representation $h : \mathbb{S} \to \mathbf{GL}(V)_{\mathbb{R}}$ such that the weight homomorphism w_h is defined over $\mathbb{Q}$ (and so $w_h : \mathbb{G}_m(\mathbb{Q}) \to \mathrm{GL}(V_{\mathbb{Q}})$ is a character of $\mathrm{GL}(V_{\mathbb{Q}})$).[1]

Remark. For a homomorphism of $\mathbb{Q}$-algebraic groups $h : \mathbb{S} \to \mathbb{G}$, w_h is a homomorphism defined over $\mathbb{R}$, of $\mathbb{Q}$-algebraic groups. Thus, it is a question whether w_h is defined over $\mathbb{Q}$.

Definition 1.4. We say that *a real (rational) Hodge structure* (V,h) *is of weight* n if the weight homomorphism w_h satisfies the following condition :

$$w_h(r)v = r^{-n}v \quad \forall\ r \in \mathbb{R}^*, \forall v \in V_{\mathbb{R}}.$$

Remark. The definition is equivalent to say there is a decomposition $V_{\mathbb{C}} = \oplus_{p+q=n} V^{p,q}$. If $V^{p,q} = 0$ in the decomposition when $pq < 0$, we say V *is pure of weight* n.

Examples 1.5. There are typical examples of Hodge structures:

a) Let (V_1,h_1) and (V_2,h_2) be two rational Hodge structures. The tensor product Hodge structure is $(V_1\otimes V_2, h_1\otimes h_2)$. The dual Hodge structure $(V_1^\vee, h_1^\vee)$ is defined by $V_1^\vee = \mathrm{Hom}(V_1,\mathbb{Q})$,

[1] We show the equivalence of two definitions of the Hodge structure : Because h is a homomorphism of real algebraic groups, the following two sets

$$\{v \in V_{\mathbb{C}} \mid h(\zeta^{-1}(a,b))v = a^p b^q v,\ \forall\ (a,b) \in \mathbb{C}^*\times\mathbb{C}^*\}$$

$$\{v \in V_{\mathbb{C}} \mid h(z)v = z^p \overline{z}^q v,\ \forall\ z \in \mathbb{C}^* = \mathbb{S}(\mathbb{R}^*)\}$$

coincide. Thus, using the following rule :

$$v \in V^{p,q}_{\mathbb{C}} \Longleftrightarrow h(z)v = z^p\cdot\overline{z}^q v, \quad \forall z \in \mathbb{C}^* = \mathbb{S}(\mathbb{R}^*),$$

we pass from the one definition to the another : there is a weight gradation on $V_{\mathcal{A}}$ ($\mathcal{A} = \mathbb{Q}$ or $\mathbb{R}$), i.e.,

$$V_{\mathcal{A}} = \bigoplus V_n \quad , \quad V_n \otimes \mathbb{C} = \bigoplus_{p+q=n} V^{p,q}.$$

Here $V_n = \{v \in V_{\mathcal{A}} \mid w_h(r)v = r^{-n}v\, \forall r \in \mathbb{R}^*\}$ is the Hodge structure of weight n.

i.e., $(h_1^\vee(s)f)(v) := f(h_1(s^{-1})v)$, $s \in \mathbb{S}, f \in V_1^\vee, v \in V_1$. Similarly, $\mathrm{Hom}(V_1, V_2)$ and $V_1^{\otimes n} \otimes (V_1^\vee)^{\otimes m}$ $\forall (n,m) \in \mathbb{Z}^2$ are rational Hodge structures.

b) *Hodge structures of Tate-Type.* Let $V = \mathbb{Q}(1) = 2\pi\sqrt{-1}\mathbb{Q}$. Define $h : \mathbb{S} \to \mathrm{GL}(V)_\mathbb{R}$ by

$$h(z)v = (z\overline{z})^{-1}v, \quad \forall z \in \mathbb{C}^* = S(\mathbb{R}) \text{ and } \forall v \in V_\mathbb{R}.$$

Then, V has a rational Hodge structure of weight -2 with $V = V^{-1,-1}$. Similarly, let $\mathbb{Q}(m) = (2\pi\sqrt{-1})^m\mathbb{Q}$. Then, $\mathbb{Q}(m)$ is a rational Hodge structure of weight $-2m$ as the homomorphism $h : \mathbb{S} \to \mathrm{GL}(\mathbb{Q}(m))_\mathbb{R}$ is given by

$$h(z)v = (z\overline{z})^{-m}v \quad \forall z \in \mathbb{C}^* = S(\mathbb{R}), \forall v \in \mathbb{R}(m) := \mathbb{Q}(m) \otimes \mathbb{R}.$$

Therefore, $\mathrm{Sym}^m\mathbb{Q}(1) = \mathbb{Q}(m)$.

c) Let V be a rational vector space. Assume $V_\mathbb{R}$ has a complex structure, i.e., there is a $J \in \mathrm{End}(V_\mathbb{R})$ with $J^2 = \mathrm{Id}_{V_\mathbb{R}}$. Then, $\sqrt{-1} \mapsto J, 1 \mapsto \mathrm{Id}_\mathbb{R}$ gives a homomorphism

$$h : \mathbb{C} \to \mathrm{End}(V_\mathbb{R})$$

of $\mathbb{R}$-algebras. The restriction $h|_{\mathbb{C}^*} : \mathbb{C}^* \to \mathrm{GL}(V_\mathbb{R})$ gives rise to a homomorphism of algebraic groups $h : \mathbb{S} \to \mathbf{GL}(V)_\mathbb{R}$. The homomorphism h is defined over $\mathbb{Q}$, and it determines a pure rational Hodge structure on V of weight 1.

Classifying spaces of polarized Hodge structures. In this paper, we only consider pure Hodge structures with some weight.

Let (V, h) be a rational Hodge structure. The Weil operator is

$$C = h(\sqrt{-1}) \in h(U^1(\mathbb{R})) \subset \mathrm{Hdg}(\mathbb{R}) \subset \mathrm{Aut}(V_\mathbb{R}),$$

and so $C(v) = i^{p-q}v$ for all $v \in V^{p,q}$, C^2 acts on V as $(-1)^{\mathrm{weght}(V)}\mathrm{Id}_V$.

Definition 1.6. Let $\mathcal{A}$ be a ring equal to one of $\mathbb{Q}$ and $\mathbb{R}$.

1. *A morphism between $\mathcal{A}$-Hodge structures* (V, h) *and* (V', h') is a linear transformation of $\mathcal{A}$-vector spaces which is $\mathbb{S}$-equivariant after tensoring with $\mathbb{R}$, i.e., there exists a $g \in \mathrm{Hom}_\mathcal{A}(V, V')$ with
$$g_\mathbb{R}(h(s)v) = h'(s)(g_\mathbb{R}(v)) \text{ for } \forall s \in \mathbb{S}(\mathbb{R}) \text{ and } \forall v \in V_\mathbb{R}.$$
2. *A morphism of pure Hodge structures of type* (p, q) is a morphism $\phi : H_1 \to H_2$ of $\mathcal{A}$-Hodge structures such that $\phi(H_1^{r,s}) \subset H_2^{r+p,s+q}$. Moreover, if ϕ is of type (l, l) we say the morphism ϕ is *of weight* $2l$.
3. A Hodge structure is called *irreducible* if its Hodge substructures are only 0 and itself.

Lemma 1.7 ([7])**.** *Let $\mathcal{A}$ be a ring equal to one of $\mathbb{Q}$ and $\mathbb{R}$.*

a) Two $\mathcal{A}$-Hodge structures $(V, h), (V, h')$ are equivalent if and only if $\exists g \in \mathrm{GL}_\mathcal{A}(V)$ satisfies
$$g_\mathbb{R}(h(s)v) = h'(s)(g_\mathbb{R}(v)) \text{ for } \forall s \in \mathbb{S}(\mathbb{R}) \text{ and } v \in V_\mathbb{R}.$$

b) A morphism of pure Hodge structures of type (p, q) is a strict morphism, that is,
$$\phi(H_1^{r,s}) = \phi(H_1) \cap H_2^{r+p,s+q}.$$
For a weight-2l morphism,it should be $\phi(F_1^p) = \phi(H_1) \cap F_2^{p+l}$.

c) The category of pure Hodge structures is abelian, i.e., it is an additive category including the kernel, the cokernel and the image of any morphisms between two pure Hodge structures.

Definition 1.8. Let V (resp. $V_{\mathbb{R}}$) be a pure rational (resp. real) Hodge structure of weight n and let C be its Weil operator. Let ψ (resp. $\psi_{\mathbb{R}}$) be a bilinear on V. If the bilinear form $V_{\mathbb{R}} \times V_{\mathbb{R}} \to \mathbb{R}$ which is defined by

$$(v, w) \longmapsto \psi_{\mathbb{R}}(Cv, w)$$

is symmetric and positive-definite, then ψ (resp. ψ) is called a *polarization of the rational (real) Hodge structure V (resp. $V_{\mathbb{R}}$)*, and so the corresponding triple (V, h, ψ) (resp. $(V_{\mathbb{R}}, h, \psi_{\mathbb{R}})$) is called *a polarized rational (resp. real) Hodge structure of weight n*. Moreover, if the triple (V, h, ψ) has a $\mathbb{Z}$-lattice $V_{\mathbb{Z}}$ such that the restriction of ψ to $V_{\mathbb{Z}} \times V_{\mathbb{Z}}$ takes values in $\mathbb{Z}$, then the tripe $(V_{\mathbb{Z}}, h, \psi)$ is a *polarized $\mathbb{Z}$-Hodge structure of weight n*.

In this paper, for convenience we use one sign Q to denote ψ, $\psi_{\mathbb{R}}$ and $\psi_{\mathbb{C}}$. We then have:

Lemma 1.9. *Let (V, h) be a pure Hodge structure of weight n and ψ a biliner form on V. Then (V, h, ψ) is polarized if and only if $Q = \psi_{\mathbb{C}}$ is $(-1)^n$-symmetric, and satisfies the Hodge-Riemann bilinear relations:*

$$Q(V^{p,q}, V^{r,s}) = 0 \text{ unless } r = n-p, s = n-q;\ (\sqrt{-1})^{p-q}Q(x, \overline{x}) > 0\ \forall x \in V^{p,q}.$$

Let V be a pure Hodge structure of weight $n \geq 0$ and denote $F^p = \oplus_{r \geq p} V^{r,n-r}$. Then we have a *Hodge filtration $F^\bullet$* on $V_{\mathbb{C}}$, i.e., there is a decreasing filtration

$$V_{\mathbb{C}} = F^0 \supset F^1 \supset \cdots \supset F^p \supset F^{p+1} \cdots \supset F^n \supset 0$$

on $V_{\mathbb{C}}$ such that

$$V_{\mathbb{C}} = F^p \bigoplus \overline{F^{n-p+1}}. \tag{1.9.1}$$

Conversely, any decreasing filtration $F^\bullet$ on $V_{\mathbb{C}}$ satisfying 1.9.1 recovers a Hodge decomposition of $V_{\mathbb{C}}$ by setting $V^{p,q} = F^p \cap \overline{F^q}$. Therefore, a Hodge filtration $F^\bullet$ determines a pure Hodge structure $\{V^{p,q}\}$ of weight n. Furthermore, we have :

Corollary 1.10. *Assume that the Hodge structure V is polarized and is pure of weight $n \geq 0$. Let C be Weil operator. Then, the Hodge-Riemannian bilinear relations in (1.9) are equivalent to the following conditions:*

$$Q(F^p, F^{n-p+1}) = 0 \forall p \text{ with } 0 \leq p \leq n;\ Q(Cv, \overline{v}) > 0\ \forall v \in V_{\mathbb{C}}.$$

Now, fix a finite dimension complex vector space $V_{\mathbb{C}} = V_{\mathbb{R}} \otimes \mathbb{C}$ with a real structure and fix Hodge numbers $\{h^{p,q}\}_{(p,q) \in \mathbb{Z}^2}$ such that

i. $h^{p,q} = h^{q,p} \geq 0$, all $h^{p,q}$ are integers, and $\dim_{\mathbb{C}} V_{\mathbb{C}} = \sum h^{p,q}$;
ii. $h^{p,q} = 0$ if $p + q \neq n$, or if $p + q = n$ but $pq < 0$.

Denote $\breve{\mathfrak{D}}$ to be the set of all decreasing filtrations $\{F^\bullet\}$ on $V_{\mathbb{C}}$ with

$$h^p := \dim_{\mathbb{C}} F^p = \sum_{i \geq p} h^{i,n-i}.$$

Example 1.11. Let $z_1, z_2, \cdots .z_m$ be coordinates on $V_{\mathbb{C}}$ and $F_0^\bullet$ a point in $\breve{\mathfrak{D}}$ with $F_0^p = \{z \in V_{\mathbb{C}} \mid z_1 = \cdots = z_{h^p} = 0\}$. Then, $\mathrm{SL}(V_{\mathbb{C}})$ acts transitively on $\breve{\mathfrak{D}}$ and the stationary subgroup P of $\{F_0^\bullet\}$ consists of unimodular matrices satisfying the following conditions:

i. along main diagonal, there are square blocks of dimension $h^0 - h^1, \cdots, h^{n-1} - h^n, h^n$;
ii. below these blocks there are zeros,and above them the elements are arbitrary.

Therefore, P contains a Borel subgroup in $\mathrm{SL}(V_{\mathbb{C}})$ and then it is a parabolic subgroup. Thus, $\breve{\mathfrak{D}} \subset \prod_{p=0}^k \mathrm{Gr}(h^p, V_{\mathbb{C}})$ and $\breve{\mathfrak{D}} \simeq \mathrm{SL}(V_{\mathbb{C}})/P$ is a a complex projective manifold (cf.[4]).

Let $\mathfrak{D}$ be a subset of $\breve{\mathfrak{D}}$ consisting of all Hodge filtrations on V_C with the fixed Hodge numbers $\{h^{p,q}\}_{(p,q)\in\mathbb{Z}^2}$. Then $\mathfrak{D}$ is a Hausdroff open subset of $\breve{\mathfrak{D}}$, it is the *classifying space of pure Hodge structure of weight n with the fixed Hodge numbers* $\{h^{p,q}\}_{(p,q)\in\mathbb{Z}^2}$.

Assume that $V_{\mathbb{C}}$ has a rational structure with $V_{\mathbb{C}} = V_{\mathbb{Q}} \otimes \mathbb{C}$ and there is a non-degenerate $\mathbb{Q}$-bilinear form on $V_{\mathbb{Q}}$ such that it is $(-1)^n$-symmetric when it is extend to $V_{\mathbb{C}}$. Let D be the set of all equivalent classes defined over $\mathbb{Q}$ in $\mathfrak{D}$ polarized by Q. We call D *period domain.* D is actually the *classifying space of all $\mathbb{Q}$-Hodge structures Hodge structure pure of weight n polarized by Q with the fixed Hodge numbers* $\{h^{p,q}\}_{(p,q)\in\mathbb{Z}^2}$.

Corollary 1.12. *Let $\breve{D}$ be the subset of $\breve{\mathfrak{D}}$ consisting of all points satisfying*

$$Q(F^p, F^{n-p+1}) = 0 \,\forall\, 0 \le p \le n.$$

Then, $\breve{D}$ is a closed subvariety of $\breve{\mathfrak{D}}$, hence it is a compact complex manifold. Moreover, $\breve{D}$ is the compact dual of D, and D is a Hausdroff open subset of $\breve{D}$.

Let G be an $\mathbb{Q}$-algebraic subgroup of $\mathbf{GL}(V_{\mathbb{Q}})$ given by

$$G(\mathcal{R}) = \{\gamma \in \mathrm{GL}(V_{\mathcal{R}}) \mid Q(\gamma(v), \gamma(w)) = Q(v, w) \;\forall\; v, w \in V_{\mathcal{R}}\}$$

for $\mathcal{R}$ be any ring between $\mathbb{Q}$ and $\mathbb{C}$. Clearly, G is a semisimple $\mathbb{Q}$-algebraic group. If V has a $\mathbb{Z}$ structure such that Q is defined over $\mathbb{Z}$ then we have a lattice $G(\mathbb{Z})$.

Lemma 1.13. *$G(\mathbb{C})$ acts transitively on $\breve{D}$, so that $\breve{D}$ is a complex homogenous space. Moreover, $G(\mathbb{R})$ is a semi-simple real Lie group and it acts on D transitively, thus D is also a complex homogenous space.*

Let $h \in \breve{D}$ be a filtration. Denote

$$B := G_h(\mathbb{C}) = \{g \in G(\mathbb{C}) \mid gF_h^p = F_h^p \;\forall\; 0 \le p \le n\}.$$

B is a parabolic subgroup of the complex Lie group $G(\mathbb{C})$ and is the isotropy group of h. Therefore, the Baily-Borel theorem says that $G(\mathbb{C})/B$ has a structure of complex projective variety and the identification $G(\mathbb{C})/B \xrightarrow{\simeq} \breve{D}$ is a complex analytic isomorphism, and so it is algebraic by the GAGA principal. Similarly, $G(\mathbb{R})$ acts on D transitively and the isotropy group of h in D is $K := G_h(\mathbb{R}) = G(\mathbb{R}) \cap B$. K is then a compact real subgroup of $G(\mathbb{R})$, and $D = G(\mathbb{C})/B$ is a Hausdroff open subset of $\breve{D}$, the embedding $D \subset \breve{D}$ corresponds to the inclusion $G(\mathbb{R})/K = G(\mathbb{R})/(G(\mathbb{R}) \cap B) \subset G(\mathbb{C})/B$. But, in general $\breve{D}$ and D are not symmetric spaces.

Example 1.14 (The period domain of rank 2 pure Hodge structures of weight 1)**.** Let $H = H^{0,1} \oplus H^{1,0}$ with $h^{0,1} = h^{1,0} = 1$. Then $G(\mathbb{C}) = \mathrm{SL}(2, \mathbb{C})$, $G(\mathbb{R}) = \mathrm{SL}(2, \mathbb{R})$, $\breve{D} = \mathbb{P}^1$ and $D = U \subset \mathbb{P}^1$ is the upper half-plane.

2. Mumford-Tate Groups

Mumford-Tate groups of Hodge structures.

Definition 2.1. Let (V,h) be a pure rational Hodge structure of weigh n.

1. The *Mumford-Tate group* $\mathrm{MT}(V,h)$ of (V,h) is the smallest $\mathbb{Q}$-algebraic subgroup G of $\mathbf{GL}(V)$ such that h factors through $G_{\mathbb{R}}$, i.e.,
$$h(\mathbb{S}(\mathbb{R})) \subset G(\mathbb{R}).$$
2. The Hodge-Tate group (or special Mumford-Tate group) $\mathrm{Hdg}(V,h)$ of (V,h) is the smallest $\mathbb{Q}$-algebraic subgroup G of $\mathbf{GL}(V)$ defined over $\mathbb{Q}$ such that
$$h(U^1(\mathbb{R})) \subset G(\mathbb{R}).$$

Remark. $\mathrm{MT}(V,h)$ is the $\mathbb{Q}$-algebraic group generated by $\{\sigma(h(\mathbb{S}(\mathbb{R})))\}_{\sigma\in\mathrm{Aut}(\mathbb{R}/\mathbb{Q})}$, as well as $\mathrm{Hdg}(V,h)$ is the $\mathbb{Q}$-algebraic group generated by $\{\sigma(h(U^1(\mathbb{R})))\}_{\sigma\in\mathrm{Aut}(\mathbb{R}/\mathbb{Q})}$.

It is obvious that $h(U^1(\mathbb{R}))$ is a connected compact subgroup of $\mathbb{R}^*$, and so $\mathrm{Hdg}(V,h)$ is a irreducible algebraic group. Let $\det_{\mathbb{R}} : \mathbf{GL}(V)_{\mathbb{R}} \to \mathbb{G}_m$ be the determinant homomorphism. Then
$$\det_{\mathbb{R}}(h(U^1(\mathbb{R}))) \equiv 1,$$
so that $\mathrm{Hdg}(V,h)$ is a Q-algebraic subgroup of $\mathbf{SL}(V)$. The natural multiplication map
$$\mathbf{mult} : \mathrm{Hdg}(V,h)\times\mathbb{G}_m \longrightarrow \mathrm{MT}(V,h), \quad (u,w)\longmapsto u\cdot w$$
is a surjective homomorphism because the image of $\mathbb{R}$-points contains $h(\mathbb{C}^*)$. Thus there is a short sequence of $\mathbb{Q}$-algebraic groups :
$$1 \longrightarrow \mathrm{Ker} \longrightarrow \mathrm{Hdg}(V,h)\times\mathbb{G}_m \xrightarrow{\mathbf{mult}} \mathrm{MT}(V,h) \longrightarrow 1.$$
Since $\mathrm{Hdg}(V,h)\subset\mathbf{SL}(V)$, the group Ker is finite and so **mult** is an isogeny of algebraic groups. Altogether, we have :

Lemma 2.2. *There are known properties of Mumford-Tate group:*

a) $\mathrm{MT}(V,h)$ *and* $\mathrm{Hdg}(V,h)$ *are both connected algebraic group defined over* $\mathbb{Q}$, *and* $\mathrm{MT}(V,h)$ *is reductive if and only if* $\mathrm{Hdg}(V,h)$ *is reductive. Moreover,* $\mathrm{Hdg}(V,h)$ *is a normal subgroup of* $\mathrm{MT}(V,h)$ *such that*
$$\dim_{\mathbb{Q}}\mathrm{MT}(V,h) = \dim_{\mathbb{Q}}\mathrm{Hdg}(V,h)+1.$$
Assume that the weight of (V,h) *is not zero. Then* $\mathrm{MT}(V,h)$ *contains the homothety group* $\mathbb{G}_m$, *and so*
$$\mathrm{MT}(V,h) = \mathbb{G}_m\mathrm{Hdg}(V,h),$$
$$\dim_{\mathbb{Q}}\mathrm{MT}(V,h)\cap\mathbf{SL}(V) = \dim_{\mathbb{Q}}\mathrm{Hdg}(V,h).$$
Therefore, $\mathrm{Hdg}(V,h)$ *is the identity component of* $\mathrm{MT}(V,h)\cap\mathbf{SL}(V)$. *In particular, the index*
$$[\mathrm{MT}(V,h)(\mathbb{C})\cap\mathrm{SL}(V_{\mathbb{C}}) : \mathrm{Hdg}(V,h)(\mathbb{C})] < \infty.$$

b) Let W be a $\mathbb{Q}$-subspace of V. W is $\mathrm{Hdg}(V,h)$*-invariant if and if $W_{\mathbb{R}}$ is $U^1(\mathbb{R})$-invariant. In particular,*

$$V^{\mathrm{Hdg}(V,h)} = V_{\mathbb{R}}^{U^1(\mathbb{R})} \cap V.$$

Let σ be an endomorphism of V. Then σ commutes with $\mathrm{Hdg}(V,h)$ *if and only if σ commutes with $h(U^1(\mathbb{R}))$, and then σ preserves the Hodge decomposition.*

Altogether, we have a $\mathbb{Q}$-algebra

$$\mathrm{End}^{Hdg(V,h)}(V) = \{\gamma \in \mathrm{GL}_{\mathbb{Q}}(V) \mid \gamma \text{ commutes with } \mathrm{Hdg}(V,h)\},$$

and for any $V_{\mathbb{C}}^{p,q} \neq 0$ there is a nontrivial homomorphism

$$\mathrm{End}^{\mathrm{Hdg}(V,h)}(V) \otimes \mathbb{C} \longrightarrow \mathrm{End}_{\mathbb{C}} V_{\mathbb{C}}^{p,q}.$$

Remark. Let V be a rational Hodge structure of weight n, $\psi : V \times V \to \mathbb{Q}$ a bilinear form. It is obvious any $\mathbb{R}$-bilinear form $\psi_{\mathbb{R}} : V_{\mathbb{R}} \times V_{\mathbb{R}} \to \mathbb{R}$ is $U^1(\mathbb{R})$-invariant (and so $\psi_{\mathbb{C}}$ is $U^1(\mathbb{R})$-invariant) if and only if

$$\psi' = (2\pi i)^{-n} \psi_{\mathbb{R}} : V_{\mathbb{R}} \times V_{\mathbb{R}} \to \mathbb{R}(-n)$$

is $\mathbb{S}$-invariant, and so ψ is $\mathrm{Hdg}(V,h)$-invariant and $\psi' = (2\pi i)^{-n}\psi_{\mathbb{R}}$ is $\mathrm{MT}(V,h)$-invariant.

Proposition 2.3 (cf. [10])**.** *Let (V,h,Q) be a polarized pure rational Hodge structure of weigh n. Then the Hodge-Tate group* $\mathrm{Hdg}(V,h)$ *is reductive, so is the Mumford-Tate group* $\mathrm{MT}(V,h)$*.*

Proof. Let $A(V)$ be $\mathrm{Aut}^0(V,Q)$, $C = h(\sqrt{-1})$ be the Weil operator of h. Since Q is $(-1)^n$-symmetry, then

$$A(V) = \mathrm{Sp}(V,Q) \text{ or } A(V) = \mathrm{SO}(V,Q),$$

and so $\mathrm{Hdg}(V,h) \subset A(V)$. Let $\sigma \in \mathrm{Inn}(\mathrm{Hdg}(V,h)_{\mathbb{R}})$ defined by $\sigma(g) = C^{-1}gC$ for any $g \in \mathrm{Hdg}(V,h)_{\mathbb{R}}$. σ is an involution of $\mathrm{Hdg}(V,h)_{\mathbb{R}}$ as $C^2 = (-1)^n \mathrm{Id}_{V_{\mathbb{R}}}$. Let H_σ be an algebraic group defined by

$$H_\sigma(\mathcal{R}) = \{g \in \mathrm{Hdg}(V,h)(\mathcal{R}) \mid \sigma(g) = g\}, \text{ for any ring } \mathbb{Q} \subset \mathcal{R} \subset \mathbb{C}.$$

Then H_σ is a real algebraic group and

$$\mathrm{Hdg}(V,h)_{\mathbb{C}} \simeq H_\sigma \otimes_{\mathbb{R}} \mathbb{C}.$$

Since $Q(\cdot, C\cdot)$ is a positive-definite form on $V_{\mathbb{R}}$ with

$$Q(u, Cw) = Q(h(u), C(h(w))),$$

$H_\sigma(\mathbb{R})$ is a compact real Lie group, and so $\mathrm{Hdg}(V,h)$ is reductive. Here σ is in fact the Cartan involution of $\mathrm{Hdg}(V,h)$. □

The following results characters Mumford-Tate groups and Hodge-Tate groups :

Theorem 2.4. *Let (V,h,Q) be polarized rational Hodge structure. The Mumford-Tate group* $\mathrm{MT}(V,h)$ *is the largest $\mathbb{Q}$-algebraic subgroup of* $\mathbf{GL}(V)$ *fixing all Hodge tensors, i.e., tensors of Hodge type $(0,0)$ in*

$$\bigoplus_{(n_1,n_2)\in\mathbb{N}^2} V^{\otimes n_1} \otimes (V^\vee)^{\otimes n_2}.$$

Theorem 2.5. *Let (V, h, Q) be a polarized pure rational Hodge structure of weight r. Then $\mathrm{Hdg}(V, h)$ is the largest $\mathbb{Q}$-algebraic subgroup of $\mathbf{GL}(V)$ fixing all tensors*

$$\eta \in V^{\otimes n_1} \otimes (V^\vee)^{\otimes n_2}$$

of Hodge type $(\frac{1}{2}r(n_1 - n_2), \frac{1}{2}r(n_1 - n_2))$ for all $(n_1, n_2) \in \mathbb{N}^2$.

In particular, if V is pure of weight 1, $\mathrm{Hdg}(V, h)$ is then the largest $\mathbb{Q}$-algebraic subgroup of $\mathbf{GL}(V)$ fixing all Hodge tensors

$$\eta \in (V^{\otimes 2n})^{n,n}\, \forall n \in \mathbb{N}.$$

In order to prove 2.4 and 2.5, we first introduce some knowledge of representation: Let G be any reductive algebraic group defined over a field k of characteristic zero, and $(V_\alpha)_{\alpha\in\Gamma}$ (here Γ is an index set) be the family of finite-dimensional k-representations of G, i.e., we have injective homomorphisms $G \hookrightarrow \mathrm{GL}(V_\alpha)$ for $\forall \alpha \in \Gamma$, and so

$$G \hookrightarrow \prod \mathrm{GL}(V_\alpha)$$

is injective too. For any $(m, n) \in \mathbb{N}^\Gamma \times \mathbb{N}^\Gamma$ (here $m \in \mathbb{N}^\Gamma$ means only finite of columns are nonzero), we denote

$$T^{m,n} := \bigotimes_{\alpha\in\Gamma} V_\alpha^{\otimes m(\alpha)} \otimes (V_\alpha^\vee)^{\otimes n(\alpha)},$$

which is again a representation of G. For any subgroup H of G, denote

$$T^H := \{t \mid t \in T^{m,n} \text{ for some } (m, n) \in \mathbb{N}^\Gamma \times \mathbb{N}^\Gamma \text{ and } t \text{ is fixed by } H\},$$

$$H' := \{h \in G \mid h(t) = t\ \forall\ t \in T^H\},$$

It is obvious that $H \subset H'$. On the other hand, we have :

Lemma 2.6 (Chevalley's theorem cf. [10])**.** *Let G be a reductive algebraic group over a field of characteristic zero. Then, we have :*

- *a) Any finite-dimensional representation of G is contained in a finite direct sum of representations $T^{m.n}$.*
- *b) Any subgroup H of G is the stabilizer of line D in some finite-dimensional representation of G.*
- *c) Let $X_k(G) = \mathrm{Hom}_k(G, \mathbb{G}_m)$ be the set of all characters of G. Assume that H is reductive or $X_k(G) \to X_k(H)$ is surjective. Then $H = H'$.*

Proof of the theorems 2.4 and 2.5. For 2.4 : Let $T^{m,n} = V^{\otimes m} \otimes (V^\vee)^{\otimes n}$ for any $(m, n) \in \mathbb{N}^2$. Then $t \in T^{m,n}$ is of $(0, 0)$ type if and only if t is fixed by $h(\mathbb{C}^*)$, i.e., t is fixed by $\mathrm{MT}(V, h)$. Let P be the largest algebraic subgroup of $\mathbf{GL}(V)$ fixing all tensors of Hodge type $(0, 0)$ in

$$V^{\otimes m} \otimes (V^\vee)^{\otimes n} \text{ for some } (m, n) \in \mathbb{N}^2.$$

Then $P = \mathrm{MT}(V, h)'$. Since $\mathrm{MT}(V, h)$ is reductive, by Chevalley's theorem $P = \mathrm{MT}(V, h)$.

The assertions in 2.5 can be proved by similar methods in proving 2.4. ☐

Mumford-Tate groups of variations of Hodge structures. At first, we shall define the *Mumford-Tate group for any polarized variation of rational Hodge structures* by the results of 2.4 and 2.5.

Let $(\mathbb{V}, \nabla, Q)$ be a polarized variation of pure rational Hodge structures of weight n over a connected complex manifold M. Let s_0 be a base point of M and $V = \mathbb{V}_{s_0}$. The VHS $(\mathbb{V}, \nabla)$ corresponds to a representation

$$\rho : \pi_1(M, s_0) \longrightarrow \mathrm{Aut}(V, Q).$$

For each $s \in M$, there is a Hodge structure (V, h_s) and Mumford-Tate group

$$\mathrm{MT}_s = \mathrm{MT}(V, h_s) \subset \mathbf{GL}(V).$$

Lemma 2.7 (cf.[9])**.** *Let $(\mathbb{V}, \nabla, Q)$ be a polarized variation of pure rational Hodge structures of weight n over a connected complex manifold (M, s_0) and $V = \mathbb{V}_{s_0}$. Then there exists a nowhere dense (complex topology) set $\sum$ of M such that:*

a) For any $s \in U := M \setminus \sum$, MT_s is constant group in $\mathbf{GL}(V)$. We denote this constant group to be $\mathrm{MT}(\mathbb{V})$, call it the Mumford-Tate group of $\mathbb{V}$.
b) On the other hand, for each $s \in \sum$,

$$\mathrm{MT}_s \subsetneq \mathrm{MT}(\mathbb{V}).$$

Sketch of the proof. In 2.4, we have shown that η is a Hodge tensor of (V, h) if and only if η is fixed by $\mathrm{MT}(V, h)$. Thus, by the theorem 2.4, we obtain :

$$\mathrm{MT}_s \neq \mathrm{MT}_{s'} \Longleftrightarrow T^{\mathrm{MT}_s} \neq T^{\mathrm{MT}_{s'}},$$

i.e., they have different spaces of fixed tensors and here "=" is in the meaning of parallel transform under ∇.

Given $\eta \in T^{m,n}$ for some $(m.n)$, then

$$B_\eta = \{s \in M(\mathbb{C}) \mid \eta \text{ is Hodge tensor of } \mathrm{MT}_s\}$$

has scheme structure and to be subvariety of M.

Let $\sum$ be the sum of all such B_η where $B_\eta \neq M$, then $\sum$ will be at most the union of countable subvarieties of M because V is rational vector space.

(b) is obvious. □

Remark. Similarly, we can define the *Hodge-Tate group* $\mathrm{Hdg}(\mathbb{V})$ for any VHS $(\mathbb{V}, \nabla)$. The largest set U of (a) is nowhere dense in M and is called the *Hodge generic locus,* and $\sum = \mathbb{M} \setminus U$ is called *Hodge exceptional locus.*

Now, we choose s_0 in the Hodge generic locus. Then the variation of polarized $\mathbb{Q}$-Hodge structures $(\mathbb{V}, Q)$ corresponds to a representation

$$\rho : \pi_1(M, s_0) \longrightarrow \mathrm{Aut}(V, Q)$$

where $(V, h) = (\mathbb{V}_{s_0}, h_{s_0})$. The *algebraic monodromy group* G^{Mon} is defined to be the Zariski closure of image of ρ in $A(V) := \mathrm{Aut}(V, Q)$. Thus by Deligne's complete reducibility (cf. [7]) G^{Mon} is a reductive $\mathbb{Q}$-algebraic subgroup of $A(V)$. The *connected algebraic monodromy group* G_0^{Mon} is the identity component of G^{Mon}, it is also a reductive $\mathbb{Q}$-algebraic group.

Lemma 2.8. *Let $(\mathbb{V}, \nabla, Q)$ is a polarized $\mathbb{Q}$-VHS with a representation*

$$\rho : \pi_1(M, s_0) \longrightarrow \mathrm{Aut}(V, Q).$$

Suppose that $\mathbb{V} \simeq \mathbb{V}_{\mathbb{Z}} \otimes \mathbb{Q}$ for an integral variation of Hodge structures $V_{\mathbb{Z}}$. Then there is a subgroup $\mathcal{G}$ of monodromy group $\Gamma := \rho(\pi_1(M, s_0))$ such that it has a finite index in $\mathrm{MT}(\mathbb{V})(\mathbb{Q})$, the set of Q-points of Mumford-Tate group of $\mathbb{V}$.

Proof. Choose s_0 in the Hodge generic locus of VHS $\mathbb{V}$ and let $V = \mathbb{V}_{s_0}$. Let (n_1, n_2) be any pair in $\mathbb{N}^2$ and denote the Q-space

$$W^{(n_1,n_2)} := (V^{n_1} \otimes (V^{\vee})^{n_2})^{0,0}.$$

Then, we have an induced VHS $\mathbb{W}^{(n_1,n_2)}$ with $W^{(n_1,n_2)} = \mathbb{W}_{s_0}^{(n_1,n_2)}$, and the corresponding representation

$$\rho : \pi_1(M, s_0) \longrightarrow \mathrm{Aut}(W^{(n_1,n_2)}, Q)$$

is induced from $\rho : \pi_1(M, s_0) \to \mathrm{Aut}(V, Q)$ (It is why we still use ρ to represent two representations). Actually, we have a commutative diagram

$$\begin{array}{ccc} \pi_1(M, s_0) & \xrightarrow{\quad\rho\quad} & \mathrm{Aut}(W^{(n_1,n_2)}, Q) \\ & {\scriptstyle\rho}\searrow \qquad \nearrow & \\ & \mathrm{Aut}(V, Q) & \end{array}.$$

It is obvious that $\mathbb{W}^{(n_1,n_2)}$ also has a $\mathbb{Z}$-structure induced from $\mathbb{V}_Z$, thus the representation factors though

$$\begin{array}{ccc} \pi_1(M, s_0) & \xrightarrow{\quad\rho\quad} & \mathrm{Aut}(W^{(n_1,n_2)}, Q) \\ & {\scriptstyle\rho_{\mathbb{Z}}}\searrow \qquad \nearrow & \\ & \mathrm{Aut}(W_{\mathbb{Z}}^{(n_1,n_2)}, Q) & \end{array}.$$

where $W_{\mathbb{Z}}^{(n_1,n_2)} = (V_Z^{n_1} \otimes (V_Z^{\vee})^{n_2})^{0,0}$. The polarization on $W^{(n_1,n_2)}$ is induced from the polarization on V and is again a definite quadratic form and $\pi_1(M, s_0)$ preserves it. Thus $\rho(\pi_1(M, s_0))$ is a finite subgroup in $\mathrm{Aut}(W^{(n_1,n_2)}, Q)$ as $O(W_{\mathbb{Z}}^{(n_1,n_2)})$ is a finite group, that is equivalent to say $\pi_1(M.s_0)$ acts on $W^{(n_1,n_2)}$ through a finite quotient.

Denote $G_{(n_1,n_2)}$ be the largest $\mathbb{Q}$-subgroup of $\mathbf{GL}(V)$ which acts trivially on $W^{(n_1,n_2)}$, then $\rho^{-1}(G_{(n_1,n_2)}(\mathbb{Q}))$ is subgroup of $\pi_1(M, s_0)$ of finite index. We then have

$$\mathrm{MT}(V, h) = \bigcap_{\text{all } (n_1,n_2) \in \mathbb{N}^2} G_{(n_1,n_2)},$$

In fact $\mathrm{MT}(V, h)$ is the intersection of only finitely many $G_{(n_1,n_2)}$'s as all G_{n_2,n_2} are noetherian spaces. Therefore, $\rho^{-1}(\mathrm{MT}(V, h)(\mathbb{Q}))$ is a subgroup of $\pi_1(M, s_0)$ with a finite index. □

Remark. The lemma 2.8 is cited in [9], here we give a detailed proof .

Theorem 2.9. *Let $(\mathbb{V}, \nabla, Q)$ be a polarized $\mathbb{Q}$-VHS with a representation*

$$\rho : \pi_1(M, s_0) \longrightarrow \mathrm{Aut}(V, Q).$$

Suppose that $\mathbb{V} \simeq \mathbb{V}_{\mathbb{Z}} \otimes \mathbb{Q}$ *for an integral variation of Hodge structures* $V_{\mathbb{Z}}$*. Then,* G_0^{Mon} *is a* $\mathbb{Q}$*-algebraic subgroup of* $\mathrm{MT}(\mathbb{V})$*. Moreover,*

$$G_0^{\mathrm{Mon}} \subset \mathrm{Hdg}(\mathbb{V}).$$

Proof. Here we use the same notations in 2.8. Then, by 2.7 and 2.8 we obtain that there is a subgroup $\mathcal{G}$ of the geometric monodromy group Γ such that it has a finite index in $\mathrm{MT}(\mathbb{V})(\mathbb{Q})$. Then,

$$G_0^{\mathrm{Mon}}(\mathbb{Q}) \subset \mathcal{G}.$$

□

Remarks. In [24], there are more general results. We note that if a VHS $\mathbb{V}$ comes from relative cohomology groups of any geometry family over M, $\mathbb{V}$ has a natural $\mathbb{Z}$-variation of Hodge structure $\mathbb{V}_{\mathbb{Z}}$ on M and a representation

$$\rho : \pi_1(M, s_0) \to \mathrm{Aut}(\mathbb{V}_{\mathbb{Z},s_0}, Q).$$

To end this subsection, we introduce some natural questions.

Question 2.10. Let $(\mathbb{V}, \nabla, Q)$ be a polarized variation of pure rational Hodge structures of weight n over a connected complex manifold (M, s_0) and $V = \mathbb{V}_{s_0}$. We always have :

$$G_0^{\mathrm{Mon}} \subset \mathrm{Hdg}(\mathbb{V}) \subset A(V). \tag{2.10.1}$$

Thus, we have questions as follows :

(1) For which VHS $\mathbb{V}$, $G_0^{\mathrm{Mon}} = \mathrm{Hdg}(\mathbb{V})$?
(2) For which VHS $\mathbb{V}$, $\mathrm{Hdg}(\mathbb{V}) = A(V)$?

Example 2.11. Here are examples for the above questions:

(1) Let $\mathbb{V}$ be a polarized weight one VHS from a family of Abelian varieties over a curve. Viehweg and Zuo recently showed : If the Arakelov–Yau inequality of that family attains the bound, then $G_0^{\mathrm{Mon}} = \mathrm{Hdg}(\mathbb{V})$ (cf. [24]).
(2) For a non-isotrivial Lefschetz pencil of odd-dimensional varieties with at least one singular fiber, there is an absolutely irreducible VHS $(\mathbb{V}, Q)$ corresponding to the vanishing cycles space. By the Kazhdan–Margulis theorem, one has (cf. [8])

$$\mathrm{Hdg}(\mathbb{V}) = A(V) = \mathrm{Sp}(V, Q).$$

If one of " ⊂ " becomes " = " in the question 2.10, what type should the base manifold be? Moreover, can the base manifold move freely in a moduli space, i.e., can the corresponding family deform freely?

Big monodromies and rigid families of Calabi-Yau manifolds. Let us recall a result about rigidity for the analogue Shafarevich conjecture for families of Calabi-Yau manifolds:

Theorem 2.12 ([30, 31])**.** *Any non-isotrivial Lefschetz pencil of Calabi-Yau manifolds of odd dimension is rigid.*

As exposed in [30, 31], the proof of the theorem 2.12 depends heavily on the special properties of Lefschetz pencils : the local system of vanishing cycles is absolute irreducible under the action of fundamental group $\pi_1(\mathbb{P}^1 \backslash S)$ where S is the set of singular values, i.e., the local system of vanishing cycles has a big monodromy (Kazdan-Magulis theorem).

Initiated by this result, we have a little generalization.

Definition 2.13. Let $\mathbb{V}$ be a local system of $\mathbb{Q}$-vector space on a quasi-projective manifold T with monodromy representation

$$\rho : \pi_1(Y, y_0) \to \mathrm{GL}(V),\ V := \mathbb{V}_{y_0}.$$

Let G^{mon} be the algebraic monodromy group. Assume that V carries a non-degenerate bilinear form Q which is symmetric or anti-symmetric and which is preserved by the monodromy group. The monodromy is called *big* if the connected algebraic monodromy group G_0^{mon} acts irreducibly on $\mathbb{V}_{\mathbb{C}}$.

It is clear that suppose $\mathbb{V}$ be a local system on a irreducible smooth variety T with big monodromy and $\pi : Y' \to Y$ be étale covering,then $\pi^*\mathbb{V}$ is also of big monodromy.

Theorem 2.14. *Let $f : \mathcal{X} \to M$ be a non-isotrivial smooth family of n-dimensional Calabi-Yau manifolds over a quasi-projective manifold M and let $R^n_{prim} f_* \mathbb{Q}$ carry the nature polarized VHS. Suppose that there is a sub $\mathbb{Q}$-VHS $\mathbb{V}$ of $R^n_{prim} f_* \mathbb{Q}$ such that $\mathbb{V}$ has big monodromy and the first Hodge piece of $R^n_{prim} f_* \mathbb{Q}$ belongs to the local system $\mathbb{V}_{\mathbb{C}}$. Then the family $f : \mathcal{X} \to M$ must be rigid.*

For a smooth family $f : \mathcal{X} \to M$, there is a natural polarized variation of Hodge Structure on M, i.e., a holomorphic vector bundle $(R^n_{prim} f_*(\mathbb{C}), \nabla)$ where ∇ is the Gauss-Manin connection. We then obtain a unique Higgs bundle $(E, \overline{\partial}, \theta)$ from the VHS $R^n_{prim} f_*(\mathbb{C})$ (cf. [21]). Here $(E, \overline{\partial})$ is holomorphic bundle under the holomorphic structure $\overline{\partial}$ and θ is so called *Higgs field* such that $\overline{\partial}(\theta) = 0$ and $\theta \wedge \theta = 0$. Analyzing the Higgs bundles over any product varieties and using *the generalized Donaldson-Uhlenbeck-Yau-Simpson correspondence* (cf. [29, 31]), we obtain :

Lemma (Faltings, Jost-Yau, Peters cf. [29, 31])**.** *Let $f : \mathcal{X} \to M$ be a non-isotrivial smooth polarized family of n-folds, if the family f is not rigid, then there is a non-zero flat section*

$$\sigma \in \mathrm{End}(R^n_{prim} f_*(\mathbb{C}))^{-1,1}.$$

Moreover, the Zariski tangent space of the deformation space of f is into $\mathrm{End}(R^n_{prim} f_*(\mathbb{C}))^{-1,1}$.

Proof of 2.14. Assume that the assertion is not true, we have the nontrivial extension family

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\subset} & \mathfrak{X} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ M \times \{0\} & \xrightarrow{\subset} & M \times T^0 \end{array} .$$

where T^0 is a smooth quasi-projective curve.

We obtain a unique Higgs bundle (E, θ) from the VHS $R^n_{prim} f_*(\mathbb{C})$, then by the above lemma we obtain a splitting of the Higgs bundle (cf. [31]) :

$$(E, \theta) = \mathrm{Ker}(\sigma) \bigoplus (\mathrm{Ker}(\sigma))^{\perp}.$$

Therefore, along M we always have $E^{0,n} \in \mathrm{Ker}(\sigma)$. Because of the non-triviality of the deformation of f, the map $\sigma : E^{n,0}|_M \to E^{n-1,1}|_M$ must be injective at some point s_0 of M, and so at s_0

$$E^{n,0} \nsubseteq \mathrm{Ker}(\sigma) \ \text{ and } \ E^{0,n} \subset \mathrm{Ker}(\sigma).$$

On the other hand, the representation $\rho : \pi_1(M, s_0) \to \mathrm{Aut}(\mathbb{V}_{\mathbb{C},s_0})$ is irreducible, so $E^{n,0}$ and $E^{0,n}$ are all in $\mathbb{V}_{\mathbb{C}}$. It is a contradiction.

□

Recently, Viehweg–Zuo obtained a characterization of certain Shimura curves in a moduli stack of Abelian varieties, and obtained some rigid Shimura curves in a moduli stack of even-dimensional polarized Calabi–Yau manifolds. It seems these results open a way for us to study the relation between Mumford–Tate groups of families and their *rigidity*.

3. Calabi-Yau Varieties with Complex Multiplication

CM-type points in moduli spaces of polarized Abelian varieties. Let $V_{\mathbb{R}}$ be a finite dimensional vector space over $\mathbb{R}$. The following sets are natural one to one correspondent:

a) the complex structure on $V_{\mathbb{R}}^{\vee}$;
b) the complex structure on $V_{\mathbb{R}}$;
c) the homomorphism of $\mathbb{R}$-algebra $h : \mathbb{C}^* \to \mathrm{GL}(V_{\mathbb{R}})$.

Let (V, h) be a pure rational $\mathbb{Q}$-Hodge structure of weight one. Its dual $V^{\vee}$ is also a rational $\mathbb{Q}$-Hodge structure and as a rational vector space it has at least one integer lattice. Let Λ be an integer lattices of $V^{\vee}$. If the rank of V is $2g$, we obtain a complex torus

$$A = V_{\mathbb{R}}^{\vee}/\Lambda$$

of dimensional g with a complex structure induced from the Weil operator $C = h(\sqrt{-1})$ on V. There is a canonical isomorphism (cf. [14]) :

$$H^i(A \times \cdots \times A, \mathbb{Q}) \cong \wedge^i(V \oplus \cdots \oplus V). \tag{3.0.1}$$

A result of Lefschetz (cf. [14]) says that (V, h) is polarized if and only the induced complex torus A is an Abelian variety. On the other hand, for any Abelian variety A, the cohomology group $H^1(A, \mathbb{Q})$ is a natural polarized pure $\mathbb{Q}$-VHS of weight one. Altogether, we have :

Proposition 3.1. *The contra-covariant functor*

$$A \longmapsto (H^1(A, \mathbb{Q}), h, Q)$$

defines an equivalence from the category of Abelian varieties over $\mathbb{C}$ up to isogeny to the category of polarized pure $\mathbb{Q}$-Hodge structures of weight one.

Moreover, let $(V^{\vee}, h, Q)$ be a polarized pure $\mathbb{Q}$-VHS of weight one and Λ be an integer lattice of $V_{\mathbb{R}}^{\vee} \cong \mathbb{R}^{2g}$. Then, $Q(\Lambda, \Lambda) \subset \mathbb{Z}$ and we can choose suitable generators of Λ as a basis of $V_{\mathbb{R}}$ such that :

i. Under this basis, the matrix

$$\begin{pmatrix} 0 & \mathfrak{L} \\ \mathfrak{L} & 0 \end{pmatrix}$$

represents the bilinear form Q where $\mathfrak{L} = \mathrm{diag}(e_1, \cdots, e_g)$;

ii. all e_i are positive integers with $e_1|e_2|\cdots|e_n$, and they are independent of the choosing of the generators of Λ.

We also say such $\mathfrak{L}$ is a *polarization*, in fact it determines an ample line $\mathcal{L}$on the complex torus $A = V_{\mathbb{R}}^{\vee}/\Lambda$ so that $(A, \mathcal{L})$ is a polarized Abelian variety. The integer $(e_1 \cdots e_g)^2$ is called the *degree of* $\mathfrak{L}$, or the *degree of* Q *under* Λ. Let Q'be another polarization of $V_{\mathbb{R}}^{\vee}$, we have a respective polarized Abelian variety $(A', \mathcal{L}')$. We say $Q \sim Q'$ if $(A, \mathcal{L}) \cong (A', \mathcal{L}')$.

Definition 3.2. Denote the contra-covariant functor in 3.1 to be ι. For any Abelian variety A, The Hodge-Tate group of A is defined by

$$\mathrm{Hdg}(A) := \mathrm{Hdg}(V, h)$$

where $(V, h, Q) = \iota(A)$ is the corresponding polarized pure $\mathbb{Q}$-Hodge structure of weight one.

Let X be a Abelian variety. Any endomorphism of X can be naturally realized as an endomorphism of V defined over $\mathbb{Q}$ where $(V, h, Q) = \iota(X)$. Thus,

$$\mathrm{End}^0(X) := \mathrm{End}(X) \otimes \mathbb{Q} = \{g \in \mathrm{Hom}(V, V)_{\mathbb{Q}} \mid g_{\mathbb{R}} \text{ commutes with } h(U^1(\mathbb{R}))\},$$

and so we have :

$$\mathrm{End}^0(X) = \{g \in \mathrm{Hom}(V, V)_{\mathbb{Q}} \mid gg' = g'g \ , \forall g' \in \mathrm{Hdg}(X) \ \} = \mathrm{End}^{\mathrm{Hdg}(V,h)}(V). \tag{3.2.1}$$

Definition 3.3. Let Y be a compact Kähler manifold. Its *Hodge ring* is defined as :

$$H_0^*(Y) = H^*(Y, \mathbb{Q}) \bigcap \sum_{p=0}^{\dim_{\mathbb{C}} Y} H^{p,p}(Y).$$

By 2.5 and 3.0.1, we have :

Corollary 3.4. *Let X be an Abelian variety. Then, we have a natural representation of* $\mathrm{Hdg}(X)$ *on $H^*(X^k, \mathbb{Q})$ which is defined over $\mathbb{Q}$. Moreover, for any positive integer k the Hodge ring $H_0^*(X^k)$ of X^k is a ring consisting of all* $\mathrm{Hdg}(X)$*-invariant elements in $H^*(X, \mathbb{Q})$.*

Let X be an Abelian variety of dimension g, and denote $\mathfrak{D} := \mathrm{End}^0(X) = \mathrm{End}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. Assume that X is simple. We then have :

$$\mathrm{End}(X) \overset{\hookrightarrow}{\longrightarrow} M_{2g}(\mathbb{Z}).$$

Moreover, $\mathfrak{D}$ is a division algebra defined over $\mathbb{Q}$ with a *Rosati-involution* $'$,i.e., for any $0 \neq x \in \mathfrak{D}$ $\mathrm{Trace}_{\mathfrak{D}/\mathbb{Q}}(xx') > 0$. Then, $\mathfrak{D}$ is a finite-dimensional simple algebra over $\mathbb{Q}$. Let $\mathbb{K}$ be the central of $\mathfrak{D}$, it is a subfield of $\mathfrak{D}$. By Albert's classification theorem of simple central algebra (cf. [14]), we have :

$$d^2 = [\mathfrak{D} : \mathbb{K}], \ e = [\mathbb{K} : \mathbb{Q}], \ \text{ and } \ ed|2g \tag{3.4.1}$$

Definition 3.5 (Abelian varieties with complex multiplication (CM-type))**.** Let X be an abelian variety. Then X is isogenous to $X_1^{n_1} \times \ldots \times X_k^{n_k}$ such that each X_i is simple and

$$n_1 g_1 + \cdots + n_k g_k = g.$$

a) In case that X is simple, using the notations above we say X is of CM type only if

$$ed = 2g.$$

b) In case that X is not simple, we say X is of CM type only if each isogenous component X_i of X is of CM type.

Proposition 3.6 (cf. [14]). *Let X be an Abelian variety of CM-type.*

a) If X is simple, then $\mathbb{K}$ is equal to $\mathfrak{D}$ and $d = 1$. Moreover, $\mathbb{K}$ must be a totally imaginary quadratic extension of a totally real field $\mathbb{K}^0$ of degree g over $\mathbb{Q}$, i.e.,

$$\mathrm{End}^0(X) \cong \mathbb{K}^0(\sqrt{-m}) \text{ for some positive integer } m.$$

b) If X is not simple, then

$$\mathrm{End}^0(X) \cong M_{n_1}(\mathbb{K}_1^0(\sqrt{-m_1})) \oplus \cdots \oplus M_{n_k}(\mathbb{K}_k^0(\sqrt{-m_k}))$$

where each $\mathbb{K}_i^0$ is a totally real field of degree g_i over $\mathbb{Q}$ and each $m_i \in \mathbb{K}_i^0$ is a positive integer. $\mathrm{End}^0(X)$ is a semi-simple algebra over $\mathbb{Q}$. Denote

$$\mathcal{R} = (Id_{n_1} \otimes \mathbb{K}_1^0(\sqrt{-m_1})) \oplus \cdots \oplus (Id_{n_k} \otimes \mathbb{K}_k^0(\sqrt{-m_k})).$$

Then $\mathcal{R}$ is a semi-simple and commutative $\mathbb{Q}$-subalgebra of $\mathrm{End}^0(X)$ with $[\mathcal{R} : \mathbb{Q}] = 2g$.

Altogether, we have:

Corollary 3.7. *An Abelian variety X is of CM-type if and only if there is a semi-simple commutative $\mathbb{Q}$-subalgebra $\mathcal{R}$ of $\mathrm{End}^0(X)$ such that $[\mathcal{R} : \mathbb{Q}] \geqslant 2g$ (hence $[\mathcal{R} : \mathbb{Q}] = 2g$).*

Moreover, there is a deep relation between complex multiplication and Hodge-Tate groups :

Theorem 3.8 (Mumford cf. [10],[16]). *Let X be an Abelian variety. Then, X is of CM-type if and only if its Hodge-Tate group $\mathrm{Hdg}(X)$ is a torus algebraic group, i.e., some power of $\mathbb{G}_m$.*

Sketch of the proof. The proof follows as :

1. Suppose that X is of CM type. Then, by 3.7 there is a commutative semi-simple $\mathbb{Q}$-subalgebra $\mathcal{R}'$ in $\mathrm{End}^0(X)$ with $[\mathcal{R}' : \mathbb{Q}] = 2g$. By the Zorn-lemma, we assume $\mathcal{R}'$ is maximal. Due to the formula in 3.2.1, we have
$$\mathrm{End}^0(X) = \mathrm{End}^{\mathrm{Hdg}(V,h)}(V),$$
where $(V, h, Q) = \iota(X)$. We claim that $\mathrm{Hdg}(X)$ is in a subset of $\mathcal{R}'$ consisting of all units of $\mathcal{R}'$. Otherwise, we would have a commutative Q-subalgebra $\mathcal{R}''$ of $\mathrm{End}^0(X)$ such that $[\mathcal{R}'' : \mathbb{Q}] > 2g$. Therefore, the reductive $\mathrm{Hdg}(X)$ $\mathbb{Q}$-algebraic group is connected and commutative, it must be a torus algebraic group.

2. Conversely, suppose $\mathrm{Hdg}(X)$ is torus $\mathbb{Q}$-algebraic group. Then, $\mathrm{Hdg}(X)$ can be diagonaliable as an algebraic subgroup of $\mathrm{GL}(V)$, and so $\mathrm{Hdg}(X)$ is in the center of $\mathrm{Hom}(V, V)$. On the other hand, by 3.2.1
$$\mathrm{End}^0(X) = \mathrm{End}^{\mathrm{Hdg}}(V) = \mathrm{Hom}(V, V)_{\mathbb{Q}}.$$
Let $\mathcal{R}$ be a diagonal subalgebra of $\mathrm{Hom}(V, V)$ (which depends on the bases that digitalize the $\mathrm{Hdg}(X)$). Then, we have :
$$[\mathcal{R} : \mathbb{Q}] = \dim_{\mathbb{Q}} V = 2 \dim_{\mathbb{C}} X.$$

□

Let X be an Abelian variety and $(V,h,Q) := \iota(X)$. There is a period domain D containing (V,h,Q). As we just present the result that D is a complex manifold and classify each point corresponds to a polarized pure rational Hodge structure (V,h',Q) of weight one. Let $f : \mathcal{X} \to M$ be any non-isotrivial smooth family of polarized Abelian varieties over a smooth varieties M. We have a natural pure $\mathbb{Q}$-VHS of weight one $\mathbb{V} = R^1 f_*(\mathbb{Q}_{\mathcal{X}})$ polarized by Q and $\mathbb{V}$ has a $\mathbb{Z}$-structure such that $\mathbb{V} = \mathbb{V}_{\mathbb{Z}} \otimes \mathbb{Q}$. Let $V = \mathbb{V}_{s_0}$ for some $s_0 \in M$, the local system $\mathbb{V}$ corresponds to a representation

$$\rho_f : \pi_1(M, s_0) \to \mathrm{Aut}(V,Q).$$

Let $\Gamma_f := \rho_f(\pi_1(M,s_0))$, then $\Gamma_f \subset \mathrm{Aut}(V_{\mathbb{Z}}, Q)$. By the global Torelli theorem, the period map

$$\Phi : M \longrightarrow D/\Gamma_f$$

is a holomorphic immersion from M to the analytic space D/Γ_f.

Definition 3.9. Let A be an complex Abelian variety with genus g. For any positive integer n, denote $A[n] = \ker(n\cdot : A \longrightarrow A)$. Then, we have finite isomorphisms

$$A[n] \xrightarrow{\cong} (\mathbb{Z}/n\mathbb{Z})^{2g},$$

and we call any such isomorphism to be *a level n-structure of A*.

Let S be noetherian scheme over C. An *Abelian scheme* over S is a group scheme $g : X \to S$ such that g is smooth and proper. The following theorem is due to Mumford :

Theorem 3.10 ([17])**.** *For any positive integer g, d, n, let $\mathcal{A}_{g,n,d}$ be a moduli functor*

$$\{ \text{ Scheme of finite type over } \mathbb{C}\} \longrightarrow \{Sets\}$$

defined by

$$Y \longmapsto \{ \text{ Abelian scheme over } Y, \text{ with a level } n\text{-structure and a } d^2\text{-degree polarization } \}/\cong .$$

Then, the functor $\mathcal{A}_{g,n,d}$ has a coarse moduli $A_{g,n,d}$ which is a quasi-projective scheme over $\mathbb{C}$. Moreover, if $n \geq 3$ the moduli functor F is fine, i.e., it is represented by $A_{g,n,d}$, and so we have a universal family $\pi : \mathfrak{X} \to A_{g,n,d}$.

Corollary 3.11 ([17])**.** *Notions as in 3.10, we have :*

$$A_{g,n,d} \xrightarrow{\cong} \coprod_{\mathfrak{L} \text{ with } \deg(\mathcal{L})=d} \mathfrak{D}_{n,\mathfrak{L}}$$

where $\mathfrak{D}_{n,\mathfrak{L}} \longleftrightarrow \{$ Abelian variety polarized by $\mathfrak{L}$ with a level n-structure$\}$.

Remark. Each $\mathfrak{D}_{n,\mathfrak{L}}$ is a Siegel moduli space, moreover it is a *Shimura variety.*

Lemma 3.12. *Notions as in 3.10 and the paragraph above, let $n \geq 3$ and $\Gamma = \rho(\pi_1(A_{g,n,d}))$. Then, by global Torelli theorem $A_{g,n,d}$ is a ramified covering over D/Γ. Here D/Γ has a unique algebraic structure to be a quasi-projective variety by Baily–Borel's theorem.*

Therefore, there is a beautiful result:

Theorem 3.13 (Shimura-Mumford-Deligne)**.** *Notions as in 3.10 and 3.11, we have that the set of CM points of D/Γ is Zariski dense. Then, the set of CM points of $A_{g,n,d}$ is Zariski dense in $A_{g,n,d}$, and so the set of CM points of $\mathfrak{D}_{n,\mathfrak{L}}$ is Zariski dense in $\mathfrak{D}_{n,\mathfrak{L}}$ where $\mathfrak{L} = \mathrm{diag}(e_1, \cdots, e_g)$ is a fixing polarization.*

The idea of the proof of 3.13 is actually implied in [10] and [16]. We generalize this result and obtain the theorem 3.17.

Remark. Generally, one can consider the moduli problem with more structure (for example, type of endomorphism $\mathrm{End}^0(X)$), and so any CM point is a zero dimensional Shimura variety. It is clear that in every period domain there exists a CM-type point.

The generalized Shimura-Mumford-Deligne theorem for polarized VHSs.

Definition 3.14. A polarized rational Hodge structure (V, h) is of CM type if the Hodge-Tate group $\mathrm{Hdg}(V, h)$ is commutative.

Remark. This definition is motivated by the Hodge theoretic proof of Hodge-Tate groups for Abelian varieties : an Abelian variety X is of CM-type if and only if $\mathrm{Hdg}(H^1(X, \mathbb{Q}), h)$ is an algebraic torus.

Let D be a Hermitian symmetric domain of type $G(\mathbb{R})/K_\infty$ where G is a connected semisimple algebraic group defined over $\mathbb{Q}$ and K_∞ is the maximal compact $\mathbb{R}$-subgroup of $G(\mathbb{R})$ (certainly, $G(\mathbb{R})/K_\infty = G(\mathbb{R})^0/K^0_\infty$). Choose a $\mathbb{Z}$-structure $G_\mathbb{Z}$ on G, let Γ be a arithmetic lattice, i.e., a subgroup of $G(\mathbb{Q}) \cap G(\mathbb{R})^0$ and commensurable to $G_\mathbb{Z}(\mathbb{Z})$.

Proposition 3.15. *The Baily–Borel theorem says that* $\Gamma\backslash D$ *has a unique algebraic structure of quasi-projective variety over* $\mathbb{C}$. *Moreover, we have:*

a) $\Gamma\backslash D$ *is smooth if* Γ *is a torsion-free lattice.*

b) $\Gamma\backslash D$ *is a projective variety if and only if* $\mathrm{Hom}(G, \mathbb{G}_m) = 0$.

Remark. The condition that $\mathrm{Hom}(G, \mathbb{G}_m) = 0$ is equivalent to that $\Gamma\backslash D$ is of finite volume and $G(\mathbb{Q})$ contains no unipotent element other than the identity (cf. [12]).

Assume that G has no simple compact factor G_i over $\mathbb{Q}$ (so that $G_i(\mathbb{R})$ is compact in the Hausdroff topology) and Γ contains a congruence group. Then $\Gamma\backslash D$ is a *connected Shimura variety.*

Proposition 3.16 (Deligne cf. [12])**.** *Any irreducible Hermitian symmetric domain can be regarded as a connected component of a period domain of a suitable polarized VHS.*

We have a generalization of Shimura-Mumford-Deligne' density theorem:

Theorem 3.17 (Density of CM points)**.** *Let D is a period domain of some polarized of pure weight. Assume D has type as $G(\mathbb{R})/K$ where G is a connected semisimple algebraic group defined over $\mathbb{Q}$ and K is the maximal compact $\mathbb{R}$-subgroup of $G(\mathbb{R})$.*

Then, for any subgroup

$$\Gamma \subset G(\mathbb{Q}) \cap G(\mathbb{R})^0$$

which is commensurable to $G_\mathbb{Z}(\mathbb{Z})$, the set of CM points in the algebraic variety $\Gamma\backslash D$ is Zariski dense in $\Gamma\backslash D$.

Remark. Borcea checked the case for any polarized Hodge structure of type Calabi-Yau threefold-like, i.e., any polarized pure Hodge structure of weight 3 with $h^{1,0} = 1$ (cf. [3]). Here we generalize the Shimura-Mumford-Deligne' theorem to any polarized pure VHS.

Thus, by Deligne's result 3.16 and the density theorem 3.17, we have:

Corollary 3.18. *Any connected Shimura variety has a Zariski dense set of CM points.*

We are going to prove the density theorem. At first, we have:

Lemma 3.19. *Let H be a connected linear algebraic group defined over $\mathbb{Q}$. Assume that H is reductive. Then, every $H(\mathbb{R})$-conjugate class of maximal $\mathbb{R}$-algebraic torus in H contains an algebraic torus T which can be defined over $\mathbb{Q}$.*

Proof. It is sufficient to prove that for any maximal tori T' in H defined over $\mathbb{R}$ there exists a $g \in H(\mathbb{R})$ such that $T = g^{-1}T'g$ is a maximal tori defined over $\mathbb{Q}$, we show the proof as follows :

a) Let T_1 be a maximal torus in H defined over $\mathbb{R}$ and $t_1 \in T_1$ be an element. The following properties are known (cf. [4, 12.2]):
   i. t_1 is *regular* if and only if $\mathrm{Cent}_H(t_1)^0 = H^{T_1}$.
   ii. The set U of the regular elements in T_1 is Zariski dense open in T_1.S
   Because $T_1(\mathbb{R}) = (\mathbb{R}^*)^{\dim T_1}$ and $\dim_{\mathbb{R}}(T_1 - U) < \dim_{\mathbb{R}} T_1$, the set $U_1(\mathbb{R})$ is nonempty and is Hausdroff open in $T_1(\mathbb{R})$.
b) By the assumption that H is reductive, we have $H^{T_1} = T_1$ and any Cartan subgroup of H coincide with a maximal torus (cf. [4, 13.17]), here a Cartan subgroup is the centralizer of a maximal torus. Then, for any $t' \in H(\mathbb{R})$, if it is regular then $T' = \mathrm{Cent}_H(t')^0$ is a Cartan group and T' is a maximal torus in H defined over $\mathbb{R}$. Moreover, if $t' \in U(\mathbb{R})$ then $\mathrm{Cent}^0(t') = T_1$.
c) Let
$$V = \bigcup_{g \in H(\mathbb{R})} gU(\mathbb{R})g^{-1}.$$
   We have a everywhere regular map $H(\mathbb{R}) \times V \to H(\mathbb{R})$ defined by $(g, x) \mapsto gxg^{-1}$; then V is Hausdroff open set in $H(\mathbb{R})$. As H is a linear group defined over $\mathbb{Q}$, it is obvious that the Hausdroff closure of $H(\mathbb{Q})$ contains the identity component $H(\mathbb{R})^0$, then $H(\mathbb{Q}) \cap V$ is not empty. Altogether, we have a regular element $t \in H(\mathbb{Q})$, $T := \mathrm{Cent}_H(t)^0$ is an algebraic torus defined over $\mathbb{Q}$, and T is in the $H(\mathbb{R})$-conjugate class of T_1.

□

Remark. Let $t_1 \in U(\mathbb{R})$ and $B \subset U$ be any Zariski open neighborhood of t_1. Denote $B(\mathbb{R}) = U(\mathbb{R}) \cap B$. Then
$$\mathcal{B} := \bigcup_{g \in H(\mathbb{R})} gB(\mathbb{R})g^{-1}$$
is an Hausdroff open set in $H(\mathbb{R})$, and so we can choose an regular $t \in H(\mathbb{Q}) \cap \mathcal{B}$ close to t_1.

Proof of the theorem 3.17: Let (V, h, ψ) is an arbitrary point in the period domain D. Then,
$$H := \mathrm{Hdg}(V, h) \subset G.$$

1. Let $K_\infty := \mathrm{Cent}_H(h(U^1(\mathbb{R})))^0$ where U^1 is defined in the proposition 1.1. K_∞ is an algebraic subgroup of $H_{\mathbb{R}}$ defined over $\mathbb{R}$. By a result of Borel and Springer (cf. [5, p26]), K_∞ itself has at least one maximal algebraic torus defined over $\mathbb{R}$; let $T_1 \subset K_\infty$ be one of maximal algebraic torus in K_∞.

2. Actually, T_1 is a maximal algebraic torus of $H_{\mathbb{R}}$: Since $h(U^1(\mathbb{R})) \subset \mathrm{Cent}_H(K_\infty)$, we have $h(U^1(\mathbb{R})) \subset T_1$. If there exists an $\mathbb{R}$-torus $T_1' \subset H_{\mathbb{R}}$ containing T_1, then T_1' centralizes $h(U^1(\mathbb{R}))$; hence $T_1' \subset K_\infty$ and so $T_1 = T_1'$.
3. By the lemma 3.19, there is a $g \in H(\mathbb{R})$ such that

$$T = g^{-1}T_1 g$$

is defined over $\mathbb{Q}$. Since $h(U^1(\mathbb{R})) \subset T_1$, then

$$gh(U^1(\mathbb{R}))g^{-1} \subset T.$$

On the other hand, $gh(U^1(\mathbb{R}))g^{-1}$ determines one rational Hodge structure (V, ghg^{-1}). Therefore,

$$\mathrm{Hdg}(V, ghg^{-1}) \subset T,$$

i.e., (V, ghg^{-1}) is a CM point in the domain D.
4. Moreover, T_1 is $\mathrm{Cent}_H(t_1)^0$ for some regular element $t_1 \in H(\mathbb{R})$. Let $\mathcal{J}$ be any Zariski neighborhood of t_1 in G. Let $B = \mathcal{J} \cap U$ where U is the set of regular elements of G in T_1. By the remark of 3.19, B is not empty Zariski open in T_1, and we can choose an regular element $t \in G(\mathbb{Q}) \cap \mathcal{B}$ such that $T = \mathrm{Cent}(t)^0$ is a maximal algebraic torus defined over $\mathbb{Q}$. Altogether, the set of CM points of $\Gamma \backslash D$ is Zariski dense.

□

Corollary 3.20. *Let V_1, V_2 be two polarized rational Hodge structures. Then $V_1 \otimes V_2$ is of CM type if and only if both V_1 and V_2 are of CM type.*

Analogue André-Oort problems for families of polarized Calabi-Yau varieties.

Definition 3.21. A polarized Calabi-Yau manifold (X, L) is of CM type if its period point is of CM-type, i.e., the Hodge-Tate group of $H^{\dim X}_{prim}(X, \mathbb{Q})$ is commutative where

$$H^{\dim X}_{prim}(X, \mathbb{Q}) := \ker(H^{\dim X}(X, \mathbb{C}) \xrightarrow{\wedge c_1(L)} H^{\dim X+2}(X, \mathbb{C})) \cap H^{\dim X}(X, \mathbb{Q}).$$

Remark. This definition is compatible with the original definition of CM al elliptic curve when $\dim_{\mathbb{C}} X = 1$, compatible with the definition in [3] and [27] for Calabi-Yau threefolds. However, one can generalize the formal definition to any polarized projective varieties.

Besides the motivations by pure mathematics for us to study Calabi-Yau manifolds of CM-type, there is a deep theorem in the string theory : a Calabi–Yau sigma model is completely characterized by its complex and Kähler moduli, i.e., a Calabi–Yau sigma model is determined by a mirror pair of Calabi–Yau manifolds (M, W) where M is a Calabi–Yau manifold and W is its mirror. Recently, Gukov–Vafa showed a deep and interesting phenomena :

Theorem 3.22 (cf. [11])**.** *A Calabi-Yau sigma model (M, W) is an RCFT (Rational Conformal Field Theory) if and only if M and W admit complex multiplication over a same number field.*

This result push us to study Calabi-Yau manifolds of CM-type throughly.

Example 3.23 (CM-type Calabi-Yau threefolds cf. [3])**.** Let h_1 and h_2 be rational polarized Hodge structures on V with weights w_1 and w_2. The Hodge structure $h_3 = h_1 \otimes h_2$ is CM if and only if h_1 and h_2 are CM . Let E_1, E_2, E_3 be three elliptic curves. The group $G_4 = \mathbb{Z}_2 \times \mathbb{Z}_2$ acts on $E_1 \times E_2 \times E_3$ by the involution on $E_1 \times E_2$ and by the identity on the third factor so that the action preserves the holomorphic 3-form on the product variety. The quotient variety $E_1 \times E_2 \times E_3/G_4$ has a Calabi–Yau threefold resolution X with Euler number 96 and $h^{2,1}(X) = 3$. Let h_i be the Hodge structure of $H^1(E_i, \mathbb{C})\,\forall i$. The Hodge structure of $H^3(X, \mathbb{C})$ is then $h_1 \otimes h_2 \otimes h_3$, i.e. the G_4- invariant part of $H^3(E_1 \times E_2 \times E_3, \mathbb{C})$. Hence X is CM if and only if all E_i are CM.

Now, let us indicate some open questions.

Let $f : \mathfrak{X} \to M$ be any smooth family of polarized Calabi-Yau manifolds over a quasi-projective manifold and $h : M \to D/\Gamma$ be its period map. It is known that $dh(T_M)$ is in the horizonal tangent space of the period domain. We have some efficient results as 3.17 for period domains of VHSs, but it is very difficult to find out necessary conditions for the family f such that the set of CM points is Zariski dense in M. Therefore, it is still far away knowing that whether the set of CM points is Zariski dense in the moduli space of polarized Calabi-Yau manifolds, or dense into a subvariety with dimension ≥ 1. For moduli spaces of higher dimension algebraic manifolds, we have the following significant result by Viehweg, and so the algebraic quasi-projective coarse moduli scheme exists for the set of Calabi-Yau manifolds with a fixed polarization.

Theorem 3.24 (cf. [23])**.** *Let h be a fixed polynomial of degree n with $h(\mathbb{Z}) \subset \mathbb{Z}$. Define a moduli functor*

$$\begin{aligned}\mathcal{M}_h(Y) := \; \{(f : \mathcal{X} \longrightarrow Y, \mathcal{L}) \mid \; & f \text{ flat, projective and } \mathcal{L} \text{ invertible, relatively ample} \\ & \text{over } Y, \text{ such that : for all } p \in Y(\mathbb{C})\ \mathcal{X}_p = f^{-1}(p) \text{ is a projective} \\ & \text{manifold with semi-ample canonical bundle and } \chi(\mathcal{L}|_{\mathcal{X}_p}) = h \qquad \}/\sim .\end{aligned}$$

Then, the moduli functor $\mathcal{M}_h$ is bounded by the Matsusaka Big theorem, and there exists a quasi-projective coarse moduli scheme M_h for $\mathcal{M}_h$, of finite type over $\mathbb{C}$. Moreover, if

$$\omega_\Gamma^\delta = \mathcal{O}_\Gamma \,\forall\, \Gamma \in \mathcal{M}_h(\mathbb{C})$$

for one integer $\delta > 0$, then for some $p > 0$ there exists an ample line bundle $\lambda^{(p)}$ on M_h such that $\phi_g^* \lambda^{(p)} = g_* \omega_{\mathcal{X}/Y}^{\delta \cdot p}$ for any family $(g : \mathcal{X} \to Y, \mathcal{L}) \in \mathcal{M}_h(Y)$ with moduli morphism $\phi_g : Y \to M_h$.

It is believed that the set of CM points is dense in a moduli space only if the moduli space itself is of the form G/K. For example, the moduli spaces of elliptic curves, polarized Abelian varieties and polarized $K3$ surfaces are of this type, so that we can predict such a moduli space will have a dense set of CM (RCFT) points. Unfortunately, moduli spaces of polarized Calabi–Yau manifolds may not be homogeneous spaces and the deformation space of a Calabi–Yau manifold is into the horizonal tangent space of the corresponding period domain; thus we now can say little about the distribution of CM points in the moduli space of Calabi–Yau manifolds.

For sufficiently large genus g ($g \geq 4$), it is conjectured by Coleman that there should be only a finite number of Riemann surfaces C of genus g admitting CM (cf. [6]), and in order that a (sub)family of varieties contain a dense set of CM points, the base should be a Shimura (sub)variety due to the André–Oort conjecture.

Conjecture 3.25. [André–Oort cf. [1, 2],[18]] Assume S is a *connected Shimura variety.* Let $Z \subset S$ be an irreducible algebraic subvariety. Then Z contains a Zariski dense set of CM points if and only if Z is a *Shimura subvariety* of S.

Remark. Moonen showes : Let Z be an irreducible subvariety of a *Shimura variety.* Z is a *Shimura subvariety* if and only if it contains a CM point and the completion of the local ring of any smooth point of Z is formally linear (cf. [13]).

The André-Oort conjecture is thus a motivation for us to find necessary conditions for a smooth family of Calabi-Yau manifolds with an open dense set of CM points in the base. However, moduli spaces of polarized Calabi-Yau manifolds are like homogeneous spaces. Hence, it is reasonable to consider the analogue of the André-Oort conjecture for moduli spaces of Calabi–Yau manifolds. We believe this is a meaningful question. Suppose that the conjecture holds and the set of CM points is open dense in a quasi-projective curve C in a moduli space M_h of Calabi–Yau manifolds. Then, C should be a Shimura curve and so the induced family $f : \mathcal{X} \to C$ should be rigid, because any Shimura curve has a model defined over a number field and then it can not move freely in the moduli space M_h. We have interest in studying the relation between the distribution of CM points in moduli spaces and the *rigidity* problem for Shafarevich problems (cf. [25],[31]).

Example 3.26 (cf. [25]). Recently, Viehweg–Zuo constructed a nontrivial family of Calabi-Yau manifolds such that the closed fibers are Calabi-Yau manifolds of CM-type over a Zariski dense set. Precisely, they obtained a family $g : \mathcal{Z} \to S$ of quintic hypersurfaces in $\mathbb{CP}^4$ such that S is finite dominant over a ball quotient (which is a Shimura variety) and S has a dense set of CM points. Furthermore, they got an important counterexample for the *rigidity* part of the Shafarevich problem by showing that there exists a product of moduli spaces of hypersurfaces of degree d in $\mathbb{P}^n$ and that this product can be embedded into the moduli space of hypersurfaces of degree d in $\mathbb{P}^N$ for some $N > n$.

We have interest in the following questions, and we hope to make progress in further.

Question 3.27. Let X be a Calabi–Yau n-fold of CM-type. Assume that X is fibered by $(n-1)$-dimensional Calabi–Yau varieties over $\mathbb{P}^1$. Is the set of CM points dense in $\mathbb{P}^1$? Even in the case $n = 3$, i.e., for $K3$ fibrations, this does not seem to be known.

Question 3.28. Fix a pair (C, S) where C is a smooth projective curve and $S \subset C$ is a finite subset. Let $f : \mathcal{X} \to C$ be a non-isotrivial family of Calabi–Yau varieties smooth over $C - S$ and $\mathfrak{E}$ be the set of CM points in $C \setminus S$. Does there exist a number N independent of f such that the family f would be rigid if $\#\mathfrak{E} > N$?

In the coming paper, we will focus studying on the analogue André-Oort problem for families of polarized Calabi-Yau varieties and construct examples.

4. Appendix : A Geometric Introduction to Shimura Curves

Denote

$$\mathbf{PSL}(2,\mathbb{R}) = \{z \mapsto T(z) = \frac{az+b}{cz+d} | \; ad-bc=1\}.$$

$T \in \mathbf{PSL}(2,\mathbb{R})$ is elliptic if $\mathrm{Tr}(T) := |a+d| < 2$, parabolic if $\mathrm{Tr}(T) = 2$ and hyperbolic otherwise. A *Fuchsian group* is defined to be a discrete subgroup of $\mathbf{PSL}(2,\mathbb{R})$. Let $\mathcal{H}$ be the upper half plane, or Poincaré disk.

Theorem 4.1. *A subgroup* Γ *of* $\mathbf{PSL}(2,\mathbb{R})$ *is a Fuchsian group if and only if it acts properly discontinuously on* $\mathcal{H}$.

Let $\mathbb{F}$ be a totally real number field of degree n and φ_i $(1 \le i \le n)$ be the n distinct embeddings of $\mathbb{F}$ into $\mathbb{R}$. We assume that $\varphi_1 = \mathrm{id}$.

Definition 4.2 (Archimedean ramification)**.** Let $A = (\frac{l,q}{\mathbb{F}})$ be a quaternion algebra over a totally real number field $\mathbb{F}$. A possesses *Archimedean ramification* if A is unramified at φ_1 and ramified at the other infinite places $\varphi_i (2 \le i \le n)$, i.e. there exist $\mathbb{R}$-isomorphisms ρ_i with

$$\rho_1 : A^{\varphi_1} \otimes \mathbb{R} \to M(2,\mathbb{R}),\ \rho_i : A^{\varphi_i} \otimes \mathbb{R} \to \mathbb{H}\ (2 \le i \le n).$$

Remark. $A = (\frac{l,q}{\mathbb{F}})$ is a subalgebra of $M(2,\mathbb{F}(\sqrt{l}))$, i.e. an $\mathbb{F}$-linear injective homomorphism $\phi: \; A \hookrightarrow M(2,\mathbb{F}(\sqrt{l}))$ satisfying

$$\phi(x) = g_x = \begin{bmatrix} x_0 + x_1\sqrt{l} & x_2 + x_3\sqrt{l} \\ q(x_2 + x_3\sqrt{l}) & x_0 - x_1\sqrt{l} \end{bmatrix}.$$

Theorem 4.3. *Let A be a quaternion algebra over a totally real number field $\mathbb{F} \neq \mathbb{Q}$. Assume $\mathbb{F}$ possesses Archimedean ramification. Then A is a division algebra.*

Example 4.4. (Criteria for division algebra). Let A be a quaternion algebra and Nrd be the norm function on A.

(i) A is a division algebra $\Longleftrightarrow$ $\mathrm{Nrd}(x) = 0$ only at $x = 0$.
(ii) $A = (\frac{a,b}{\mathbb{F}})$ is a division algebra if A is not isomorphic to $M(2,\mathbb{F})$.

Let $\mathcal{O}$ be an order of A and $\mathcal{O}^1$ be the group of units with reduced norm 1 in $\mathcal{O}$; then $\rho_1(\mathcal{O}^1)$ is a subgroup of $\mathbf{SL}(2,\mathbb{R})$ and $\Gamma(A,\mathcal{O}) = \rho_1(\mathcal{O}^1)/\{+1,-1\}$ is a subgroup of $\mathbf{PSL}(2,\mathbb{R})$. One has

$$\mathrm{Nrd}(x) = \det(\rho_1(x)),\ \mathrm{Trd}(x) = \mathrm{Tr}(\rho_1(x)),$$

$$\varphi_i(\mathrm{Nrd}(x)) = \mathrm{Nrd}_{\mathbb{H}}(\rho_i(x)),\ \varphi_i(\mathrm{Trd}(x)) = \mathrm{Trd}_{\mathbb{H}}(\rho_i(x)).$$

Proposition 4.5. $\Gamma(A,\mathcal{O})$ *is a Fuchsian group of the first kind, i.e., the volume* $\mu(\mathcal{H}/\Gamma(A,\mathcal{O}))$ *is finite.*

Definition 4.6. One says that a Fuchsian group Γ is *derived from a quaternion algebra* A if Γ is a subgroup of $\Gamma(A,\mathcal{O})$ of finite index. Moreover, the Fuchsian group Γ is called *arithmetic* if Γ is commensurable with $\Gamma(A,\mathcal{O})$.

Theorem 4.7 (Takeuchi cf. [22]). *Let* Γ *be a Fuchsian group. Assume* Γ *is of the first kind. Then* Γ *is* derived from a quaternion algebra A *over a totally real number field* $\mathbb{F}$ *if and only if* Γ *satisfies the following conditions :*

1. *Let* k_1 *be the field* $\mathbb{Q}(\mathrm{Tr}(T) \mid T \in \Gamma)$ *generated by the set* $\mathrm{Tr}(T)$ *over* $\mathbb{Q}$. *Then* k_1 *is an algebraic number field of finite degree, and* $\mathrm{Tr}(T)$ *is contained in* $\mathcal{O}_{k_1}$ *the ring of integers of* k_1.
2. *For any embedding* φ *of* k_1 *into* $\mathbb{C}$ *such that* $\varphi \neq \mathrm{Id}$, $\varphi(\mathrm{Tr}(\Gamma))$ *is bounded in* $\mathbb{C}$.

Remarks. (Results related to the arithmeticity of Fuchsian groups).

1. Γ is *arithmetic* if and only if $\{T^2 | T \in \Gamma\}$ is *derived from a quaternion algebra.*
2. Let $A(\Gamma)$ be a vector space spanned by Γ over k_1 in $M(2, \mathbb{R})$; then $A(\Gamma)$ is a quaternion algebra over k_1. Moreover, Γ satisfies the condition (1) and the submodule $\mathcal{O}(\Gamma)$ of $A(\Gamma)$ spanned by Γ over $\mathcal{O}_{k_1}$ is an order of $A(\Gamma)$.
3. Let Γ be a Fuchsian group with $\mu(\mathcal{H}/\Gamma) < \infty$. Assume Γ satisfies the conditions (1) and (2). Then,
   a) k_1 is a totally real number field. Moreover, if φ is not the identity, then $\varphi(\mathrm{Tr}(T))$ is contained in $[-2, 2]$;
   b) $A(\Gamma)$ satisfies the Archimedean ramified condition.

Example 4.8. (Quaternion algebras over $\mathbb{Q}$).

(i) Let Γ be a Fuchsian group with $\mu(\mathcal{H}/\Gamma) < \infty$. Then Γ is *derived from a quaternion algebra* A over $\mathbb{Q}$ if and only if $\mathrm{Tr}(T) \in \mathbb{Z}\ \forall T \in \Gamma$.

(ii) If $T \in \mathbf{PSL}(2, \mathbb{R})$ is elliptic and $\mathrm{Tr}(T) \in \mathbb{Z}$, T is then of order 2 or 3.

(iii) By (i) and (ii), a Fuchsian group *derived from a quaternion algebra* over $\mathbb{Q}$ only has order 2 or 3 elliptic elements.

Theorem 4.9 (Shimura curve cf. [20]). *Suppose a Fuchsian group* Γ *is derived from a division quaternion algebra. Then* $\mathcal{H}/\Gamma$ *is a projective curve, and it is called Shimura curve.*

Example 4.10. Let A be a quaternion algebra over a totally real number field $\mathbb{F} \neq \mathbb{Q}$. Assume that A satisfies the Archimedean ramified condition. Then A is a division algebra, and thus $\mathcal{H}/\Gamma(A, \mathcal{O})$ is compact.

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