Notes on Shimura curves

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July 10, 2005

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1 Shimura curves

1.1 Arithmetic of quaternion algebras

Let B be a quaternion algebra over a field F i.e. B is a central simple algebra of dimension 4 over F. Let $'$ denote the main involution of B. Thus

$$tr(\alpha) = \alpha + \alpha', Nm(\alpha) = \alpha\alpha'$$

where tr and Nm denote the reduced trace and reduced norm.

Lemma 1.1 *Every inner automorphism of B commutes with the main involution*

Proof: Indeed $\alpha^{-1}\beta'\alpha = (\alpha^{-1}\beta\alpha)'$ since $\alpha\alpha' \in F$ and hence commutes with β'. □

Lemma 1.2 *Let $\rho \in B, Nm(\rho) \neq 0$. Let $\alpha^* = \rho^{-1}\alpha'\rho$. Then $\alpha \to \alpha^*$ is an involution of B if and only if $\rho^2 \in F$. In fact, every involution of B is obtained in this way.*

Proof: By the previous lemma, $(\alpha^*)^* = \rho^{-2}\alpha\rho^2$. Thus $(\alpha^*)^* = \alpha$ for all α $\iff$ $\rho^2 \in F$. Conversely, $\alpha \to (\alpha^*)'$ is an automorphism of B that preserves the center, hence is inner. Thus there exists an invertible element $\rho \in B$ such that $(\alpha^*)' = \rho^{-1}\alpha\rho$ for all α. Then $\alpha^* = (\rho^{-1}\alpha\rho)^{-1} = \rho^{-1}\alpha'\rho$. $\square$

Let us now restrict to the case $F = \mathbb{Q}$. B is said to be indefinite if it satisfies any one of the following equivalent conditions:
(i) B contains a real quadratic field.
(ii) $B \otimes \mathbb{R} \simeq M_2(\mathbb{R})$.
(iii) The norm form on B is an *indefinite* quadratic form $/\mathbb{Q}$.

Of particular interest are involutions that are *positive*, i.e. that satisfy $tr(\alpha\alpha^*) > 0$ for $\alpha \neq 0$. (Motivation: Later we will be interested in abelian varieties A with multiplication by B and polarizations on A such that the associated Rosati involution preserves B. It must then restrict to a positive involution on B.)

Lemma 1.3 *Let B be an indefinite quaternion algebra over $\mathbb{Q}$, and $*$ an involution corresponding to an element $\rho \in B$ as in the previous lemma. Then $*$ is a positive involution $\iff$ $\rho^2 < 0$.*

Proof: Suppose $*$ is a positive involution. First note that if $\rho \in \mathbb{Q}$ then $\alpha^* = \alpha'$ and $tr(\alpha\alpha^*) = tr(\alpha\alpha') = 2Nm(\alpha)$. But this is impossible since $tr(\alpha\alpha^*) > 0$ on the one hand while B is indefinite on the other. Thus $\rho \notin \mathbb{Q}$. Since $\rho^2 \in \mathbb{Q}$, $\rho' = -\rho$ and $\rho\rho^* = -\rho^2$. Now $tr(\rho\rho^*) > 0 \Rightarrow \rho^2 < 0$. Conversely, suppose $\rho^2 < 0$. Let $K = \mathbb{Q}(\rho)$, so that K is an imaginary quadratic field. Now we can pick $\tau \in B$ such that $\tau^2 \in \mathbb{Q}$ and $\tau^{-1}\rho\tau = -\rho$. Thus $B = K + K\tau$ and $\tau^{-1}a\tau = a' = \bar{a}$ for $a \in K$. Now, for $a, b \in K$, $\alpha = a + b\tau$,

$$N\alpha = (a + b\tau)(a + b\tau)' = Nm(a) - Nm(b)\tau^2$$

Since B is indefinite, $\tau^2 > 0$. Now, $tr(\alpha\alpha^*) = tr(\alpha\rho^{-1}\alpha'\rho) = 2(Nm(a) + Nm(b)\tau^2) > 0$, so $*$ is positive, as required. $\square$

Definition 1.4 *An* **order** *in B is a subring $\mathfrak{o}$ which is a free $\mathbb{Z}$ module of rank 4. If $\mathfrak{o}$ is an order, a left (resp. right) $\mathfrak{o}$-ideal is a $\mathbb{Z}$-lattice $\mathfrak{a}$ in B such that $\mathfrak{o}\mathfrak{a} = \mathfrak{a}$ (resp. $\mathfrak{a}\mathfrak{o} = \mathfrak{a}$.)*

An order is said to be maximal if it is not contained strictly in any other order.

Proposition 1.5 *(i) All maximal orders are conjugate in B.*
(ii) Let $\mathfrak{o}$ be a maximal order, and $\mathfrak{a}$ a left (resp. right) $\mathfrak{o}$-ideal. Then $\mathfrak{a}$ is a principal $\mathfrak{o}$-ideal. i.e. $\mathfrak{a} = \mathfrak{o}\alpha$ (resp. $\mathfrak{a} = \alpha\mathfrak{o}$) for some $\alpha \in B$.
(iii) Let $\mathfrak{o}$ be a maximal order. Then there exists an element $\gamma \in \mathfrak{o}$, such that $Nm(\gamma) = -1$.

1.2 Abelian surfaces with multiplication by B

With these preliminaries we begin the study of abelian surfaces with multiplication by B. Let us fix, once and for all, an isomorphism

$$\Phi_\infty : B \otimes \mathbb{R} \simeq M_2(\mathbb{R})$$

Let A be an abelian surface, $i : B \hookrightarrow End^0(A)$ be an embedding and

$$0 \to \Lambda \to T_e(A) \to A \to 0$$

be the canonical complex uniformization of A. Now B acts faithfully on the tangent space $T_e(A)$ which is a 2-dimensional $\mathbb{C}$-vector space. By the Artin-Wedderburn theorem, $B \otimes \mathbb{C}$ has a unique simple module V up to isomorphism, which is of rank 2. Hence the representation of B on $T_e(A)$ is equivalent to Φ_∞ (tensored up to $\mathbb{C}$.) In other words, we may pick a basis for $T_e(A)$ with respect to which the matrix of $\alpha \in B$ is $\Phi_\infty(\alpha) \in M_2(\mathbb{R})$. Let $\mathfrak{o} = i^{-1}(End(B))$, so that $\mathfrak{o}$ is an order in B. Now $\Phi_\infty(\mathfrak{o})\Lambda \subseteq \Lambda$. Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \Lambda$ and consider the injective map

$$\theta : \mathfrak{o} \to \Lambda, \theta(\alpha) = \Phi_\infty(\alpha)x$$

Set $\Lambda' = \theta(\mathfrak{o})$. Also say that a vector $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ satisfies condition (N) if

$$(N) : y_2 \neq 0 \text{ and } Im(y_1/y_2) \neq 0$$

Lemma 1.6 *Λ' is a discrete subgroup of $\mathbb{C}^2$ of rank 4 if and only if x satisfies condition (N).*

Proof: Let $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ be a basis of $\mathfrak{o}$ over $\mathbb{Z}$. Set $\beta_i = \Phi_\infty(\alpha_i)$. Suppose x satisfies condition (N). We claim that the vectors $\beta_i x$ are linearly independent over $\mathbb{R}$, which will imply that Λ' generates $\mathbb{C}^2$ over $\mathbb{R}$. Indeed, suppose $\sum_i r_i\beta_i x = 0$ for some $r_i \in \mathbb{R}$. Then $Cx = 0$ where $C = \sum_i r_i\beta_i \in M_2(\mathbb{R})$. But now the conditions $x_2 \neq 0, Im(x_1/x_2) \neq 0$ force $C = 0$. Now the vectors β_i are certainly linearly independent over $\mathbb{R}$ (in fact even over $\mathbb{C}$) hence each $r_i = 0$, as required.

Conversely, suppose either $x_2 = 0$ or $x_2 \neq 0$ and $Im(x_1/x_2) = 0$. Then we can find a non-zero matrix $R \in M_2(\mathbb{R})$ satisfying $Rx = 0$. Expand R in terms of the β_i : $R = \sum_i r_i\beta_i$. Then $\sum_i r_i\beta_i x = 0$, whence the vectors $\beta_i x$ are not $\mathbb{R}$-linearly independent. Thus $\mathbb{R}\Lambda' \neq \mathbb{C}^2$ in this case. □

Of course, since $\mathbb{R}\Lambda = \mathbb{C}^2$, we can pick a vector $x \in \Lambda$ satisfying $x_2 \neq 0$ and $Im(x_1/x_2) \neq 0$. Then Λ', being a lattice, has finite index in Λ. Pick a positive integer m such that $m\Lambda \subseteq \Lambda'$. Now define a mapping

$$\theta' : \Lambda \to \mathfrak{o}, \theta'(y) = \theta^{-1}(my)$$

Let $\mathfrak{m}$ denote the image of this map. Then $\mathfrak{m}$ is a left ideal in $\mathfrak{o}$. Further, if $\alpha\mathfrak{m} \subseteq \mathfrak{m}$ for some $\alpha \in B$, then $\Phi_\infty(\alpha)\Lambda \subseteq \Lambda$, hence $\alpha \in \mathfrak{o}$. Thus $\mathfrak{m}$ is a *proper* left $\mathfrak{o}$-ideal and $\theta(\mathfrak{m}) = m\Lambda$ i.e. $\Lambda = \{\Phi_\infty(\alpha)y | \alpha \in \mathfrak{m}\}$ where $y = \frac{1}{m}x$. We summarize these in the following proposition.

Proposition 1.7 *Suppose A is an abelian surface and $i : B \hookrightarrow End^0(A)$ is an embedding. Let $\mathfrak{o} = i^{-1}(End(B))$. Then $\mathfrak{o}$ is an order in B. Also there exists a proper left $\mathfrak{o}$-ideal $\mathfrak{m}$, a vector $y \in \mathbb{C}^2$ satisfying condition (N) and a complex analytic isomorphism*

$$\mathbb{C}^2/\Phi_\infty(\mathfrak{m})y \simeq A$$

commuting with the action of B (by Φ_∞ on the left and by i on the right.)

In fact the converse is also true, namely every CM torus of the form $\mathbb{C}^2/\Phi_\infty(\mathfrak{m})y$ with y satisfying condition (N) is an abelian variety. But to prove that we will need to construct a Riemann form on $\mathbb{C}^2/\Phi_\infty(\mathfrak{m})$. Thus we begin by studying polarizations on A.

Suppose $\lambda : A \to \hat{A}$ be a polarization. We assume that the associated Rosati involution preserves $i(B)$. Thus it restricts to an involution $*$ on B. Note that $Tr(\alpha) = 2tr(\alpha)$ where Tr denotes the trace of α on the l-adic Tate module (or rational homology), and tr denotes the reduced trace in B. Since the Rosati involution is positive, it follows that $*$ is a positive involution on B. By a previous proposition, there exists an element $\rho \in B$, $\rho^2 \in \mathbb{Q}$, $\rho^2 < 0$, such that

$$\alpha^* = \rho^{-1}\alpha'\rho$$

where $\alpha \to \alpha'$ is the main involution of B. Set

$$E_0(\alpha, \beta) = E(\Phi_\infty(\alpha)y, \Phi_\infty(\beta)y)$$

where E is the Riemann form associated to λ. Thus E_0 is integer valued on $\mathfrak{m}$. Since $\xi \to E_0(\xi, 1)$ is a $\mathbb{Q}$-linear map of B into $\mathbb{Q}$, there exists a unique element $\tau \in B$ such that $E_0(\xi, 1) = tr(\tau\xi)$.

Proposition 1.8

$$\tau = c\rho$$

for some $c \in \mathbb{Q}$.

Proof: Notice that

$$E_0(\alpha, \beta) = E_0(\beta^*\alpha, 1) = tr(\tau\beta^*\alpha) = tr(\tau\rho^{-1}\beta'\rho\alpha)$$

Since E is skew-symmetric,

$$\begin{aligned} tr(-\tau\beta) &= E_0(-\beta,1) = E_0(1,\beta) = tr(\tau\rho^{-1}\beta'\rho) \\ &= tr(\rho^{-1}\beta\rho\tau')(\text{ since } tr(ab') = tr(ba')) \\ &= tr(\rho\tau'\rho^{-1}\beta) \end{aligned}$$

Thus $\rho\tau'\rho^{-1} = -\tau$, hence $(\rho^{-1}\tau)' = -\tau'\rho^{-1} = \rho^{-1}\tau$, whence $\rho^{-1}\tau \in \mathbb{Q}$, as required. $\square$

This yields the following

Proposition 1.9 *In the setting of Prop. 1.7, any Riemann form on* $\mathbb{C}^2/\Phi_\infty(\mathfrak{m})y$ *is of the form*

$$E(\Phi_\infty(\alpha)y, \Phi_\infty(\beta)y) = ctr(\rho\alpha\beta')$$

for all $\alpha, \beta \in B$.

Conversely, given an isomorphism $\Phi_\infty : B \otimes \mathbb{R} \to M_2(\mathbb{R})$, a lattice $\mathfrak{m}$ in $\mathfrak{o}$, and a vector $y \in \mathbb{C}^2$ satisfying condition (N), set

$$\Lambda = \{\Phi_\infty(\alpha)y | \alpha \in \mathfrak{m}\}$$

From a previous proposition, Λ is a lattice in $\mathbb{C}^2$. We shall now show that the complex torus $\mathbb{C}^2/\Lambda$ is an abelian variety. Pick $\rho \in B$, with $\rho^2 \in \mathbb{Q}$ and $\rho^2 < 0$. Consider the form

$$E(\Phi_\infty(\alpha)y, \Phi_\infty(\beta)y) = c \cdot tr(\rho\alpha\beta') \tag{1}$$

for a non-zero rational number c. Note that this defines E on $\mathbb{C}^2 \times \mathbb{C}^2$ since $\alpha \mapsto \Phi_\infty(\alpha)y$ gives an isomorphism $\Phi_\infty(B_\mathbb{R}) \simeq \mathbb{C}^2$.

Proposition 1.10 *For c a suitable integer, (1) gives a Riemann form on* $\mathbb{C}^2/\Lambda$.

Proof: We need to check that E is skew-symmetric, integer valued on Λ and that the associated Hermitian form is positive definite (equivalently that the $\mathbb{R}$-bilinear form $(y_1, y_2) \mapsto E(y_1, iy_2) > 0$ is symmetric and positive definite.) First note that there is a unique $\mu \in B_\mathbb{R}$ such that

$$iy = \Phi_\infty(\mu)y$$

We must have $\mu^2 = -1$ and $\mu' = -\mu$. Suppose $\rho^2 = -s$ and set $\rho_1 = \rho/\sqrt{s}$. Then ${\rho_1}^2 = -1$ and $tr(\rho_1) = 0$. By the Skolem-Noether theorem, there exists $\gamma \in GL_2(\mathbb{R})$ such that

$$\mu = \gamma\rho_1\gamma^{-1}$$

Now

$$E(\Phi_\infty(\alpha)y, i\Phi_\infty(\beta)y) = c \cdot tr(\rho\alpha\mu'\beta')$$

and

$$tr(\rho\alpha\mu'\beta') = tr(\beta\mu\alpha'\rho') = tr(\rho'\beta\mu\alpha) = tr(\rho\beta\mu'\alpha)$$

whence $E(y_1, iy_2)$ is symmetric. Next we compute

$$\begin{aligned} E(\Phi_\infty(\alpha)y, i\Phi_\infty(\alpha)y) &= c \cdot tr(\rho\alpha\mu'\alpha') \\ &= -c \cdot tr(\rho\alpha\gamma\rho_1\gamma^{-1}\alpha') \\ &= -\frac{c}{\sqrt{s}} \cdot tr(\alpha\gamma'^{-1}\rho'\gamma'\alpha'\rho')(\text{ using } tr(x) = tr(x')) \\ &= \frac{c}{\sqrt{s}} Nm(\gamma)^{-1} \cdot s \cdot tr(\alpha\gamma\rho^{-1}\gamma'\alpha'\rho) \\ &= c\sqrt{s}Nm(\gamma)^{-1} \cdot tr((\alpha\gamma)(\alpha\gamma)^*) \end{aligned}$$

Now all we need to do is pick c a sufficiently divisible integer so that $tr(\rho\alpha\beta')$ is integer valued on $\mathfrak{m}$ and the sign of c equal to the sign of $Nm(\gamma)$. $\square$

NOTE: If we replace ρ by $-\rho$, $Nm(\gamma)$ will change sign and so will c. Thus by picking ρ such that $\gamma \in GL_2(\mathbb{R})^+$, we may ensure that $c > 0$.

Finally, note that that if we set $\mathfrak{o} = \{x \in B, x\mathfrak{m} \subseteq \mathfrak{m}\}$, $\Phi_\infty(\alpha)$ preserves Λ and gives an endomorphism of $A = \mathbb{C}^2/\Lambda$. Denoting this endomorphism by $i(\alpha)$, we get an embedding $i : \mathfrak{o} \hookrightarrow End(A)$. Since

$$tr(\rho\alpha(\xi\beta)') = tr(\rho\alpha\beta'\xi') = tr(\xi'\rho\alpha\beta') = tr(\rho(\rho^{-1}\xi'\rho)\alpha\beta') = tr(\rho\xi^*\alpha\beta)$$

we see that the Rosati involution restricts to $*$ on B.

1.3 Families of abelian surfaces

Let us specialize the discussion of the previous section to the case $\mathfrak{o}$= the maximal order in B. Since every left $\mathfrak{o}$-ideal $\mathfrak{m}$ is principal, we may pick μ such that $\mathfrak{m} = \mathfrak{o}\mu$. Then

$$\Lambda = \{\Phi_\infty(\alpha)\tilde{y} | \alpha \in \mathfrak{o}\}$$

where $\tilde{y} = \Phi_\infty(\mu)y$. Set $\tilde{y} = \begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{bmatrix}$. We may assume that $\tilde{y}_2 \neq 0, Im(\tilde{y}_1/\tilde{y}_2) > 0$, replacing $\tilde{y}$ if necessary with $\Phi_\infty(\gamma)\tilde{y}$ for some $\gamma \in \mathfrak{o}^\times$ with $Nm(\gamma) = -1$. Now put $\tau = \tilde{y}_1/\tilde{y}_2$ and set

$$\Lambda_\tau = \{\Phi_\infty(\alpha) \begin{bmatrix} \tau \\ 1 \end{bmatrix} | \alpha \in \mathfrak{o}\}$$

The linear map $\mathbb{C}^2 \to \mathbb{C}^2, x \mapsto \tilde{y}_2^{-1} x$ gives an isomorphism of complex tori

$$\frac{\mathbb{C}^2}{\Phi_\infty(\mathfrak{o}) \begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{bmatrix}} \simeq \frac{\mathbb{C}^2}{\Phi_\infty(\mathfrak{o}) \begin{bmatrix} \tau \\ 1 \end{bmatrix}} = \frac{\mathbb{C}^2}{\Lambda_\tau}$$

that commutes with the $\mathfrak{o}$ action.

Now, we let τ vary in the upper half plane $\mathfrak{H}$. In this way we obtain an analytic family of Abelian varieties A_τ on $\mathfrak{H}$. More precisely, one takes the quotient $\mathcal{A}$ of $\mathbb{C}^2 \times \mathfrak{H}$ for the equivalence relation

$$(z_1, \tau_1) \equiv (z_2, \tau_2) \iff \tau_1 = \tau_2, \text{ and } z_1 - z_2 \in \Lambda_{\tau_1} = \Lambda_{\tau_2}$$

and shows the following:

(i) $\mathcal{A}$ admits a structure of a complex analytic manifold such that the projection map to $\mathfrak{H}$ and the "zero-section" $\tau \mapsto [(0, \tau)]$ are analytic maps, the former being a proper submersion.

(ii) The fibre over τ, $\mathcal{A}_\tau$ is identified with $\mathbb{C}^2/\Lambda_\tau$ and thus has the structure of a complex torus; the group operations on $\mathcal{A}_\tau$ vary analytically with τ.

(iii) The action of $\alpha \in \mathfrak{o}$ on $\mathcal{A}_\tau$ varies analytically with τ.

One can do more: fixing a ρ and c as in the previous section, one can construct a relative polarization $\mathcal{L} \to \mathcal{A} \times_{\mathfrak{H}} \mathcal{A}$, such that on the fibre over τ, $\mathcal{L}$ restricts to a polarization $\mathcal{L}_\tau$ which via the identification $\mathcal{A}_\tau = \mathbb{C}^2/\Lambda_\tau$ corresponds to the polarization on the latter torus with associated Riemann form

$$E\left(\Phi_\infty(\alpha) \begin{bmatrix} \tau \\ 1 \end{bmatrix}, \Phi_\infty(\beta) \begin{bmatrix} \tau \\ 1 \end{bmatrix}\right) = c \cdot tr(\rho\alpha\beta')$$

Of particular interest is the case $\rho^2 = -D$. Then taking $c = \frac{1}{D}$ gives a principal polarization on $\mathcal{A}_\tau$. Thus each fibre $(\mathcal{A}_\tau, \mathcal{L}_\tau, i_\tau)$ has a structure of an abelian surface with principal polarization and endomorphisms such that the Rosait involution restricts to the involution $*$. There is the natural notion of equivalence or isomorphism of two such structures.

Proposition 1.11 *Let $\tau_1, \tau_2 \in \mathfrak{H}$. Then $(\mathcal{A}_{\tau_1}, \mathcal{L}_{\tau_1}, i_{\tau_1})$ and $(\mathcal{A}_{\tau_2}, \mathcal{L}_{\tau_2}, i_{\tau_2})$ are isomorphic $\iff$ $\tau_1 = \gamma \cdot \tau_2$ for some $\gamma \in \mathfrak{o}^1$.*

Proof: Suppose $\lambda : \mathcal{A}_{\tau_1} \to \mathcal{A}_{\tau_2}$ is an isomorphism. λ corresponds to a linear mapping from $\mathbb{C}^2$ to $\mathbb{C}^2$ sending Λ_{τ_1} isomorphically onto λ_{τ_2}. Since λ commutes with the action of $\mathfrak{o}$ (which is irreducible), λ must be a scalar. Set $\mathfrak{r}_1 = \begin{bmatrix} \tau_1 \\ 1 \end{bmatrix}, \mathfrak{r}_2 = \begin{bmatrix} \tau_2 \\ 1 \end{bmatrix}$. Since $\lambda\mathfrak{r}_1 \in \Lambda_{\tau_2}$, we have $\lambda\mathfrak{r}_1 = \Phi_\infty(\gamma)\mathfrak{r}_2$ for some $\gamma \in \mathfrak{o}$. Thus

$$\Phi_\infty(\mathfrak{o})\mathfrak{r}_2 = \Lambda_{\tau_2} = \lambda\Lambda_{\tau_1} = \Phi_\infty(\mathfrak{o}\gamma)\mathfrak{r}_2$$

which implies that $\mathfrak{o} = \mathfrak{o}\gamma$, whence $\gamma \in \mathfrak{o}^\times$. Clearly, $\tau_1 = \gamma \cdot \tau_2$, hence $Nm(\gamma) > 0$. This implies that $\gamma \in \mathfrak{o}^1$. Conversely, suppose $\tau_1 = \gamma \cdot \tau_2$ with $\gamma \in \mathfrak{o}^1$. If $\Phi_\infty(\gamma) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, set $\lambda = c\tau_2 + d$, so that $\lambda\tau_1 = a\tau_1 + b$. Then λ gives an isomorphism $\mathcal{A}_{\tau_1} \simeq \mathcal{A}_{\tau_2}$ commuting with the action of $\mathfrak{o}$. The polarization $\mathcal{L}_2$ must correspond via this isomorphism to a principal polarization on $\mathcal{A}_{\tau_1}$ whose associated Rosati involution restricts to $*$ on B. But there is a unique such polarization, namely $\mathcal{L}_1$. Thus λ indeed gives an isomorphism between $(\mathcal{A}_{\tau_1}, \mathcal{L}_{\tau_1}, i_{\tau_1})$ and $(\mathcal{A}_{\tau_2}, \mathcal{L}_{\tau_2}, i_{\tau_2})$. □

Now let $\Gamma = \mathfrak{o}^1$. Using the previous proposition, one may show that the natural action of Γ on $\mathfrak{H}$ (via Φ_∞) extends to an action of Γ on $\mathcal{A}$; taking the quotient gives us a family of abelian surfaces with PE structure over $X = \mathfrak{H}/\Gamma$. We denote this family by $\mathcal{A}_X \to X$. X is a compact (if $B \neq M_2(\mathbb{Q})$) complex manifold of dimension 1, hence has the unique structure of an algebraic curve $/\mathbb{C}$. It is a theorem of Shimura that X admits a *canonical model* over $\mathbb{Q}$. We shall not prove this fact - instead we explain how one might construct canonical models for some related Shimura curves, namely those with added level structure.

Let N be an integer coprime to D. Pick a primitive Nth root of unity ζ. This is equivalent to fixing an isomorphism $\mu_N \simeq \mathbb{Z}/N\mathbb{Z}$. Now define a level-N structure on $(A, i, \mathcal{L})$ to be an isomorphism

$$\phi : \mathfrak{o}/N\mathfrak{o} \simeq A[N] \tag{2}$$

commuting with the action of $\mathfrak{o}$ and such that the Weil pairing on $A[N] \times A[N] \to \mu_N \simeq \mathbb{Z}/N\mathbb{Z}$ associated to $\mathcal{L}$ goes over to the pairing $(x, y) \mapsto \zeta^{E(\alpha,\beta)}$, where $E(\alpha, \beta) = \frac{1}{D} tr(\rho\alpha\beta')$. A level N-structure on an analytically varying family is the choice of a level N-structure ϕ_s on each fibre varying analytically with the parameters s on the base space. The family $\mathcal{A} \to \mathfrak{H}$ carries a canonical level N-structure

$$\phi_\tau : \mathfrak{o}/N\mathfrak{o} \simeq \mathcal{A}_\tau[N] = \Phi_\infty(\frac{1}{N}\mathfrak{o})\mathfrak{r}_\tau / \Phi_\infty(\mathfrak{o})\mathfrak{r}_\tau$$

given by $\phi_\tau(1) = \frac{1}{N}\mathfrak{r}_\tau = \begin{bmatrix} \tau/N \\ 1/N \end{bmatrix}$. Here $\mathfrak{r}_\tau = \begin{bmatrix} \tau \\ 1 \end{bmatrix}$. It is easy to check that $(\mathcal{A}_{\tau_j}, i_{\tau_j}, \mathcal{L}_{\tau_j}, \phi_{\tau_j})$ for $j = 1, 2$ are isomorphic structures if and only if $\tau_1 = \gamma \cdot \tau_2$ for some $\gamma \in \Gamma_N$, where

$$\Gamma_N = \{\gamma \in \mathfrak{o}^1, \gamma \equiv 1 \mod N\mathfrak{o}\}$$

Thus taking the quotient by Γ_N gives a PEL family $\mathcal{A}_N \to X_N$ where $X_N = \mathfrak{H}/\Gamma_N$.

Our next goal is to construct a model for X_N over $\mathbb{Q}(\zeta)$. Let W denote the symplectic Shimura variety classifying abelian surfaces equipped with principal polarization, and level N-structure (defined as in (2) above but without the action of $\mathfrak{o}$.) We get then a canonical map

$$f : X_N \to W$$

coming from forgetting the $\mathfrak{o}$-action on the fibres of X_N. It is not hard to show that for N large enough, this map is generically injective. If $\tilde{X}_N$ denotes its image, X_N is the normalisation of $\tilde{X}_N$. Suppose $x = (A, i, \mathcal{L}, \phi)$ is a point on X_N, and let $y = (A, \mathcal{L}, \phi)$ be the image $f(x)$ of x in $\tilde{X}_N$. Let $\sigma \in Aut(\mathbb{C}/\mathbb{Q}(\zeta_n))$. Then $y^\sigma = (A^\sigma, \mathcal{L}^\sigma, \phi^\sigma)$ is the image $f(x^\sigma)$ of x^σ. This shows that $\tilde{X}_N$ is preserved by $Aut(\mathbb{C}/\mathbb{Q}(\zeta))$ - further, if x and y are CM points, it is clear that y^σ is another CM point on $\tilde{X}_N$. Now setting $T =$ the set of CM points on $\tilde{X}_N$ and applying the descent criterion Thm. 6.2.10 of [1] we see that $\tilde{X}_N$ and hence X_N descend to curves $\tilde{Y}$ and Y respectively over $\mathbb{Q}(\zeta)$. (Note that CM points in W are known to be algebraic.) Further, for any $t \in T_K \subset T$ (T_K being the CM points corresponding to any quartic CM type (K', Φ) with reflex field $= K$ imaginary quadratic), the main theorem of CM describes the action of $Aut(\mathbb{C}/K)$ (and hence of $Aut(\mathbb{C}/K \cdot \mathbb{Q}(\zeta))$) on t. By Thm. 6.2.9 and Thm. 6.2.11 of [1], the models $\tilde{Y}$ and Y are uniquely characterised by this last property. The curve Y is the *canonical* model of $X_N/\mathbb{Q}(\zeta)$.

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