LOCALIZATION AND DUALITY

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Abstract. We describe some recent results motivated by physical dualities and their proofs by localization methods. These include some closed formulas for some Hodge integrals, calculations of Gromov-Witten invariants in arbitrary degree and arbitrary genus for some open Calabi-Yau threefolds, and their identifications with invariants of framed moduli spaces of torsion free sheaves.

1. Introduction

Duality in the physics literature means the equivalence of two different physical theories. In a quantum theory, one is often concerned with the partition function defined by a Feynman integral over some configuration space which is usually infinite dimensional, hence is often not mathematically well-defined. However often the intrinsic symmetries of the theory enables one to reduce the partition function to an integral over a finite dimensional moduli space. This is where rigorous mathematical definitions can be made and concrete calculations can be carried out. When the physical duality involves two such theories, sometimes it is possible to mathematically verify the answers. Since the relevant different physical theories may involve different branches of mathematics to set up, the duality of the physical theories then suggests sometimes surprising connections between the corresponding mathematical branches.

We will report on some recent mathematical results that verify some physical conjectures predicted by duality. Our main technical tool is the localization techniques applied to various moduli spaces. Localization is a technique in geometry that computes global invariants by local contributions. The global invariants are often expressed as integrals over the whole space, but when there is symmetry, one can often compute them by summing up contributions from fixed point components of the symmetry. The idea can be traced back to the Poincaré-Hopf theorem on vector fields, the Lefschetz fixed point formula and Chern's famous proof of the Gauss-Bonnet theorem. It has been developed in both differential geometry and algebraic geometry into a powerful method.

In general localization applied to moduli spaces leads us naturally to combinatorial objects such as partitions of numbers and summations over labeled graphs. It is exactly this connection that relates the geometry and topology of moduli spaces to the rich mathematics of representations and integrable systems.

Since we will present results that have strong physics backgrounds, it might be helpful to list below some physical theories and their corresponding mathematical counterparts. In dimension 2:

2D quantum gravity	Deligne-Mumford moduli spaces
Topological 2D Yang-Mills theory	Moduli spaces of flat connections on Riemann surfaces
Wess-Zumino-Witten model	Representations of Kac-Moody algebras

In dimension 3:

Chern-Simons theory	Jones/HOMFLY polynomials
Witten 3-manifold invariants	Reshetikhin-Turaev invariants

In dimension 4:

Topological Yang-Mills theory	Donaldson theory
Instantons in 4D YM	Self-dual connections
Noncommutative 4D instantons	Torsionfree sheaves

In dimension 6, on a Calabi-Yau 3-fold,

Type IIA topological string theory	Gromov-Witten theory
Type IIB topological string theory	Variation of Hodge structures

We will present results that establish the following relations. In 2D and 3D (§4):

$$\boxed{\text{Hodge integrals on Deligne-Mumford moduli spaces}}$$
$$\updownarrow$$
$$\boxed{\text{Colored HOMFLY polynomials of the Hopf link}}$$

In complex 3D and real 3D (§2):

$$\boxed{\text{Gromov-Witten invariants of toric Fano local Calabi-Yau 3-folds}}$$
$$\updownarrow$$
$$\boxed{\text{Colored HOMFLY polynomials of the Hopf link}}$$

In complex 3D and complex 2D (§6):

$$\boxed{\text{Gromov-Witten invariants of toric Fano local Calabi-Yau 3-folds}}$$
$$\updownarrow$$
$$\boxed{\text{Equivariant indices of the framed moduli spaces of torsionfree sheaves on } \mathbb{C}^2}$$

We see in these results that the following contributions of three great Chinese scientists are related to each other: Chern-Weil theory, Chern-Simons theory, Yang-Mills theory, Yang-Baxter equations, Calabi-Yau geometries.

As is clear from the above brief introduction, the work of Professor Shiing-Shen Chern plays an important role in many of the relevant mathematical and physical theories. His passaway is a great loss to many of us both mathematically and personally. I dedicate this article to his memory.

2. Calabi-Yau Geometries and Chern-Simons Link Invariants

2.1. **Gromov-Witten invariants of local Calabi-Yau spaces.** Let S be a toric Fano surface (e.g. $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$), we are interested in the Gromov-Witten invariants of open Calabi-Yau 3-fold given by the total space of its canonical line bundle K_S. They can be defined as follows. Let $\beta \in H_2(S,\mathbb{Z})$, and let $\overline{\mathcal{M}}_{g,0}(S,\beta)$ be the moduli space of genus g stable maps to S of degree β. An element of this space is an equivalence class of holomorphic maps $f : \Sigma \to S$ such that Σ is a *connected* curve with nodes which has arithmetic genus g, $f_*([\Sigma]) = \beta$, and f has finite automorphisms. On this space there is a canonically defined bundle $K_{g,\beta}$ whose fiber at $f : \Sigma \to S$ is given by $H^1(\Sigma, f^*K_S)$. The rank of this bundle is the same as the virtual dimension of $\overline{\mathcal{M}}_{g,0}(S,\beta)$, so one can define the Gromov-Witten invariants by

$$n_{g,\beta} = \int_{[\overline{\mathcal{M}}_{g,0}(S,\beta)]^{\mathrm{virt}}} e(K_{g,\beta}).$$

We are interested in the generating series:

$$F_{K_S} = \sum_{\beta \neq 0 \in H_2(S,\mathbb{Z})} n_{g,\beta} \lambda^{2g-2} e^{\beta},$$

here e^{β} denotes an element in the Novikov ring of $H_2(S,\mathbb{Z})$, i.e.,

$$e^{\beta_1} \cdot e^{\beta_2} = e^{\beta_1+\beta_2}.$$

Here we are considering the *instanton part* of the free energy, the corresponding instanton part of the partition function is given by:

$$Z_{K_S} = \exp F_{K_S}.$$

2.2. **Chern-Simons link invariants of the Hopf link.** Following the famous work of Witten [57] on Chern-Simons theory and link invariants, the colored HOMFLY polynomial for $SU(N)$ can be given as follows. The level k integrable highest weight representations of the Kac-Moody algebra $\widehat{\mathfrak{su}}(N)$ are labeled by partitions of length $< N$ and width $\leq k$. Denote their characters by $\chi_\mu(\tau)$, then by the modular properties of such representations one has

$$\chi_\mu(-\frac{1}{\tau}) = \sum_\nu S_{\mu\nu}(\tau)\chi_\nu(\tau).$$

Then the colored HOMFLY polynomial of the Hopf link is given by

$$W_{\mu\nu} = \frac{S_{\mu\nu}}{S_{(0)(0)}}.$$

It can expressed in terms of Schur functions. Such expressions can be mathematically established by skein theory (Morton-Lukac [46]) or by representation theory of quantum groups and Hecke algebras (Lin-Zheng [36]). What concerns us here is the large N and k behavior of $W_{\mu\nu}$ and the following leading term:

$$\mathcal{W}_{\mu\nu}(q) = s_\mu(q^\rho)s_\nu(q^{\mu+\rho}),$$

where for a partition ν,

$$q^{\nu+\rho} = (q^{\nu_1-\frac{1}{2}}, \dots, q^{\nu_i-i+\frac{1}{2}}, \dots).$$

When $\nu = (0)$, one gets the leading term of the quantum dimension which can also be defined in the representation theory of quantum groups:

$$\mathcal{W}_\mu(q) = \mathcal{W}_{\mu(0)}(q) = s_\mu(q^\rho).$$

2.3. **Gromov-Witten invariants in terms of link invariants.** Now we come to a remarkable conjecture made by Aganagic-Mariño-Vafa [3] based on duality with Chern-Simons theory. This conjecture states that one can express F_{K_S} in terms of the colored HOMFLY polynomials of the Hopf link. Iqbal [22] interpreted this conjecture in terms of Feynman rules. The reader can consult e.g. [44] for the sequence of physical ideas in a series of papers [59, 15, 17, 53] that lead to this conjecture, and many related references in the physics literature.

Consider the image of the moment map of the torus action on the toric surface S. This is a convex polygon whose vertices are images of the fixed points and whose edges are images of invariant divisors. From the edges one can read the information on the weight decomposition at the fixed point. We also add external edges which encode also the weights of K_S. For example, the following are the diagrams for the Hirzebruch surfaces $\mathbb{F}_m$ $(m = 0, 1, 2)$.

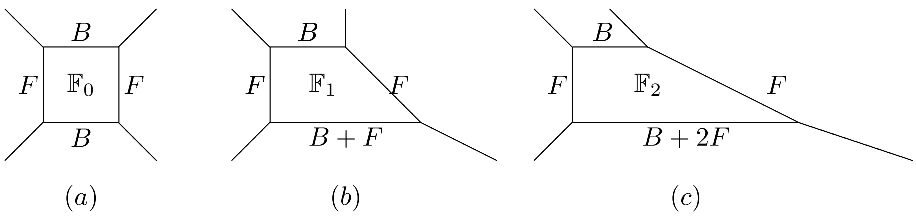

Figure 1

Give the k internal edges a cyclic labeling as E_i counterclockwise. Denote by s_i the self-intersection number of E_i in S, and write $t_i = e^{E_i}$. Then we have

Theorem 2.1. *We have*

$$Z_{K_S} = \exp F_{K_S} = \prod_{i\in\mathbb{Z}_k}\sum_{\nu_i} e^{\sqrt{-1}\kappa_{\nu_i}s_i\lambda/2}((-1)^{s_i}t_i)^{|\nu_i|}\mathcal{W}_{\nu_i,\nu_{i-1}}(e^{\sqrt{-1}\lambda}), \tag{1}$$

for a partition μ,

$$\kappa_\mu = \sum_{i=1}^{l(\mu)} \mu_i(\mu_i - 2i + 1).$$

This result was conjectured in [3, 22] and proved in [64]. Such a result was first brought to my attention by Kefeng Liu. The efforts to prove it initiated my work in this area. See [54, 27] for recent work on Gopakumar-Vafa conjecture [16] for toric Fano local Calabi-Yau geometries based on this result.

There is a similar result for the resolved conifold $\mathcal{O}(-1)\oplus\mathcal{O}(-1)\to\mathbb{P}^1$. The corresponding diagram is:

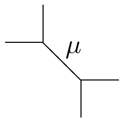

Figure 2

Let $\mathcal{O}(-1)^g_d$ be the bundle on $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1,d)$ whose fiber at a map $f : C \to \mathbb{P}^1$ representing a point in the moduli space is given by

$$H^1(C, f^*\mathcal{O}(-1)).$$

Define

$$K^g_d = \int_{[\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1,d)]^{vir}} e(\mathcal{O}(-1)^g_d \oplus \mathcal{O}(-1)^g_d),$$

and its generating series:

$$F(q,Q) = \sum_{d>0}\sum_{g\geq 0} K^g_d \lambda^{2g-2} Q^d.$$

Also set $Z_{4D}(q;Q) = \exp F(q,Q)$. Then one has

$$Z_{4D}(q,Q) = \sum_{\mu} (-1)^{|\mu|}\mathcal{W}_\mu(q)\mathcal{W}_{\mu'}(q)Q^{|\mu|}, \tag{2}$$

where $\mathcal{W}_\mu(q) = \mathcal{W}_{\mu(0)}(q) = s_\mu(q)$ is the invariant for the unknot.

More recently Aganagic *et al* [2] introduced a *topological vertex* defined in terms of $\mathcal{W}_{\mu\nu}$. This is used to compute Gromov-Witten invariants for Calabi-Yau geometries associated to more complicated diagrams, e.g. the nonplanar diagram obtained by gluing the two horizontal edges with each other of the following diagram:

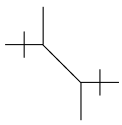

Figure 3

which gives the partition function

$$Z_{5D}(q,Q,Q_m) = \sum_{\mu\in\mathcal{P}}(-Q)^{|\mu|}\sum_{\nu\in\mathcal{P}}(-Q_m)^{|\nu|}C_{\nu^t\mu(0)}(q)C_{\nu\mu^t(0)}(q). \tag{3}$$

or glue furthermore the vertical edges with each other:

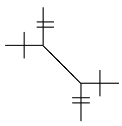

Figure 4

which gives the partition function

$$Z_{6D}(Q,Q_m,Q_1,q) = \sum_{\mu,\nu,\eta\in\mathcal{P}}(-Q)^{|\mu|}(-Q_1)^{|\nu|}(-Q_m)^{|\eta|}C_{\mu\nu\eta}(q)C_{\mu^t\nu^t\eta^t}(q); \tag{4}$$

or diagrams of the form:

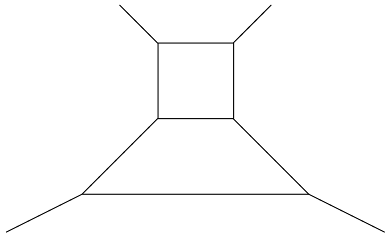

Figure 5

A mathematical theory of the topological vertex has been developed in [30].

3. Localizations on Moduli Spaces of Stable Maps and Hodge Integrals

In this and the next two sections we will explain our proof of Theorem 2.1. We summarize our strategy as follows. In this section we apply virtual localization [29, 18] on $\overline{\mathcal{M}}_{g,0}(S,\beta)$. This leads us to some special Hodge integrals involving partitions of numbers and summation over graphs. In §4 we show by localization on the relative moduli space introduced in [34] to a suitable space with these partitions as boundary conditions, one can prove closed formulas for such Hodge integrals in terms of $\mathcal{W}_{\mu\nu}$. In §5 we briefly describe how summations over graphs naturally appear when using localization techniques and a combinatorial trick called the chemistry of graphs [64] that handles such summations.

3.1. **Localization on moduli spaces of stable maps.** When there are symmetries, one can often compute global invariants by local contributions from fixed points. This idea is called the localization. It dates back to the Poincaré-Hopf theorem and Chern's proof of the Gauss-Bonnet theorem. It is developed by Borel, Atiyah, Bott, Segal, Berline, Vergne and many others. As an example let T be a torus acting on a vector bundle $\pi : V \to M$ which covers a T-action on M, suppose both E and M are oriented and M is compact, and the rank of V is equal to the dimension of M. We are interested in the Euler number $\int_M e(V)$. This can be computed by considering the integral of the equivariant Euler class $e_T(V)$, and we have

$$\int_M e(V) = \int_M e_T(V) = \sum_F \int_F \frac{e_T(V)|_F}{e_T(F/M)},$$

where the sum is taken over all fixed point components F, $e_T(F/M)$ is the equivariant Euler class of F in M.

Now we take M to be the moduli space of stable maps $\overline{\mathcal{M}}_{g,0}(S,\beta)$. For this we need the modification using virtual fundamental class [29, 18]. The fixed points are parameterized by labeled graphs, and the computations of the equivariant Euler classes lead to Hodge integrals.

3.2. **Hodge integrals.** Let $\overline{\mathcal{M}}_{g,n}$ denote the Deligne-Mumford moduli stack of stable curves of genus g with n marked points. The Hodge bundle $\mathbb{E}$ is a rank g vector bundle over $\overline{\mathcal{M}}_{g,n}$ whose fiber over $[(C, x_1, \ldots, x_n)] \in \overline{\mathcal{M}}_{g,n}$ is $H^0(C, \omega_C)$, where ω_C is the dualizing sheaf of C. Let $\mathbb{L}_i$ be the line bundle over $\overline{\mathcal{M}}_{g,n}$ whose fiber over $[(C, x_1, \ldots, x_n)] \in \overline{\mathcal{M}}_{g,n}$ is the cotangent line $T^*_{x_i}C$ at the i-th marked point x_i. A Hodge integral is an integral of the form

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{j_1} \cdots \psi_n^{j_n} \lambda_1^{k_1} \cdots \lambda_g^{k_g} \tag{5}$$

where $\psi_i = c_1(\mathbb{L}_i)$ is the first Chern class of $\mathbb{L}_i$, and $\lambda_j = c_j(\mathbb{E})$ is the j-th Chern class of the Hodge bundle.

The study of Hodge integrals is an important part of the intersection theory on $\overline{\mathcal{M}}_{g,n}$. The famous Witten Conjecture/Kontsevich Theorem says that a suitable generating series of integrals of the form

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{j_1} \cdots \psi_n^{j_n} \tag{6}$$

is a τ-function of the KdV hierarchy, and this recursively determines such integrals. It is known that integrals of the form (5) can be reduced to the simpler form (6) and there is a computer program to compute them in this way.

Faber [9] made some interesting conjectures on Hodge integrals. Some of these conjectures are related by Getzler and Pandharipande [13] to the Virasoro constraint conjecture for Gromov-Witten invariants in various dimensions. Faber and Pandhripande in a series of papers [10, 11, 12] used localizations on $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, d)$ to prove some of these conjectures. For example, they have proved the following identities:

$$\sum_{g\geq 0} \lambda^{2g} \int_{\overline{\mathcal{M}}_{g,1}} \frac{\Lambda_g^\vee(a)}{1-\psi_1} = \left(\frac{\sin(\lambda/2)}{\lambda/2}\right)^{a-1}, \tag{7}$$

$$\int_{\overline{\mathcal{M}}_g} \lambda_{g-2}\lambda_{g-1}\lambda_g = \frac{1}{2(2g-2)!} \frac{|B_{2g-2}|}{2g-2} \frac{|B_{2g}|}{2g}, \tag{8}$$

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1} \cdots \psi_n^{k_n} \lambda_g = \binom{2g+n-3}{k_1, \ldots, k_n} \frac{2^{2g-1}-1}{2^{2g-1}} \frac{|B_{2g}|}{(2g)!}, \tag{9}$$

where B_n are Bernoulli numbers, and

$$\Lambda_g^\vee(a) = \sum_{i=0}^{g} (-1)^i a^{g-i} \lambda_i.$$

The following identity

$$\sum_{g\geq 0} \lambda^{2g-2} \int_{\overline{\mathcal{M}}_{g,1}} \frac{\Lambda_g^\vee(1)\Lambda_g^\vee(a)\Lambda_g^\vee(b)}{1-\psi_1} = \left(\frac{\sin(\lambda/2)}{\lambda/2}\right)^{a+b}$$

is obtained in a similar fashion in [56]. Another important result on Hodge integrals is the ELSV formula [8]:

$$H_{g,\mu} = \frac{(2g-2+|\mu|+l(\mu))!}{|\operatorname{Aut}(\mu)|} \prod_{i=1}^{l(\mu)} \frac{\mu_i^{\mu_i}}{\mu_i!} \int_{\overline{\mathcal{M}}_{g,l(\mu)}} \frac{\Lambda_g^\vee(1)}{\prod_{i=1}^{l(\mu)}(1-\mu_i\psi_i)};$$

where $H_{g,\mu}$ is the Hurwitz number associated to the partition $\mu = (\mu_1, \dots, \mu_{l(\mu)})$.

As we will see below to prove the Hodge integral identities needed for the proof of Theorem 2.1, localizations on relative moduli spaces seem to be more powerful. The reason is that those identities involve partitions of numbers which naturally come from boundary conditions for relative moduli spaces.

3.3. **Some Hodge integrals from localizations on $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, d)$.** We use the following $T(= S^1)$-action on $\mathbb{P}^1$:

$$e^{\sqrt{-1}t} \cdot [z^0 : z^1] = [z_0 : e^{\sqrt{-1}t} z_1].$$

This induces T-actions on $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, d)$. For $k \in \mathbb{Z}$, a lifting of the T-action to $L_k = \mathcal{O}_{\mathbb{P}^1}(k)$ is determined by the weights a_0 of $L_k|_{[1:0]}$ and a_1 of $L_k|_{[0:1]}$. It is easy to see that $a_0 - a_1 = k$. We say L_k is given the weights $[a_0, a_1]$. For example, the induced action on $T\mathbb{P}^1$ has weights $[1, -1]$, the cotangent bundle has weights $[-1, 1]$. For the two copies of $\mathcal{O}(-1)$, we use weights $[p, -p-1]$ and $[-p-1, p]$ respectively. Now the localization of

$$K_d^g = \int_{[\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1,d)]_T^{vir}} e_T(\mathcal{O}(-1)_d^g \oplus \mathcal{O}(-1)_d^g)$$

gives rise to the following Hodge integral [62]:

$$\begin{aligned} G_\mu(\lambda;\tau) = & -\frac{\sqrt{-1}^{l(\mu)}}{z_\mu}[\tau(\tau+1)]^{l(\mu)-1} \cdot \prod_{i=1}^{l(\mu)} \frac{\prod_{a=1}^{\mu_i-1}(\mu_i\tau + a)}{\mu_i!} \\ & \cdot \sum_{g\geq 0} \lambda^{2g-2} \int_{\overline{\mathcal{M}}_{g,l(\mu)}} \frac{\Lambda_g^\vee(1)\Lambda_g^\vee(\tau)\Lambda_g^\vee(-1-\tau)}{\prod_{i=1}^{l(\mu)} \frac{1}{\mu_i}\left(\frac{1}{\mu_i} - \psi_i\right)}, \end{aligned} \tag{10}$$

Here $\mu = (\mu_1 \geq \mu_2 \geq \mu_{l(\mu)} > 0)$ is a partition of d. Such integrals first appear in the calculations by Katz and Liu [26].

3.4. **Hodge integrals from localizations on $\overline{\mathcal{M}}_{g,0}(S, \beta)$.** Similarly, by applying localization to

$$n_{g,\beta} = \int_{[\overline{\mathcal{M}}_{g,0}(S,\beta)]_T^{\mathrm{virt}}} e_T(K_{g,\beta})$$

I am led to the following Hodge integrals:

(11)

$$\begin{aligned} G_{\mu^+,\mu^-}(\lambda;\tau) = & -\frac{(\sqrt{-1}\lambda)^{l(\mu^+)+l(\mu^-)}}{z_{\mu^+}\cdot z_{\mu^-}}[\tau(\tau+1)]^{l(\mu^+)+l(\mu^-)-1} \\ & \cdot \prod_{i=1}^{l(\mu^+)} \frac{\prod_{a=1}^{\mu_i^+-1}(\mu_i^+\tau + a)}{\mu_i^+!} \cdot \prod_{i=1}^{l(\mu^-)} \frac{\prod_{a=1}^{\mu_i^- -1}(\mu_i^-\frac{1}{\tau} + a)}{\mu_i^-!} \\ & \cdot \sum_{g\geq 0}\lambda^{2g-2}\int_{\overline{\mathcal{M}}_{g,l(\mu^+)+l(\mu^-)}} \frac{\Lambda_g^\vee(1)\Lambda_g^\vee(\tau)\Lambda_g^\vee(-\tau-1)}{\prod_{i=1}^{l(\mu^+)}\frac{1}{\mu_i^+}\left(\frac{1}{\mu_i^+}-\psi_i\right)\prod_{j=1}^{l(\mu^-)}\frac{\tau}{\mu_i^-}\left(\frac{\tau}{\mu_j^-}-\psi_{l(\mu^+)+j}\right)}, \end{aligned}$$

where λ, τ are variables, $(\mu^+, \mu^-) \in \mathcal{P}^2_+$, the set of pairs of partitions which are not both empty. We will call the Hodge integrals in $G_{\mu^+,\mu^-}(\lambda;\tau)$ the *two-partition Hodge integrals.*

In a more recent work [30] where a mathematical theory for the topological vertex is developed, a similar kind of Hodge integrals which involve three partitions naturally appear when applying localization techniques.

4. Proof of Hodge Integral Identities by Localizations

In the previous section we have seen localization on moduli spaces of stable maps leads one naturally to some Hodge integrals hence it is desirable to evaluate them. In this section we will briefly describe how this can be achieved by localization again, but this time on relative moduli spaces. The interested reader can consult [38, 39, 30] and the survey papers [37, 41] for details.

4.1. **Mariño-Vafa formula.** Hodge integrals of the form (10) can be evaluated by the Mariño-Vafa formula. This formula arises in the duality between relative Gromov-Witten invariants of the resolved conifold and the invariant of the unknot [45]. The leading term of the former was computed by Katz and Liu [26], it is given by $G_\mu(\lambda, \tau)$ in §3.3. Consider its generating series:

$$G^\bullet(\lambda;\tau;p) \quad = \quad \exp\left(\sum_\mu G_\mu(\lambda;\tau)p_\mu\right), \quad p_\mu = p_{\mu_1}\cdots p_{\mu_l}.$$

The latter is given by the following generating series of the leading terms of the quantum dimensions W_μ:

$$R^\bullet(\lambda;\tau;p) \quad = \quad \sum_\nu q^{\kappa_\nu \tau/2}\mathcal{W}_\nu(q)s_\nu.$$

Here s_ν are the Schur functions, they are related to the Newton functions p_μ by the characters of irreducible representations of the symmetric groups:

$$\langle s_\mu, p_\nu\rangle = \chi_\mu(\nu),$$

where χ_μ denotes the character of the irreducible representation indexed by μ, and $\chi_\mu(\nu)$ denotes its value on the conjugacy class indexed by ν. The remarkable formula conjectured by Mariño and Vafa [45] states that

$$G^\bullet(\lambda;\tau;p) = R^\bullet(\lambda;\tau;p). \tag{12}$$

Note the left-hand side of this formula encodes the geometric information of the Hodge integrals of the Deligne-Mumford moduli spaces while the right-hand side encodes the algebraic information from representation theory of symmetric groups, Kac-Moody algebras, quantum groups and Hecke algebras. In physics, the left-hand side comes from 2D quantum gravity and the right-hand side comes from 2D Yang-Mills theory.

Here is one way to understand the Mariño-Vafa formula. One can think of $G^\bullet(\lambda;\tau;p)$ as a vector in the space of symmetric functions. Its coefficients expanded in the basis p_μ are given by the Hodge integrals, while its coefficients expanded in the basis s_μ are given by the leading terms of quantum dimensions.

The Mariño-Vafa formula was proved in [38] (see also [48]). The basic ingredients are a system of ordinary differential equations called the *cut-and-join equations* and localizations on relative moduli spaces $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1,\mu)$ introduced in [34]. The strategy was proposed in [61] and carried out jointly with Melissa Liu and Kefeng Liu in [38]. The motivation for such a proof comes from comparison with the ELSV formula. It is well-known by Burnside formula that Hurwitz numbers can be expressed in terms of representations of symmetric groups. One can think of the Mariño-Vafa formula as a deformation of the ELSV formula. It is known that the Hurwitz numbers satisfy the cut-and-join equations, by combinatorial method [14] and by geometric method [33] using the symplectic version of relative moduli spaces [32, 21]. The motivation of applying localization on the algebro-geometric relative moduli spaces comes from comparing [26] with [35] and from [19].

Originally I tried to prove the Mariño-Vafa formula by localizations on $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1,d)$ and got the special cases for $|\mu|=2$ and 3 in [60] but it became too complicated to do so in higher degrees. The treatment in [48] uses localization on $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1,d)$ and the complicated proof of (9) in [12].

Many earlier results on Hodge integrals can be derived from the Mariño-Vafa formula. See [40] for the derivations of (7)-(9). Mariño and Vafa conjectured that one can recover all Hodge integrals with at most three λ classes from their formula. Lu [42] has shown that this is true with the exception of only one class of Hodge integrals.

4.2. **Two-partition formula.** Motivated by the Mariño-Vafa formula I conjectured the following formula in [63] for two-partition Hodge integrals:

$$G^\bullet(\lambda;p^+,p^-;\tau)=R^\bullet(\lambda;p^+,p^-;\tau) \tag{13}$$

where

$$\begin{aligned}
G^\bullet(\lambda;p^+,p^-;\tau) &= \exp\left(\sum_{(\mu^+,\mu^-)\in\mathcal{P}^2_+} G_{\mu^+,\mu^-}(\lambda;\tau)p^+_{\mu^+}p^-_{\mu^-}\right),\\
R^\bullet(\lambda;p^+,p^-;\tau) &= \sum_{\nu^\pm} e^{\sqrt{-1}(\kappa_{\nu^+}\tau+\kappa_{\nu^-}\tau^{-1})\lambda/2}\mathcal{W}_{\nu^+,\nu^-}(e^{\sqrt{-1}\lambda})s^+_{\nu^+}s^-_{\nu^-},\\
p^\pm_\mu &= p^\pm_{\mu_1}\cdots p^\pm_{\mu_h}.
\end{aligned}$$

This formula is proved in [39] by localization on the relative moduli spaces of stable maps to $\mathbb{P}^1\times\mathbb{P}^1$ blown up at a point.

In a recent paper [5], Diaconescu and Florea conjectured a relation between three-partition Hodge integrals and the topological vertex [2]. In [30] we prove a formula for three-partition Hodge integrals in terms of the topological vertex.

4.3. **Relationship with integrable hierarchies.** The relationship between Hodge integrals with link invariants has a straightforward consequence of relating them to integrable hierarchies. This is a result of the appearance of Schur functions in the generating series, and the fact that $\mathcal{W}_\mu$ can also be expressed in terms of skew Schur functions [63]. As a results one can use the boson-fermion correspondence to express them in terms of free fermionics. Hence by Kyoto school's approach to integrable systems, one sees that the generating series of the one-partition, two-partition and three-partition Hodge integrals are the τ-functions of the KP, 2-Toda and 3-component KP hierarchies respectively [65, 1]. Note here we do not use a matrix model to establish a connection with integrable hierarchies (even though using the explicit formula one can write down such a model), while in Kontsevich' proof [28] of the Witten conjecture [58] matrix model plays an important role.

5. Summation over Graphs Arising in Localization

We explain in this section how graphs arise in localization on moduli spaces. We will also describe the technique developed in [62, 64] to take summations over these graphs.

5.1. **Toric Fano surfaces and cyclic graphs.** Let S be a toric Fano surface with associated $T = U(1)^2$-action. The image $\mu(S)$ of the moment map $\mu : S \to \mathbb{R}^2$ of the T-action is a convex polygon whose vertices are the images of the fixed points and whose edges are the images of the invariant divisors (see e.g. [4]). We give these vertices and edges cyclic labels. For example, the following pictures correspond to $\mathbb{F}_m$, $m = 0, 1, 2$ respectively:

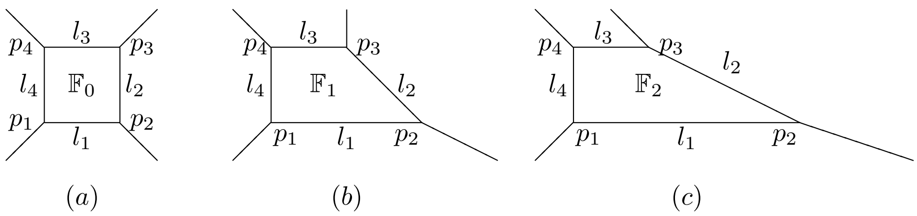

Figure 6

Denote the weights of $T_{p_i}S$ of the torus action corresponding to the two invariant divisors l_i and l_{i-1} by u_i^+ and u_i^- respectively.

5.2. **Fixed points on $\overline{\mathcal{M}}_{g,0}(S,d)$.** For $d \in H_2(S,\mathbb{Z})$, denote by $\overline{\mathcal{M}}_{g,0}(S,d)$ the moduli space of stable maps of genus g to S of class d. The T-action induces T-actions on $\overline{\mathcal{M}}_{g,0}(S,d)$. The fixed point components of $\overline{\mathcal{M}}_{g,n}(S,d)^T$ are very easy to describe. They are in one-to-one correspondence with a set $G_g(S,d)$ of decorated graphs described below. Each vertex v of the graph $\Gamma \in G_g(S,d)$ is assigned an index $i(v) \in S^T$, and a genus $g(v)$. The valence $\mathrm{val}(v)$ of v is the number of edges incident at v. If two vertices u and v are joined by an edge e, then $i(u) \neq i(v)$, and e is assigned a "degree"

$$\delta(e) = d_e[e] \in H_2(X,\mathbb{Z}).$$

Denote by $E(\Gamma)$ the set of edges of Γ, $V(\Gamma)$ the set of vertices of Γ. The genus of the graph is given by

$$g(\Gamma) = 1 - |V(\Gamma)| + |E(\Gamma)|.$$

The decorations of Γ are required to satisfy the following conditions:

$$\sum_{e\in E(\Gamma)} \delta_e = d, \qquad \sum_{v\in V(\Gamma)} g(v) + g(\Gamma) = g.$$

Let $f : C \to S$ represent a fixed point. Then each vertex v corresponds to a connected component C_v of genus $g(v)$, with $\mathrm{val}(v)$ nodal points. The component C_v is mapped by f to the fixed point $i(v)$. When $2g(v)-2+\mathrm{val}(v) < 0$, C_v is simply a point. There are only two cases when this happens: $g(v) = 0$ and $\mathrm{val}(v) = 1$, $g(v) = 0$ and $\mathrm{val}(v) = 2$. Each edge e corresponds to a component C_e of C which is isomorphic to $\mathbb{P}^1$. Each C_e is mapped to an invariant divisor l_i with degree d_e.

Define

$$\overline{\mathcal{M}}_\Gamma = \prod_{v\in V(\Gamma)} \overline{\mathcal{M}}_{g(v),val(v)}.$$

In this product, $\overline{\mathcal{M}}_{0,1}$ and $\overline{\mathcal{M}}_{0,2}$ are interpreted as points. There is a natural morphism

$$\tau_\Gamma : \overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, d)^T.$$

Its image is $\overline{\mathcal{M}}_\Gamma/A_\Gamma$, where for A_Γ we have an exact sequence:

$$0 \to \prod_{e\in E(\Gamma)} \mathbb{Z}_{d_e} \to A_\Gamma \to \mathrm{Aut}(\Gamma) \to 1.$$

Given a graph Γ in $G_g(X, d)$, we call the labelled graph obtained from Γ by ignoring the markings of $g(v)$ of the vertices the type of Γ. Denote by $G(X, d)$ the set of types of graphs in $G_g(X, d)$.

5.3. **Feynman rule for a type of graphs.** Now by applying localization one can get [64]:

$$F(\lambda) = \sum_{g\geq 0} \lambda^{2g-2} \sum_{\Gamma\in G_g(\mathbb{Z}_k,d)} \frac{1}{|A_\Gamma|} \prod_{v\in V(\Gamma)} w_v \cdot \prod_{e\in E(\Gamma)} w_e,$$

where

$$w_v = z_{\mu^+(v)} \cdot z_{\mu^-(v)} \cdot G_{\mu^+(v),\mu^-(v)}(u^+_{i(v)}, u^-_{i(v)}),$$
$$w_e = ((-1)^{s_e} e_{l_i})^{d_e}.$$

5.4. **Chemistry of graphs.** In the above we have reduced the problem of computing the free energy to a problem of summing over some graphs. Motivated by quantum field theory where Feynman diagrams and Feynman rules are common, I have developed in [62, 64] a method to take such summations by using the theory of free bosons. The basic idea is to associate to each edge in the toric diagram of S a system of free bosons, and use vacuum expectation and Wick theorem to produce from the toric diagram all graph types that appear in localization. In a more recent paper [1], the authors used free fermions to study the topological vertex. This is related to my work by boson-fermion correspondence.

6. Local Calabi-Yau Geometries and Yang-Mills Theory

In this section we explain the relationship between the topological string partition functions computed above and some invariants of the moduli spaces in 4D Yang-Mills theory.

6.1. **Geometric engineering.** An interesting way to establish relationships between Gromov-Witten invariants of Calabi-Yau spaces and invariants of Yang-Mills theory is an idea called *geometric engineering.* In string theory physicists start with a theory on $\mathbb{R}^4 \times X$ where X is a Calabi-Yau space. In what they call the low energy effective theory of this theory (a supergravity theory in this case) they get a theory which contains both gravitational fields and gauge fields. By suitably choosing the Calabi-Yau spaces, one can in principle obtain gauge theories on $\mathbb{R}^4$ with various gauge groups. See [25] and the references therein for more details. Mathematically this leads to the possibility of identifying the generating series of Gromov-Witten invariants of some local Calabi-Yau geometries with the generating series of some invariants of some Yang-Mills theory on $\mathbb{R}^4$ which we regard as $\mathbb{C}^2$. As we explained in §2, the former can be mathematically computed. For the latter, Nekrasov studied some partition functions in the context of moduli spaces of noncommuative instantons in the physics literature. He conjectured his partition functions can be identified with topological string partition functions of local Calabi-Yau geometries obtained as A_n-fibered over $\mathbb{P}^1$. Such moduli spaces correspond to the framed moduli spaces of torsion free sheaves on $\mathbb{C}^2$ and have been studied by Nakajima and Yoshioka [49]. In two papers Iqbal and Kashini-Poor [23, 24] proposed some combinatorial identities which lead to the proof of Nekrasov's conjecture. These identities are proved in [6, 7, 66]. In a more recent work [30] this connection between Calabi-Yau geometries and gauge theory is used to studied the Gopakumar-Vafa invariants. Nekrasov [51] also conjectured a relationship with the Seiberg-Witten prepotential. See [49, 52] for the proof.

6.2. **Some gauge theoretical invariants by localization.** Let $M(N,k)$ denote the framed moduli space of torsion free sheaves on $\mathbb{P}^2$ with rank N and $c_2 = k$. The framing means a trivialization of the sheaf restricted to the line at infinity. In particular when $N = 1$ we get the Hilbert scheme $(\mathbb{C}^2)[k]$. See [49] for details.

As proved in [49], $M(N,k)$ is a nonsingular variety of dimension $2Nk$. The maximal torus T of $GL_N(\mathbb{C})$ together with the torus action on $\mathbb{P}^2$ induces an action on $M(N,k)$. As shown in [49], the fixed points are isolated and parameterized by N-tuples of partitions $\vec{\mu} = (\mu^1, \cdots, \mu^N)$ such that $\sum_i |\mu^i| = k$. The weight decomposition of the tangent bundle of $TM(N,k)$ at a fixed point $\vec{\mu}$ is given by

$$\sum_{\alpha,\gamma=1}^{N} e_\gamma e_\alpha^{-1}\Big(\sum_{(i,j)\in\mu^\alpha} t_1^{-((\mu^\gamma)^t_j - i)} t_2^{\mu_i^\alpha - j+1} + \sum_{(i,j)\in\mu^\gamma} t_1^{(\mu^\alpha)^t_j - i+1} t_2^{-(\mu_i^\gamma - j)}\Big), \tag{14}$$

where $t_1, t_2 \in \mathbb{C}^* \times \mathbb{C}^*$, and $e_\alpha \in T$.

The space $M(N,k)$ has the following remarkable property. Let E be an equivariant coherent sheaf on it. Even though $H^i(M(N,k), E)$ might be infinite-dimensional, the weight spaces of the induced torus action on it are finite-dimensional, hence it makes sense to define the equivariant index. In other words, if

$$H^i(M(N,k),E) = \sum V_\nu$$

is the weight decomposition of $H^i(M(N,k),E)$, then

$$\dim V_\nu < \infty$$

for all weight ν. Hence one can define

$$\mathrm{ch}\, H^i(M(N,k),E) = \sum (\dim V_\nu) e^\nu.$$

Furthermore, one can compute the equivariant index by localization[49]:

$$\chi(M(N,k),E) = \sum_{i=0}^{2Nk} (-1)^i \,\mathrm{ch}\; H^i(M(N,k),E) = \sum_{\vec{\mu}} \mathrm{ch}\left(\frac{i_{\vec{\mu}}^* E}{\wedge_{-1} T_{\vec{\mu}}^* M(N,k)} \right).$$

The partition functions of Nekrasov are equivariant indices of some naturally defined bundles on $M(n,k)$ ([20, 55, 31]).

6.3. **Hilbert schemes and symmetric products.** We use the $n=1$ cases to illustrate the idea. See [31] for details and other cases. The framed moduli spaces $M(1,k)$ are the Hilbert schemes $(\mathbb{C}^2)^{[k]}$ of points on $\mathbb{C}^2$. The partition functions Z_{4D}, Z_{5D} and Z_{6D} can be related to equivariant χ_0, χ_y and elliptic genera of $(\mathbb{C}^2)^{[k]}$ (cf. [20, 31]).

Since the Hilbert-Chow morphism $(\mathbb{C}^2)^{[k]} \to (\mathbb{C}^2)^{(k)}$ to the k-th symmetric product of $\mathbb{C}^2$ is a crepant resolution, one expects invariants of $(\mathbb{C}^2)^{[k]}$ can be identified with the corresponding orbifold invariants of $(\mathbb{C}^2)^{(k)}$. The latter has natural infinite product expressions hence one can extract from them Gopakumar-Vafa invariants. For example, by localization, one has

$$\sum_{n=0}^\infty Q^n \chi_0((\mathbb{C}^2)^{[n]})(t_1,t_2) = \sum_{n=0}^\infty Q^n \sum_{|\mu|=n} \frac{1}{\prod_{e\in\mu}(1-t_1^{-l(e)}t_2^{a(e)+1})(1-t_1^{l(e)+1}t_2^{-a(e)})}.$$

This is a sum over partitions. It can be used to compare with (2) to get:

$$Z_{4D}(q,Q) = \sum_{n=0}^\infty Q^n \chi_0((\mathbb{C}^2)^{[n]})(q,q^{-1})$$

On the other hand,

$$\sum_{n=0}^\infty Q^n \chi_0((\mathbb{C}^2)^{(n)})(t_1,t_2) = \exp(\sum_{n\geq 1} \frac{Q^n}{n(1-t_1^n)(1-t_2^n)}).$$

This can be written as an infinite product. Since one has

$$\chi_0((\mathbb{C}^2)^{[n]})(t_1,t_2) = \chi_0((\mathbb{C}^2)^{(n)})(t_1,t_2),$$

one can get:

$$\sum_{n=0}^\infty Q^n \chi_0((\mathbb{C}^2)^{[n]})(q,q^{-1}) = \frac{1}{\prod_{m\geq 1}(1-q^mQ)^m}.$$

This gives a geometric explanation of

$$Z_{4D}(q, Q) = \frac{1}{\prod_{m\geq 1}(1-q^m Q)^m}.$$

In [31] we extract the Gopakumar-Vafa invariants of Z_{5D} (Z_{6D}) from the equivariant χ_y (elliptic) genera by similar method. We also propose to do the same for other local Calabi-Yau geometries related to geometric engineering.

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