Two themes in geometric quantization

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Abstract

We give a brief survey of our work (joint with Youliang Tian) on the theme of

“geometric quantization commutes with symplectic reduction”

and (joint with Xiaonan Ma) on the theme of

Toeplitz quantization and symplectic reduction.
§1. Quantization on symplectic manifolds

Let \((M, \omega)\) be a closed symplectic manifold. Let \(J\) be an almost complex structure on \(TM\) such that

\[ g^{TM}(v, w) = \omega(v, Jw) \]

defines a Riemannian metric on \(TM\).

Let \(E\) be a Hermitian vector bundle over \(M\) admitting a Hermitian connection \(\nabla^E\).

Then one can construct a (twisted) spin\(^c\) Dirac operator

\[ D^E : \Gamma \left( \Lambda^0,*(T^*M) \otimes E \right) \rightarrow \Gamma \left( \Lambda^0,*(T^*M) \otimes E \right). \]

**Remark.** When \((M, \omega, J)\) is Kähler and \(E\) is a holomorphic vector bundle over \(M\), one has

\[ D^E = \sqrt{2} \left( \overline{\partial}^E + \left( \overline{\partial}^E \right)^* \right). \]
Let $D^E_{\pm}$ be the restriction of $D^E$:

$$D^E_{\pm} : \Gamma \left( \bigwedge^{0, \text{even}} \Lambda (T^* M) \otimes E \right) \to \Gamma \left( \bigwedge^{0, \text{odd}} \Lambda (T^* M) \otimes E \right).$$

Then

$$D^E = D^E_{+} + D^E_{-}, \quad (D^E_{+})^* = D^E_{-}.$$  

Define the **quantization space** of $E$ to be the formal difference

$$Q(E) = \left( \ker D^E_{+} \right) - \left( \text{coker } D^E_{+} \right)$$

$$= \left( \ker D^E_{+} \right) - \left( \ker D^E_{-} \right).$$

It can be viewed as an element in $K(\cdot)$ (point view due to Bott), and does not depend on the choice of $J$ and the metric and connection on $E$. 
Atiyah-Singer index theorem.

\[
\dim Q(E) = \text{ind } D_E^\pm = \langle \text{Td}(TM) \text{ch}(E), [M] \rangle \\
= \int_M \det \left( \frac{e^{\frac{\sqrt{-1} R^{TM}}{2\pi}}}{1 - e^{-\frac{\sqrt{-1} R^{TM}}{2\pi}}} \right) \text{Tr} \left[ \exp \left( \frac{\sqrt{-1} R^E}{2\pi} \right) \right],
\]

where \( R^{TM} \) is the curvature of the Levi-Civita connection associated to \( g^{TM} \), \( R^E = (\nabla^E)^2 \) is the curvature of \( \nabla^E \).

**Remark.** When \((M, \omega, J)\) is Kähler and \(E\) is holomorphic, then

\[
Q(E) = H^{0,\text{even}}(M, E) - H^{0,\text{odd}}(M, E).
\]
§2. Hamiltonian action and symplectic reduction

Let $G$ be a compact connected Lie group. Let $\mathfrak{g}$ be the Lie algebra of $G$.

Assume $G$ acts on $(M, \omega)$ in a Hamiltonian way, and preserves $J$.

Then there exists a $G$-equivariant moment map

$$\mu : M \to \mathfrak{g}^*$$

such that for any $V \in \mathfrak{g}$, one has

$$i_V \omega = d\langle \mu, V \rangle,$$

where we use the same notation $V \in \Gamma(TM)$ to denote the vector field on $M$ generated by $V \in \mathfrak{g}$.

**Definition.** The Marsden-Weinstein symplectic reduction space $M_G$ is defined to be

$$M_G = \mu^{-1}(0)/G.$$
Basic assumption: $0 \in g^*$ is a regular value of the moment map $\mu : M \to g^*$.

Then $\mu^{-1}(0)$ is a closed manifold.

For simplicity, also assume that $G$ acts on $\mu^{-1}(0)$ freely, then $M_G$ is a closed manifold and carries an induced symplectic form $\omega_G$.

Moreover, $J$ induces an almost complex structure $J_G$ on $TM_G$ such that $\omega_G(v, J_Gw)$ determines a Riemannian metric $g^{TM_G}$ on $TM_G$.

Remark. If $(M, \omega, J)$ is Kähler, then $(M_G, \omega_G, J_G)$ is also Kähler.
3. Pre-quantization and symplectic reduction

Let $L$ be an Hermitian line bundle over $M$ carrying an Hermitian connection $\nabla^L$ such that

$$\frac{\sqrt{-1}}{2\pi} (\nabla^L)^2 = \omega.$$  

When such an $L$ exists, we call $(M, \omega)$ pre-quantizable, and call $L$ the pre-quantized line bundle.

We assume the existence of $L$ now.

We make the assumption that the Hamiltonian $G$ action lifts to an action on $L$, which preserves the Hermitian metric and Hermitian connection on $L$. 
Then $L$ descends to a pre-quantized line bundle $L_G$ over $M_G$ carrying a canonically induced Hermitian metric and Hermitian connection $\nabla^{L_G}$.

**Remark.** When $(M, \omega, J)$ is Kähler and $L$ is a holomorphic line bundle over $M$, then $L_G$ is also holomorphic over $M_G$. 
§4. Geometric quantization commutes with symplectic reduction

Continue the discussion above.

Then the canonical spin$^c$ Dirac operator $D^L$ commutes with the induced $G$-action on $\Lambda^{(0,*)}(T^*M) \otimes L$. Thus, $G$ preserves $\ker D^L_{\pm}$.

Let $(\ker D^L_{\pm})^G$ denote the $G$-invariant part in $\ker D^L_{\pm}$.

Define the geometric quantization of $L$ to be

$$Q(L)^G = (\ker D^L_+)^G - (\ker D^L_-)^G.$$ 

Also recall that the quantization of $L_G$ on $M_G$ is defined by

$$Q(L_G) = (\ker D^L_{+G}) - (\ker D^L_{-G}).$$
The Guillemin-Sternberg conjecture:

$$\dim Q(L)^G = \dim Q(L_G).$$  \hfill (*)

**Remark.** Tautologically, the above means “*Geometric quantization commutes with symplectic reduction*”.

**Remark.** Guillemin-Sternberg first proved in 1982 that when \((M, \omega, J)\) is Kähler and \(L\) is holomorphic,

$$\dim H^{(0,0)}(M, L)^G = \dim H^{(0,0)}(M_G, L_G)$$

and proposed \((*)\) as a conjecture.

When \(G\) is abelian, \((*)\) was first proved by Guillemin (1995) in a special case, and later in general by Meinrenken (JAMS 1996) and Vergne (DMJ 1996) independently.
The remaining non-abelian case was proved by Meinrenken (Adv. in Math. 1998) by using the technique of *symplectic cut* of Lerman.

There are also approaches of Duistermaart-Guillemin-Meinrenken-Wu (for circle actions) and Jeffrey-Kirwan (for non-abelian group actions with certain extra conditions).

**Remark.** All the above approaches use the Atiyah-Bott-Segal-Singer equivariant index theorem in an essential way: first relate $\dim Q(L)^G$ to quantities on the fixed point set of the $G$-action, and then try to relate the later to quantities on the symplectic quotient (through symplectic cut or through the Jeffrey-Kirwan-Witten non-abelian localization formulas).

**Natural question.** Whether there is an approach relating $\dim Q(L)^G$ directly to $\dim Q(L_G)$?
§5. A direct analytic approach
(with Youliang Tian)

We try to put the problem into the framework of an analytic Morse theory, analogous to what Witten did in the usual (real) case.

Let $g$ be equipped with an $\text{Ad}G$-invariant metric. Set

$$\mathcal{H} = |\mu|^2.$$  

Let $X^\mathcal{H}$ be the associated Hamiltonian vector field, i.e.,

$$i_{X^\mathcal{H}}\omega = d\mathcal{H}.$$  

**Definition** (Tian-Zhang, 1998) For any $T \in \mathbb{R}$, set

$$D_T^L = D^L + \frac{\sqrt{-1}T}{2}c(X^\mathcal{H}):
\Gamma \left( \Lambda^0,*(T^*M) \otimes L \right) \to \Gamma \left( \Lambda^0,*(T^*M) \otimes L \right).$$
Remark. If $(M, \omega, J)$ is Kähler and $L$ is holomorphic, then one has

$$D^L_T = \sqrt{2} \left( e^{-\frac{TH}{2}} \bar{\partial} L e^{\frac{TH}{2}} + e^{\frac{TH}{2}} \left( \bar{\partial} L \right)^* e^{-\frac{TH}{2}} \right).$$

This is an analogue of the Witten deformation in Morse theory, but now in a non-abelian context.

By using this deformation, one can then apply the analytic localization technique of Bismut-Lebeau to complete the proof of the Guillemin-Sternberg conjecture.

There are also many immediate generalizations arriving from this analytic approach (e.g. to the family case and to the case of manifolds with boundary).
§6. Toeplitz quantization on Kähler manifolds

From now on, we assume \((M, \omega, J)\) is Kähler.

Now for \(p \gg 0\), in view of the Kodaira vanishing theorem, we consider the space of holomorphic sections \(H^{0,0}(M, L^p)\).

Let \(\Pi_p\) denote the orthogonal projection from \(L^2(L^p)\) to \(H^{0,0}(M, L^p)\).

For any \(f \in C^\infty(M)\), consider the Toeplitz operators

\[
T_p(f) = \Pi_p f \Pi_p : H^{0,0}(M, L^p) \to H^{0,0}(M, L^p).
\]
We now state two results due to Bordemann-Meinrenken-Schlichenmaier, concerning the asymptotic behavior of $T_p(f)$ as $p \to +\infty$.

**Theorem** (B-M-S, 1994) One has

$$\lim_{p \to +\infty} \|T_p(f)\| = \|f\|_{\infty},$$

$$[T_p(f), T_p(g)] = \frac{T_p(\{f, g\})}{\sqrt{-1}p} + O(p^{-2}),$$

as $p \to +\infty$.

Now we want to describe a generalization of the second result to the framework of geometric quantization.
§7. Toeplitz quantization and symplectic reduction

We now assume that a connected compact Lie group acts on \((M, \omega, J, L)\) in a Hamiltonian way as before.

Let \(H^0,0(M, L)^G\) be the \(G\)-invariant part of \(H^0,0(M, L)\).

Let \(i : \mu^{-1}(0) \hookrightarrow M\) denote the canonical embedding. We assume as before that 0 is a regular value of \(\mu\) and \(G\) acts on \(\mu^{-1}(0)\) freely. Then

\[
\pi : \mu^{-1}(0) \to M_G
\]

is a principal fibration with fiber \(G\).

For any \(x \in M_G\), we define the potential function

\[
h(x) = \sqrt{\text{vol}(\pi^{-1}(x))}.
\]
With this potential function, one has an isometric identification

$$\pi_G = h\pi_* : C^\infty(L^p|_{\mu^{-1}(0)})^G \to C^\infty(L_G^p),$$

where $C^\infty(L^p|_{\mu^{-1}(0)})^G$ denotes the space of $G$-invariant smooth sections of $L^p|_{\mu^{-1}(0)}$.

By the classical result of Guillemin-Sternberg, one knows that for any $p > 0$,

$$\sigma_p = \pi_G^i : H^{0,0}(M, L^p)^G \to H^{0,0}(M_G, L_G^p)$$

is an isomorphism.

For any $p > 0$, let $P_p^G$ denote the orthogonal projection from $L^2(L^p)$ to $H^{0,0}(M, L^p)^G$. Set

$$\sigma_p^G = \sigma_p P_p^G : L^2(L^p) \to H^{0,0}(M_G, L_G^p).$$

Let

$$(\sigma_p^G)^* : H^{0,0}(M_G, L_G^p) \to L^2(L^p)$$

denote the adjoint of $\sigma_p$. 
Let $\Pi_{G,p} : L^2(L^p_G) \rightarrow H^{0,0}(M_G, L^p_G)$ denote the natural orthogonal projection.

**Theorem** (Ma-Zhang, 2005). For any $f \in C^\infty(M)$, let $f^G$ denotes the associated $G$-invariant function. Then

$$\mathcal{T}_p(f) = p^{-\frac{\dim G}{2}} \sigma_p^G f(\sigma_p^G)^* : H^{0,0}(M_G, L^p_G) \rightarrow H^{0,0}(M_G, L^p_G)$$

is a $p$-family of Toeplitz operators with principal symbol $2^{\frac{\dim G}{2}} \frac{f^G}{h^2}(x)$. That is,

$$\mathcal{T}_p(f) = \Pi_{G,p} 2^{\frac{\dim G}{2}} \frac{f^G}{h^2} \Pi_{G,p} + O(1/p)$$

as $p \rightarrow +\infty$. In particular, $p^{-\frac{\dim G}{2}} \sigma_p^G (\sigma_p^G)^*$ is a family of Toeplitz operator with principal symbol $2^{\frac{\dim G}{2}} \frac{2}{h^2}$. 
By combining with the result of Bordemann-Meinrenken-Schlichenmaier, one gets

**Corollary** For any $f, g \in C^\infty(M)$, one has

$$[T_p(f), T_p(g)] = \frac{2^\text{dim } G}{\sqrt{-1}p} \Pi_{G,p} \left\{ \frac{f^G}{\hbar^2}, \frac{g^G}{\hbar^2} \right\} \Pi_{G,p}$$

$$+ O(p^{-2}).$$

**Remark** One can view this corollary as a generalization of the Bordemann-Meinrenken-Schlichenmaier theorem in the framework of geometric quantization.
On the other hand, if one defines the unitary operator
\[ \Sigma_p = (\sigma_p^G)^* (\sigma_p^G(\sigma_p^G)^*)^{-1/2} : H^{0,0}(M_G, L^p_G) \rightarrow L^2(L^p), \]
then one has the following result:

**Theorem** (Ma-Zhang, 2005) For any \( f \in C^\infty(M), \)
\[ T_p^G(f) = \Sigma_p^* f \Sigma_p : H^{0,0}(M_G, L^p_G) \rightarrow H^{0,0}(M_G, L^p_G) \]
is a Toeplitz operator on \( M_G \) with principal symbol \( f^G. \)

**Remark** When \( G = T^k \) is a torus, this theorem was first proved by Charles.

**Remark** The proof of the above theorems relies on the asymptotic behavior as \( p \to +\infty \) of the \((G\text{-invariant})\) Bergman kernel associated to \( P_p^G : L^2(L^p) \rightarrow H^{0,0}(M, L^p)^G. \)
§8. The asymptotic expansion of the $G$-invariant Bergman kernel

Let $P^G_p(x, y)$ with $x, y \in M$ denote the smooth kernel of the orthogonal projection $P^G_p$ with respect to $d\text{vol}_M$.

We now describe some behavior of $P^G_p(x, y)$, as $p \to +\infty$.

Let $U$ be an arbitrary (fixed) small open $G$-invariant neighborhood of $\mu^{-1}(0)$.

**Theorem** (Ma-Zhang, 2005) For any $x, y \in M \setminus U$, as $p \to +\infty$,

$$|P^G_p(x, y)| = O(p^{-\infty}).$$

**Remark** Paoletti proved such an estimate for $x, y \in M \setminus (U \cup R)$ with $R$ the set of unstable points of the $G$-action.
This result shows that when $p \to +\infty$, $P^G_p(x, y)$ “localizes” near $\mu^{-1}(0)$ (and thus close to $M_G$).

Since $G$ acts on $\mu^{-1}(0)$ freely, we may assume that $U$ is small enough so that

$$\pi : U \to U/G$$

is a principal fibration with fiber $G$.

Since $P^G_p(x, y)$ is $G$-invariant, we can push it to $U/G$ and “consider” $P^G_p(x, y)$ for $x, y \in U/G$.

While approximately, we identify $U/G$ with the normal bundle $N_G$ to the submanifold $M_G \subset U/G$. So we tautologically denote an element in $U/G$ by $Z$ in the normal spaces.
Let \( h \) be the function on \( U/G \) such that for any \( x \in U/G \),

\[
h(x) = \sqrt{\text{vol}(\pi^{-1}(x))}.
\]

**Theorem** (Ma-Zhang, 2005) There exist polynomials \( Q_r(Z) \) in \( Z \) with the same parity as \( r \), and with

\[
Q_0(Z) = 1,
\]

such that there exists \( C' > 0 \) such that for any \( l, m, m' \in \mathbb{N} \), there exists \( C > 0 \) such that for \( x_0 \in M_G \), \( Z \in N_G \), \( |Z| \leq \varepsilon_0 \),

\[
\left| p^{-n + \frac{n_0}{2}} h^2(Z) P^G_p(Z, Z) - 2^{-\frac{n_0}{2}} \exp\left( -2\pi p|Z|^2 \right) \sum_{r=0}^{l} Q_r(\sqrt{p}Z)p^{-r} \right|_{C^m(M_G)} \\
\leq Cp^{-(l+1)/2}(1 + \sqrt{p}|Z|)^{-m'} + O(p^{-\infty}),
\]

where \( n_0 = \dim G \).
If we take $Z = 0$ in this Theorem, then we get for $x_0 \in M_G$, 

**Corollary** (Ma-Zhang, 2005) One has 

$$Q^{(0)}(0) = 1$$ 

and 

$$\left| p^{-n+\frac{n_0}{2}} h^2(x_0) P^G_p(x_0, x_0) \right| - 2^{\frac{n_0}{2}} \sum_{r=0}^{l} Q^{(2r)}(0) p^{-r} \left| C^m(M_G) \right| \leq C p^{-l-1}.$$ 

**Remark** One uses the above Theorems and Corollary to get the Toeplitz properties in the geometric quantization framework.
§9. Basic idea of the proof

The proofs of the main results use techniques adapting from the works of Bismut-Lebeau, Dai-Liu-Ma and Ma-Marinescu.

One key step is to deform the Laplacian of the spin\(^c\) Dirac operator by a Casimir type operator.

To be more precise, let \( \{K_i\} \) be an orthonormal basis of the Lie algebra \( g \) of \( G \). Let \( L_{K_i} \) denote the induced infinitesimal action on \( \Gamma(L^p) \).

We define

\[
\mathcal{L}_p = \left(D^{L^p}\right)^2 - p \sum_{i=1}^{\dim G} L_{K_i} L_{K_i}.
\]
The following spectral gap property plays a key role.

**Theorem** (Ma-Zhang, 2005) There exist $\nu, C > 0$ such that for any $p \in \mathbb{N}$,

$$\text{ker } \mathcal{L}_p = \left( \ker D^{L^p} \right)^G,$$

$$\text{Spec}(\mathcal{L}_p) \subset \{0\} \cup [2\nu\nu - C, +\infty).$$

**Remark** When $G = \{1\}$, this result is due to Bismut-Vasserot and has played an essential role in the works of Dai-Liu-Ma and Ma-Marinescu on the asymptotic expansion of Bergman kernels.

**Remark** Indeed, many of the above result can be extended to the symplectic manifold case, and this is essentially done in the works of Dai-Liu-Ma and Ma-Marinescu in the non-equivariant case, and by Ma-Zhang in the geometric quantization framework.
Thank you!