Monomial Hopf algebras over fields of positive characteristic *

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Abstract

This paper can be looked as a continues work of [3] where the authors classified all the Hopf structures on monomial coalgebras over a field of characteristic zero and containing all roots of unity. Let k be a field of characteristic p. We give a necessary and sufficient condition for the monomial coalgebra $C_d(n)$ to admitting Hopf structure. We give all graded Hopf structure on $C_d(n)$ and construct a Hopf algebras filtration on it which will help us to discuss Andruskiewitsch-Schneider conjecture. At last, we give a description of all monomial Hopf algebras.

1 Introduction

There are several works to construct neither commutative nor cocommutative Hopf algebras via quivers (see [3][4][7]). An advantage for this construction is that a natural basis consisting of paths is available, and one can relate the properties of a quiver to the ones of the corresponding Hopf structures.

In [3], the authors have classified all the finite-dimensional Hopf structures on a monomial algebra, or equivalently, on a monomial coalgebra over a field of characteristic zero and containing all roots of unity. As a continuous work of [3], we want to classify Hopf structures on a monomial coalgebra when the characteristic of the field is positive.

On the one hand, we note that there do exist Hopf structures on monomial coalgebra when the characteristic of the field is *not* zero (see Example 2.1). On the other hand,

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we note that there exists essential difference on the monomial Hopf structures when the characteristic of the base field is different. For example, we can get examples of finite-dimensional monomial (of course pointed) Hopf algebras which can *not* generated by group-like and primitive elements when characteristic of the base field is p. But, if the characteristic is zero, we can not get such examples (see Section 3). These facts stimulate us to write this paper.

Just like in [3], our main task is to study the Hopf structures on $C_d(n)$, where $C_d(n)$ is the sub-coalgebra of path coalgebra kZ_n^c with basis the set of paths of length strictly smaller than d (see Section 2). It turns out that the coalgebra $C_d(n)$ admits a Hopf structure if and only if there exists a d_0-th primitive root of unity $q \in k$ with $d_0|n$ and $r \geq 0$ such that $d = p^r d_0$, where p is the characteristic of k(Theorem 3.4). By this conclusion, we can get a Hopf algebras filtration for $C_d(n)$ which will help us to discuss Andruskiewitsch-Schneider conjecture. We give all the graded (with length grading) Hopf structures on $C_d(n)$ (see Theorem 3.5). As for non-graded case, unfortunately, we can not give them all. But we show there dose exist non-graded structures on $C_d(n)$ (see Example 3.1).

Then, we discuss any monomial Hopf algebra H, it shows that its every indecomposable component as coalgebras is isomorphic to $C_d(n)$ ($d \geq 2$) or the field k simultaneous (see Lemma 4.1). At last, by a theorem of Montgomery (Theorem 3.2 in [9]), we can describe the structure of monomial Hopf algebras.

2 Preliminaries

Throughout this paper, k denotes a field of characteristic p. By an algebra we mean a finite-dimensional associative k-algebra with identity element.

For completeness, we recall some definitions, notations and results in [3].

Quivers considered here are always finite. Given a quiver $Q = (Q_0, Q_1)$ with Q_0 the set of vertices and Q_1 the set of arrows, denote by kQ, kQ^a, and kQ^c, the k-space with basis the set of all paths in Q, the path algebra of Q, and the path coalgebra of Q, respectively. Note that they are all graded with respective to length grading. For $\alpha \in Q_1$, let $s(\alpha)$ and $t(\alpha)$ denote respectively the starting and ending vertex of α.

Recall that the comultiplication of the path coalgebra kQ^c is defined by

$$\Delta(p) = \sum_{\beta\alpha=p} \beta \otimes \alpha = \alpha_l \cdots \alpha_1 \otimes s(\alpha_1) + \sum_{i=1}^{l-1} \alpha_l \cdots \alpha_{i+1} \otimes \alpha_i \cdots \alpha_1 + t(\alpha_l) \otimes \alpha_l \cdots \alpha_1$$

for each path $p = \alpha_l \cdots \alpha_1$ with each $\alpha_i \in Q_1$; and $\varepsilon(p) = 0$ for $l \geq 1$ and 1 if $l = 0$ (Note that $l = 0$ means p is a vertex). This is a pointed coalgebra.

Let C be a coalgebra. The set of group-like elements is defined to be

$$G(C) := \{c \in C | \Delta(c) = c \otimes c,\ c \neq 0\}$$

It is clear $\varepsilon(c) = 1$ for $c \in G(C)$. For $x, y \in G(C)$, denote by

$$P_{x,y}(C) := \{c \in C | \Delta(c) = c \otimes x + y \otimes c\}$$

the set of x, y-primitive elements in C. It is clear that $\varepsilon(c) = 0$ for $c \in P_{x,y}(C)$. Note that $k(x - y) \subseteq P_{x,y}(C)$. An element $c \in P_{x,y}(C)$ in non-trivial if $c \notin k(x - y)$. Clearly, $G(kQ^c) = Q_0$ and (Lemma 1.1 in [3])

Lemma 2.1 *For $x, y \in Q_0$, we have $P_{x,y}(C) = y(kQ_1)x \oplus k(x-y)$ where $y(kQ_1)x$ denotes the k-space spanned by all arrows from x to y. In particular, there is a non-trivial element in kQ^c if and only if there is an arrow from x to y in Q.*

An ideal I of kQ^a is *admissible* if $J^N \subseteq I \subseteq J^2$ for some positive integer $N \geq 2$, where J is the ideal generated by all arrows. An algebra A is *elementary* if $A/R \cong k^n$ as algebras for some n, where R is the Jacobson radical of A. For an elementary algebra A, there is a (unique) quiver Q, and an admissible ideal I of kQ^a, such that $A \cong kQ^a/I$. (See [2]).

An algebra A is *monomial* if there exists an admissible ideal I generated by some paths in Q such that $A \cong kQ^a/I$. Dually, the authors of [3] gave the definition of *monomial coalgebras.*

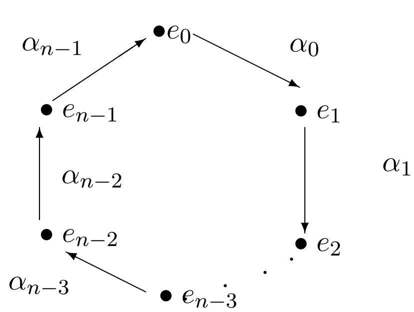
Definition 2.1 *A subcoalgebra C of kQ^c is called monomial provided that the following conditions are satisfied:*

(1) C contains all vertices and arrows in Q;

(2) C is contained in subcoalgebra $C_d(Q) := \oplus_{i=0}^{d-1} kQ(i)$ for some $d \geq 2$, where $Q(i)$ is the set of all paths of length i in Q;

(3) C has a basis consisting of paths.

Consider the following quiver.



We denote this quiver by Z_n and call it the *basic cycle of length n* . Denote by p_i^l the path in Z_n of length l starting at e_i. Thus we have $p_i^0 = e_i$ and $p_i^1 = \alpha_i$.

For each n-th root $q \in k$ of unity, Cibils and Rosso [4] have defined a graded Hopf algebra structure $kZ_n(q)$ (with length grading) on the path coalgebra kZ_n^c by

$$p_i^l \cdot p_j^m = q^{jl} \begin{pmatrix} m+l \\ l \end{pmatrix}_q p_{i+j}^{l+m}$$

with antipode S mapping p_i^l to $(-1)^l q^{-\frac{l(l+1)}{2}-il} p_{n-l-i}^l$, where $\begin{pmatrix} m+l \\ l \end{pmatrix}_q$ is the Gaussian binomial coefficient defined by $\begin{pmatrix} m+l \\ l \end{pmatrix}_q := \frac{(l+m)!_q}{l!_q m!_q}$ where $l!_q = 1_q \cdots l_q,\ l_q := 1 + q + \cdots + q^{l-1}$.

In the following, denote $C_d(Z_n)$ by $C_d(n)$. That is, $C_d(n)$ is the subcoalgebra of kZ_n^c with basis the set of all paths of length strictly less than d.

Clearly, if $\begin{pmatrix} m+l \\ l \end{pmatrix}_q \equiv 0$ for all $0 < m, l < d$ and $m + l \geq d$, then $C_d(n)$ will be a graded sub-Hopf algebra of $kZ_n(q)$.

Example 2.1 *Let q be a d_0-th primitive root of unity with $d_0|n$. Assume $q \in k$. In next section (Proposition 3.3), we will prove that if $d = p^t d_0$ for some nonnegative integer t, then $\begin{pmatrix} d \\ l \end{pmatrix}_q = 0$ for all $0 < l < d$. By a standard identity about Gaussian binomial coefficients (See [8]), that is,*

$$\begin{pmatrix} n \\ k \end{pmatrix}_q = \begin{pmatrix} n-1 \\ k-1 \end{pmatrix}_q + q^k \begin{pmatrix} n-1 \\ k \end{pmatrix}_q$$

we have $\begin{pmatrix} m+l \\ l \end{pmatrix}_q = 0$ for all $0 < m, l < d$ and $m + l \geq d$. Therefore, by discussion above, $C_{p^t d_0}(n)$ is a graded sub-Hopf algebra of $kZ_n(q)$. We denote this Hopf algebra by $C(d_0, t, n, q)$

Next conclusion (Lemma 2.3 in [3]) shows the importance of $C_d(n)$.

Lemma 2.2 *Let A be an indecomposable monomial coalgebra. Then A is coFrobenius (i.e. A^* is Frobenius) if and only if $A = k$ or $A \cong C_d(n)$ for some positive integers n and d, with $d \geq 2$.*

The following lemma (Lemma 3.3 in [3]) is needed in our proof of Theorem 3.4.

Lemma 2.3 *Suppose that there is a Hopf algebra structure on $C_d(n)$. Then up to a Hopf algebra isomorphism we have*

$$p_i^l \cdot p_j^m \equiv q^{jl} \begin{pmatrix} m+l \\ l \end{pmatrix}_q p_{i+j}^{l+m} \quad (mod\ C_{l+m}(n))$$

for $0 \leq i, j \leq n-1$, and for $l, m \leq d-1$, where $q \in k$ is an n-th root of unity.

3 Hopf structures on $C_d(n)$ and Andruskiewitsch-Schneider conjecture

The aim of section is to give an equivalent condition for $C_d(n)$ to admitting Hopf structures (Theorem 3.4), and then classify all the graded Hopf structures on $C_d(n)$. At last, we will construct a Hopf algebras filtration of $C_d(n)$ which will help us to discuss Andruskiewitsch-Schneider conjecture.

By a direct analysis from the definition of the Gaussian binomial coefficients we have

Lemma 3.1 *Let q be above. Then* $\begin{pmatrix} m+l \\ l \end{pmatrix}_q = 0$ *if and only if*

$$\left[\frac{m+l}{d_0}\right]+\left[\frac{m+l}{pd_0}\right]+\left[\frac{m+l}{p^2d_0}\right]+\cdots+\left[\frac{m+l}{p^id_0}\right]+\cdots$$
$$-(\left[\frac{m}{d_0}\right]+\left[\frac{m}{pd_0}\right]+\left[\frac{m}{p^2d_0}\right]+\cdots+\left[\frac{m}{p^id_0}\right]+\cdots)$$
$$-(\left[\frac{l}{d_0}\right]+\left[\frac{l}{pd_0}\right]+\left[\frac{l}{p^2d_0}\right]+\cdots+\left[\frac{l}{p^id_0}\right]+\cdots)>0$$

Lemma 3.2 *Let $m>1$ be a positive integer. Then*

$$[m]+\left[\frac{m}{p}\right]+\left[\frac{m}{p^2}\right]+\cdots+\left[\frac{m}{p^i}\right]+\cdots$$
$$-([n]+\left[\frac{n}{p}\right]+\left[\frac{n}{p^2}\right]+\cdots+\left[\frac{n}{p^i}\right]+\cdots)$$
$$-([m-n]+\left[\frac{m-n}{p}\right]+\left[\frac{m-n}{p^2}\right]+\cdots+\left[\frac{m-n}{p^i}\right]+\cdots)>0$$

for all $0<n<m$ if and only if $m=p^t$ for $t\geq 1$

Proof: For simplicity, denote

$$[m]+\left[\frac{m}{p}\right]+\left[\frac{m}{p^2}\right]+\cdots+\left[\frac{m}{p^i}\right]+\cdots$$
$$-([n]+\left[\frac{n}{p}\right]+\left[\frac{n}{p^2}\right]+\cdots+\left[\frac{n}{p^i}\right]+\cdots)$$
$$-([m-n]+\left[\frac{m-n}{p}\right]+\left[\frac{m-n}{p^2}\right]+\cdots+\left[\frac{m-n}{p^i}\right]+\cdots)$$

by $I_{m,n}$

"If Part: " Clearly, $\left[\frac{m}{p^i}\right]-\left[\frac{n}{p^i}\right]-\left[\frac{m-n}{p^i}\right]\geq 0$ for all $i\in N$. So, in order to prove the conclusion, it is enough to find one $j\in N$ such that $\left[\frac{m}{p^j}\right]-\left[\frac{n}{p^j}\right]-\left[\frac{m-n}{p^j}\right]>0$. In fact, let $j=t$, $1=\left[\frac{p^t}{p^t}\right]>\left[\frac{n}{p^t}\right]+\left[\frac{m-n}{p^t}\right]=0$ for all $0<n<m$. Thus, we proved the sufficiency.

"Only if Part: " Clearly, $p \leq m$. At first, we claim that $p|m$. Otherwise, assume $m = kp + r$ with $k \geq 1$ and $0 < r < p$. Let $n = kp$, then it is easy to see that $I_{m,n} = 0$ now. It is contradict to the assumption.

Thus, generally, let $m = p^r(a_l p^l + \cdots + a_1 p + a_0)$ where $r \geq 1$ and $a_i < p$ for $i = 1, 2, \ldots, l$. Let $n = a_0 p^r$ and then $m - n = p^r(a_l p^l + \cdots + a_1 p)$. Then, for any $0 \leq j \leq r$, $\left[\frac{m}{p^j}\right] = p^{r-j}(a_l p^l + \cdots + a_1 p + a_0)$, $\left[\frac{m-n}{p^j}\right] = p^{r-j}(a_l p^l + \cdots + a_1 p)$ and $\left[\frac{n}{p^j}\right] = a_0 p^{r-j}$. This implies $\left[\frac{m}{p^j}\right] = \left[\frac{m-n}{p^j}\right] + \left[\frac{n}{p^j}\right]$ when $j \leq r$. If $j > r$, then $m = p^j(a_l p^{l-(j-r)} + \cdots + a_{j-r}) + a_{j-r-1}p^{j-1} + \cdots + a_0 p^r$. But, $a_{j-r-1}p^{j-1} + \cdots + a_0 p^r \leq (p-1)p^{j-1} + \cdots + (p-1)p^r = p^j - p^r < p^j$. Thus $\left[\frac{m}{p^j}\right] = a_l p^{l-(j-r)} + \cdots + a_{j-r}$, $\left[\frac{m-n}{p^j}\right] = a_l p^{l-(j-r)} + \cdots + a_{j-r}$ and $\left[\frac{n}{p^j}\right] = 0$. This implies $\left[\frac{m}{p^j}\right] = \left[\frac{m-n}{p^j}\right] + \left[\frac{n}{p^j}\right]$ for $j > r$. Summarizing above discussion, we have $\left[\frac{m}{p^j}\right] = \left[\frac{n}{p^j}\right] + \left[\frac{m-n}{p^j}\right]$ for all j and thus $I_{m,n} = 0$. It is contradict to assumption. Therefore we know that $a_0 = 0$ or $a_i = 0$ for all $l \leq i \leq 1$. We claim there is only one $a_i \neq 0$. In fact, if $a_0 \neq 0$, then above conclusion asserts $a_i = 0$ for all $l \leq i \leq 1$. If $a_0 = 0$, then repeat above discussion shows that $a_1 = 0$ or $a_i = 0$ for all $l \leq i \leq 2$. So, at last, we have a unique a_i such that $m = a_i p^{r+i}$.

If $a_i > 1$, then we can write $a_i = l_1 + l_2$ with $l_1 l_2 \neq 0$. Let $n = l_1 p^{r+i}$, then it is easy to see that $I_{m,n} = 0$. It is also contradict to the assumption. Thus $m = p^{r+i}$ and we get the desire conclusion. $\square$

With these preparations, we can give the following conclusion which will help us to give our main results (Theorem 3.4) in this section.

Proposition 3.3 *Let $q \in k$ be a d_0-th primitive root of unity. Then* $\begin{pmatrix} d \\ n \end{pmatrix}_q = 0$ *for all* $0 < n < d$ *if and only if* $d = p^r d_0$ *for some nonnegative integer* r.

Proof: For simplicity, denote

$$\left[\frac{m}{d_0}\right] + \left[\frac{m}{pd_0}\right] + \left[\frac{m}{p^2 d_0}\right] + \cdots + \left[\frac{m}{p^i d_0}\right] + \cdots$$
$$-(\left[\frac{n}{d_0}\right] + \left[\frac{n}{pd_0}\right] + \left[\frac{n}{p^2 d_0}\right] + \cdots + \left[\frac{n}{p^i d_0}\right] + \cdots)$$
$$-(\left[\frac{m-n}{d_0}\right] + \left[\frac{m-n}{pd_0}\right] + \left[\frac{m-n}{p^2 d_0}\right] + \cdots + \left[\frac{m-n}{p^i d_0}\right] + \cdots)$$

by $I_{m,n,q}$.

"If Part " Similarly to the proof of Lemma 3.2, $1 = \left[\frac{p^r d_0}{p^r d_0}\right] > \left[\frac{n}{p^r d_0}\right] + \left[\frac{p^r d_0 - n}{p^t}\right] = 0$ for all $0 < n < p^r d_0$. That's to say, $I_{d,n,q} = 0$ for all $n < d$ and thus $\begin{pmatrix} d \\ n \end{pmatrix}_q = 0$ for all $0 < n < d$ according to Lemma 3.1.

"Only if Part " Clearly, $d \geq d_0$. We claim $d_0|d$. If not, $d = kd_0 + r$ with $k \geq 1$ and $0 < r < d_0$. Let $n = kd_0$, then it is easy to see that $I_{d,n,q} = 0$ and thus $\begin{pmatrix} d \\ n \end{pmatrix}_q \neq 0$ by Lemma 3.1. It is contradict to the assumption.

So, we now have $\frac{d}{d_0}$ is an positive integer and denote it by m. If $m = 1$, then $d = p^0 d_0$.If $m > 1$, then Lemma 3.1 and Lemma 3.2 assert that $m = p^r$ and thus $d = md_0 = p^r d_0$ for some $r \geq 1$. Therefore, $d = p^r d_0$ for some $r \geq 0$. □

Theorem 3.4 *$C_d(n)$ admits a Hopf algebra structure if and only if there exist a d_0-th primitive root of unity $q \in k$ with $d_0|n$ such that $d = p^r d_0$ for some $r \geq 0$.*

Proof: "If Part " By Proposition 3.3, $\begin{pmatrix} d \\ n \end{pmatrix}_q = 0$ for all $0 < n < d$. So, Example 2.1 implies the sufficiency.

"Only if Part " If there is a Hopf structure on $C_d(n)$, then Lemma 2.3 implies there is a n-th root of unity $q \in k$ such that

$$p_i^l \cdot p_j^m \equiv q^{jl} \begin{pmatrix} m+l \\ l \end{pmatrix}_q p_{i+j}^{l+m} \pmod{C_{l+m}(n)}$$

There is no harm to assume that q is a d_0-th primitive root of unity. Since the length of all paths in $C_d(n)$ is strictly less than d, $\begin{pmatrix} m+l \\ l \end{pmatrix}_q = 0$ for all $0 < l, m < d$. In particular, $\begin{pmatrix} d \\ n \end{pmatrix}_q = 0$ for all $0 < n < d$. Thus, by Proposition 3.3, $d = p^r d_0$ for $r \geq 0$. □

Theorem 3.5 *Any graded Hopf structure (with length grading) on $C_d(n)$ is isomorphic to some $C(d_0, t, n, q)$, where $C(d_0, t, n, q)$ is given in Example 2.1.*

Proof: By Lemma 2.3 and the proof of Theorem 3.4 we see that any graded Hopf structure (with length grading) is isomorphic to $C(d_0, t, n, q)$ for some d_0-th primitive root of unity q with $d_0|n$ and $d = p^t d_0$. □

The following example will show that there exist non-graded Hopf structures on $C_d(n)$. But, unfortunately, we can not give a complete classification in this case.

Example 3.1 *We give a non-graded Hopf structure on $C_{pd_0}(n)$. Let $q \in k$ be a d_0-th primitive root of unity and $d_0|n$. We also denote p_i^l the path in Z_n of length l staring at e_i. Define, for $s_1 d_0 + r_1 \leq pd_0$ and $s_2 d_0 + r_2 \leq pd_0$,*

$$p_i^{s_1 d_0 + r_1} p_j^{s_2 d_0 + r_2} = 0 \quad \text{if } r_1 + r_2 \geq d_0$$

and

$$p_i^{s_1d_0+r_1}p_j^{s_2d_0+r_2} = q^{r_1j}\begin{pmatrix}(s_1+s_2)d_0+r_1+r_2\\ s_1d_0+r_1\end{pmatrix}_q p_{i+j}^{(s_1+s_2)d_0+r_1+r_2}$$

if $r_1+r_2<d_0$ *and* $(s_1+s_2)d_0+r_1+r_2<pd_0$
and

$$p_i^{s_1d_0+r_1}p_j^{s_2d_0+r_2} = q^{r_1j}\frac{((d_0)!_q)^p((s_1+s_2)d_0+r_1+r_2)!_q}{(s_1d_0+r_1)!_q(s_2d_0+r_2)!_q}(p_{i+j}^{(s_1+s_2-p)d_0+r_1+r_2}-p_{i+j+pd_0}^{(s_1+s_2-p)d_0+r_1+r_2})$$

if $r_1+r_2<d_0$ *and* $(s_1+s_2)d_0+r_1+r_2\geq pd_0$ *with*

$$S(p_i^l) := (-1)^l q^{-\frac{l(l+1)}{2}-il}p_{n-l-i}^l$$

for $l\leq pd_0$. *This is indeed a Hopf algebra with identity element* $p_0^0=e_0$ *and note that it is not graded with respect to length grading. We can see that, as an algebra, it is generated by* p_1^0, p_0^1 *and* $p_0^{d_0}$. *An advantage of this construction is that we have a natural basis. We can also get this Hopf algebra through generators and relations.*

Let n, d_0, p, q *be above. We define* $A(n,d_0,p,q)$ *as follows. As an algebra, it is generated by* g,x,y *with relations*

$$g^n=1,\quad x^{d_0}=0,\quad y^p=1-g^{pd_0},\quad xg=qgx,\quad yg=gy,\quad yx=xy$$

Its comultiplication Δ, *counit* ε *and the antipode defined by*

$$\Delta(g)=g\otimes g,\quad \Delta(x)=x\otimes 1+g\otimes x,\quad \Delta(y)=y\otimes 1+g^{d_0}\otimes y+\Sigma_{i=1}^{d_0-1}\frac{1}{(d_0-i)!_q(i)!_q}g^ix^{d_0-i}\otimes x^i$$

$$\varepsilon(g)=1,\quad \varepsilon(x)=\varepsilon(y)=0$$

$$S(g)=g^{n-1},\quad S(x)=-g^{n-1}x,\quad S(y)=-g^{n-d_0}y$$

Through tedious but straightforward computation, we can prove $A(n,d_0,p,q)$ *is indeed a Hopf algebra. We can also see that* $A(n,d_0,p,q)\cong C_{pd_0}(n)$ *as Hopf algebras by* $g\mapsto p_1^0$, $x\mapsto p_0^1$ *and* $y\mapsto p_0^{d_0}$.

Let $q\in k$ be a d_0-th primitive root of unity with $d_0|n$, then Theorem 3.4 implies that we have a series of Hopf algebras

$$C_{d_0}(n)\subsetneq C_{pd_0}(n)\subsetneq C_{p^2d_0}(n)\subsetneq\cdots\subsetneq C_{p^id_0}(n)\subsetneq\cdots\qquad(*)$$

Noth that, if $d_0=1$, $C_1(n)$ is indeed *not* a monomial coalgebra since it does not contain any arrow. But it is a Hopf algebra and clearly isomorphic to a group algebra.

If $d_0\geq 2$, then $C_{d_0}(n)$ contains all vertices and arrows. By Lemma 2.1, all group-like and primitive elements of $kZ_n(q)$ lie in $C_{d_0}(n)$. Thus any path β whose length is not less than d_0 can not be generated by group-like and primitive elements since $\beta\notin C_{d_0}(n)$. Therefore, if $d_0\geq 2$, then for any $t\geq 1$, $C_{p^td_0}$ can not be generated by group-like and primitive elements as Hopf algebras. This supplies many counter-examples for the following Andruskiewitsch-Schneider Conjecture (see [1]) when the characteristic of field is positive.

Andruskiewitsch-Schneider Conjecture Let H be a finite-dimensional pointed Hopf algebra over a algebraically closed field of characteristic zero, then it is generated by group-like and primitive elements.

When the characteristic of k is zero, above Hopf algebras serious $(*)$ will not happen. In fact, in [3], the authors have shown (See the proof of Theorem 3.1 of [3]) that $d = d_0$, that is $C_d(n) = C_{d_0}(n)$ now. In this case, we can not deny above conjecture since $C_{d_0}(n)$ is indeed generated by group-like and primitive elements (See Theorem 3.6 in [3]).

4 On monomial Hopf algebras

The main aim of this section is to discuss the structures of monomial Hopf algebras. Recall that a Hopf algebra is monomial if it is monomial as coalgebra. We firstly prove a result which is similar to Theorem 5.1 in [3].

Lemma 4.1 *Let C be a monomial coalgebra. Then C admits a Hopf algebra structure if and only if $C \cong k \oplus \cdots \oplus k$ as a coalgebra, or*

$$C \cong C_d(n) \oplus \cdots \oplus C_d(n)$$

as a coalgebra for some $d = p^r d_0 \geq 2$ with $d_0 | n$ and there exists a d_0-th primitive root of unity $q \in k$.

The proof of this lemma is also similar to that of Theorem 5.1 in [3]. For completeness, we write it out.

Proof: "If Part " By assumption, we have $C = C_1 \oplus \cdots \oplus C_l$ as a coalgebra, where each $C_i \cong C_1$ as coalgebras for $1 \leq i \leq l$ and C_1 admits a Hopf structure H_1 by Theorem 3.4. Then $H_1 \otimes kG$ is a Hopf structure on C, where G is any group of order l. This gives the sufficiency.

"Only if Part " Let C be a monomial coalgebra admitting a Hopf structure. Since a finite-dimensional Hopf algebra is coFrobenius, it follows follows from Lemma 2.2 that as a coalgebra C has the form $C = C_1 \oplus \cdots \oplus C_l$ with each C_i indecomposable as coalgebra, and $C_i = k$ or $C_i = C_{d_i}(n_i)$ for some n_i and $d_i \geq 2$.

We claim that if there exists a $C_i = k$, then $C_j = k$ for all j. In fact, otherwise, let $C_j = C_d(n)$ for some j. Let α be an arrow in C_j from x to y. Let h be the unique group-like element in $C_i = k$. Since the set $G(C)$ of the group-like elements of C forms a group, it follows that there exists an element $g \in G(C)$ such that $h = gx$. Then $g\alpha$ if a h, gy-primitive element in C. But according to the coalgebra decomposition $C = C_1 \oplus \cdots \oplus C_l$ with $C_i = kh$, C has no h, gy-primitive elements. A contradiction.

Thus, by claim above, if $C \neq k \oplus \cdots \oplus k$, then C is of the form

$$C = C_{d_1}(n_1) \oplus \cdots \oplus C_{d_l}(n_l)$$

as coalgebras, with each $d_i \geq 2$. Assume that the identity element 1 of $G(C)$ is contained in $C_1 = C_{d_1}(n_1)$. It follows from a theorem of Montgomery (Theorem 3.2 in [9]) that C_1 is a sub-Hopfalgebra of C, and that

$$g_i^{-1} C_{d_i}(n_i) = C_{d_i}(n_i) g_i^{-1} = C_{d_1}(n_1)$$

for any $g_i \in G(C_{d_i}(n_i))$ and for each i. By comparing the numbers of group-like elements in $C_{d_i}(n_i)$ and in $C_{d_1}(n_1)$ we have $n_i = n_1 = n$ for each i. While by comparing the k-dimensions we see that $d_i = d_1 = d$ for each i. Now, since $C_1 = C_d(n)$ is a Hopf algebra, it follows Theorem 3.4, there exist a d_0-th primitive root of unity $q \in k$ with $d_0|n$ and $r \geq 0$ such that $d = p^r d_0$. $\square$

Theorem 4.2 *Let H be a non-semisimple monomial Hopf algebra over k. Then there exist a d_0-th primitive root of unity $q \in k$ with $d_0|n$, $r \geq 0$ and $d = p^r d_0 \geq 2$ such that*

$$H \cong C_d(n) \oplus \cdots \oplus C_d(n)$$

as coalgebras and

$$H \cong C_d(n) \#_\sigma k(G/N)$$

as Hopf algebras, where $G = G(H)$ and $N = G(C_d(n))$.

Proof: By Theorem 3.2 in [9] and Lemma 4.1 above, we can get this conclusion directly. $\square$

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