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A MATHEMATICAL THEORY OF THE TOPOLOGICAL VERTEX

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Abstract. We develop a mathematical theory of the topological vertex, an algorithm proposed by M. Aganagic, A. Klemm, M. Mariño, and C. Vafa on effectively computing Gromov-Witten invariants of toric Calabi-Yau threefold.

1. Introduction

In [1], M. Aganagic, A. Klemm, M. Mariño and C. Vafa proposed an algorithm to compute Gromov-Witten invariants in all genera of any toric Calabi-Yau threefold. By virtual localization [11], Gromov-Witten invariants of a toric Calabi-Yau threefold can be reduced to Hodge integrals, which can be computed recursively [29, 16, 9]. However, the algorithm proposed in [1] does not involve Hodge integrals and is significantly more effective. It can be summarized as follows.

O1. There exist certain open Gromov-Witten invariants counting holomorphic maps from bordered Riemann surfaces to $\mathbb{C}^3$ with boundary mapped to three particular Lagrangian submanifolds. The *topological vertex* $C_{\vec{\mu}}(\lambda;\mathbf{n})$ is a generating function of such invariants, where $\vec{\mu}=(\mu^1,\mu^2,\mu^3)$ is a triple of partitions and $\mathbf{n}=(n_1,n_2,n_3)$ is a triple of integers.

O2. The Gromov-Witten invariants of any toric Calabi-Yau threefold can be expressed in terms of $C_{\vec{\mu}}(\lambda;\mathbf{n})$ by explicit gluing algorithms.

O3. By the duality between Chern-Simons theory and Gromov-Witten theory, the topological vertex is given by

$$C_{\vec{\mu}}(\lambda;\mathbf{n}) = q^{\left(\sum_{i=1}^{3}\kappa_{\nu^i} n_i\right)/2}\mathcal{W}_{\vec{\mu}}(q) \tag{1.1}$$

where $q=e^{\sqrt{-1}\lambda}$, and $\mathcal{W}_{\vec{\mu}}(q)$ is a combinatorial expression related to the Chern-Simons link invariants of a particular link. (The precise definitions of κ_μ and $\mathcal{W}_{\vec{\mu}}(q)$ are given in Section 3.)

To justify the above algorithm mathematically, one encounters the following difficulties. First of all, open Gromov-Witten theory beyond disc instantons is still in a very primitive stage, and in particular, a mathematical definition of the invariants in O1 is not known. Secondly, the gluing algorithms in O2 contradict geometric intuition because a three dimensional submanifold does not split a six dimensional manifold into two pieces. Finally, the duality in O3 is not well-understood mathematically.

In this paper, we overcome the above difficulties by developing a mathematical theory of the topological vertex based on relative Gromov-Witten theory [19, 14, 15, 17, 18]. Our results can be summarized as follows.

R1. We introduce the notion of formal toric Calabi-Yau (FTCY) graphs, which is a refinement and generalization of the graph associated to a toric Calabi-Yau threefold. An FTCY graph Γ determines a relative FTCY threefold $Y_\Gamma^{\mathrm{rel}} = (\hat{Y}, \hat{D})$, where $\hat{Y}$ is a formal scheme with at most normal crossing singularities, $\hat{D}$ is a possibly disconnected smooth divisor in $\hat{Y}$, and

$$\det\left(\Omega_{\hat{Y}}(\log \hat{D})\right) \cong \mathcal{O}_{\hat{Y}}.$$

R2. We define *formal relative Gromov-Witten invariants* for relative FTCY threefolds (**Theorem 5.7**). These invariants include the Gromov-Witten invariants of (smooth) toric Calabi-Yau threefolds as special cases.

R3. We show that the formal relative Gromov-Witten invariants in R2 satisfy the degeneration formula of relative Gromov-Witten invariants of projective varieties (**Theorem 7.5**).

R4. Any smooth relative FTCY threefold can be degenerated to indecomposable ones, whose isomorphism classes are determined by a triple of integers $\mathbf{n} = (n_1, n_2, n_3)$. By degeneration formula, the formal relative Gromov-Witten invariants in R2 can be expressed in terms of the generating function $\tilde{C}_{\vec{\mu}}(\lambda; \mathbf{n})$ of formal relative Gromov-Witten invariants of an indecomposable FTCY threefold (**Proposition 7.4**). This degeneration formula agrees with the gluing algorithms described in O2.

R5. We derive that (**Proposition 6.5**, **Theorem 8.1**)

$$\tilde{C}_{\vec{\mu}}(\lambda; \mathbf{n}) = q^{\left(\sum_{i=1}^3 \kappa_{\nu^i} n_i\right)/2} \tilde{\mathcal{W}}_{\vec{\mu}}(q), \tag{1.2}$$

where $\tilde{\mathcal{W}}_{\vec{\mu}}(q)$ is a combinatorial expression defined in Section 3.3. The expression $\tilde{\mathcal{W}}_{\vec{\mu}}(q)$ coincides with $\mathcal{W}_{\vec{\mu}}(q)$ when one of the partitions is empty (**Corollary 8.8**) and in all the low degree cases that have been checked.

By virtual localization, the formal relative Gromov-Witten invariants $\tilde{C}_{\vec{\mu}}(\lambda; \mathbf{n})$ that we define here can be expressed in terms of Hodge integrals (**Proposition 6.6**). Combined with (1.2) we obtain a formula of three-partition Hodge integrals (**Theorem 8.2**):

$$G^\bullet_{\vec{\mu}}(\lambda; \mathbf{w}) = \sum_{|\nu^i| = |\mu^i|} \prod_{i=1}^3 \frac{\chi_{\nu^i}(\mu^i)}{z_{\mu^i}} q^{\frac{1}{2}\left(\sum_{i=1}^3 \kappa_{\nu^i} \frac{w_{i+1}}{w_i}\right)} \tilde{\mathcal{W}}_{\vec{\nu}}(q) \tag{1.3}$$

where $G^\bullet_{\vec{\mu}}(\lambda; \mathbf{w})$ is a generating function of three-partition Hodge integrals, $\mathbf{w} = (w_1, w_2, w_3)$, $w_3 = -w_1 - w_2$, $w_4 = w_1$ (see Section 3 for precise definitions). This generalizes a formula of two-partition Hodge integrals (**Theorem 8.7**) proved in [23].

An important class of examples are local toric Calabi-Yau threefolds, by which we mean the total space of the canonical line bundle of a projective toric Fano surface (e.g. $\mathcal{O}(-3) \to \mathbb{P}^2$). In this case, only $\tilde{C}_{\mu^1, \mu^2, \emptyset}(\lambda; \mathbf{n})$ (or two-partition Hodge integrals) are involved. The algorithm in this case was described in [2]; explicit formula was given in [13] and derived in [32] by localization, using the formula of two-partition Hodge integrals.

It is worth mentioning that, assuming the existence of $C_{\vec{\mu}}(\lambda; \mathbf{n})$ and the validity of open string virtual localization, E. Diaconescu and B. Florea related $C_{\vec{\mu}}(\lambda; n_1, n_2, n_3)$ (at certain fractional n_i) to three-partition Hodge integrals, and derived the gluing algorithms in O2 by localization [6].

The rest of this paper is organized as follows. In Section 2, we give an overview of the theory of the topological vertex, and state the main results of this paper. In Section 3, we recall some definitions and previous results, and introduce some generating functions. R1 is carried out in Section 4. R2 is carried out in Section 5; the case when the relative FTCY threefold is indecomposable gives the mathematical definition of topological vertex, and the proof of its invariance (Theorem 5.9) is given in Appendix A. In Section 6, we express the topological vertex in terms of three-partition Hodge integrals and double Hurwitz numbers. In Section 7, we establish R3 and R4. In Section 8, we derive the combinatorial expression in R5. Some examples of the identity $\mathcal{W}_{\vec{\mu}}(q) = \tilde{\mathcal{W}}_{\vec{\mu}}(q)$ are listed in Appendix B.

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2. The theory of the topological vertex

In this section, we give an overview of the theory of the topological vertex and state the main results in this paper.

2.1. **Gromov-Witten invariants of Calabi-Yau threefolds.** We begin with the Gromov-Witten invariants (of not necessarily connected domains) of a general Calabi-Yau threefold Z, i.e., a smooth complex algebraic variety of dimension three with $c_1(T_Z) = 0$. Let

$$\mathcal{M}^\bullet_\chi(Z, d)$$

be the moduli space of stable morphisms $u : X \longrightarrow Z$ that have not necessarily connected domains X, have $2\chi(\mathcal{O}_X) = \chi$ and have fundamental classes $f_*([X]) = d \in H_2(Z;\mathbb{Z})$. It is a Deligne-Mumford stack with a perfect obstruction theory of virtual dimension zero. When $\mathcal{M}^\bullet_\chi(Z, d)$ is proper (which is true for any $d \in H_2(Z;\mathbb{Z})$ when Z is projective), the perfect obstruction theory defines a virtual fundamental 0-cycle, and the Gromov-Witten invariant is, by definition,

$$\deg[\mathcal{M}^\bullet_\chi(Z, d)]^{\mathrm{vir}} = \int_{[\mathcal{M}^\bullet_\chi(Z,d)]^{\mathrm{vir}}} 1 \in \mathbb{Q}. \tag{2.1}$$

The topological vertex is an algorithm of computing Gromov-Witten invariants of *toric* Calabi-Yau threefolds. Recall that a Calabi-Yau threefold Z is toric if it contains the algebraic torus $(\mathbb{C}^*)^3$ as an open dense subset, and the action of $(\mathbb{C}^*)^3$ on itself extends to Z. Note that toric Calabi-Yau threefolds are noncompact.

2.2. **The traditional algorithm.** Let Z be a smooth toric Calabi-Yau threefold. The $(\mathbb{C}^*)^3$ action on Z induces a $(\mathbb{C}^*)^3$ action on the moduli space, and we can apply virtual localization [11] to transform the degree on the left hand side of (2.1) to an integral over the fixed loci of $(\mathbb{C}^*)^3$ or any subtorus of $(\mathbb{C}^*)^3$.

For the case we are interested, the fixed loci can be described fairly easily. Under very mild assumption on Z, one can specify a distinguished subtorus $T \subset (\mathbb{C}^*)^3$ of rank 2 (see Section 4.1 for details). Let $T_\mathbb{R} \cong U(1)^2$ be the maximal compact subgroup of T. The geometry of Z^1, the union of 1-dimensional T orbit closures, can be described by the planar trivalent graph Γ which is the image of Z^1 under the moment map of the $T_{\mathbb{R}}$-action on Z. Each vertex of Γ corresponds to a T fixed point of Z, and each edge of $e \in \Gamma$ corresponds to an irreducible component C^e of Z^1. Let $E_i(\Gamma)$ denote the set of edges e of Γ such that C^e is a projective line, and let $V(\Gamma)$ denote the set of vertices of Γ.

Let
$$F_d^{\bullet Z}(\lambda) = \sum_{\chi} \lambda^{-\chi} \deg[\mathcal{M}_\chi^\bullet(Z,d)]^{\mathrm{vir}}.$$
be the generating function of disconnected Gromov-Witten invariants of degree d in all genera. Applying virtual localization, we obtain (see Section 7 for convention):

$$F_d^{\bullet Z}(\lambda) = \sum_{\sum_{e\in E_i(\Gamma)} |\nu^e|[C^e]=d} \prod_{e\in E_i(\Gamma)} (-1)^{n^e|\nu^e|} z_{\nu^e} \prod_{v\in V(\Gamma)} \sqrt{-1}^{\ell(\vec{\nu}^v)} G^\bullet_{\vec{\nu}^v}(\lambda;\mathbf{w}_v) \tag{2.2}$$

where $G^\bullet_{\vec{\mu}}(\lambda;\mathbf{w})$ is a generating function of Hodge integrals defined in Section 3.2. Hodge integrals can be computed recursively [29, 16, 9].

2.3. **The algorithm of AKMV.** The topological vertex proposed by AKMV is an algorithm of computing $F_d^{\bullet Z}(\lambda)$ for any toric Calabi-Yau threefold Z (as long as it is defined). We have summarized the algorithm of AKMV as O1 – O3 at the beginning of Section 1. Let $C_{\vec{\mu}}(\lambda;\mathbf{n})$ be the generating function of open Gromov-Witten invariants described in O1, and let
$$C_{\vec{\mu}}(\lambda) = C_{\vec{\mu}}(\lambda;\mathbf{0})$$
be the *topological vertex at the standard framing* in [1]. In our notation, the gluing formula in O2 reads

$$F_d^{\bullet Z}(\lambda) = \sum_{\sum_{e\in E_i(\Gamma)} |\nu^e|[C^e]=d} \prod_{e\in E_i(\Gamma)} (-1)^{(n^e+1)|\nu^e|} q^{\kappa_{\nu^e} n^e/2} \prod_{v\in V(\Gamma)} C_{\vec{\nu}^v}(\lambda), \tag{2.3}$$

where $q = e^{\sqrt{-1}\lambda}$. By (1.1) in O3,

$$C_{\vec{\mu}}(\lambda) = \mathcal{W}_{\vec{\mu}}(q), \tag{2.4}$$

where $\mathcal{W}_{\vec{\mu}}(q)$ is defined in Section 3.3. Therefore, the algorithm gives the following closed formula for $F_d^{\bullet Z}(\lambda)$:

$$F_d^{\bullet Z}(\lambda) = \sum_{\sum_{e\in E_i(\Gamma)} |\nu^e|[C^e]=d} \prod_{e\in E_i(\Gamma)} (-1)^{(n^e+1)|\nu^e|} q^{\kappa_{\nu^e} n^e/2} \prod_{v\in V(\Gamma)} \mathcal{W}_{\vec{\nu}^v}(\lambda). \tag{2.5}$$

The right hand side of (2.5) is a finite sum involving representations of symmetric groups. The numbers computed by the formula (2.5) agree with those computed by the traditional algorithm in all the examples that have been checked.

2.4. **Interpretation from relative Gromov-Witten theory.** Upon a closer examination of the gluing formula (2.3), one sees that it is reminiscent to the gluing formula of Gromov-Witten invariants [15, 19, 18]. Recall that given a smooth variety W_1 that degenerates (specializes) to a singular scheme W_0 that is a union of two smooth varieties Y_1 and Y_2 intersecting along a smooth divisor D, the Gromov-Witten invariants of W_1 can be computed using the relative Gromov-Witten invariants of pairs (Y_1,D) and (Y_2,D):

$$GW(W_1) = \sum_{\bullet} GW^{\mathrm{rel}}((Y_1,D),\bullet) * GW^{\mathrm{rel}}((Y_2,D),\bullet). \tag{2.6}$$

Here $*$ is a bilinear operation and $\bullet$ stands for certain combinatoric data associated to the splitting of stable maps to W_0 into pairs of relative stable maps to (Y_1, D) and (Y_2, D). Comparing with (2.3), should $C_{\vec{\mu}}(\lambda)$ be interpreted as a generating function of relative Gromov-Witten invariants, then (2.3) could be proved by applying the gluing formula (2.6).

On the surface, one can not degenerate a toric Calabi-Yau threefold to a union of relative Calabi-Yau threefolds so that the relative Gromov-Witten invariants of each component is exactly the topological vertex proposed in [1]. On the other hand, the formula (2.2) is the result of localization along the fixed loci

$$\mathcal{M}^{\bullet}_{\chi}(Z,d)^{T_{\mathbb{R}}} = \mathcal{M}^{\bullet}_{\chi}(Z,d)^{T},$$

which is contained in the moduli space $\mathcal{M}^{\bullet}_{\chi}(Z^1,d)$ of stable morphisms to Z^1 of identical topological types. To incorporate the obstruction theory of $\mathcal{M}^{\bullet}_{\chi}(Z,d)$, we shall work with stable maps to the formal completion of Z along Z^1, denoted $\hat{Z}$. Namely, we consider the moduli of stable morphisms to $\hat{Z}$ of the given topological type and show that its formal Gromov-Witten invariants—the ones computed using the virtual localization formula—is equal to the Gromov-Witten invariants of Z; we then degenerate $\hat{Z}$ into a formal scheme with normal crossing singularities, with each irreducible component the formal completion of the three coordinate axes in $\mathbb{A}^3$. Afterwards, we apply the gluing formula to this degeneration to derive a gluing formula similar to that in [1], with each entry the formal relative Gromov-Witten invariants of the indecomposable relative formal Calabi-Yau threefold.

2.5. **Main results.**

R1. The formal scheme $\hat{Z}$ and the T action on $\hat{Z}$ can be recovered from the trivalent graph Γ which is the image of Z^1 under the moment map of the $T_{\mathbb{R}}$-action. We introduce the notation of formal toric Calabi-Yau (FTCY) graphs which generalize such graphs. By a procedure similar to recovering $\hat{Z}$ from Γ, to each FTCY graph Γ we associate a relative FTCY threefold which is a pair

$$\mathrm{Y}^{\mathrm{rel}}_{\Gamma} = (\hat{Y}, \hat{D})$$

where $\hat{Y}$ is a formal scheme with at most normal crossing singularities, $\hat{D}$ is a possibly disconnected smooth divisor in $\hat{Y}$, and

$$\det\left(\Omega^1_{\hat{Y}}(\log \hat{D})\right) \cong \mathcal{O}_{\hat{Y}}.$$

A FTCY graph is regular if its associated relative FTCY threefolds is smooth.

When Γ comes from an honest toric Calabi-Yau threefold Z, we have $\hat{Z} = \hat{Y} \setminus \hat{D}$. The precise definitions of FTCY graphs and detailed construction of relative FTCY threefolds are given in Section 4.

R2. Let

$$\mathcal{M}^{\bullet}_{\chi,\vec{d},\vec{\mu}}(\hat{Y}^{\mathrm{rel}}_{\Gamma}, \hat{L})$$

be moduli space of relative stable morphisms $u: X \longrightarrow \hat{Y}_{\mathbf{m}}$ that have not necessarily connected domain X with $2\chi(\mathcal{O}_X) = \chi$, fundamental class $(\pi_{\mathbf{m}} \circ u)_*[X] = \vec{d} \in H_2(\hat{Y};\mathbb{Z})$, and ramification pattern $\vec{\mu}$ along $\hat{D}$, such that $u(X) \cap \hat{D} \subset \hat{L}$, where $\hat{L}$ is a T-invariant divisor $\hat{L}$ of $\hat{D}$. Here $\hat{Y}_{\mathbf{m}}$ is an expanded target with a morphism $\pi_{\mathbf{m}}: \hat{Y}_{\mathbf{m}} \longrightarrow \hat{Y}$. (Please consult Section 5.1 for precise definitions.) We have

Proposition 2.1. *$\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(\hat{Y}^{\mathrm{rel}}_\Gamma,\hat{L})$ is a separated formal Deligne-Mumford stack with a perfect obstruction theory of virtual dimension zero.*

The restriction of the perfect obstruction theory $[\mathcal{T}^1\to\mathcal{T}^2]$ to the fixed points set of the T-action on $\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(\hat{Y}^{\mathrm{rel}}_\Gamma,\hat{L})$ decomposes into the fixing part $[\mathcal{T}^{1,f}\to\mathcal{T}^{2,f}]$ and the moving part $[\mathcal{T}^{1,m}\to\mathcal{T}^{2,m}]$. The fixing part is a perfect obstruction theory on the fixed points set $\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(\hat{Y}^{\mathrm{rel}}_\Gamma,\hat{L})^T$, which is a proper Deligne-Mumford stack, so there is a virtual fundamental cycle

$$[\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(\hat{Y}^{\mathrm{rel}}_\Gamma,\hat{L})^T]^{\mathrm{vir}}\in A_*\Big(\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(\hat{Y}^{\mathrm{rel}}_\Gamma,\hat{L})^T\Big).$$

The *formal relative Gromov-Witten invariants* of $\hat{Y}^{\mathrm{rel}}_\Gamma$ are defined by

$$F^{\bullet\Gamma}_{\chi,\vec{d},\vec{\mu}}(u_1,u_2)=\frac{1}{|\operatorname{Aut}(\vec{\mu})|}\int_{[\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(\hat{Y}^{\mathrm{rel}}_\Gamma,\hat{L})^T]^{\mathrm{vir}}}\frac{e^T(\mathcal{T}^{2,m})}{e^T(\mathcal{T}^{1,m})}\in\mathbb{Q}(u_2/u_1) \tag{2.7}$$

where $H^*_T(\mathrm{pt};\mathbb{Q})=\mathbb{Q}[u_1,u_2]$, and $\operatorname{Aut}(\vec{\mu})$ is a finite group determined by $\vec{\mu}$. (See Section 5.4 for the precise meaning of the symbol $\int$ in this context.)

If $\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(\hat{Y}^{\mathrm{rel}}_\Gamma,\hat{L})$ were a proper Deligne-Mumford stack then its obstruction theory would define a virtual cycle

$$[\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(\hat{Y}^{\mathrm{rel}}_\Gamma,\hat{L})]^{\mathrm{vir}}\in A_0(\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(\hat{Y}^{\mathrm{rel}}_\Gamma,\hat{L}))$$

and

$$F^{\bullet\Gamma}_{\chi,\vec{d},\vec{\mu}}(u_1,u_2)=\frac{1}{|\operatorname{Aut}(\vec{\mu})|}\deg[\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(\hat{Y}^{\mathrm{rel}}_\Gamma,\hat{L})]^{\mathrm{vir}}\in\mathbb{Q}$$

would be a *topological* invariant independent of u_1,u_2. Although $\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(\hat{Y}^{\mathrm{rel}}_\Gamma,\hat{L})$ is not a proper Deligne-Mumford stack, we will show that

Theorem 5.7. *The function $F^{\bullet\Gamma}_{\chi,\vec{d},\vec{\mu}}(u_1,u_2)$ is independent of u_1,u_2; hence is a rational number depending only on Γ, χ, $\vec{d}$ and $\vec{\mu}$.*

Therefore, we may write $F^{\bullet\Gamma}_{\chi,\vec{d},\vec{\mu}}$ instead of $F^{\bullet\Gamma}_{\chi,\vec{d},\vec{\mu}}(u_1,u_2)$. These are new topological invariants which determine Gromov-Witten invariants of all toric Calabi-Yau threefolds and contain more refined information. More precisely, if Γ is associated to an honest toric Calabi-Yau threefold Z then

$$F^{\bullet Z}_{\chi,d}(\lambda)=\sum_{i_*(\vec{d})=d}F^{\bullet\Gamma}_{\chi,\vec{d},\vec{\emptyset}}(\lambda)$$

where $i_*:H_2(Z^1;\mathbb{Z})\longrightarrow H_2(Z;\mathbb{Z})$ is the surjective homomorphism induced by the inclusion $i:Z^1\longrightarrow Z$, and $\vec{\emptyset}$ corresponds to the condition that the image curve is disjoint from the divisor $\hat{D}$.

Theorem 5.7 also provides a definition of open Gromov-Witten invariants for toric Calabi-Yau threefolds (in all genera, with any number of holes). The values of the invariants $F^{\bullet\Gamma}_{\chi,\vec{d},\vec{\mu}}$ agree with physicists' prediction of the corresponding open Gromov-Witten invariants in all the cases that have been checked.

Topological vertex. The graph of a topological vertex is determined by a triple of integers $\mathbf{n} = (n_1, n_2, n_3)$ and an ordered basis (w_1, w_2) of $\mathbb{Z}^{\oplus 2}$. Let $\Gamma_{\mathbf{n};w_1,w_2}$ denote such a graph. It turns out that

$$F^{\bullet\Gamma_{\mathbf{n};w_1,w_2}}_{\vec{d},\vec{\mu}}(\lambda)$$

depends only on the topological data $\mathbf{n}$, not on the equivariant data (w_1, w_2), of the FTCY graph $\Gamma_{\mathbf{n};w_1,w_2}$, and $\vec{d}$ is determined by $\vec{\mu} = (\mu^1, \mu^2, \mu^3)$. Set

$$\tilde{F}^{\bullet}_{\vec{\mu}}(\lambda; \mathbf{n}) = (-1)^{\sum_{i=1}^3 (n_i - 1)|\mu^i|} \sqrt{-1}^{\ell(\vec{\mu})} F^{\bullet\Gamma_{\mathbf{n};w_1,w_2}}_{\vec{\mu}}(\lambda; \mathbf{n}).$$

We derive the following expression by virtual localization:

Proposition 6.3.

$$\tilde{F}^{\bullet}_{\vec{\mu}}(\lambda; \mathbf{n}) = \sum_{|\nu^i| = |\mu^i|} G^{\bullet}_{\vec{\nu}}(\lambda; w_1, w_2, w_3) \prod_{i=1}^3 z_{\nu^i} \Phi^{\bullet}_{\nu^i,\mu^i}\left(\sqrt{-1}\left(n_i - \frac{w_{i+1}}{w_i}\right)\lambda\right). \tag{6.10}$$

In Proposition 6.3, $G^{\bullet}_{\vec{\mu}}(\lambda; w_1, w_2, w_3)$ is the generating function of three-partition Hodge integrals defined in Section 3.2, $w_3 = -w_1 - w_2$, $w_4 = w_1$, and $\Phi^{\bullet}_{\nu,\mu}(\lambda)$ is the generation function of double Hurwitz numbers defined in Section 3.4. Note that the right hand side of (6.10) is valid for any complex numbers n_1, n_2, n_3.

The generating function $\tilde{C}_{\vec{\mu}}(\lambda; \mathbf{n})$ which corresponds to the topological vertex at the framing $\mathbf{n} = (n_1, n_2, n_3)$ is related to $\tilde{F}^{\bullet}_{\vec{\nu}}(\lambda; \mathbf{n})$ by change of basis of partitions:

$$\tilde{C}_{\vec{\mu}}(\lambda; \mathbf{n}) = \sum_{|\nu^i| = |\mu^i|} \tilde{F}^{\bullet}_{\vec{\nu}}(\lambda; \mathbf{n}) \prod_{i=1}^3 \chi_{\mu^i}(\nu^i).$$

We have

Proposition 6.5.

$$\tilde{C}_{\vec{\mu}}(\lambda; \mathbf{n}) = e^{\sqrt{-1}(\sum_{i=1}^3 \kappa_{\mu^i} n_i)\lambda/2} \tilde{C}_{\vec{\mu}}(\lambda; \mathbf{0}).$$

Set $\tilde{C}_{\vec{\mu}}(\lambda) = \tilde{C}_{\vec{\mu}}(\lambda; \mathbf{0})$, which corresponds to the topological vertex at the standard framing, and let $q = e^{\sqrt{-1}\lambda}$. Then

Proposition 6.6. *We have*

$$\tilde{C}_{\vec{\mu}}(\lambda) = q^{-\frac{1}{2}(\sum_{i=1}^3 \kappa_{\mu^i} \frac{w_{i+1}}{w_i})} \sum_{|\nu^i| = |\mu^i|} G^{\bullet}_{\vec{\nu}}(\lambda; \mathbf{w}) \prod_{i=1}^3 \chi_{\mu^i}(\nu^i). \tag{6.15}$$

$$G^{\bullet}_{\vec{\mu}}(\lambda; \mathbf{w}) = \sum_{|\nu^i| = |\mu^i|} \prod_{i=1}^3 \frac{\chi_{\nu^i}(\mu^i)}{z_{\mu^i}} q^{\frac{1}{2}(\sum_{i=1}^3 \kappa_{\nu^i} \frac{w_{i+1}}{w_i})} \tilde{C}_{\vec{\nu}}(\lambda). \tag{6.16}$$

Equation (6.16) is a structure theorem of three-partition Hodge integrals.

R3. The formal Gromov-Witten invariants $F^{\bullet\Gamma}_{\vec{d},\vec{\mu}}(\lambda)$ satisfy a gluing formula analogous to (2.6):

Theorem 7.5 (gluing formula). *Let Γ be a FTCY graph, and let Γ_2 and Γ^2 be its full smoothing and its full resolution, respectively. Let $(\vec{d},\vec{\mu})$ be an effective class of Γ_2. Then*

$$F^{\bullet\Gamma_2}_{\vec{d},\vec{\mu}}(\lambda)=F^{\bullet\Gamma}_{\vec{d},\vec{\mu}}(\lambda)=\sum_{\vec{\sigma}\in I_\Gamma(\vec{d},\vec{\mu})} z_{\vec{\sigma}}F^{\bullet\Gamma^2}_{\vec{d}_\Gamma,\vec{\mu}\sqcup\vec{\sigma}}(\lambda).$$

The relative FTCY threefold $\hat{Y}^{\mathrm{rel}}_{\Gamma_2}$ is obtained by smoothing all the normal crossing singularities of $\hat{Y}^{\mathrm{rel}}_{\Gamma}$, and $\hat{Y}^{\mathrm{rel}}_{\Gamma^2}$ is the normalization of $\hat{Y}^{\mathrm{rel}}_{\Gamma}$. Please consult Section 5 and Section 7 for the definitions involved in Theorem 7.5.

R4. One can always degenerate a regular FTCY graph Γ to a FTCY graph $\bar{\Gamma}$ such that $\bar{\Gamma}_2=\Gamma$ and $\bar{\Gamma}^2$ is a union of indecomposable graphs $\Gamma_{\mathbf{n};w_1,w_2}$. Proposition 6.5 and the gluing formula (Theorem 7.5) allow us to express $F^{\bullet\Gamma}_{\vec{d},\vec{\mu}}(\lambda)$ in terms of $\tilde{C}_{\vec{\mu}}(\lambda)$.

Proposition 7.4. *Let Γ be a regular FTCY graph. Then*

$$\begin{aligned}F^{\bullet\Gamma}_{\vec{d},\vec{\mu}}(\lambda) &= \sum_{\vec{\nu}\in T(\vec{d},\vec{\mu})}\prod_{\bar{e}\in E(\Gamma)}(-1)^{(n^e+1)\vec{d}(\bar{e})}e^{-\sqrt{-1}\kappa_{\nu^e}n^e\lambda/2}\\ &\cdot\prod_{v\in V_3(\Gamma)}\tilde{C}_{\vec{\nu}^v}(\lambda)\prod_{v\in V_1(\Gamma),v_0(e)=v}\frac{\chi_{\nu^e}(\mu^v)}{\sqrt{-1}^{\ell(\mu^v)}z_{\mu^v}}\end{aligned}$$

Please consult Section 7 for the definitions involved in Proposition 7.4. Note that Proposition 6.6 provides a link between (2.2) and Proposition 7.4.

R5. It remains to evaluate $\tilde{C}_{\vec{\mu}}(\lambda)$. We derive the following closed formula for $\tilde{C}_{\vec{\mu}}(\lambda)$:

Theorem 8.1. *Let $\vec{\mu}\in\mathcal{P}^3_+$. Then*

$$\tilde{C}_{\vec{\mu}}(\lambda)=\tilde{\mathcal{W}}_{\vec{\mu}}(q)$$

where $q=e^{\sqrt{-1}\lambda}$, and $\tilde{\mathcal{W}}_{\vec{\mu}}(q)$ is the combinatorial expression defined in Section 3.3.

Theorem 8.1 implies

$$\tilde{\mathcal{W}}_{\mu^1,\mu^2,\mu^3}(q)=\tilde{\mathcal{W}}_{\mu^2,\mu^3,\mu^1}(q)=\tilde{\mathcal{W}}_{\mu^3,\mu^1,\mu^2}(q). \tag{2.8}$$

because the cyclic symmetry is obvious from the definition of $\tilde{C}_{\vec{\mu}}(\lambda)$. On the other hand, it is hard to verify (2.8) directly from the definition of $\tilde{\mathcal{W}}_{\vec{\mu}}(q)$ given in Section 3.3.

In Section 8, we will show that

$$\tilde{\mathcal{W}}_{\vec{\mu}}(q)=\mathcal{W}_{\vec{\mu}}(q) \tag{2.9}$$

when one of the partitions is empty, where $\mathcal{W}_{\vec{\mu}}(q)$ is the combinatorial expression given by AKMV (recall O3 from Section 1). We strongly believe that (2.9) holds in general — A. Klemm has checked all the cases where $|\mu^i|\leq 6$ by computer. We list some examples in Appendix B.

Conclusion. The above results give the following closed formula for the formal relative Gromov-Witten invariants of any smooth relative FTCY threefold.

Theorem 2.2.

$$\begin{aligned} F^{\bullet\Gamma}_{\vec{d},\vec{\mu}}(\lambda) &= \sum_{\vec{\nu}\in T(\vec{d},\vec{\mu})} \prod_{\bar{e}\in E(\Gamma)} (-1)^{(n^e+1)\vec{d}(\bar{e})} e^{-\sqrt{-1}\kappa_{\nu^e} n^e \lambda/2} \\ &\quad \cdot \prod_{v\in V_3(\Gamma)} \tilde{\mathcal{W}}_{\vec{\nu}^v}(\lambda) \prod_{v\in V_1(\Gamma), v_0(e)=v} \frac{\chi_{\nu^e}(\mu^v)}{\sqrt{-1}^{\ell(\mu^v)} z_{\mu^v}} \end{aligned}$$

In particular, we have a closed formula for Gromov-Witten invariants of any smooth toric Calabi-Yau threefold. We also obtain the following closed formula of three-partition Hodge integrals.

Theorem 8.2 (Formula of three-partition Hodge integrals).

$$G^\bullet_{\vec{\mu}}(\lambda; \mathbf{w}) = \sum_{|\nu^i|=|\mu^i|} \prod_{i=1}^3 \frac{\chi_{\nu^i}(\mu^i)}{z_{\mu^i}} q^{\frac{1}{2}(\sum_{i=1}^3 \kappa_{\nu^i} \frac{w_{i+1}}{w_i})} \tilde{\mathcal{W}}_{\vec{\nu}}(\lambda)$$

3. Definitions and previous results

In this section, we recall some definitions and previous results, and introduce some generating functions.

3.1. **Partitions.**

Recall that a *partition* μ of a nonnegative integer d is a sequence of positive integers

$$\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_h > 0)$$

such that $d = \mu_1 + \ldots + \mu_h$. We write $\mu \vdash d$ or $|\mu| = d$, and call $\ell(\mu) = h$ the *length* of the partition. Let $\emptyset$ denote the empty partition, the unique partition such that $|\emptyset| = \ell(\emptyset) = 0$.

For each positive integer j, define

$$m_j(\mu) = |\{i : \mu_i = j\}|.$$

Then

$$|\operatorname{Aut}(\mu)| = \prod_j m_j(\mu)!.$$

The *transpose* of μ is a partition μ^t defined by

$$m_i(\mu^t) = \mu_i - \mu_{i+1}.$$

Note that

$$|\mu^t| = |\mu|, \ \ (\mu^t)^t = \mu, \ \ \ell(\mu^t) = \mu_1.$$

A partition μ corresponds to a conjugacy class in S_d, the permutation group of $d = |\mu|$ elements. Let z_μ be the cardinality of the centralizer of any element in this conjugacy class. Then

$$z_\mu = a_\mu |\operatorname{Aut}(\mu)|,$$

where $a_\mu = \mu_1 \cdots \mu_{\ell(\mu)}$.

Define

$$\kappa_\mu = \sum_{i=1}^{\ell(\mu)} \mu_i(\mu_i - 2i + 1).$$

Note that $\kappa_{\mu^t} = -\kappa_\mu$.

Let $\mathcal{P}$ denote the set of partitions, and let

$$\mathcal{P}_+ = \mathcal{P} - \{\emptyset\}, \quad \mathcal{P}^2_+ = \mathcal{P}^2 - \{(\emptyset,\emptyset)\}, \quad \mathcal{P}^3_+ = \mathcal{P}^3 - \{(\emptyset,\emptyset,\emptyset)\}.$$

Given $\vec{\mu} = (\mu^1,\mu^2,\mu^3) \in \mathcal{P}^3$, define

$$\ell(\vec{\mu}) = \sum_{i=1}^{3} \ell(\mu^i), \quad \mathrm{Aut}(\vec{\mu}) = \prod_{i=1}^{3} \mathrm{Aut}(\mu^i).$$

3.2. **Three-partition Hodge integrals.** Let $\overline{\mathcal{M}}_{g,n}$ denote the Deligne-Mumford moduli stack of stable curves of genus g with n marked points. Let $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ be the universal curve, and let ω_π be the relative dualizing sheaf. The Hodge bundle

$$\mathbb{E} = \pi_*\omega_\pi$$

is a rank g vector bundle over $\overline{\mathcal{M}}_{g,n}$ whose fiber over $[(C, x_1, \ldots, x_n)] \in \overline{\mathcal{M}}_{g,n}$ is $H^0(C, \omega_C)$. Let $s_i : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n+1}$ denote the section of π which corresponds to the i-th marked point, and let

$$\mathbb{L}_i = s_i^*\omega_\pi$$

be the line bundle over $\overline{\mathcal{M}}_{g,n}$ whose fiber over $[(C, x_1, \ldots, x_n)] \in \overline{\mathcal{M}}_{g,n}$ is the cotangent line $T^*_{x_i}C$ at the i-th marked point x_i. A Hodge integral is an integral of the form

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{j_1}\cdots\psi_n^{j_n}\lambda_1^{k_1}\cdots\lambda_g^{k_g}$$

where $\psi_i = c_1(\mathbb{L}_i)$ is the first Chern class of $\mathbb{L}_i$, and $\lambda_j = c_j(\mathbb{E})$ is the j-th Chern class of the Hodge bundle.

Let w_1, w_2, w_3 be formal variables, and let $w_4 = w_1$. Write $\mathbf{w} = (w_1, w_2, w_3)$. For $\vec{\mu} = (\mu^1,\mu^2,\mu^3) \in \mathcal{P}^3_+$, let

$$d^1_{\vec{\mu}} = 0, \quad d^2_{\vec{\mu}} = \ell(\mu^1), \quad d^3_{\vec{\mu}} = \ell(\mu^1) + \ell(\mu^2).$$

Define

(3.1)
$$G_{g,\vec{\mu}}(\mathbf{w}) = \frac{(-\sqrt{-1})^{\ell(\vec{\mu})}}{|\mathrm{Aut}(\vec{\mu})|} \prod_{i=1}^{3}\prod_{j=1}^{\ell(\mu^i)} \frac{\prod_{a=1}^{\mu^i_j-1}(\mu^i_j w_{i+1} + a w_i)}{(\mu^i_j-1)!\, w_i^{\mu^i_j-1}} \int_{\overline{\mathcal{M}}_{g,\ell(\vec{\mu})}} \prod_{i=1}^{3} \frac{\Lambda^\vee_g(w_i) w_i^{\ell(\vec{\mu})-1}}{\prod_{j=1}^{l(\mu^i)}(w_i(w_i - \mu^i_j \psi_{d^i_{\vec{\mu}}+j}))}.$$

where

$$\Lambda^\vee_g(u) = u^g - \lambda_1 u^{g-1} + \cdots + (-1)^g\lambda_g.$$

We call $G_{g,\vec{\mu}}(\mathbf{w})$ a *three-partition Hodge integral*. We have the following cyclic symmetry:

$$G_{g,\mu^1,\mu^2,\mu^3}(w_1,w_2,w_3) = G_{g,\mu^2,\mu^3,\mu^1}(w_2,w_3,w_1) = G_{g,\mu^3,\mu^1,\mu^2}(w_3,w_1,w_2) \quad (3.2)$$

Note that

$$\sqrt{-1}^{\ell(\vec{\mu})} G_{g,\vec{\mu}}(\mathbf{w}) \in \mathbb{Q}(w_1,w_2,w_3)$$

is homogeneous of degree 0, so

$$G_{g,\vec{\mu}}(w_1, w_2, -w_1-w_2) = G_{g,\vec{\mu}}(1, \frac{w_2}{w_1}, -1-\frac{w_2}{w_1}).$$

Let

$$G_{g,\vec{\mu}}(\tau) = G_{g,\vec{\mu}}(1,\tau,-\tau-1).$$

Then

$$G_{g,\mu^1,\mu^2,\mu^3}(\tau) = G_{g,\mu^2,\mu^3,\mu^1}(-1-\frac{1}{\tau}) = G_{g,\mu^3,\mu^1,\mu^2}(\frac{-1}{\tau+1})$$

Introduce variables λ, $p^i = (p^i_1, p^i_2, \ldots)$, $i = 1,2,3$. Given a partition μ, define

$$p^i_\mu = p^i_1 \cdots p^i_{\ell(\mu)}$$

for $i = 1,2,3$. In particular, $p^i_\emptyset = 1$. Write

$$\mathbf{p} = (p^1, p^2, p^3), \quad \mathbf{p}_{\vec{\mu}} = p^1_{\mu^1} p^2_{\mu^2} p^3_{\mu^3}.$$

Define generating functions

$$\begin{aligned}
G_{\vec{\mu}}(\lambda;\mathbf{w}) &= \sum_{g=0}^{\infty} \lambda^{2g-2+\ell(\vec{\mu})} G_{g,\vec{\mu}}(\mathbf{w}) \\
G(\lambda;\mathbf{p};\mathbf{w}) &= \sum_{\vec{\mu}\in\mathcal{P}^3_+} G_{\vec{\mu}}(\lambda;\mathbf{w})\mathbf{p}_{\vec{\mu}} \\
G^\bullet(\lambda;\mathbf{p};\mathbf{w}) &= \exp(G(\lambda;\mathbf{p};\mathbf{w})) = 1 + \sum_{\vec{\mu}\in\mathcal{P}^3_+} G^\bullet_{\vec{\mu}}(\lambda;\mathbf{w})\mathbf{p}_{\vec{\mu}} \\
G^\bullet_{\vec{\mu}}(\lambda;\mathbf{w}) &= \sum_{\chi\in 2\mathbb{Z},\chi\leq 2\ell(\vec{\mu})} \lambda^{-\chi+\ell(\vec{\mu})} G^\bullet_{\chi,\vec{\mu}}(\mathbf{w})
\end{aligned}$$

$$\begin{aligned}
G_{\vec{\mu}}(\lambda;\tau) &= \sum_{g=0}^{\infty} \lambda^{2g-2+\ell(\vec{\mu})} G_{g,\vec{\mu}}(\tau) \\
G(\lambda;\mathbf{p};\tau) &= \sum_{\vec{\mu}\in\mathcal{P}^3_+} G_{\vec{\mu}}(\lambda;\tau)\mathbf{p}_{\vec{\mu}} \\
G^\bullet(\lambda;\mathbf{p};\tau) &= \exp(G(\lambda;\mathbf{p};\tau)) = 1 + \sum_{\vec{\mu}\in\mathcal{P}^3_+} G^\bullet_{\vec{\mu}}(\lambda;\tau)\mathbf{p}_{\vec{\mu}} \\
G^\bullet_{\vec{\mu}}(\lambda;\tau) &= \sum_{\chi\in 2\mathbb{Z},\chi\leq 2\ell(\vec{\mu})} \lambda^{-\chi+l(\vec{\mu})} G^\bullet_{\chi,\vec{\mu}}(\tau)
\end{aligned}$$

3.3. **Representations of symmetric groups.** Let χ_ν denote the irreducible character of $S_{|\nu|}$ indexed by ν, and let $\chi_\nu(\mu)$ be the value of χ_ν on the conjugacy class determined by the partition μ. Recall that the Schur functions are related to Newton functions by

$$s_\mu(x) = \sum_{|\nu|=|\mu|} \frac{\chi_\mu(\nu)}{z_\nu} p_\nu(x)$$

where $x = (x_1, x_2, \ldots)$ are formal variables such that

$$p_i(x) = x_1^i + x_2^i + \cdots .$$

The Littlewood-Richardson coefficients $c^\eta_{\mu\nu}$ are given by

$$s_\mu s_\nu = \sum_\eta c^\eta_{\mu\nu} s_\eta.$$

The coefficients $c^\eta_{\mu\nu}$ are nonnegative integers. The skew Schur functions are defined by

$$s_{\eta/\mu} = \sum_\nu c^\eta_{\mu\nu} s_\nu.$$

Definition 3.1.

$$\mathcal{W}_\mu(q) = q^{\kappa_\mu/4} \prod_{1\leq i<j\leq l(\mu)} \frac{[\mu_i-\mu_j+j-i]}{[j-i]} \prod_{i=1}^{l(\mu)} \prod_{v=1}^{\mu_i} \frac{1}{[v-i+l(\mu)]}, \tag{3.3}$$

where

$$[m] = q^{m/2} - q^{-m/2}.$$

$$\mathcal{W}_{\mu,\nu}(q) = q^{|\nu|/2}\mathcal{W}_\mu \cdot s_\nu(\mathcal{E}_\mu(q,t)), \tag{3.4}$$

where

$$\mathcal{E}_\mu(q,t) = \prod_{j=1}^{l(\mu)} \frac{1+q^{\mu_j-j}t}{1+q^{-j}t} \cdot \left(1 + \sum_{n=1}^{\infty} \frac{t^n}{\prod_{i=1}^n (q^i-1)}\right).$$

$$\mathcal{W}_{\mu^1,\mu^2,\mu^3}(q) = q^{\kappa_{\mu^2}/2+\kappa_{\mu^3}/2} \sum_{\rho^1,\rho^3} c^{\mu^1(\mu^3)^t}_{\rho^1(\rho^3)^t} \frac{\mathcal{W}_{(\mu^2)^t\rho^1}(q)\mathcal{W}_{\mu^2(\rho^3)^t}(q)}{\mathcal{W}_{\mu^2}(q)}, \tag{3.5}$$

where

$$c^{\mu^1(\mu^3)^t}_{\rho^1(\rho^3)^t} = \sum_\eta c^{\mu^1}_{\eta\rho^1} c^{(\mu^3)^t}_{\eta(\rho^3)^t}.$$

$$\tilde{\mathcal{W}}_{\rho^1,\rho^2,\rho^3}(q) \tag{3.6}$$

$$= q^{-(\kappa_{\rho^1}-2\kappa_{\rho^2}-\frac{1}{2}\kappa_{\rho^3})/2} \sum c^{\nu^+}_{(\nu^1)^t\rho^2} c^{\rho^1}_{(\eta^1)^t\nu^1} c^{\rho^3}_{\eta^3(\nu^3)^t} q^{(-2\kappa_{\nu^+}-\frac{\kappa_{\nu^3}}{2})/2} \mathcal{W}_{\nu^+,\nu^3}(q) \frac{1}{z_\mu} \chi_{\eta^1}(\mu)\chi_{\eta^3}(2\mu).$$

We have the following identities (see [33]):

$$\mathcal{W}_{\mu,\nu,\emptyset}(q) = \mathcal{W}_{\emptyset,\mu,\nu}(q) = \mathcal{W}_{\nu,\emptyset,\mu}(q) = q^{\kappa_\nu/2}\mathcal{W}_{\mu,(\nu)^t}(q). \tag{3.7}$$

$$\mathcal{W}_{\mu,\nu}(q) = \mathcal{W}_{\nu,\mu}(q). \tag{3.8}$$

$$\mathcal{W}_{\mu,\emptyset}(q) = \mathcal{W}_\mu(q). \tag{3.9}$$

3.4. **Double Hurwitz numbers.** Let μ^+, μ^- be partitions of d. Let $H^\bullet_{\chi,\mu^+,\mu^-}$ denote the weighted counts of Hurwitz covers of the sphere of the type (μ^+,μ^-) by possibly disconnected Riemann surfaces of Euler characteristic χ. Define a generating function

$$\Phi^\bullet_{\mu^+,\mu^-}(\lambda) = \sum_\chi \lambda^{-\chi+\ell(\mu^+)+\ell(\mu^-)} \frac{H^\bullet_{\chi,\mu^+,\mu^-}}{(-\chi+\ell(\mu^+)+\ell(\mu^-))!}.$$

The following is a special case of the Burnside formula of Hurwitz numbers:

$$\Phi^\bullet_{\mu^+,\mu^-}(\lambda) = \sum_{|\nu|=d} e^{\kappa_\nu\lambda/2} \frac{\chi_\nu(\mu^+)}{z_{\mu^+}} \frac{\chi_\nu(\mu^-)}{z_{\mu^-}}. \tag{3.10}$$

Using the orthogonality of characters

$$\sum_\rho \frac{\chi_\mu(\rho)\chi_\nu(\rho)}{z_\rho} = \delta_{\mu\nu} \tag{3.11}$$

it is straightforward to check that (3.10) implies the following two identities:

$$\Phi^\bullet_{\mu^+,\mu^-}(\lambda_1+\lambda_2) = \sum_{|\nu|=d} \Phi^\bullet_{\mu^+,\nu}(\lambda_1) z_\nu \Phi^\bullet_{\nu,\mu^-}(\lambda_2) \tag{3.12}$$

$$\Phi^\bullet_{\mu^+,\mu^-}(0) = \frac{\delta_{\mu^+\mu^-}}{z_{\mu^+}} \tag{3.13}$$

Equation (3.12) is a sum formula for double Hurwitz numbers, and Equation (3.13) gives the initial values for the double Hurwitz numbers.

Introduce variables $p^+ = (p_1^+, p_2^+, \ldots)$, $p^- = (p_1^-, p_2^-, \ldots)$. For a partition μ, define

$$p^\pm_\mu = p^\pm_{\mu_1} \cdots p^\pm_{\mu_{\ell(\mu)}}.$$

Define a generating function

$$\Phi^\bullet(\lambda; p^+, p^-) = 1 + \sum_{d=1}^\infty \sum_{|\mu^\pm|=d} \Phi^\bullet_{\mu^+,\mu^-}(\lambda) p^+_{\mu^+} p^-_{\mu^-}.$$

and differential operators

$$C^\pm = \sum_{j,k} (j+k) p_j^\pm p_k^\pm \frac{\partial}{\partial p^\pm_{j+k}}, \quad J^\pm = \sum_{j,k} jk p^\pm_{j+k} \frac{\partial^2}{\partial p_j^\pm \partial p_k^\pm}.$$

We have the following cut-and-join equations for double Hurwitz numbers:

$$\frac{\partial \Phi^\bullet}{\partial \lambda} = \frac{1}{2}(C^+ + J^+)\Phi^\bullet = \frac{1}{2}(C^- + J^-)\Phi^\bullet. \tag{3.14}$$

Actually, $\Phi^\bullet(\lambda; p^+, p^-)$ is the unique solution to the cut-and-join equations (3.14) with the initial value

$$\Phi^\bullet(0; p^+, p^-) = 1 + \sum_{\mu \in \mathcal{P}^+} \frac{p^+_\mu p^-_\mu}{z_\mu}.$$

4. Relative Formal toric Calabi-Yau threefolds

In this section, we will introduce formal toric Calabi-Yau (FTCY) graphs, and construct relative FTCY threefolds.

4.1. **Toric Calabi-Yau threefolds.** Let Z be a smooth toric Calabi-Yau threefold. Let Z^1 be the union of all one-dimensional $(\mathbb{C}^*)^3$ orbit closures in Z, and let Z^0 be the union of $(\mathbb{C}^*)^3$ fixed points. We assume that Z^1 is connected and Z^0 is nonempty; under this mild assumption, there is a distinguished subtorus $T \subset (\mathbb{C}^*)^3$ defined as follows. Let $p \in Z^0$ be a fixed point of the $(\mathbb{C}^*)^3$ action. Then $(\mathbb{C}^*)^3$ acts on T_pZ and $\Lambda^3 T_pZ$, where T_pZ is the tangent space of Z at p. The action of $(\mathbb{C}^*)^3$ on the complex line $\Lambda^3 T_pZ$ gives an irreducible character $\alpha : (\mathbb{C}^*)^3 \to \mathbb{C}^*$. By the Calabi-Yau condition and the connectedness of Z^1, the character α is independent of choice of the fixed point p. Define $T = \mathrm{Ker}\alpha \cong (\mathbb{C}^*)^2$.

Let $T_\mathbb{R} \cong U(1)^2$ be the maximal compact subgroup of T, and let

$$\mu : Z \to \mathfrak{t}_\mathbb{R}^\vee$$

be the moment map of the $T_\mathbb{R}$-action on Z, where $\mathfrak{t}_\mathbb{R}$ is the Lie algebra of $T_\mathbb{R}$. Note that $\mathfrak{t}_\mathbb{R}^\vee$, the dual of $\mathfrak{t}_\mathbb{R}$, is canonically isomorphic to $\Lambda_T \otimes_\mathbb{Z} \mathbb{R}$, where

$$\Lambda_T = \mathrm{Hom}(T, \mathbb{C}^*) \cong \mathbb{Z}^{\oplus 2}$$

is the group of irreducible characters of T. The image of Z^1 under μ is a planar trivalent graph Γ. Some examples are shown in Figure 1.

The map μ induces a one-to-one correspondence between irreducible components of Z^1 and edges of Γ, and also between T fixed points in Z and vertices of Γ.

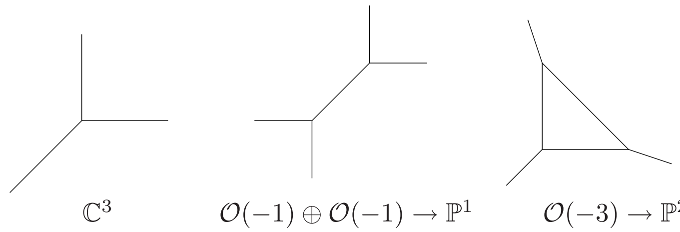
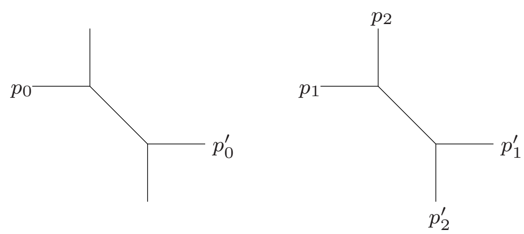


Figure 1



p_i is identified with p_i'

Figure 2. Locally planar trivalent graphs

Moreover, the action of T on an irreducible component of Z^1 is determined by the slope of the corresponding edge.

4.2. **FTCY graphs.** In this subsection, we will define formal toric Calabi-Yau graphs, which are generalizations of planar trivalent graphs associated to toric Calabi-Yau threefolds. For example, we will consider graphs which are only locally planar (Figure 2).

Definition 4.1. *A graph* Γ *consists of a set of* oriented edges $E^o(\Gamma)$*, a set of* vertices $V(\Gamma)$*, an* orientation reversing map $\mathfrak{rev}: E^o(\Gamma) \to E^o(\Gamma)$*, an* initial vertex map $\mathfrak{v}_0: E^o(\Gamma) \to V(\Gamma)$ *and a* terminal vertex map $\mathfrak{v}_1: E^o(\Gamma) \to V(\Gamma)$*, that satisfy*

(1) $\mathfrak{rev}$ *is a fixed point free involution;*
(2) *both* $\mathfrak{v}_0$ *and* $\mathfrak{v}_1$ *are surjective and* $\mathfrak{v}_1 = \mathfrak{v}_0 \circ \mathfrak{rev}$.

We say Γ *is* weakly trivalent *if* $|\mathfrak{v}^{-1}(v)| \leq 3$ *for* $v \in V(\Gamma)$.

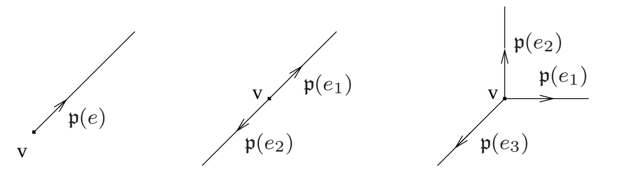
For simplicity, we will abbreviate $\mathfrak{rev}(e)$ to $-e$. Note that then the equivalence classes $E(\Gamma) = E^o(\Gamma)/\{\pm 1\}$ is the set of edges of Γ in the ordinary sense. We denote by $V_1(\Gamma)$, $V_2(\Gamma)$ and $V_3(\Gamma)$ the set of univalent, bivalent and trivalent vertices of Γ.

Definition 4.2. *A* locally planar lattice graph *is a weakly trivalent graph* Γ *together with a* position map

$$\mathfrak{p} : E^o(\Gamma) \longrightarrow \mathbb{Z}^{\oplus 2} - \{0\}$$

so that $\mathfrak{p}(-e) = -\mathfrak{p}(e)$. *We say* Γ *is a* formal Calabi-Yau graph *if in addition the position map satisfies the following requirement (see Figure 3):*

(1) $\mathfrak{p}(E^o(\Gamma)) \subset \mathbb{Z}^{\oplus 2}_{pri}$*, where* $\mathbb{Z}^{\oplus 2}_{pri}$ *is the set of primitive lattice points in* $\mathbb{Z}^{\oplus 2}$*;*
(2) *for bivalent vertices* $v \in V_2(\Gamma)$ *with* $\mathfrak{v}_0^{-1}(v) = \{e_1, e_2\}$, $\mathfrak{p}(e_1) = -\mathfrak{p}(e_2)$*;*



$\mathfrak{v}_0^{-1}(v) = \{e\}$ $\quad$ $\mathfrak{v}_0^{-1}(v) = \{e_1, e_2\}$ $\quad$ $\mathfrak{v}_0^{-1}(v) = \{e_1, e_2, e_3\}$

Figure 3

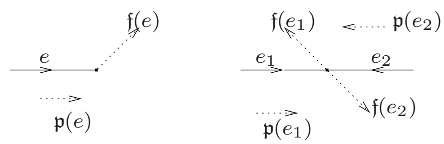


Figure 4

(3) *for trivalent vertices* $v \in V_3(\Gamma)$ *with* $\mathfrak{v}_0^{-1}(v) = \{e_1, e_2, e_3\}$*, any two vectors in* $\{\mathfrak{p}(e_1), \mathfrak{p}(e_2), \mathfrak{p}(e_3)\}$ *form an integral basis of* $\mathbb{Z}^{\oplus 2}$*, and*

$$\mathfrak{p}(e_1) + \mathfrak{p}(e_2) + \mathfrak{p}(e_3) = 0.$$

Let

$$E_{1f}(\Gamma) = \mathfrak{v}_1^{-1}(V_1(\Gamma) \cup V_2(\Gamma))$$

A formal toric Calabi-Yau (FTCY) graph *is a formal Calabi-Yau with a* framing map

$$\mathfrak{f} : E_{1f}(\Gamma) \to \mathbb{Z}^{\oplus 2} - \{0\}$$

so that (see Figure 4)

(1) $\mathfrak{p}(e) \wedge \mathfrak{f}(e) = u_1 \wedge u_2$*, where we fix an ordered integral basis* (u_1, u_2) *of* $\mathbb{Z}^{\oplus 2}$*;*
(2) $\mathfrak{f}(e_1) + \mathfrak{f}(e_2) = 0$ *if* $\{e_1, e_2\} = \mathfrak{v}_1^{-1}(v)$ *for some* $v \in V_2(\Gamma)$*.*

We say Γ *is a* regular *FTCY graph if it has no bivalent vertex.*

For later convenience, we introduce some notation. Given a FTCY graph, let

$$E_{ip} = \mathfrak{v}_i^{-1}(V_3(\Gamma)), \quad E_{if} = \mathfrak{v}_i^{-1}(V_1(\Gamma) \cup V_2(\Gamma)),$$

where $i = 0, 1$. We will define

$$\mathfrak{p}_i : E_{ip}(\Gamma) \to \mathbb{Z}^{\oplus 2} - \{0\}, \quad \mathfrak{f}_i : E_{if}(\Gamma) \to \mathbb{Z}^{\oplus 2} - \{0\}$$

for $i = 0, 1$, and $\vec{n} : E^{\mathrm{o}}(\Gamma) \to \mathbb{Z}$ (see Figures 5, 6, 7, 8).

(1) Given $e \in E_{0p}(\Gamma)$, there exists a unique $e_0 \in E^{\mathrm{o}}(\Gamma)$ such that

$$\mathfrak{v}_0(e_0) = \mathfrak{v}_0(e), \quad \mathfrak{p}(e) \wedge \mathfrak{p}(e_0) = u_1 \wedge u_2.$$

Define $\mathfrak{p}_0(e) = \mathfrak{p}(e_0)$.

(2) Given $e \in E_{1p}(\Gamma)$, there exists a unique $e_1 \in E^{\mathrm{o}}(\Gamma)$ such that

$$\mathfrak{v}_0(e_1) = \mathfrak{v}_1(e), \quad \mathfrak{p}(e) \wedge \mathfrak{p}(e_1) = u_1 \wedge u_2.$$

Define $\mathfrak{p}_1(e) = \mathfrak{p}(e_1)$.

(3) Given $e \in E_{of}(\Gamma)$, we have $-e \in E_{1f}(\Gamma)$. Define $\mathfrak{f}_0(e) = -\mathfrak{f}(-e)$.
(4) Given $e \in E_{1f}(\Gamma)$, define $\mathfrak{f}_1(e) = \mathfrak{f}(e)$.
(5) Given $e \in E^{\mathrm{o}}(\Gamma)$, define $n^e \in \mathbb{Z}$ as follows.

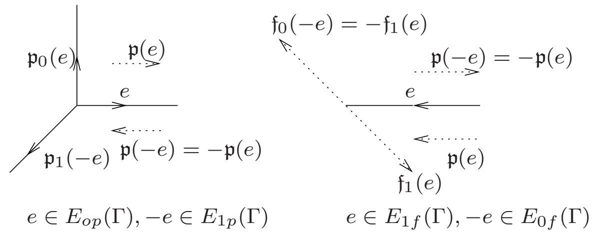


FIGURE 5

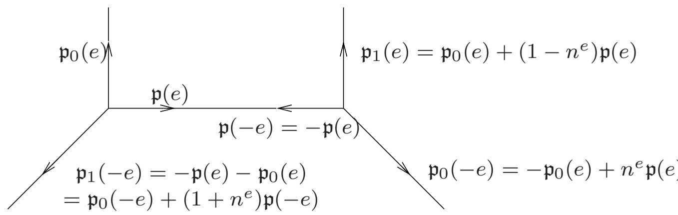


FIGURE 6. $e, -e \in E_{pp}(\Gamma)$

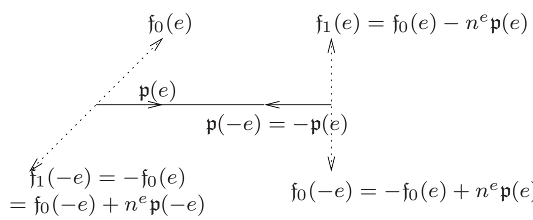


FIGURE 7. $e, -e \in E_{ff}(\Gamma)$

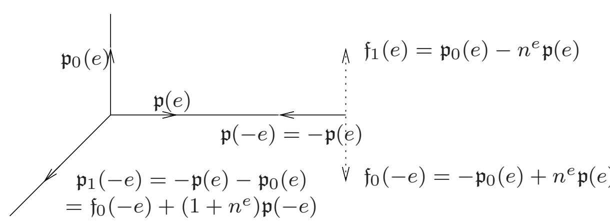


FIGURE 8. $e \in E_{pf}(\Gamma), -e \in E_{fp}(\Gamma)$

(a) $e \in E_{pp}(\Gamma) = E_{0p}(\Gamma) \cap E_{1p}(\Gamma)$. Then $\mathfrak{p}_1(e) = \mathfrak{p}_0(e) + (1-n^e)\mathfrak{p}(e)$.
(b) $e \in E_{ff}(\Gamma) = E_{0f}(\Gamma) \cap E_{1p}(\Gamma)$. Then $\mathfrak{f}_1(e) = \mathfrak{f}_0(e) - n^e\mathfrak{p}(e)$.
(c) $e \in E_{fp}(\Gamma) = E_{0f}(\Gamma) \cap E_{1p}(\Gamma)$. Then $\mathfrak{p}_1(e) = \mathfrak{f}_0(e) + (1-n^e)\mathfrak{p}(e)$.
(d) $e \in E_{pf}(\Gamma) = E_{0p}(\Gamma) \cap E_{1f}(\Gamma)$. Then $\mathfrak{f}_1(e) = \mathfrak{p}_0(e) - n^e\mathfrak{p}(e)$.

The map $\vec{n} : E^o(\Gamma) \to \mathbb{Z}$ is given by $e \mapsto n^e$.

Note that $\mathfrak{p}_0, \mathfrak{p}_1, \mathfrak{f}_0, \mathfrak{f}_1, \vec{n}$ are all determined by Γ. They satisfy the following properties.

(1) For $e \in E_{op}(\Gamma)$, we have $\mathfrak{p}(e) + \mathfrak{p}_0(e) + \mathfrak{p}_1(-e) = 0$.

(2) For $e \in E_{1p}(\Gamma)$, we have $-\mathfrak{p}(e) + \mathfrak{p}_0(-e) + \mathfrak{p}_1(e) = 0$.
(3) For $e \in E_{0f}(\Gamma)$, we have $\mathfrak{f}_0(e) + \mathfrak{f}_1(-e) = 0$.
(4) For $e \in E_{1f}(\Gamma)$, we have $\mathfrak{f}_1(e) + \mathfrak{f}_0(-e) = 0$.
(5) For $e \in E^{\mathrm{o}}(\Gamma)$, we have $\vec{n}(-e) = -\vec{n}(e)$.

4.3. **Operations on FTCY graphs.** In this subsection, we define four operations on FTCY graphs: smoothing, degeneration, normalization, and gluing.

Let Γ be a FTCY graph and let $v \in V_2(\Gamma)$ be a bivalent vertex. The smoothing of Γ along v is a new graph obtained by combining the two edges e_1 and e_2 of $\mathfrak{v}_0^{-1}(v)$ a single edge.

Definition 4.3. *The* smoothing *of* Γ *along* $v \in V_2(\Gamma)$ *is a graph* Γ_v *that has vertices* $V(\Gamma) - \{v\}$*, oriented edges* $E^{\mathrm{o}}(\Gamma) - \{\pm e_2\}$*; its initial vertices* $\tilde{\mathfrak{v}}_0$*, terminal vertices* $\tilde{\mathfrak{v}}_1$*, position map* $\tilde{\mathfrak{p}}$ *and framing map* $\tilde{\mathfrak{f}}$ *are identical to those of* Γ *except* $\tilde{\mathfrak{v}}_1(e_1) = \mathfrak{v}_1(e_2)$.

The reverse of the above construction is called a degeneration.

Definition 4.4. *Let* Γ *be a FTCY graph and let* $e \in E^{\mathrm{o}}(\Gamma)$ *be an edge. We pick a lattice point* $\mathfrak{f}_0 \in \mathbb{Z}^{\oplus 2}$ *so that* $\mathfrak{p}(e) \wedge \mathfrak{f}_0 = u_1 \wedge u_2$*. The* degeneration *of* Γ *at* e *with framing* $\mathfrak{f}_0$ *is a graph* $\Gamma_{e,\mathfrak{f}_o}$ *whose edges are* $E^{\mathrm{o}}(\Gamma) \cup \{\pm e_1, \pm e_2\} - \{\pm e\}$ *and whose vertices are* $V(\Gamma) \cup \{v_0\}$*; its initial vertices* $\tilde{\mathfrak{v}}_0$*, terminal vertices* $\tilde{\mathfrak{v}}_1$*, position map* $\tilde{\mathfrak{p}}$ *and framing map* $\tilde{\mathfrak{f}}$ *are identical to those of* Γ *except*

$$\tilde{\mathfrak{v}}_0(e_1) = \mathfrak{v}_0(e),\ \tilde{\mathfrak{v}}_1(e_1) = \tilde{\mathfrak{v}}_0(e_2) = v_0,\ \tilde{\mathfrak{v}}_1(e_2) = \mathfrak{v}_1(e),$$
$$\tilde{\mathfrak{p}}(e_1) = \tilde{\mathfrak{p}}(e_2) = \mathfrak{p}(e),\ \tilde{\mathfrak{f}}(e_1) = \mathfrak{f}_0,\ \tilde{\mathfrak{f}}(-e_2) = -\mathfrak{f}_0.$$

The next operation is the normalization of a graph at a bivalent vertex; and its inverse operation.

Definition 4.5. *Let* Γ *be a FTCY graph and let* $v \in V_2(\Gamma)$ *be a bivalent vertex. The* normalization *of* Γ *at* v *is a graph* Γ^v *whose edges are the same as that of* Γ *and whose vertices are* $V(\Gamma) \cup \{v_1, v_2\} - \{v\}$*; its initial vertices* $\tilde{\mathfrak{v}}_0$*, terminal vertices* $\tilde{\mathfrak{v}}_1$*, position map* $\tilde{\mathfrak{p}}$ *and framing map* $\tilde{\mathfrak{f}}$ *are identical to that of* Γ *except*

$$\tilde{\mathfrak{v}}_1(e_1) = v_1,\ \tilde{\mathfrak{v}}_1(e_2) = v_2$$

where e_1 *and* e_2 *are the two edges in* $\mathfrak{v}_1^{-1}(v)$.

The reverse of the above operation is called gluing.

Definition 4.6. *Let* Γ *be a formal toric Calabi-Yau graph and let* $v_1, v_2 \in V_1(\Gamma)$ *be two univalent vertices of* Γ*. Let* $\mathfrak{f}_i = \mathfrak{f}(e_i)$*, where* $\{e_i\} = \mathfrak{v}_1^{-1}(v_i)$*. Suppose* $\mathfrak{p}(e_1) = -\mathfrak{p}(e_2)$ *and* $\mathfrak{f}_1 = -\mathfrak{f}_2$*. We then identify* v_1 *and* v_2 *to become a single vertex, and keep the framing* $\mathfrak{f}(e_i) = \mathfrak{f}_i$*. The resulting graph* Γ^{v_1,v_2} *is called the* gluing *of* Γ *at* v_1 *and* v_2.

Figure 9 shows examples of the above four operations.

It is straightforward to generalize smoothing and normalization to subset A of $V_2(\Gamma)$. Given $A \subset V_2(\Gamma)$, let Γ_A denote the smoothing of Γ along A, and let Γ^A denote the normalization of Γ along A. There are surjective maps

$$\pi_A : E(\Gamma) \to E(\Gamma_A), \quad \pi^A : V(\Gamma^A) \to V(\Gamma).$$

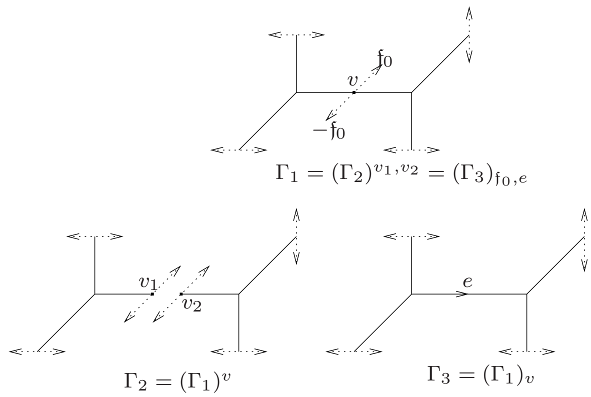


FIGURE 9

4.4. **Relative FTCY threefolds.** In this subsection we will introduce relative formal toric Calabi-Yau threefolds.

A formal toric Calabi-Yau graph Γ determines a formal relative Calabi-Yau threefold with a T-action, where $T \cong (\mathbb{C}^*)^2$. It is a threefold $\hat{Y}$, possibly with normal crossing singularities, coupled with a relative divisor $\hat{D} \subset \hat{Y}$, so that $Y^{\mathrm{rel}}_{\Gamma} = (\hat{Y}, \hat{D})$ is a formal relative Calabi-Yau threefold:

$$\wedge^3 \Omega_{\hat{Y}}(\log \hat{D}) \cong \mathcal{O}_{\hat{Y}}. \tag{4.1}$$

The pair $(\hat{Y}, \hat{D})$ admits a T-action so that the action on $\Lambda^3 T_p \hat{Y}$ is trivial for any fixed point p. As a set, the scheme $\hat{Y}$ is a union of $\mathbb{P}^1$'s, each associated to an edge of Γ; two $\mathbb{P}^1$ intersect exactly when their associated edges share a common vertex; the normal bundle to each $\mathbb{P}^1$ in $\hat{Y}$ and the T-action on $\hat{Y}$ are dictated by the data encoded in the graph Γ.

In the following construction, we will use the notation introduced in Section 4.2.

4.4.1. *Edges.* Given $e \in E^o(\Gamma)$, let T acts on $\mathbb{P}^1$ by

$$t \cdot [X, Y] = [\mathfrak{p}(e)(t)X, Y]$$

for $t \in T$. Here we view $\mathfrak{p}(e)$ as an element in

$$\Lambda_T = \mathrm{Hom}(T, \mathbb{C}^*).$$

The two fixed points are $q_0 = [0,1]$ and $q_1 = [1,0]$. We construct a T-equivariant line bundle $L^e \to \mathbb{P}^1$ by specifying the degree and the T action on the fibers at the two fixed points $q_0 = [0,1]$ and $q_1 = [1,0]$ (see Figures 6, 7, 8).

e	$\deg L^e$	$L^e_{q_0}$	$L^e_{q_1}$
$E_{pp}(\Gamma)$	$n^e - 1$	$\mathfrak{p}_0(e)$	$\mathfrak{p}_1(e)$
$E_{ff}(\Gamma)$	n^e	$\mathfrak{f}_0(e)$	$\mathfrak{f}_1(e)$
$E_{fp}(\Gamma)$	$n^e - 1$	$\mathfrak{f}_0(e)$	$\mathfrak{p}_1(e)$
$E_{pf}(\Gamma)$	n^e	$\mathfrak{p}_0(e)$	$\mathfrak{f}_1(e)$

Let $\Sigma(e)$ be the formal completion of the total space of $L_e \oplus L_{-e}$ along the zero section. The T-actions on L_e and on L_{-e} induce a T-action on $\Sigma(e)$. By construction, there is a T-isomorphism

$$\Sigma(e) \cong \Sigma(-e) \tag{4.2}$$

that sends $q_0 \in \Sigma(e)$ to $q_1 \in \Sigma(-e)$ and sends the first summand L_e in $N_{\mathbb{P}^1/\Sigma(e)}$ to the second summand in $N_{\mathbb{P}^1/\Sigma(-e)}$.

It is clear that

$$\wedge^3 \Omega_{\Sigma(e)} \cong p^* \mathcal{O}_{\mathbb{P}^1}(c), \tag{4.3}$$

where $p: \Sigma(e) \to \mathbb{P}^1$ is the projection, and

$$c = \begin{cases} 0, & e \in E_{pp}(\Gamma), \\ 1, & e \in E_{pf}(\Gamma) \cup E_{fp}(\Gamma), \\ 2, & e \in E_{ff}(\Gamma). \end{cases}$$

4.4.2. *Trivalent vertices.* Given $v \in V_3(\Gamma)$, let $\mathfrak{v}_0^{-1}(v) = \{e_1, e_2, e_3\}$, where the ordered is chosen such that $\mathfrak{p}(e_1) \wedge \mathfrak{p}(e_2) = u_1 \wedge u_2$. We define

$$\Sigma(v) = \operatorname{Spec} \mathbb{C}[\![x_1, x_2, x_3]\!]$$

with a T-action defined via

$$t \cdot x_i = \mathfrak{p}(e_i)(t) x_i.$$

for $t \in T$.

To glue the formal scheme $\Sigma(e_1)$, $\Sigma(e_2)$ and $\Sigma(e_3)$, we need to introduce the gluing morphisms

$$\psi_{e_k v} : \Sigma(v) \longrightarrow \Sigma(e_k).$$

First, we let $\hat{\Sigma}(e_k)$ be the formal completion of $\Sigma(e_k)$ along $q_0 \in \mathbb{P}^1 \subset \Sigma(e_k)$. $\hat{\Sigma}(e_k)$ is a formal T-scheme and is T-isomorphic to the T-scheme

$$\operatorname{Spec} \mathbb{C}[\![y_1, y_2, y_3]\!]; \quad t \cdot y_i = \mathfrak{p}(e_{i+k})(t) y_i$$

such that $L_{q_0}^{e_k}$, $L_{q_1}^{-e_k}$, $T_{q_0}\mathbb{P}^1$ are mapped to $\mathbb{C}\frac{\partial}{\partial y_1}$, $\mathbb{C}\frac{\partial}{\partial y_2}$, $\mathbb{C}\frac{\partial}{\partial y_3}$, respectively. Here we agree that $e_{k+3} = e_k$ for $k = 1, 2, 3$. The gluing morphism $\psi_{e_k v}$ is the composite of

$$\operatorname{Spec} \mathbb{C}[\![y_1, y_2, y_3]\!] \longrightarrow \hat{\Sigma}(e_k) \longrightarrow \Sigma(e_k) \tag{4.4}$$

with the T-isomorphism

$$\hat{\Sigma}(v) \equiv \operatorname{Spec} \mathbb{C}[\![x_1, x_2, x_3]\!] \longrightarrow \operatorname{Spec} \mathbb{C}[\![y_1, y_2, y_3]\!]$$

that is defined by $y_i \mapsto x_{k+i}$, where we agree that $x_{i+3} = x_i$ for $i = 1, 2, 3$.

Using the morphisms $\psi_{e_k v}$, we can glue $\Sigma(e_1)$ and $\Sigma(e_2)$ and then glue $\Sigma(e_3)$ onto it via the cofiber products

$$\begin{array}{ccc} \Sigma(e_1) & \longrightarrow & \Sigma(e_1) \coprod_{\Sigma(v)} \Sigma(e_2) \\ \uparrow & & \uparrow \\ \Sigma(v) & \longrightarrow & \Sigma(e_2) \end{array} \qquad \begin{array}{ccc} \Sigma(e_3) & \longrightarrow & \Sigma(e_1) \coprod_{\Sigma(v)} \Sigma(e_2) \coprod_{\Sigma(v)} \Sigma(e_3) \\ \uparrow & & \uparrow \\ \Sigma(v) & \longrightarrow & \Sigma(e_1) \coprod_{\Sigma(v)} \Sigma(e_2) \end{array}$$

Since the gluing map $\psi_{e_k v}$ are T-equivalent, the T-actions on $\Sigma(e_k)$ descend to the glued scheme.

4.4.3. *Bivalent vertices.* Next we glue $\Sigma(e_1)$ and $\Sigma(e_2)$ in case $\{e_1, e_2\} = \mathfrak{v}_0^{-1}(v)$ for some $v \in V_2(\Gamma)$. We have $e_1, e_2 \in E_{0f}(\Gamma)$, and

$$\mathfrak{f}_0(e_1) + \mathfrak{f}_0(e_2) = 0$$

Let $\Sigma(v)$ be the formal T-scheme

$$\Sigma(v) = \operatorname{Spec} \mathbb{C}[\![x_1, x_2]\!], \quad t \cdot x_i = \mathfrak{f}_0(e_i)(t) x_i;$$

we let the gluing morphism $\psi_{e_k v}$ be the composite of (4.4) with the T-morphism

$$\operatorname{Spec} \mathbb{C}[\![x_1, x_2]\!] \longrightarrow \operatorname{Spec} \mathbb{C}[\![y_1, y_2, y_3]\!]$$

that is defined via $y_3 \mapsto 0$, y_1 and y_2 map to x_1 and x_2 respectively in case $k = 1$ and to x_2 and x_1 respectively in case $k = 2$. We can glue $\Sigma(e_1)$ and $\Sigma(e_2)$ along $\Sigma(v)$ via the cofiber product as before.

4.4.4. *Univalent vertices.* Lastly, we consider the case $\mathfrak{v}_0(e) = v$ for some $v \in V_1(\Gamma)$. Let $\Sigma(v)$ be the formal T-scheme

$$\Sigma(v) = \operatorname{Spec} \mathbb{C}[\![x_1, x_2]\!], \quad t \cdot x_1 = \mathfrak{f}_0(e)(t) x_1, \quad t \cdot x_2 = \mathfrak{f}_1(-e)(t) x_2;$$

and define ψ_{ev} as $\psi_{e_1 v}$ in Section 4.4.3. We let $\hat{D}^v$ be the image divisor $\psi_{ev}(\Sigma(v)) \subset \Sigma(e)$ and consider it as part of the relative divisor of the formal Calabi-Yau scheme Y_Γ^{rel} we are constructing.

Let $L(v) \subset \Sigma(v)$ be the divisor defined by $x_2 = 0$, and let $\hat{L}^v = \psi_{ev}(L(v)) \subset \hat{D}^v$.

4.4.5. *Final step.* Now it is standard to glue $\Sigma(e)$, $e \in E^{\mathrm{o}}(\Gamma)$, to form the scheme $\hat{Y}$. We first form the disjoint union

$$\coprod_{e \in E^{\mathrm{o}}(\Gamma)} \Sigma(e);$$

because of (4.2), the orientation reversing map $E^{\mathrm{o}}(\Gamma) \to E^{\mathrm{o}}(\Gamma)$ defines a fixed point free involution

$$\tau : \coprod_{e \in E^{\mathrm{o}}(\Gamma)} \Sigma(e) \longrightarrow \coprod_{e \in E^{\mathrm{o}}(\Gamma)} \Sigma(e);$$

we define $\tilde{Y}$ be its quotient by τ. Next, for each trivalent vertex v of $\mathfrak{v}_0^{-1}(v) = \{e_1, e_2, e_3\}$, we glue $\Sigma(e_1), \Sigma(e_2), \Sigma(e_3)$ along $\Sigma(v)$; for each bivalent vertex v of $\mathfrak{v}_0^{-1}(v) = \{e_1, e_2\}$, we glue $\Sigma(e_1)$ and $\Sigma(e_2)$ along $\Sigma(v)$. We denote by $\hat{Y}$ the resulting scheme after completing all the gluing associated to all trivalent and bivalent vertices. The T-action on each $\Sigma(e)$ descends to a T-action on $\hat{Y}$. Finally, for each univalent vertex v with $e = \mathfrak{v}_0^{-1}(v)$, we let $\hat{D}^v \subset \Sigma(e)$ be the divisor defined in Section 4.4.4. The union of all such $\hat{D}^v$ form a divisor $\hat{D}$ that is the relative divisor of $\hat{Y}$. Since $\hat{D}$ is invariant under T, the pair $Y_\Gamma^{\mathrm{rel}} = (\hat{Y}, \hat{D})$ is a T-equivariant formal scheme. Because of (4.3), we have

$$\wedge^3 \Omega_{\hat{Y}}(\log \hat{D}) \cong \mathcal{O}_{\hat{Y}};$$

hence $Y_\Gamma^{\mathrm{rel}} = (\hat{Y}, \hat{D})$ is a formal toric Calabi-Yau scheme.

Following the construction, the scheme $\hat{Y}_\Gamma^{\mathrm{rel}}$ is smooth away from the images $\psi_{ev}(\Sigma(v))$ associated to bivalent vertices v, and has normal crossing singularities there. So $\hat{Y}_\Gamma^{\mathrm{rel}}$ is smooth iff Γ is a regular. The relative divisor $\hat{D}$ is the union of smooth divisor $\hat{D}^v$ indexed by $v \in V_0(\Gamma)$. Within each divisor $\hat{D}^v$ there is a divisor $\hat{L}^v \subset \hat{D}^v$ defined as in Section 4.4.4.

For later convenience, we introduce some notation. Let $\bar{e}$ denote the equivalence class $\{e, -e\}$ in $E(\Gamma)$, and let $C^{\bar{e}}$ denote the projective line in $\hat{Y}$ coming from the $\mathbb{P}^1$ in $\Sigma(e)$. For $v \in V_1(\Gamma)$, let z^v denote the point in $\hat{D}^v$ coming from the closed point q_0 in $\Sigma(e)$, where $\mathfrak{v}_0(e) = v$.

5. Definition of formal relative Gromov-Witten invariants

In this section, we will define relative Gromov-Witten invariants of relative FTCY threefolds; the case when the relative FTCY threefold is indecomposable gives the mathematical definition of topological vertex.

5.1. **Moduli spaces of relative stable morphisms.** Let Γ be a FTCY graph and let $\hat{Y}^{\mathrm{rel}}_\Gamma = (\hat{Y}, \hat{D})$ be its associated scheme. The degrees and the ramification patterns of relative stable morphisms to $\hat{Y}^{\mathrm{rel}}_\Gamma$ are characterized by *effective classes* of Γ:

Definition 5.1. *Let Γ be a FTCY graph. An* effective class *of Γ is a pair of functions $\vec{d}: E(\Gamma) \to \mathbb{Z}_{\geq 0}$ and $\vec{\mu}: V_1(\Gamma) \to \mathcal{P}$ that satisfy*

(1) *$|\vec{\mu}(v)| = \vec{d}(\bar{e})$ if $v \in V_1(\Gamma)$ and $\mathfrak{v}_1(e) = v$;*
(2) *$\vec{d}(\bar{e}_1) = \vec{d}(\bar{e}_2)$ if $v \in V_2(\Gamma)$ and $\mathfrak{v}_0^{-1}(v) = \{e_1, e_2\}$.*

We write μ^v for $\vec{\mu}(v)$, $d^{\bar{e}}$ for $\vec{d}(\bar{e})$.

To show that an effective class does characterize a relative stable morphism, a quick review of its definition is in order.

Recall that an ordinary relative morphism u to $(\hat{Y}, \hat{D})$ consists of

- a possibly disconnected nodal curve X
- distinct smooth points $\{q^v_j \mid v \in V_0(\Gamma), 1 \leq j \leq \ell(\mu^v)\}$ in X such that each connected component of X contains at least one of these points,
- a morphism $u: X \to \hat{Y}$

so that

- $u^{-1}(\hat{D}^v) = \sum_{j=1}^{\ell(\mu^v)} \mu^v_j q^v_j$ for some positive integers μ^v_j;
- u is *pre-deformable* along the singular loci
$$\coprod_{v \in V_2(\Gamma)} \Sigma(v)$$
of $\hat{Y}^{\mathrm{rel}}_\Gamma$, i.e, if $v \in V_2(\Gamma)$ and $\mathfrak{v}_0^{-1}(v) = \{e_1, e_2\}$, then $u^{-1}(\Sigma(v))$ consists of nodes of X, and for each $y \in u^{-1}(\Sigma(v))$, $u|_{u^{-1}(\Sigma(e_1))}$ and $u|_{u^{-1}(\Sigma(e_2))}$ have the same contact order to $\Sigma(v)$ at y;
- u coupled with the marked points q^v_i is a stable morphism in the ordinary sense.

Unless otherwise specified, all the stable morphisms in this paper are with not necessarily connected domain.

Since
$$H_2(\hat{Y}; \mathbb{Z}) = \bigoplus_{\bar{e} \in E(\Gamma)} \mathbb{Z}[C^{\bar{e}}],$$
the morphism u defines a map $\vec{d}: E(\Gamma) \to \mathbb{Z}$ via
$$u_*([X]) = \sum_{\bar{e} \in E(\Gamma)} \vec{d}(\bar{e})[C^{\bar{e}}]. \tag{5.1}$$

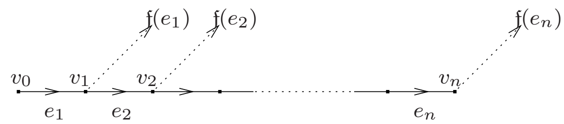


FIGURE 10. A flat chain of length n

The integers μ_i^v form a partition

$$\mu^v = (\mu_1^v, \cdots, \mu_{\ell(\mu^v)}^v)$$

and the map $\vec{\mu}: V_1(\Gamma) \to \mathcal{P}$ is

$$\vec{\mu}(v) = \mu^v.$$

With this definition, the requirement (1) in Definition 5.1 follows from (5.1) and (2) holds since u is pre-deformable.

To define relative stable morphisms to $\hat{Y}_\Gamma^{\mathrm{rel}}$, we need to work with the expanded schemes of $\hat{Y}_\Gamma^{\mathrm{rel}}$ introduced in [17]. In the case studied, they are the associated formal schemes of the expanded graphs of Γ.

Definition 5.2. *Let Γ be a FTCY graph. A* flat chain of length n *in Γ is a subgraph $\check{\Gamma} \subset \Gamma$ that has n edges $\pm e_1, \cdots, \pm e_n$, $n+1$ univalent or bivalent vertices $v_0, \cdots, v_n$ with identical framings $\mathfrak{f}$ so that*

$$\mathfrak{v}_0(e_1) = v_0;\ \mathfrak{v}_1(e_i) = \mathfrak{v}_0(e_{i+1}) = v_i\ \ i = 1, \cdots, n;\ \mathfrak{v}_1(e_n) = v_n,$$

and that all $\mathfrak{p}(e_i)$ are identical.

Definition 5.3. *A* contraction *of a FTCY graph Γ along a flat chain $\check{\Gamma} \subset \Gamma$ is the graph after eliminating all edges and bivalent vertices of $\check{\Gamma}$ from Γ, identifying the univalent vertices of $\check{\Gamma}$ while keeping their framings unchanged.*

Given a FTCY graph Γ and a function

$$\mathbf{m} : V_1(\Gamma) \cup V_2(\Gamma) \longrightarrow \mathbb{Z}_{\geq 0},$$

the expanded graph $\Gamma_{\mathbf{m}}$ is obtained by replacing each $v \in V_1(\Gamma) \cup V_2(\Gamma)$ by a flat chain $\check{\Gamma}_{m^v}^v$ of length $m^v = \mathbf{m}(v)$ with framings $\pm\mathfrak{f}(e)$, where $\mathfrak{v}_1(e) = v$. In particular $\Gamma_{\mathbf{0}} = \Gamma$, where $\mathbf{0}(v) = 0$ for all $v \in V_1(\Gamma) \cup V_2(\Gamma)$. The original graph Γ can be recovered by contracting $\Gamma_{\mathbf{m}}$ along the flat chains

$$\{\check{\Gamma}_{m^v}^v \mid v \in V_1(\Gamma) \cup V_2(\Gamma)\}.$$

We now study their associated Calabi-Yau scheme. We denote by $(\hat{Y}, \hat{D})$ the associated Calabi-Yau scheme of Γ and by $(\hat{Y}_{\mathbf{m}}, \hat{D}_{\mathbf{m}})$ that of $\Gamma_{\mathbf{m}}$. We recover the original scheme $\hat{Y}$ by shrinking the irreducible components of $\hat{Y}_{\mathbf{m}}$ associated to the flat chains that are contracted. This way we define a projection

$$\pi_{\mathbf{m}} : \hat{Y}_{\mathbf{m}} \longrightarrow \hat{Y}.$$

We define a relative automorphism of $\hat{Y}_{\mathbf{m}}$ to be an automorphism of $\hat{Y}_{\mathbf{m}}$ that is also a $\hat{Y}$-morphism; an automorphism of a relative morphism $u : X \to (\hat{Y}_{\mathbf{m}}, \hat{D}_{\mathbf{m}})$ is a pair of a relative automorphism σ of $\hat{Y}_{\mathbf{m}}$ and an automorphism h of X so that

$$u \circ h = \sigma \circ u.$$

Definition 5.4. *A relative morphism to* $\hat{Y}_\Gamma^{\mathrm{rel}}$ *is an ordinary relative morphism to* $(\hat{Y}_{\mathbf{m}}, \hat{D}_{\mathbf{m}})$ *for some* $\mathbf{m}$*; it is* stable *if its automorphism group is finite.*

Note that an effective class $(\vec{d},\vec{\mu})$ of an FTCY graph Γ can also be viewed as an effective class of any degeneration of Γ, and in particular, an effective class of $\Gamma_{\mathbf{m}}$. We fix a FTCY graph Γ, an effective class $(\vec{d},\vec{\mu})$, and an even integer χ. We then form the moduli space $\mathcal{M}^\bullet_\chi(\hat{Y}_\Gamma^{\mathrm{rel}},\vec{d},\vec{\mu})$ of all stable relative morphisms u to $\hat{Y}_\Gamma^{\mathrm{rel}}$ that satisfy

- $\chi(\mathcal{O}_X)=\chi/2$, where X is the domain curve of u;
- the associated effective class of u is $(\vec{d},\vec{\mu})$.

Since $\hat{Y}$ is a formal Calabi-Yau threefold with possibly normal crossing singularity and smooth singular loci, the moduli space $\mathcal{M}^\bullet_\chi(\hat{Y}_\Gamma^{\mathrm{rel}},\vec{d},\vec{\mu})$ is a formal Deligne-Mumford stack with a perfect obstruction theory [17, 18].

Lemma 5.5. *The virtual dimension of* $\mathcal{M}^\bullet_\chi(\hat{Y}_\Gamma^{\mathrm{rel}},\vec{d},\vec{\mu})$ *is* $\sum_{v\in V_1(\Gamma)}\ell(\mu^v)$.

Proof. The proof is straightforward and will be omitted. □

5.2. **Equivariant degeneration.** Let T act on $\mathbb{P}^1\times\mathbb{A}^1$ by

$$t\cdot([X_0,X_1],s)=([\mathfrak{p}(t)X_0,X_1],s),$$

where $\mathfrak{p}\in\Lambda_T=\mathrm{Hom}(T,\mathbb{C}^*)$. Let $\mathfrak{Y}$ be the blowup of $\mathbb{P}^1\times\mathbb{A}^1$ at $([0,1],0)$. The T-action on $\mathbb{P}^1\times\mathbb{A}^1$ can be lifted to $\mathfrak{Y}$ such that the projection $\mathfrak{Y}\to\mathbb{P}^1\times\mathbb{A}^1$ is T-equivariant; composition with the projection $\mathbb{P}^1\times\mathbb{A}^1\to\mathbb{A}^1$ gives a T-equivariant family of curves

$$\mathfrak{Y}\longrightarrow\mathbb{A}^1$$

such that $\mathfrak{Y}_s\cong\mathbb{P}^1$ for $s\neq 0$ and $\mathfrak{Y}_0\cong\mathbb{P}^1\sqcup\mathbb{P}^1$.

The above construction can be generalized as follows. Let Γ be a FTCY graph and let

$$V_2(\Gamma)=\{v_1,\ldots,v_n\}$$

Then we have a T-equivariant family

$$\hat{\mathcal{Y}}\to\mathbb{A}^n \tag{5.2}$$

such that

$$\hat{\mathcal{Y}}_{\mathbf{0}}=\hat{\mathcal{Y}}_{(0,\ldots,0)}\cong\hat{Y}_\Gamma^{\mathrm{rel}}\quad\text{and}\quad\hat{\mathcal{Y}}_{\mathbf{s}}=\hat{\mathcal{Y}}_{(s_1,\ldots,s_n)}\cong\hat{Y}^{\mathrm{rel}}_{\Gamma_{\{v_i|s_i\neq0\}}}$$

Recall that Γ_A is the smoothing of Γ along $A\subset V_2(\Gamma)$ (Section 4.3). The T-action $\mathbb{A}^n$ is trivial, and the T-action on each fiber is consistent with the one described in Section 4.

By the construction in [18], there is a T-equivariant family

$$\mathcal{M}^\bullet_\chi(\hat{\mathcal{Y}},\vec{d},\vec{\mu})\to\mathbb{A}^n \tag{5.3}$$

such that $\mathcal{M}^\bullet_\chi(\hat{\mathcal{Y}},\vec{d},\vec{\mu})_{\mathbf{s}}=\mathcal{M}^\bullet_\chi(\hat{\mathcal{Y}}_{\mathbf{s}},\vec{d},\vec{\mu})$. In particular,

$$\mathcal{M}^\bullet_\chi(\hat{\mathcal{Y}},\vec{d},\vec{\mu})_{\mathbf{0}}=\mathcal{M}^\bullet_\chi(\hat{Y}_\Gamma^{\mathrm{rel}},\vec{d},\vec{\mu}).$$

The total space $\mathcal{M}^\bullet_\chi(\hat{\mathcal{Y}},\vec{d},\vec{\mu})$ is a formal Deligne-Mumford stack with a perfect obstruction theory $[\mathbf{T}^1\to\mathbf{T}^2]$ of virtual dimension

$$\sum_{v\in V_1(\Gamma)}\ell(\mu^v)+|V_2(\Gamma)|.$$

For each $v \in V_2(\Gamma)$ there is line bundle $\mathbf{L}^v$ over $\mathcal{M}^\bullet_\chi(\hat{\mathcal{Y}},\vec{d},\vec{\mu})$ with a section $s^v : \mathcal{M}^\bullet_\chi(\hat{\mathcal{Y}},\vec{d},\vec{\mu}) \longrightarrow \mathbf{L}^v$ such that

$$\mathcal{M}^\bullet_\chi(\hat{Y}^{\mathrm{rel}}_\Gamma,\vec{d},\vec{\mu}) = \mathcal{M}^\bullet_\chi(\hat{\mathcal{Y}},\vec{d},\vec{\mu})_{\mathbf{0}}$$

is the zero locus

$$\{s^v = 0 \mid v \in V_2(\Gamma)\} \subset \mathcal{M}^\bullet_\chi(\hat{\mathcal{Y}},\vec{d},\vec{\mu}).$$

The pair $(\mathbf{L}^v, s^v)$ corresponds to $(\mathbf{L}_0, \mathbf{r}_0)$ in [18, Section 3].

5.3. **Perfect obstruction theory.** Let Γ be a FTCY graph, and let $(\vec{d},\vec{\mu})$ be an effective class of Γ. We briefly describe the perfect obstruction theory on $\mathcal{M}^\bullet_\chi(\hat{Y}^{\mathrm{rel}}_\Gamma,\vec{d},\vec{\mu})$ constructed in [18].

Let $\mathcal{M}^\bullet_\chi(\hat{\mathcal{Y}},\vec{d},\vec{\mu}) \to \mathbb{A}^{|V_2(\Gamma)|}$, $[\mathbf{T}^1 \to \mathbf{T}^2]$, and $\{\mathbf{L}^v \mid v \in V_2(\Gamma)\}$ be defined as in Section 5.2. Let $[\tilde{\mathcal{T}}^1 \to \tilde{\mathcal{T}}^2]$ be the perfect obstruction theory on $\mathcal{M}^\bullet_\chi(\hat{Y}^{\mathrm{rel}}_\Gamma,\vec{d},\vec{\mu})$. Let $u : (X,\mathbf{q}) \longrightarrow (\hat{Y}_{\mathbf{m}},\hat{D}_{\mathbf{m}})$ represent a point in $\mathcal{M}^\bullet_\chi(\hat{Y}^{\mathrm{rel}}_\Gamma,\vec{d},\vec{\mu}) \subset \mathcal{M}^\bullet_\chi(\hat{\mathcal{Y}},\vec{d},\vec{\mu})$, where

$$\mathbf{q} = \{q^v_j \mid v \in V_1(\Gamma), 1 \le j \le \ell(\mu^v)\}.$$

We have the following exact sequence of vector spaces at u:

$$0 \to \tilde{\mathcal{T}}^1_u \to \mathbf{T}^1_u \to \bigoplus_{v\in V_2(\Gamma)} \mathbf{L}^v_u \to \tilde{\mathcal{T}}^2_u \to \mathbf{T}^2_u \to 0. \tag{5.4}$$

We will describe $\mathbf{T}^1_u$, $\mathbf{T}^2_u$, and $\mathbf{L}^v_u$ explicitly. When Γ is a regular FTCY graph, i.e., $V_2(\Gamma) = \emptyset$, the line bundles $\mathbf{L}^v$ do not arise, and $\mathcal{M}^\bullet_\chi(\hat{\mathcal{Y}},\vec{d},\vec{\mu}) = \mathcal{M}^\bullet_\chi(\hat{Y}^{\mathrm{rel}}_\Gamma,\vec{d},\vec{\mu})$.

We first introduce some notation. Given $\mathbf{m} : V_1(\Gamma) \cup V_2(\Gamma) \to \mathbb{Z}_{\ge 0}$, let $\check{\Gamma}^v_{m^v}$ be the flat chain of length $m^v = \mathbf{m}(v)$ associated to $v \in V_1(\Gamma) \cup V_2(\Gamma)$, and let

$$V(\check{\Gamma}^v_{\mathbf{m}}) = \{\bar{v}^v_0, \ldots, \bar{v}^v_{m^v}\},$$

where $\bar{v}^v_{m^v} \in V_1(\Gamma_{\mathbf{m}})$ if $v \in V_1(\Gamma)$.

Let $v \in V_1(\Gamma)$ and $0 \le l \le m^v - 1$, or let $v \in V_2(\Gamma)$ and $0 \le l \le m^v$. We define a line bundle L^v_l on the divisor $\hat{D}^v_l = \Sigma(\bar{v}_l)$ in $Y_{\mathbf{m}}$ by

$$L^v_l = N_{\hat{D}^v_l/\Sigma(e_v)} \otimes N_{\hat{D}^v_l/\Sigma(e'_v)}$$

where $\mathfrak{v}_0^{-1}(\bar{v}^v_l) = \{e_v, e'_v\}$. Note that L^v_l is a trivial line bundle on $\hat{D}^v_l$.

With the above notation, we have

$$\mathbf{L}^v_u = \bigotimes_{l=0}^{m^v} H^0(\hat{D}^v_l, L^v_l). \tag{5.5}$$

The tangent space $\mathbf{T}^1_u$ and the obstruction space $\mathbf{T}^2_u$ to $\mathcal{M}^\bullet_\chi(\hat{\mathcal{Y}},\vec{d},\vec{\mu})$ at the moduli point

$$[u : (X,\mathbf{q}) \longrightarrow (\hat{Y}_{\mathbf{m}},\hat{D}_{\mathbf{m}})]$$

are given by the following two exact sequences:

$$\begin{aligned} 0 \quad &\to \mathrm{Ext}^0(\Omega_X(R_{\mathbf{q}}),\mathcal{O}_X) \to H^0(\mathbf{D}^\bullet) \to \mathbf{T}^1_u \\ &\to \mathrm{Ext}^1(\Omega_X(R_{\mathbf{q}}),\mathcal{O}_X) \to H^1(\mathbf{D}^\bullet) \to \mathbf{T}^2_u \quad \to 0 \end{aligned} \tag{5.6}$$

$$0 \quad \to \quad H^0\left(X, u^*\left(\Omega_{Y_{\mathbf{m}}}(\log \hat{D}_{\mathbf{m}})\right)^\vee\right) \to H^0(\mathbf{D}^\bullet) \tag{5.7}$$

$$
\begin{aligned}
&\to \bigoplus_{v\in V_1(\Gamma)} \bigoplus_{l=0}^{m^v-1} H^0_{\mathrm{et}}(\mathbf{R}_l^{v\bullet}) \oplus \bigoplus_{v\in V_2(\Gamma)} \bigoplus_{l=0}^{m^v} H^0_{\mathrm{et}}(\mathbf{R}_l^{v\bullet})\\
&\to H^1\left(X, u^*\left(\Omega_{Y_{\mathbf{m}}}(\log \hat{D}_{\mathbf{m}})\right)^\vee\right) \to H^1(\mathbf{D}^\bullet)\\
&\to \bigoplus_{v\in V_1(\Gamma)} \bigoplus_{l=0}^{m^v-1} H^1_{\mathrm{et}}(\mathbf{R}_l^{v\bullet}) \oplus \bigoplus_{v\in V_2(\Gamma)} \bigoplus_{l=0}^{m^v} H^1_{\mathrm{et}}(\mathbf{R}_l^{v\bullet}) \to 0
\end{aligned}
$$

where

$$R_{\mathbf{q}} = \sum_{v\in V_1(\Gamma)} \sum_{j=1}^{\ell(\mu^v)} q_j^v,$$

(5.8)

$$H^0_{\mathrm{et}}(\mathbf{R}_l^{v\bullet}) \cong \bigoplus_{q\in u^{-1}(\hat{D}_l^v)} T_q(u^{-1}(\Sigma(e_v)))\otimes T_q^*(u^{-1}(\Sigma(e'_v))) \cong \mathbb{C}^{\oplus n_l^v},\ \ \mathfrak{v}_0^{-1}(\bar{v}_l^v) = \{e_v, e'_v\}$$

$$H^1_{\mathrm{et}}(\mathbf{R}_l^{v\bullet}) \cong H^0(\hat{D}_l^v, L_l^v)^{\oplus n_l^v} \Big/ H^0(\hat{D}_l^v, L_l^v), \tag{5.9}$$

and n_l^v is the number of nodes over $\hat{D}_l^v$. In (5.9),

$$H^0(\hat{D}_l^v, L_l^v) \to H^0(\hat{D}_l^v, L_l^e)^{\oplus n_l^v}$$

is the diagonal embedding.

We refer the reader to [18] for the definitions of $H^i(\mathrm{D}^\bullet)$ and the maps between terms in (5.6), (5.7).

5.4. **Formal relative Gromov-Witten invariants.** Usually, the relative Gromov-Witten invariants are defined as integrations of the pull back classes from the target and the relative divisor. In the case studied, the analogue is to integrate a total degree $2\sum_{v\in V_1(\Gamma)}\ell(\mu^v)$ class from the relative divisor $\hat{D}$. The class we choose is the product of the "Poincaré dual" of the divisor $\hat{L}^v \subset \hat{D}^v$, one for each marked point q_i^v. Equivalently, we consider the moduli space

$$\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(\hat{Y}_\Gamma^{\mathrm{rel}}, \hat{L}) = \left\{(u, X, \{q_j^v\}) \in \mathcal{M}^\bullet_\chi(\hat{Y}_\Gamma^{\mathrm{rel}}, \vec{d}, \vec{\mu}) \mid u(q_j^v) \in \hat{L}^v\right\}.$$

Its virtual dimension is zero. More precisely, let $[\mathcal{T}^1 \to \mathcal{T}^2]$ be the perfect obstruction theory on $\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(\hat{Y}_\Gamma^{\mathrm{rel}}, \hat{L})$, and let $[\tilde{\mathcal{T}}^1 \to \tilde{\mathcal{T}}^2]$ be the perfection obstruction theory on $\mathcal{M}^\bullet_\chi(\hat{Y}_\Gamma^{\mathrm{rel}}, \vec{d}, \vec{\mu})$. Given a moduli point

$$[u : (X, \mathbf{x}) \to (\hat{Y}_{\mathbf{m}}, \hat{D}_{\mathbf{m}})] \in \mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(\hat{Y}_\Gamma^{\mathrm{rel}}, \hat{L}) \subset \mathcal{M}^\bullet_\chi(\hat{Y}_\Gamma^{\mathrm{rel}}, \vec{d}, \vec{\mu}),$$

we have

$$\mathcal{T}_u^1 - \mathcal{T}_u^2 = \tilde{\mathcal{T}}_u^1 - \tilde{\mathcal{T}}_u^2 - \bigoplus_{v\in V_1(\Gamma)} \bigoplus_{j=1}^{\ell(\mu^v)} (N_{\hat{L}^v/\hat{D}^v})_{u(q_j^v)} \tag{5.10}$$

as virtual vector spaces.

We now define the formal relative Gromov-Witten invariants of $\hat{Y}_\Gamma^{\mathrm{rel}}$ by applying the virtual localization to the moduli scheme $\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(\hat{Y}_\Gamma^{\mathrm{rel}}, \hat{L})$. We use the equivariant intersection theory developed in [7] and the localization in [8, 11].

Since $\hat{Y}_\Gamma^{\mathrm{rel}}$ is toric, the moduli space $\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(\hat{Y}_\Gamma^{\mathrm{rel}},\hat{L})$ and its obstruction theory are T-equivariant. We consider the fixed loci of the T-action on $\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(\hat{Y}_\Gamma^{\mathrm{rel}},\hat{L})$:

$$\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(\hat{Y}_\Gamma^{\mathrm{rel}},\hat{L})^T.$$

Its coarse moduli space is projective. The virtual localization is an integration of the quotient equivariant Euler classes. When $[u]$ varies in $\mathcal{M}_a$, the vector spaces $\mathcal{T}_u^1$ and $\mathcal{T}_u^2$ form two vector bundles. We denoted them by $\mathcal{T}^1$ and $\mathcal{T}^2$. Since the obstruction theory are T-equivariant, both $\mathcal{T}^i$ are T-equivariant. We let $\mathcal{T}^{i,f}$ and $\mathcal{T}^{i,m}$ be the fixed and the moving parts of $\mathcal{T}^i$. Since fixed part $\mathcal{T}^{i,f}$ induces a perfect obstruction theory of $\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(\hat{Y}_\Gamma^{\mathrm{rel}},\hat{L})^T$, it defines a virtual cycle

$$[\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(\hat{Y}_\Gamma^{\mathrm{rel}},\hat{L})^T]^{\mathrm{vir}} \in A_*(\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(\hat{Y}_\Gamma^{\mathrm{rel}},\hat{L})^T),$$

where $A_*(\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(\hat{Y}_\Gamma^{\mathrm{rel}},\hat{L})^T)$ is the Chow group with rational coefficients.

The perfect obstruction theory $[\mathcal{T}^{1,f}\to\mathcal{T}^{2,f}]$ together with the trivial T-action defines a T-equivariant virtual cycle

$$[\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(\hat{Y}_\Gamma^{\mathrm{rel}},\hat{L})^T]^{\mathrm{vir},T} \in A_*^T(\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(\hat{Y}_\Gamma^{\mathrm{rel}},\hat{L})^T).$$

Since T acts on $\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(\hat{Y}_\Gamma^{\mathrm{rel}},\hat{L})^T$ trivially, we have

$$A_*^T(\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(\hat{Y}_\Gamma^{\mathrm{rel}},\hat{L})^T)\cong A_*(\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(\hat{Y}_\Gamma^{\mathrm{rel}},\hat{L}))\otimes\Lambda_T \tag{5.11}$$

where

$$\Lambda_T=\mathrm{Hom}(T,\mathbb{C}^*)\cong A_*^T(\mathrm{pt})\cong\mathbb{Q}[u_1,u_2].$$

Under the isomorphism (5.11), we have

$$[\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(\hat{Y}_\Gamma^{\mathrm{rel}},\hat{L})^T]^{\mathrm{vir},T}=[\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(\hat{Y}_\Gamma^{\mathrm{rel}},\hat{L})]^{\mathrm{vir}}\otimes 1.$$

The moving part $\mathcal{T}^{i,m}$ is the virtual normal bundles of $\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(\hat{Y}_\Gamma^{\mathrm{rel}},\hat{L})^T$. Let

$$e^T(\mathcal{T}^{i,m})\in A_T^*(\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(\hat{Y}_\Gamma^{\mathrm{rel}},\hat{L})^T)$$

be the T-equivariant Euler class of $\mathcal{T}^{i,m}$, where $A_T^*(\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(\hat{Y}_\Gamma^{\mathrm{rel}},\hat{L})^T)$ is the T-equivariant operational Chow group (see [7, Section 2.6]). For $i=1,2$, $e^T(\mathcal{T}^{i,m})$ lies in the subring

$$A^*(\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(\hat{Y}_\Gamma^{\mathrm{rel}},\hat{L})^T)\otimes\mathbb{Q}[u_1,u_2]\subset A_T^*(\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(\hat{Y}_\Gamma^{\mathrm{rel}},\hat{L})^T)$$

and is invertible in

$$A^*(\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(\hat{Y}_\Gamma^{\mathrm{rel}},\hat{L})^T)\otimes\mathbb{Q}[u_1,u_2]_{\mathfrak{m}}\subset A_T^*(\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(\hat{Y}_\Gamma^{\mathrm{rel}},\hat{L})^T)\otimes\mathbb{Q}[u_1,u_2]_{\mathfrak{m}}.$$

where $\mathbb{Q}[u_1,u_2]_{\mathfrak{m}}$ is $\mathbb{Q}[u_1,u_2]$ localized at the ideal $\mathfrak{m}=(u_1,u_2)$.

For later convenience, we introduce some notation. Let X be a Deligne-Mumford stack a T-action, and let X^T be the T-fixed points. Recall that

$$A_*^T(X)\otimes\mathbb{Q}[u_1,u_2]_{\mathfrak{m}}\cong A_*^T(X^T)\otimes\mathbb{Q}[u_1,u_2]_{\mathfrak{m}}\cong A_*(X^T)\otimes\mathbb{Q}[u_1,u_2]_{\mathfrak{m}}. \tag{5.12}$$

The degree of a zero cycle defines a map $\deg: A_0(X^T)\to\mathbb{Q}$. We define

$$\deg_{\mathfrak{m}}: A_d(X^T)\otimes\mathbb{Q}[u_1,u_2]_{\mathfrak{m}}\to\mathbb{Q}[u_1,u_2]_{\mathfrak{m}}$$

by

$$a\otimes b\mapsto\begin{cases}\deg(a)b & d=0,\\ 0 & d\neq 0.\end{cases}$$

This gives a ring homomorphism

$$\deg_{\mathfrak{m}}: A^T_*(X)\otimes\mathbb{Q}[u_1,u_2]_{\mathfrak{m}}\cong A_*(X^T)\otimes\mathbb{Q}[u_1,u_2]_{\mathfrak{m}}\to\mathbb{Q}[u_1,u_2]_{\mathfrak{m}}.$$

Given $c\in A^*_T(X)\otimes\mathbb{Q}[u_1,u_2]_{\mathfrak{m}}$ and $\alpha\in A^T_*(X)\otimes\mathbb{Q}[u_1,u_2]_{\mathfrak{m}}$, define

$$\int_\alpha c=\deg_{\mathfrak{m}}(c\cap\alpha)\in\mathbb{Q}[u_1,u_2]_{\mathfrak{m}}.$$

Following the lead of the virtual localization formula [11], we define

$$F^{\bullet\Gamma}_{\chi,\vec{d},\vec{\mu}}(u_1,u_2)=\frac{1}{|\operatorname{Aut}(\vec{\mu})|}\int_{[\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(\hat{Y}^{\mathrm{rel}}_\Gamma,\hat{L})^T]^{\mathrm{vir},T}}\frac{e_T(\mathcal{T}^{1,m})}{e_T(\mathcal{T}^{2,m})}\tag{5.13}$$

where we view $[\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(\hat{Y}^{\mathrm{rel}}_\Gamma,\hat{L})^T]$ as an element in $A^T_*(\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(\hat{Y}^{\mathrm{rel}}_\Gamma,\hat{L})^T)\otimes\mathbb{Q}[u_1,u_2]_{\mathfrak{m}}$. Note that

$$\frac{e_T(\mathcal{T}^{1,m})}{e_T(\mathcal{T}^{2,m})}\cap[\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(\hat{Y}^{\mathrm{rel}}_\Gamma,\hat{L})^T]^{\mathrm{vir},T}\in\left(A^T_*(\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(\hat{Y}^{\mathrm{rel}}_\Gamma,\hat{L})^T)\otimes\mathbb{Q}[u_1,u_2]_{\mathfrak{m}}\right)_0$$

where $\left(A^T_*(\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(\hat{Y}^{\mathrm{rel}}_\Gamma,\hat{L})^T)\otimes\mathbb{Q}[u_1,u_2]_{\mathfrak{m}}\right)_0$ is the degree zero part of the graded ring $A^T_*(\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(\hat{Y}^{\mathrm{rel}}_\Gamma,\hat{L})^T)\otimes\mathbb{Q}[u_1,u_2]_{\mathfrak{m}}$. Therefore,

$$F^{\bullet\Gamma}_{\chi,\vec{d},\vec{\mu}}(u_1,u_2)\in(\mathbb{Q}[u_1,u_2]_{\mathfrak{m}})^0=\mathbb{Q}(u_1/u_1)$$

where $(\mathbb{Q}[u_1,u_2]_{\mathfrak{m}})^0$ is the degree zero part of the graded ring $\mathbb{Q}[u_1,u_2]_{\mathfrak{m}}$.

Remark 5.6. *For our purpose of defining $F^{\bullet\Gamma}_{\chi,\vec{d},\vec{\mu}}(u_1,u_2)$, we may consider the equivariant Borel-Moore homology $H^T_*(\mathcal{M})=H^{T_{\mathbb{R}}}_*(\mathcal{M})$ instead of the equivariant Chow group $A^T_*(\mathcal{M})$, and consider the equivariant cohomology $H^*_T(\mathcal{M})=H^*_{T_{\mathbb{R}}}(\mathcal{M})$ instead of the equivariant operational Chow group $A^*_T(\mathcal{M})$, where $\mathcal{M}$ is any of the moduli spaces involved in the above discussions.*

If $\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(\hat{Y}^{\mathrm{rel}}_\Gamma,\hat{L})$ were a proper Deligne-Mumford stack then its obstruction theory would define a virtual cycle

$$[\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(\hat{Y}^{\mathrm{rel}}_\Gamma,\hat{L})]^{\mathrm{vir}}\in A_0\left(\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(\hat{Y}^{\mathrm{rel}}_\Gamma,\hat{L})\right)\tag{5.14}$$

and

$$F^{\bullet\Gamma}_{\chi,\vec{d},\vec{\mu}}(u_1,u_2)=\frac{1}{|\operatorname{Aut}(\vec{\mu})|}\deg[\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(\hat{Y}^{\mathrm{rel}}_\Gamma,\hat{L})]^{\mathrm{vir}}\in\mathbb{Q}$$

would be a topological invariant independent of u_1,u_2. However, (5.14) does not exist. Nevertheless, we will show that

Theorem 5.7. *The function $F^{\bullet\Gamma}_{\chi,\vec{d},\vec{\mu}}(u_1,u_2)$ is independent of u_1,u_2; hence is a rational number depending only on Γ, χ, $\vec{d}$ and $\vec{\mu}$.*

In Section 6 and Section 7, we will reduce the invariance of $F^{\bullet\Gamma}_{\chi,\vec{d},\vec{\mu}}(u_1,u_2)$ (Theorem 5.7) to the invariance of $F^{\bullet\Gamma_0}_{\chi,\vec{d},\vec{\mu}}(u_1,u_2)$ for a special topological vertex Γ^0 (Theorem 5.9).

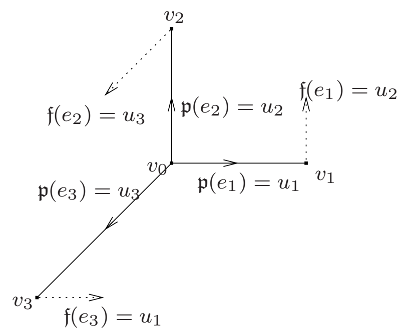


FIGURE 11. Topological vertex with the standard framing

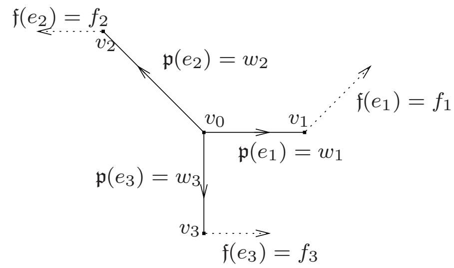


FIGURE 12. The graph of a topological vertex

Definition 5.8. *A topological vertex is a FTCY graph that has one trivalent vertex and three univalent vertices.*

The topological vertex Γ^0 (see Figure 11) is the one whose three edges e_1, e_2 and e_3 share v_0 as their initial vertices and have position vectors

$$u_1, \ u_2 \quad \text{and} \quad u_3 = -u_1 - u_2$$

and have framings

$$\mathfrak{f}(e_1) = u_2, \ \mathfrak{f}(e_2) = u_3 \quad \text{and} \quad \mathfrak{f}(e_3) = u_1.$$

Theorem 5.9. *Let Γ^0 be the topological vertex defined, let $(\vec{d}, \vec{\mu})$ be an effective class of Γ^0, and let χ be an even integer. Then the formal Gromov-Witten invariant*

$$F^{\bullet \Gamma^0}_{\chi, \vec{d}, \vec{\mu}}(u_1, u_2) \in \mathbb{Q}(u_2/u_1)$$

is a constant function in u_2/u_1.

The proof of Theorem 5.9 will be given in Appendix A.

6. Topological vertex, Hodge Integrals and double Hurwitz numbers

Let $\Gamma_{\mathbf{n};w_1,w_2}$ be the FTCY in Figure 12, where

$$f_1 = w_2 - n_1 w_1, \quad f_2 = w_3 - n_2 w_2, \quad f_3 = w_1 - n_3 w_3, \quad w_3 = -w_1 - w_2,$$

and $\mathbf{n} = (n_1, n_2, n_3) \in \mathbb{Z}^{\oplus 3}$. Any topological vertex is of this form.

In this section, we will compute

$$F^{\bullet}_{\chi,\vec{\mu}}(\mathbf{n};w_1,w_2)=F^{\bullet\Gamma_{\mathbf{n};w_1,w_2}}_{\chi,\vec{d},\vec{\mu}}(u_1,u_2) \tag{6.1}$$

To simplify the notation, we will fix $\mathbf{n}=(n_1,n_2,n_3)$ and (w_1,w_2) and write Γ instead of $\Gamma_{\mathbf{n};w_1,w_2}$.

6.1. **Torus fixed points and label notation.** In this subsection, we describe the T-fixed points in $\mathcal{M}^{\bullet}_{\chi,\vec{\mu}}(\Gamma)=\mathcal{M}^{\bullet}_{\chi,\vec{d},\vec{\mu}}(\hat{Y}^{\mathrm{rel}}_{\Gamma},\hat{L})$, and introduce the label notation. Each label corresponds a disjoint union of connected components of

$$\mathcal{M}^{\bullet}_{\chi,\vec{\mu}}(\Gamma)^T=\mathcal{M}^{\bullet}_{\chi,\vec{\mu}}(\Gamma)^{T_{\mathbb{R}}},$$

or equivalently, a collection of graphs in graph notation.

Let $\hat{Y}^{\mathrm{rel}}_{\Gamma}=(\hat{Y},\hat{D})$ be the FTCY associated to Γ, and let

$$\hat{D}^i=\hat{D}^{v_i},\quad C^i=C^{\bar{e}_i}$$

for $i=1,2,3$.

Given

$$u:(X,\mathbf{q})\longrightarrow Y^{\mathrm{rel}}_{\Gamma_{\mathbf{m}}}=\big(\hat{Y}_{\mathbf{m}},\hat{D}_{\mathbf{m}}\big)$$

which represents a point in $\mathcal{M}^{\bullet}_{\chi,\vec{\mu}}(\Gamma)^T$, let $\tilde{u}=\pi_{\mathbf{m}}\circ u:X\to\hat{Y}^{\mathrm{rel}}_{\Gamma}$, where $\pi_{\mathbf{m}}:\hat{Y}_{\mathbf{m}}\to\hat{Y}$ be the projection defined in Section 5.1. Then $\tilde{u}(X)\subset C^1\cup C^2\cup C^3$. Let z^0 and z^i be the two T fixed points on C^i, and let

$$V^i=\tilde{u}^{-1}(z^i)$$

for $i=0,1,2,3$. Let E^i be the closure of $\tilde{u}^{-1}(C^i\setminus\{z^0,z^i\})$ for $i=1,2,3$. Then E^i is a union of projective lines, and $u|_{E^i}:E^i\to C^i$ is a degree $d^i=|\mu^i|$ cover fully ramified over z^0 and z^i.

Define

$$\mathbb{P}^i(m^i)=\pi_{\mathbf{m}}^{-1}(z^i)$$

which is a point if $m^i=0$, and is a chain of m^i copies of $\mathbb{P}^1$ if $m^i>0$.

For $i=1,2,3$, let

$$\begin{aligned}\hat{u}^i=u|_{V^i}:V^i&\;\to\;\mathbb{P}^i(m^i),\\ \tilde{u}^i=u|_{E^i}:E^i&\;\to\;C_i.\end{aligned}$$

The degrees of $\tilde{u}^i$ restricted to connected components of E^i determine a partition ν^i of d^i.

For $i=0,1,2,3$, let Let $V^i_1,\ldots,V^i_{k^i}$ be the connected components of V^i, and let g^i_j be the arithmetic genus of V^i_j. (We define $g^i_j=0$ if V^i_j is a point.) Define

$$\chi^i=\sum_{j=1}^{k^i}(2-2g^i_j).$$

Then

$$-\sum_{i=0}^{3}\chi^i+2\sum_{i=1}^{3}\ell(\nu^i)=-\chi.$$

Note that $\chi^i\leq 2\min\{\ell(\mu^i),\ell(\nu^i)\}$ for $i=1,2,3$, so

$$-\chi^i+\ell(\nu^i)+\ell(\mu^i)\geq 0$$

and the equality holds if and only if $m^i=0$. In this case, we have $\nu^i=\mu^i$, $\chi^i=2\ell(\mu^i)$.

For each $i \in \{1,2,3\}$, there are two cases:
Case 1: $m^i = 0$. Then $\hat{u}^i$ is a constant map from $\ell(\mu^i)$ points to p^i.
Case 2: $m^i > 0$. Then $\hat{u}^i$ represents a point in

$$\overline{\mathcal{M}}^\bullet_{\chi^i}(\mathbb{P}^1, \nu^i, \mu^i)//\mathbb{C}^*.$$

We refer to [23, Section 5] for the definitions of $\overline{\mathcal{M}}^\bullet_\chi(\mathbb{P}^1, \nu, \mu)//\mathbb{C}^*$ and the target ψ classes ψ^0, ψ^∞.

Definition 6.1. *An* admissible label *of* $\mathcal{M}^\bullet_{\chi,\vec{\mu}}(\Gamma)$ *is a pair* $(\vec{\chi}, \vec{\nu})$ *such that*

(1) $\vec{\chi} = (\chi^0, \chi^1, \chi^2, \chi^3)$, *where* $\chi^i \in 2\mathbb{Z}$.
(2) $\vec{\nu} = (\nu^1, \nu^2, \nu^3)$, *where* ν^i *is a partition such that* $|\nu^i| = |\mu^i|$.
(3) $\chi^0 \leq 2\sum_{i=1}^3 \ell(\nu^i)$.
(4) $\chi^i \leq 2\min\{\ell(\mu^i), \ell(\nu^i)\}$ *for* $i = 1,2,3$.
(5) $-\sum_{i=0}^3 \chi^i + 2\sum_{i=1}^3 \ell(\nu^i) = -\chi$.

Let $G^\bullet_{\chi,\vec{\mu}}(\Gamma)$ *denote the set of all admissible labels of* $\mathcal{M}^\bullet_{\chi,\vec{\mu}}(\Gamma)$.

For a nonnegative integer g and a positive integer h, let $\overline{\mathcal{M}}_{g,h}$ be the moduli space of stable curves of genus g with h marked points. $\overline{\mathcal{M}}_{g,h}$ is empty for $(g,h) = (0,1),(0,2)$, but we will assume that $\overline{\mathcal{M}}_{0,1}$ and $\overline{\mathcal{M}}_{0,2}$ exist and satisfy

$$\int_{\overline{\mathcal{M}}_{0,1}} \frac{1}{1-d\psi} = \frac{1}{d^2}$$

$$\int_{\overline{\mathcal{M}}_{0,2}} \frac{1}{(1-\mu_1\psi_1)(1-\mu_2\psi_2)} = \frac{1}{\mu_1+\mu_2}$$

for simplicity of notation. Such an assumption will give the correct final results.

For a nonnegative integer g and a positive integer h, let $\overline{\mathcal{M}}^\bullet_{\chi,h}$ be the moduli of possibly disconnected stable curves C with h marked points such that

- If $C_1, \ldots, C_k$ are connected components of C, and g_i is the arithmetic genus of C_i, then
$$\sum_{i=1}^k (2-2g_i) = \chi.$$
- Each connected component contains at least one marked point.

The connected components of $\overline{\mathcal{M}}^\bullet_{\chi,h}$ are of the form

$$\overline{\mathcal{M}}_{g_1,h_1} \times \cdots \times \overline{\mathcal{M}}_{g_k,h_k}.$$

where

$$\sum_{i=1}^k (2-2g_i) = \chi, \quad \sum_{i=1}^k h_i = h.$$

The restriction of the Hodge bundle $\mathbb{E} \to \overline{\mathcal{M}}^\bullet_{\chi,h}$ to the above connected component is the direct sum of the Hodge bundles on each factor, and

$$\Lambda^\vee(u) = \prod_{i=1}^k \Lambda^\vee_{g_i}(u).$$

We define

$$\overline{\mathcal{M}}_{\vec{\chi},\vec{\nu}} = \prod_{i=0}^3 \overline{\mathcal{M}}^i_{\vec{\chi},\vec{\nu}},$$

where

$$\begin{aligned}\overline{\mathcal{M}}^0_{\vec{\chi},\vec{\nu}} &= \overline{\mathcal{M}}^\bullet_{\chi^0,\ell(\vec{\nu})}\\ \overline{\mathcal{M}}^i_{\vec{\chi},\vec{\nu}} &= \begin{cases}\{\mathrm{pt}\}, & -\chi^i+\ell(\nu^i)+\ell(\mu^i)=0\\ \overline{\mathcal{M}}^\bullet_{\chi^i}(\mathbb{P}^1,\nu^i,\mu^i)//\mathbb{C}^*, & -\chi^i+\ell(\nu^i)+\ell(\mu^i)>0\end{cases}\end{aligned}$$

For each $(\vec{\chi},\vec{\nu})\in G^\bullet_{\chi,\vec{\mu}}(\Gamma)$, there is a morphism

$$i_{\vec{\chi},\vec{\nu}}:\overline{\mathcal{M}}_{\vec{\chi},\vec{\nu}}\to\mathcal{M}^\bullet_{\chi,\vec{\mu}}(\Gamma)^T,$$

whose image $\mathcal{F}_{\vec{\chi},\vec{\nu}}$ is a union of connected components of $\mathcal{M}^\bullet_{\chi,\vec{\mu}}(\Gamma)^T$.

The morphism $i_{\vec{\chi},\vec{\nu}}$ induces an isomorphism

$$\overline{\mathcal{M}}_{\vec{\chi},\vec{\nu}}\Big/\left(\prod_{i=1}^3 A^i_{\vec{\chi},\vec{\nu}}\right)\cong\mathcal{F}_{\vec{\chi},\vec{\nu}},$$

where

$$A^i_{\vec{\chi},\vec{\nu}}=\prod_{j=1}^{\ell(\nu^i)}\mathbb{Z}_{\nu^i_j}$$

if $-\chi^i+\ell(\nu^i)+\ell(\mu^i)=0$, and

$$1\to\prod_{j=1}^{\ell(\nu^i)}\mathbb{Z}_{\nu^i_j}\to A^i_{\vec{\chi},\vec{\nu}}\to\mathrm{Aut}(\nu^i)\to 1$$

if $-\chi^i+\ell(\nu^i)+\ell(\mu^i)>0$.

The fixed points set $\mathcal{M}^\bullet_{\chi,\vec{\mu}}(\Gamma)^T$ is a disjoint union of

$$\{\mathcal{F}_{\vec{\chi},\vec{\nu}}\mid(\vec{\chi},\vec{\nu})\in G^\bullet_{\chi,\vec{\mu}}(\Gamma)\}.$$

Remark 6.2. *There are two perfect obstruction theories on $\mathcal{F}_{\vec{\chi},\vec{\nu}}$: one is the fixing part $[\mathcal{T}^{1,f}\to\mathcal{T}^{2,f}]$ of the restriction of the perfect obstruction theory on $\mathcal{M}^\bullet_{\chi,\vec{\mu}}(\Gamma)$; the other comes from the perfect obstruction theory on the moduli spaces $\overline{\mathcal{M}}^\bullet_{\chi^0,\ell(\vec{\nu})}$ and $\overline{\mathcal{M}}^\bullet_{\chi^i}(\mathbb{P}^1,\nu^i,\mu^i)//\mathbb{C}^*$. It is straightforward to check that they coincide.*

6.2. **Contribution from each label.** Let $w_3=-w_1-w_2$, $w_4=w_1$. We view w_i and f_i as elements in

$$\mathbb{Z}u_1\oplus\mathbb{Z}u_2=\Lambda_T\cong H^2_T(\mathrm{pt},\mathbb{Q}).$$

Recall that $H^*_T(\mathrm{pt};\mathbb{Q})=\mathbb{Q}[u_1,u_2]$. The results of localization calculations will involve rational functions of w_i and f_i which are elements in $\mathbb{Q}(u_1,u_2)$.

If $m^i>0$, let ψ^0_i,ψ^∞_i denote the target ψ class of $\overline{\mathcal{M}}^i_{\vec{\chi},\vec{\nu}}$. Let $N^{\mathrm{vir}}_{\vec{\chi},\vec{\nu}}$ denote the virtual bundle on $\overline{\mathcal{M}}_{\vec{\chi},\vec{\nu}}$ which is the pull back of $\mathcal{T}^{1,m}-\mathcal{T}^{2,m}$ under $i_{\vec{\chi},\vec{\mu}}$.

With the above notation and explicit description of $[\mathcal{T}^1\to\mathcal{T}^2]$ in Section 5.3, calculations similar to those in [22, Appenix A] show that

$$\frac{1}{e_T(N^{\mathrm{vir}}_{\vec{\chi},\vec{\nu}})}=\prod_{i=0}^3 A_{v_i}\prod_{i=1}^3 A_{e_i},$$

where

$$\begin{aligned} A_{v_0} &= \prod_{i=1}^3 \frac{a_{\nu^i}\Lambda^\vee(w_i)w_i^{\ell(\vec{\nu})-1}}{\prod_{j=1}^{\ell(\nu^i)}(w_i(w_i-\nu^i_j\psi^i_j))} \\ A_{v_i} &= 1, \quad -\chi^i+\ell(\nu^i)+\ell(\mu^i)=0 \\ & \quad (-1)^{\ell(\nu^i)-\chi^i/2}a_{\nu^i}\frac{f_i^{-\chi^i+\ell(\nu^i)+\ell(\mu^i)}}{-w_i-\psi^0_i}, \quad -\chi^i+\ell(\nu^i)+\ell(\mu^i)>0 \\ A_{e_i} &= (-1)^{|\nu^i|n^i+\ell(\nu^i)-|\nu^i|}\prod_{j=1}^{\ell(\nu^i)}\frac{\prod_{a=1}^{\nu^i_j-1}(w_{i+1}\nu^i_j+aw_i)}{(\nu^i_j-1)!w_i^{\nu^i_j-1}} \end{aligned}$$

Define

$$V_{\chi,\vec{\nu}}(w_1,w_2,w_3)=\frac{1}{|\operatorname{Aut}(\vec{\nu})|}\int_{\overline{\mathcal{M}}^\bullet_{\chi,\ell(\vec{\nu})}}\prod_{i=1}^3\frac{\Lambda^\vee(w_i)w_i^{\ell(\vec{\nu})-1}}{\prod_{j=1}^{\ell(\nu^i)}(w_i(w_i-\nu^i_j\psi^i_j))} \tag{6.2}$$

$$E_\nu(x,y)=\prod_{j=1}^{\ell(\nu)}\frac{\prod_{a=1}^{\nu_j-1}(y\nu_j+ax)}{(\nu_j-1)!x^{\nu_j-1}} \tag{6.3}$$

$$G^\bullet_{\chi,\vec{\nu}}(w_1,w_2,w_3)=(-\sqrt{-1})^{\ell(\vec{\nu})}V_{\chi,\vec{\nu}}(w_1,w_2,w_3)\prod_{i=1}^3 E_{\nu^i}(w_{i+1},w_i) \tag{6.4}$$

$$H^\bullet_{\chi,\nu,\mu}=\frac{(-\chi+\ell(\nu)+\ell(\mu))!}{|\operatorname{Aut}(\nu)\times\operatorname{Aut}(\mu)|}\int_{[\overline{\mathcal{M}}^\bullet_\chi(\mathbb{P}^1,\nu,\mu)//(\mathbb{C}^*)]^{\mathrm{vir}}}(\psi^0)^{-\chi+\ell(\nu)+\ell(\mu)-1} \tag{6.5}$$

It is known that $H^\bullet_{\chi,\nu,\mu}$ coincides with disconnected double Hurwitz numbers defined in Section 3.4. See [23, Section 5] for a derivation. Note that (6.4) is consistent with the definition in Section 3.2.

We have

$$\begin{aligned} & I_{\vec{\chi},\vec{\nu}}(\mathbf{n};w_1,w_2) \\ = & \int_{[\mathcal{F}_{\vec{\chi},\vec{\nu}}]^{\mathrm{vir}}}\frac{1}{e_T(N^{\mathrm{vir}}_{\vec{\chi},\vec{\nu}})} \\ = & \frac{1}{\prod_{i=1}^3|A^i_{\vec{\chi},\vec{\nu}}|}\int_{[\overline{\mathcal{M}}_{\vec{\chi},\vec{\nu}}]^{\mathrm{vir}}}\frac{1}{e_T(N^{\mathrm{vir}}_{\vec{\chi},\vec{\nu}})} \\ = & |\operatorname{Aut}(\vec{\mu})|(-1)^{\sum_{i=1}^3(n_i-1)|\mu^i|}(-\sqrt{-1})^{\ell(\vec{\mu})+\ell(\vec{\nu})}V_{\chi^0,\vec{\nu}}(w_1,w_2,w_3) \\ & \cdot\prod_{i=1}^3 E_{\nu^i}(w_i,w_{i+1})z_{\nu^i}\left(-\sqrt{-1}\frac{f_i}{w_i}\right)^{-\chi^i+\ell(\nu^i)+\ell(\mu^i)}\frac{H^\bullet_{\chi^i,\nu^i,\mu^i}}{(-\chi^i+\ell(\nu^i)+\ell(\mu^i))!} \\ = & |\operatorname{Aut}(\vec{\mu})|(-1)^{\sum_{i=1}^3(n_i-1)|\mu^i|}(-\sqrt{-1})^{\ell(\vec{\mu})}G^\bullet_{\chi^0,\vec{\nu}}(w_1,w_2,w_3) \\ & \cdot\prod_{i=1}^3 z_{\nu^i}\left(\sqrt{-1}\left(n_i-\frac{w_{i+1}}{w_i}\right)\right)^{-\chi^i+\ell(\nu^i)+\ell(\mu^i)}\frac{H^\bullet_{\chi^i,\nu^i,\mu^i}}{(-\chi^i+\ell(\nu^i)+\ell(\mu^i))!}. \end{aligned}$$

6.3. **Sum over labels.** We have

$$F^{\bullet}_{\chi,\vec{\mu}}(\mathbf{n};w_1,w_2)=\frac{1}{|\operatorname{Aut}(\vec{\mu})|}\sum_{(\vec{\chi},\vec{\nu})\in G^{\bullet}_{\chi,\vec{\mu}}(\Gamma)} I_{\vec{\chi},\vec{\nu}}(\mathbf{n};w_1,w_2).$$

Define generation functions

$$F^{\bullet}_{\vec{\mu}}(\lambda;\mathbf{n};w_1,w_2)=\sum_{\chi\in 2\mathbb{Z},\chi\leq\ell(\vec{\mu})}\lambda^{-\chi+\ell(\vec{\mu})}F^{\bullet}_{\chi,\vec{\mu}}(\mathbf{n};w_1,w_2) \tag{6.6}$$

$$\tilde{F}^{\bullet}_{\vec{\mu}}(\lambda;\mathbf{n};w_1,w_2)=(-1)^{\sum_{i=1}^{3}(n_i-1)|\mu^i|}\sqrt{-1}^{\ell(\vec{\mu})}F^{\bullet}_{\vec{\mu}}(\lambda;\mathbf{n};w_1,w_2) \tag{6.7}$$

Then

$$\tilde{F}^{\bullet}_{\vec{\mu}}(\lambda;\mathbf{n};w_1,w_2)=\sum_{|\nu^i|=|\mu^i|}G^{\bullet}_{\vec{\nu}}(\lambda;w_1,w_2,w_3)\prod_{i=1}^{3}z_{\nu^i}\Phi^{\bullet}_{\nu^i,\mu^i}\left(\sqrt{-1}\left(n_i-\frac{w_{i+1}}{w_i}\right)\lambda\right), \tag{6.8}$$

where $G^{\bullet}_{\vec{\mu}}(\lambda;w_1,w_2,w_3)$ is defined as in Section 3.2, and $\Phi^{\bullet}_{\nu,\mu}(\lambda)$ is defined as in Section 3.4.

Equations (6.8), (3.12), and (3.13) imply that

$$\tilde{F}^{\bullet}_{\vec{\mu}}(\lambda;\mathbf{n};w_1,w_2)=\sum_{|\nu^i|=|\mu^i|}\tilde{F}^{\bullet}_{\vec{\nu}}(\lambda;\mathbf{0},w_1,w_2)\prod_{i=1}^{3}z_{\nu^i}\Phi^{\bullet}_{\nu^i,\mu^i}\left(\sqrt{-1}n_i\lambda\right) \tag{6.9}$$

By Theorem 5.9,

$$F^{\bullet}_{\vec{\nu}}(\lambda;\mathbf{0};w_1,w_2)=\sum_{\chi}\lambda^{-\chi+\ell(\vec{\nu})}F^{\bullet\Gamma^0}_{\chi,\vec{d},\vec{\mu}}(w_1,w_2)$$

does not depend on w_1,w_2. So by (6.7) and (6.9), $F^{\bullet}_{\vec{\mu}}(\lambda;\mathbf{n};w_1,w_2)$ and $\tilde{F}^{\bullet}_{\vec{\mu}}(\lambda;\mathbf{n};w_1,w_2)$ do not depend on w_1,w_2. From now on, we will write

$$F^{\bullet}_{\vec{\mu}}(\lambda;\mathbf{n}),\quad \tilde{F}^{\bullet}_{\vec{\mu}}(\lambda;\mathbf{n})$$

instead of $F^{\bullet}_{\vec{\mu}}(\lambda;\mathbf{n};w_1,w_2)$, $\tilde{F}^{\bullet}_{\vec{\mu}}(\lambda;\mathbf{n};w_1,w_2)$. To summarize, for each $\vec{\mu}\in\mathcal{P}^3_+$ and each $\mathbf{n}\in\mathbb{Z}^3$, we have defined an generating function $F^{\bullet}_{\vec{\mu}}(\lambda;\mathbf{n})$ which can be expressed in terms of Hodge integrals and double Hurwitz numbers as follows.

Proposition 6.3.

$$\tilde{F}^{\bullet}_{\vec{\mu}}(\lambda;\mathbf{n})=\sum_{|\nu^i|=|\mu^i|}G^{\bullet}_{\vec{\nu}}(\lambda;w_1,w_2,w_3)\prod_{i=1}^{3}z_{\nu^i}\Phi^{\bullet}_{\nu^i,\mu^i}\left(\sqrt{-1}\left(n_i-\frac{w_{i+1}}{w_i}\right)\lambda\right), \tag{6.10}$$

Proposition 6.3 and the sum formula (3.12) of double Hurwitz numbers imply:

Corollary 6.4.

$$\tilde{F}^{\bullet}_{\vec{\mu}}(\lambda;\mathbf{n})=\sum_{|\nu^i|=|\mu^i|}\tilde{F}^{\bullet}_{\vec{\nu}}(\lambda;0)\prod_{i=1}^{3}z_{\nu^i}\Phi^{\bullet}_{\nu^i,\mu^i}\left(\sqrt{-1}n_i\lambda\right) \tag{6.11}$$

Note that (6.11) is valid for any three complex numbers n_1,n_2,n_3.

6.4. **Representation basis.** The framing dependence (6.11) is particularly simple in the representation basis used in [1]. Define $\tilde{C}_\mu(\lambda;\mathbf{n})$ by

$$\tilde{C}_{\vec{\mu}}(\lambda;\mathbf{n}) = \sum_{|\nu^i|=|\mu^i|} \tilde{F}^\bullet_{\vec{\nu}}(\lambda;\mathbf{n}) \prod_{i=1}^3 \chi_{\mu^i}(\nu^i) \tag{6.12}$$

which is equivalent to

$$\tilde{F}_{\vec{\mu}}(\lambda;\mathbf{n}) = \sum_{|\nu^i|=|\mu^i|} \tilde{C}^\bullet_{\vec{\nu}}(\lambda;\mathbf{n}) \prod_{i=1}^3 \frac{\chi_{\nu^i}(\mu^i)}{z_{\mu^i}}. \tag{6.13}$$

Then (6.11) is equivalent to

Proposition 6.5.

$$\tilde{C}_{\vec{\mu}}(\lambda;\mathbf{n}) = e^{\sqrt{-1}(\sum_{i=1}^3 \kappa_{\mu^i} n_i)\lambda/2} \tilde{C}_{\vec{\mu}}(\lambda;\mathbf{0}). \tag{6.14}$$

Define $\tilde{C}_{\vec{\mu}}(\lambda) = \tilde{C}_{\vec{\mu}}(\lambda;\mathbf{0})$, and let $q = e^{\sqrt{-1}\lambda}$. Then (6.12), (6.10), and the Burnside formula (3.10) of double Hurwitz numbers imply

Proposition 6.6. *We have*

$$\tilde{C}_{\vec{\mu}}(\lambda) = q^{-\frac{1}{2}(\sum_{i=1}^3 \kappa_{\mu^i}\frac{w_{i+1}}{w_i})} \sum_{|\nu^i|=|\mu^i|} G^\bullet_{\vec{\nu}}(\lambda;\mathbf{w}) \prod_{i=1}^3 \chi_{\mu^i}(\nu^i). \tag{6.15}$$

$$G^\bullet_{\vec{\mu}}(\lambda;\mathbf{w}) = \sum_{|\nu^i|=|\mu^i|} \prod_{i=1}^3 \frac{\chi_{\nu^i}(\mu^i)}{z_{\mu^i}} q^{\frac{1}{2}(\sum_{i=1}^3 \kappa_{\nu^i}\frac{w_{i+1}}{w_i})} \tilde{C}_{\vec{\nu}}(\lambda). \tag{6.16}$$

7. Gluing formulae of formal relative Gromov-Witten invariants

In this section, we will calculate

$$F^{\bullet\Gamma}_{\chi,\vec{d},\vec{\mu}}(u_1,u_2) \in \mathbb{Q}(u_2/u_1) \tag{7.1}$$

where Γ is a FTCY graph, and $(\vec{d},\vec{\mu})$ is an effective class of Γ. We will reduce the invariance of $F^{\Gamma\bullet}_{\chi,\vec{d},\vec{\mu}}$ (Theorem 5.7) to the invariance of the topological vertex at the standard framing (Theorem 5.9). We will derive gluing formulae for such invariants.

7.1. **Torus fixed points and label notation.** In this subsection, we describe the T-fixed points in $\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(\hat{Y}^{\mathrm{rel}}_\Gamma,\hat{L})$, and introduce the label notation.

Given a morphism

$$u:(X,\mathbf{q}) \longrightarrow (\hat{Y}_{\mathbf{m}},\hat{D}_{\mathbf{m}})$$

which represents a point in $\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(\hat{Y}^{\mathrm{rel}}_\Gamma,\hat{L})^T$, let $\tilde{u}=\pi_{\mathbf{m}}\circ u: X\to \hat{Y}$, as before. Then

$$\tilde{u}(X) \subset \bigcup_{\bar{e}\in E(\Gamma)} C^{\bar{e}}.$$

where $C^{\bar{e}}$ is defined as in Section 4.4.

Let z^v be the T fixed point associated to $v\in V(\Gamma)$, as in Section 4.4, and let

$$V^v = \tilde{u}^{-1}(z^v).$$

Let $E^{\bar{e}}$ be the closure of $\tilde{u}^{-1}(C^{\bar{e}} \setminus \{z^{\mathfrak{v}_0(e)}, z^{\mathfrak{v}_1(e)}\})$ for $\bar{e} = \{e, -e\} \in E(\Gamma)$. Then $E^{\bar{e}}$ is a union of projective lines, and $u|_{E^{\bar{e}}} : E^{\bar{e}} \to C^{\bar{e}}$ is a degree $\vec{d}(\bar{e})$ cover fully ramified over $\mathfrak{v}_0(e)$ and $\mathfrak{v}_1(e)$.

For $v \in V_1(\Gamma) \cup V_2(\Gamma)$, define

$$\mathbb{P}^v(m^v) = \pi_{\mathbf{m}}^{-1}(z^v)$$

which is a point if $m^v = 0$, and is a chain of m^v copies of $\mathbb{P}^1$ if $m^v > 0$. Let

$$\hat{u}^v = u|_{V^v} : V^v \to \mathbb{P}^v(m^v).$$

For $\bar{e} \in E(\Gamma)$, define

$$\tilde{u}^{\bar{e}} = u|_{E^{\bar{e}}} : E^{\bar{e}} \to C^{\bar{e}}.$$

The degrees of $\tilde{u}^{\bar{e}}$ restricted to connected components of $E^{\bar{e}}$ determine a partition $\nu^e = \nu^{-e}$ of $\vec{d}(\bar{e})$.

For $v \in V(\Gamma)$, let $V_1^v, \ldots, V_{k^v}^v$ be the connected components of V^v, and let g_j^v be the arithmetic genus of V_j^v. (We define $g_j^v = 0$ if V_j^v is a point.) Define

$$\chi^v = \sum_{j=1}^{k^v} (2 - 2g_j^v).$$

Then

$$-\sum_{v \in V(\Gamma)} \chi^v + \sum_{e \in E^o(\Gamma)} \ell(\nu^e) = -\chi.$$

Given $v \in V_1(\Gamma)$ with $\mathfrak{v}_1^{-1}(v) = \{e\}$, we have $\chi^v \leq 2\min\{\ell(\nu^e), \ell(\mu^v)\}$, so

$$r^v = \chi^v + \ell(\nu^e) + \ell(\mu^v) \geq 0 \tag{7.2}$$

and the equality holds if and only if $m^v = 0$. In this case, we have $\nu^e = \mu^v$, $\chi^v = 2\ell(\mu^v)$. For each $v \in V_1(\Gamma)$, there are two cases:
Case 1: $m^v = 0$. Then $\hat{u}^v$ is a constant map from $\ell(\mu^v)$ points to z^v.
Case 2: $m^v > 0$. Then $\hat{u}^v$ represents a point in

$$\overline{\mathcal{M}}^{\bullet}_{\chi^v}(\mathbb{P}^1, \nu^e, \mu^v)//\mathbb{C}^*$$

Given $v \in V_2(\Gamma)$ with $\mathfrak{v}_1^{-1}(v) = \{e, e'\}$, we have $\chi^v \leq 2\min\{\ell(\nu^e, \ell(\nu^{e'})\}$, so

$$-\chi^v + \ell(\nu^e) + \ell(\nu^{e'}) \geq 0 \tag{7.3}$$

and the equality holds if and only if $m^v = 0$. In this case, we have $\nu^e = \nu^{e'}$, $\chi^v = 2\ell(\nu^e)$. For each $v \in V_2(\Gamma)$, there are two cases:
Case 1': $m^v = 0$. Then $\hat{u}^v$ is a constant map from $\ell(\mu^v)$ points to z^v.
Case 2': $m^v > 0$. Then $\hat{u}^v$ represents a point in

$$\overline{\mathcal{M}}^{\bullet}_{\chi^v}(\mathbb{P}^1, \nu^e, \nu^{e'})//\mathbb{C}^*$$

Definition 7.1. *An* admissible label *of* $\mathcal{M}^{\bullet}_{\chi,\vec{d},\vec{\mu}}(\hat{Y}_\Gamma^{\mathrm{rel}}, \hat{L})$ *is a pair* $(\vec{\chi}, \vec{\nu})$ *such that*

(1) $\vec{\chi} : V(\Gamma) \longrightarrow 2\mathbb{Z}$. *Let* χ^v *denote* $\vec{\chi}(v)$.
(2) $\vec{\nu} : E^o(\Gamma) \longrightarrow \mathcal{P}$, *where* $\vec{\nu}(e) = \vec{\nu}(-e)$ *and* $|\vec{\nu}(e)| = \vec{d}(\bar{e})$. *Let* ν^e *denote* $\vec{\nu}(e)$.
(3) *For* $v \in V_1(\Gamma)$ *with* $\mathfrak{v}_1^{-1}(v) = \{e\}$, *we have* $\chi^v \leq 2\min\{\ell(\nu^e), \ell(\mu^v)\}$.
(4) *For* $v \in V_2(\Gamma)$ *with* $\mathfrak{v}_1^{-1}(v) = \{e, e'\}$, *we have* $\chi^v \leq 2\min\{\ell(\nu^e), \ell(\nu^{e'})\}$.
(5) *For* $v \in V_3(\Gamma)$, *define* $\ell_{\vec{\nu}}(v) = \sum_{e \in \mathfrak{v}_0^{-1}(v)} \ell(\nu^e)$. *Then* $\chi^v \leq 2l_{\vec{\nu}}(v)$.
(6) $-\sum_{v \in V(\Gamma)} \chi^v + 2\sum_{e \in E(\Gamma)} \ell(\nu^e) = -\chi$.

Let $G^\bullet_\chi(\Gamma,\vec{d},\vec{\mu})$ denote the set of all admissible labels of $\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(\hat{Y}^{\mathrm{rel}}_\Gamma,\hat{L})$.

Given $(\vec{\chi},\vec{\nu})\in G^\bullet_\chi(\Gamma,\vec{d},\vec{\mu})$, define r^v as in (7.2) and (7.3) for $v\in V_1(\Gamma)$ and $v\in V_2(\Gamma)$, respectively. We define

$$\overline{\mathcal{M}}_{\vec{\chi},\vec{\nu}}=\prod_{v\in V(\Gamma)}\overline{\mathcal{M}}^v_{\vec{\chi},\vec{\nu}}$$

where

$$\overline{\mathcal{M}}^v_{\vec{\chi},\vec{\nu}}=\begin{cases}\{\mathrm{pt}\}, & v\in V_1(\Gamma)\cup V_2,\ r^v=0,\\ \overline{\mathcal{M}}^\bullet_{\chi^v}(\mathbb{P}^1,\nu^e,\mu^v)//\mathbb{C}^*, & v\in V_1(\Gamma),\ \mathfrak{v}_1^{-1}(v)=\{e\},\ r^v>0,\\ \overline{\mathcal{M}}^\bullet_{\chi^v}(\mathbb{P}^1,\nu^e,\nu^{e'})//\mathbb{C}^*, & v\in V_2(\Gamma),\ \mathfrak{v}_1^{-1}(v)=\{e,e'\},\ r^v>0,\\ \overline{\mathcal{M}}^\bullet_{\chi^v,\ell_{\vec{\nu}}(v)}, & v\in V_3(\Gamma).\end{cases}$$

For each $(\vec{\chi},\vec{\nu})\in G^\bullet_\chi(\Gamma,\vec{d},\vec{\mu})$, there is a morphism

$$i_{\vec{\chi},\vec{\nu}}:\overline{\mathcal{M}}_{\vec{\chi},\vec{\nu}}\to\mathcal{M}^\bullet_\chi(\hat{Y}^{\mathrm{rel}}_\Gamma,\vec{d},\vec{\mu})^T,$$

whose image $\mathcal{F}_{\vec{\chi},\vec{\nu}}$ is a union of connected components of $\mathcal{M}^\bullet_\chi(\hat{Y}^{\mathrm{rel}}_\Gamma,\vec{d},\vec{\mu})^T$.

The morphism $i_{\vec{\chi},\vec{\nu}}$ induces an isomorphism

$$\overline{\mathcal{M}}_{\vec{\chi},\vec{\nu}}\Big/\left(\prod_{\bar{e}\in E(\Gamma)}A^{\bar{e}}_{\vec{\chi},\vec{\nu}}\right)\cong\mathcal{F}_{\vec{\chi},\vec{\nu}},$$

where

$$1\to\prod_{j=1}^{\ell(\nu^e)}\mathbb{Z}_{\nu^e_j}\to A^{\bar{e}}_{\vec{\chi},\vec{\nu}}\to\mathrm{Aut}(\nu^e)\to 1,$$

unless

$$\{\mathfrak{v}_0(e),\mathfrak{v}_1(e)\}\cap V_1(\Gamma)=\{v\}\neq\emptyset$$

and $r^v=0$. In this case,

$$A_{\bar{e}}=\prod_{j=1}^{\ell(\nu^e)}\mathbb{Z}_{\nu^e_j}.$$

The fixed points set $\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(\hat{Y}^{\mathrm{rel}}_\Gamma,\hat{L})^T$ is a disjoint union of

$$\{\mathcal{F}_{\vec{\chi},\vec{\nu}}\mid(\vec{\chi},\vec{\nu})\in G^\bullet_\chi(\Gamma,\vec{d},\vec{\mu})\}.$$

7.2. **Perfect obstruction theory on fixed points set.** There are two perfect obstruction theories on $\mathcal{F}_{\vec{\chi},\vec{\nu}}$: one is the fixing part $[\mathcal{T}^{1,f}\to\mathcal{T}^{2,f}]$ of the restriction of the perfect obstruction theory on $\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(\hat{Y}^{\mathrm{rel}}_\Gamma,\hat{L})$; the other comes from the perfect obstruction theory on the moduli spaces $\overline{\mathcal{M}}^\bullet_{\chi^v,\ell_{\vec{\nu}}(v)}$ and $\overline{\mathcal{M}}^\bullet_{\chi^v}(\mathbb{P}^1,\nu,\mu)//\mathbb{C}^*$. Let $[\overline{\mathcal{M}}_{\vec{\chi},\vec{\nu}}]^{\mathrm{vir}}$ denote the virtual cycle defined by $[\mathcal{T}^{1,f}\to\mathcal{T}^{2,f}]$. By inspecting the T-action on the perfect obstruction theory on $\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(\hat{Y}^{\mathrm{rel}}_\Gamma,\hat{L})$ (see [18] and the description in Section 5), we get

$$[\overline{\mathcal{M}}_{\vec{\chi},\vec{\nu}}]^{\mathrm{vir}}=\prod_{v\in V(\Gamma)}[\overline{\mathcal{M}}^v_{\vec{\chi},\vec{\nu}}]^{\mathrm{vir}}$$

where

$$[\overline{\mathcal{M}}^v_{\vec{\chi},\vec{\nu}}]^{\mathrm{vir}} = \begin{cases} [\{\mathrm{pt}\}], & v\in V_1(\Gamma)\cup V_2(\Gamma),\ r^v=0,\\ [\overline{\mathcal{M}}^\bullet_{\chi^v}(\mathbb{P}^1,\nu^e,\mu^v)//\mathbb{C}^*]^{\mathrm{vir}}, & v\in V_1(\Gamma),\ \mathfrak{v}_1^{-1}(v)=\{e\},\ r^v>0,\\ c_1(\mathbb{L})\cap[\overline{\mathcal{M}}^\bullet_{\chi^v}(\mathbb{P}^1,\nu^e,\nu^{e'})//\mathbb{C}^*]^{\mathrm{vir}}, & v\in V_2(\Gamma),\ \mathfrak{v}_1^{-1}(v)=\{e,e'\},\ r^v>0,\\ [\overline{\mathcal{M}}^\bullet_{\chi^v,\ell_{\vec{\nu}}(v)}], & v\in V_3(\Gamma), \end{cases}$$

where $\mathbb{L}$ is a line bundle on $\overline{\mathcal{M}}^v_{\vec{\chi},\vec{\nu}}$ coming from the restriction of the line bundle $\mathbf{L}^v$ on $\mathcal{M}^\bullet_\chi(\hat{\mathcal{Y}},\vec{d},\vec{\mu})$ (see Section 5.3).

We now give a more explicit description of $\mathbb{L}$. Let

$$u:(X,\mathbf{q})\longrightarrow(\mathbb{P}^1(m),p_0,p_m)$$

represent a point in $\overline{\mathcal{M}}^\bullet_{\chi^v}(\mathbb{P}^1,\nu^+,\nu^-)//\mathbb{C}^*$, where $\mathbb{P}^1(m)$ is a chain of $m>0$ copies of $\mathbb{P}^1$ with two relative divisors p_0 and p_m. Let Δ_l be the l-th irreducible component of $\mathbb{P}^1(m)$ so that $\Delta_l\cap\Delta_{l+1}=\{p_l\}$. The complex lines

$$\mathbb{L}^0_u=T_{p_0}\Delta_1,\quad \mathbb{L}^1_u=\bigotimes_{l=1}^{m-1}T_{p_l}\Delta_l\otimes T_{p_l}\Delta_{l+1},\quad \mathbb{L}^\infty_u=T_{p_m}\Delta_m.$$

form line bundles $\mathbb{L}^0,\mathbb{L}^1,\mathbb{L}^\infty$ on $\overline{\mathcal{M}}^\bullet_{\chi^v}(\mathbb{P}^1,\nu^+,\nu^-)//\mathbb{C}^*$ when we vary u in $\overline{\mathcal{M}}^\bullet_{\chi^v}(\mathbb{P}^1,\nu^+,\nu^-)//\mathbb{C}^*$. The line bundle $\mathbb{L}$ is given by

$$\mathbb{L}=\mathbb{L}^0\otimes\mathbb{L}^1\otimes\mathbb{L}^\infty.$$

Note that

$$c_1(\mathbb{L}^0)=-\psi^0,\quad c_1(\mathbb{L}^\infty)=-\psi^\infty,$$

where ψ^0,ψ^∞ are target ψ classes.

Let $\mathcal{D}$ be the divisor in $\overline{\mathcal{M}}^\bullet_{\chi^v}(\mathbb{P}^1,\nu^+,\nu^-)//\mathbb{C}^*$ which corresponds to morphisms with target $\mathbb{P}^1(m)$, $m>1$. Then $\mathbb{L}^1=\mathcal{O}(\mathcal{D})$. Let I_{χ^v,ν^+,ν^-} be the set of triples (χ^+,χ^-,σ) such that

$$\begin{aligned}&\chi^+,\chi^-\in 2\mathbb{Z},\quad \sigma\in\mathcal{P},\quad |\sigma|=|\nu^+|=|\nu^-|,\quad -\chi^++2\ell(\sigma)-\chi^-=-\chi^v\\ &-\chi^++\ell(\nu^+)+\ell(\sigma)>0,\quad -\chi^-+\ell(\sigma)+\ell(\nu^-)>0.\end{aligned}$$

For each $(\chi^+,\chi^-,\sigma)\in I_{\chi,\nu^+,\nu^-}$, there is a morphism

$$\pi_{\chi^+,\chi^-,\sigma}:\Big(\overline{\mathcal{M}}^\bullet_{\chi^+}(\mathbb{P}^1,\nu^+,\sigma)//\mathbb{C}^*\Big)\times\Big(\overline{\mathcal{M}}^\bullet_{\chi^-}(\mathbb{P}^1,\sigma,\nu^-)//\mathbb{C}^*\Big)\longrightarrow\overline{\mathcal{M}}^\bullet_{\chi^v}(\mathbb{P}^1,\nu^+,\nu^-)//\mathbb{C}^*$$

with image contained in $\mathcal{D}$. Moreover,

$$\begin{aligned}&[\overline{\mathcal{M}}^\bullet_{\chi^v}(\mathbb{P}^1,\nu^+,\nu^-)//\mathbb{C}^*]^{\mathrm{vir}}\cap c_1(\mathbb{L}^1)\\ =&\sum_{(\chi^+,\chi^-,\sigma)\in I_{\chi^v,\nu^+,\nu^-}}\frac{a_\sigma}{|\mathrm{Aut}(\sigma)|}(\pi_{\chi^+,\chi^-,\sigma})_*\Big([\overline{\mathcal{M}}^\bullet_{\chi^+}(\mathbb{P}^1,\nu^+,\sigma)//\mathbb{C}^*]^{\mathrm{vir}}\times[\overline{\mathcal{M}}^\bullet_{\chi^-}(\mathbb{P}^1,\sigma,\nu^-)//\mathbb{C}^*]^{\mathrm{vir}}\Big)\end{aligned}$$

where $a_\sigma=\sigma_1\cdots\sigma_{\ell(\sigma)}$.

7.3. **Contribution from each label.** In this subsection, we view $\mathfrak{p}(e)$ and $\mathfrak{f}(e)$ as elements in

$$\mathbb{Z}u_1\oplus\mathbb{Z}u_2=\Lambda_T\cong H^2_T(\mathrm{pt},\mathbb{Q}).$$

Recall that $H^*_T(\mathrm{pt};\mathbb{Q})=\mathbb{Q}[u_1,u_2]$. The results of localization calculations will involve rational functions of $\mathfrak{p}(e)$ and $\mathfrak{f}(e)$.

Let $N^{\mathrm{vir}}_{\vec{\chi},\vec{\nu}}$ denote the pull back of $\mathcal{T}^{1,m}-\mathcal{T}^{2,m}$ of $\mathcal{F}_{\vec{\chi},\vec{\nu}}$ under $i_{\vec{\chi},\vec{\nu}}$. Let r^v be defined as (7.2) and (7.3). For $e\in E^o(\Gamma)$, let $\bar{e}=\{e,-e\}\in E(\Gamma)$ as before.

With the above notation and the explicit description of $[T^1 \to T^2]$ in Section 5.3, calculations similar to those in [22, Appendix A] show that

$$\frac{1}{e_T(N^{\mathrm{vir}}_{\vec{\chi},\vec{\nu}})} = \prod_{v\in V(\Gamma)} A_v \prod_{\bar{e}\in E(\Gamma)} A_{\bar{e}},$$

where

$$\begin{aligned}
A_v &= 1, \quad v \in V_1(\Gamma)\cup V_2(\Gamma),\ r^v = 0\\
A_v &= (-1)^{\ell(\nu^e)-\chi^v/2} a_{\nu^e}\frac{\mathfrak{f}(e)^{r^v}}{-\mathfrak{p}(e)-\psi^0},\\
&\quad v\in V_1(\Gamma), \mathfrak{v}_1^{-1}(v)=\{e\}, r^v>0,\\
A_v &= (-1)^{\ell(\nu^e)-\chi^v/2}\frac{a_{\nu^e}a_{\nu^{e'}}\mathfrak{f}(e)^{r^v}}{(-\mathfrak{p}(e)-\psi^0)(-\mathfrak{p}(e')-\psi^\infty)},\\
&\quad v\in V_2(\Gamma), \mathfrak{v}_1^{-1}(v)=\{e,e'\}, r^v>0,\\
A_v &= \prod_{e\in\mathfrak{v}_0^{-1}(v)} \frac{a_{\nu^e}\Lambda^\vee(\mathfrak{p}(e))\mathfrak{p}(e)^{\ell_{\vec{\nu}}(v)-1}}{\prod_{j=1}^{\ell(\nu^e)}\big(\mathfrak{p}(e)(\mathfrak{p}(e)-\nu^e_j\psi^e_j)\big)}, \quad v\in V_3(\Gamma),
\end{aligned}$$

$$A_{\bar{e}} = \begin{cases} (-1)^{n^e\vec{d}(\bar{e})}E_{\nu^e}(\mathfrak{p}(e),\mathfrak{p}_0(e))E_{\nu^e}(\mathfrak{p}(-e)),\mathfrak{p}_0(-e)), & e\in E_{pp}(\Gamma)\\ (-1)^{n^e\vec{d}(\bar{e})+l(\nu^e)-\vec{d}(\bar{e})}E_{\nu^e}(\mathfrak{p}(e),\mathfrak{p}_0(e)) & e\in E_{pf}(\Gamma)\\ (-1)^{n^e\vec{d}(\bar{e})} & e\in E_{ff}(\Gamma)\end{cases}$$

Recall that $E_\nu(x,y)$ is defined by (6.3).

For $v\in V_2(\Gamma)$, we have

$$\begin{aligned}
&\int_{[\overline{\mathcal{M}}^v_{\vec{\chi},\vec{\nu}}]^{\mathrm{vir}}} \frac{\mathfrak{f}(e)^{r^v}}{(-\mathfrak{p}(e)-\psi^0)(-\mathfrak{p}(e')-\psi^\infty)}\\
=\ &\int_{[\overline{\mathcal{M}}^\bullet_{\chi^v}(\mathbb{P}^1,\nu^e,\nu^{e'})//\mathbb{C}^*]^{\mathrm{vir}}} \frac{\mathfrak{f}(e)^{r^v}c_1(\mathbb{L})}{(-\mathfrak{p}(e)-\psi^0)(\mathfrak{p}(e)-\psi^\infty)}\\
=\ &\int_{[\overline{\mathcal{M}}^\bullet_{\chi^v}(\mathbb{P}^1,\nu^e,\nu^{e'})//\mathbb{C}^*]^{\mathrm{vir}}} \frac{\mathfrak{f}(e)^{r^v}(-\mathfrak{p}(e)-\psi^0+\mathfrak{p}(e)-\psi^\infty+c_1(\mathbb{L}^1))}{(-\mathfrak{p}(e)-\psi^0)(\mathfrak{p}(e)-\psi^\infty)}\\
=\ &\int_{[\overline{\mathcal{M}}^\bullet_{\chi^v}(\mathbb{P}^1,\nu^e,\nu^{e'})//\mathbb{C}^*]^{\mathrm{vir}}} \frac{\mathfrak{f}(e)^{r^v}}{\mathfrak{p}(e)-\psi^\infty} + \int_{[\overline{\mathcal{M}}^\bullet_{\chi^v}(\mathbb{P}^1,\nu^e,\nu^{e'})//\mathbb{C}^*]^{\mathrm{vir}}} \frac{\mathfrak{f}(e)^{r^v}}{-\mathfrak{p}(e)-\psi^0}\\
&+ \sum_{(\chi^+,\chi^-,\sigma)\in I_{\chi^v,\nu^e,\nu^{e'}}} \frac{a_\sigma}{|\mathrm{Aut}(\sigma)|}\int_{[\overline{\mathcal{M}}^\bullet_{\chi^+}(\mathbb{P}^1,\nu^e,\sigma)//\mathbb{C}^*]^{\mathrm{vir}}} \frac{\mathfrak{f}(e)^{r^+_{\chi^+,\sigma}}}{-\mathfrak{p}(e)-\psi^0}\\
&\qquad\cdot\int_{[\overline{\mathcal{M}}^\bullet_{\chi^-}(\mathbb{P}^1,\sigma,\nu^{e'})//\mathbb{C}^*]^{\mathrm{vir}}} \frac{\mathfrak{f}(e)^{r^-_{\chi^-,\sigma}}}{\mathfrak{p}(e)-\psi^\infty}\\
=\ &|\mathrm{Aut}(\nu^e)\times\mathrm{Aut}(\nu^{e'})|\left(\frac{\mathfrak{f}(e)}{\mathfrak{p}(e)}\right)^{r^v} \sum_{(\chi^+,\chi^-,\sigma)\in J_{\chi^v,\nu^e,\nu^{e'}}} (-1)^{r^+_{\chi^+,\sigma}}\frac{H^\bullet_{\chi^+,\nu^e,\sigma}}{r^+_{\chi^+,\sigma}!}z_\sigma\frac{H^\bullet_{\chi^-,\sigma,\nu^{e'}}}{r^-_{\chi^-,\sigma}!}
\end{aligned}$$

where $r^+_{\chi^+,\sigma} = -\chi^+ + \ell(\nu^e)+\ell(\sigma)$, $r^-_{\chi^-,\sigma} = -\chi^- + \ell(\sigma)+\ell(\nu^{e'})$, and

$$J_{\chi^v,\nu^e,\nu^{e'}} = I_{\chi^v,\nu^e,\nu^{e'}} \cup \{(2\ell(\nu^e),\chi,\nu^e),(\chi,2\ell(\nu^{e'}),\nu^{e'})\}.$$

Given $v \in V_3(\Gamma)$, we have $\mathfrak{v}_0^{-1}(v) = \{e_1, e_2, e_3\}$, where $\mathfrak{p}(e_1) \wedge \mathfrak{p}(e_2) = u_1 \wedge u_2$. Then (e_1, e_2, e_3) is unique up to cyclic permutation. Let

$$\vec{\nu}^v = (\nu^{e_1}, \nu^{e_2}, \nu^{e_3}), \quad \mathbf{w}^v = (\mathfrak{p}(e_1), \mathfrak{p}(e_2), \mathfrak{p}(e_3)).$$

Then $V_{\chi^v, \vec{\nu}^v}(\mathbf{w}^v)$ is independent of choice of cyclic ordering of e_1, e_2, e_3, where $V_{\chi,\vec{\nu}}(w_1, w_2, w_3)$ is defined by (6.2). We have

$$\begin{aligned}
& I_{\vec{\chi},\vec{\nu}}(u_1, u_2) \\
= \;& \int_{[\mathcal{F}_{\vec{\chi},\vec{\nu}}]^{\mathrm{vir}}} \frac{1}{e_T(N^{\mathrm{vir}}_{\vec{\chi},\vec{\nu}})} \\
= \;& \frac{1}{\prod_{\bar{e}\in E(\Gamma)} |A^{\bar{e}}_{\vec{\chi},\vec{\nu}}|} \int_{[\overline{\mathcal{M}}_{\vec{\chi},\vec{\nu}}]^{\mathrm{vir}}} \frac{1}{e_T(N^{\mathrm{vir}}_{\vec{\chi},\vec{\nu}})} \\
= \;& |\operatorname{Aut}(\vec{\mu})| \prod_{\bar{e}\in E(\Gamma)} (-1)^{n^e \vec{d}(\bar{e})} z_{\nu^e} \prod_{v\in V_3(\Gamma)} V_{\chi^v,\vec{\nu}^v}(\mathbf{w}^v) \prod_{e\in E_{0p}(\Gamma)} E_{\nu^e}(\mathfrak{p}(e), \mathfrak{p}_1(e)) \\
& \cdot \prod_{v\in V_1(\Gamma), \mathfrak{v}_1(e)=v} \sqrt{-1}^{\ell(\nu^e)+\ell(\mu^v)} (-1)^{\vec{d}(\bar{e})} \cdot \left(\sqrt{-1}\frac{\mathfrak{f}(e)}{\mathfrak{p}(e)}\right)^{r^v} \frac{H^\bullet_{\chi^v,\nu^e,\mu^v}}{r^v!} \\
& \cdot \prod_{v\in V_2(\Gamma), \mathfrak{v}_1^{-1}(v)=\{e,e'\}} \left(\sqrt{-1}^{\ell(\nu^e)+\ell(\nu^{e'})} \cdot \left(\sqrt{-1}\frac{\mathfrak{f}(e)}{\mathfrak{p}(e)}\right)^{r^v} \right. \\
& \qquad \left. \cdot \sum_{(\chi^+,\chi^-,\sigma)\in J_{\chi^v,\nu^e,\nu^{e'}}} \frac{H^\bullet_{\chi^+,\nu^e,\sigma}}{r^+_{\chi^+,\sigma}!} (-1)^{\ell(\sigma)} z_\sigma \frac{H^\bullet_{\chi^-,\nu^{e'},\sigma}}{r^-_{\chi^-,\sigma}!} \right) \\
= \;& |\operatorname{Aut}(\vec{\mu})| \prod_{\bar{e}\in E(\Gamma)} (-1)^{n^e \vec{d}(\bar{e})} \prod_{v\in V_3(\Gamma)} \sqrt{-1}^{\ell_{\vec{\nu}}(v)} G^\bullet_{\chi^v,\vec{\nu}^v}(\mathbf{w}^v) \\
& \cdot \prod_{v\in V_1(\Gamma), \mathfrak{v}_1(e)=v} \sqrt{-1}^{\ell(\mu^v)+\ell(\nu^v)} (-1)^{\vec{d}(\bar{e})} \left(\sqrt{-1}\frac{\mathfrak{f}(e)}{\mathfrak{p}(e)}\right)^{r^v} \frac{H^\bullet_{\chi^v,\nu^v,\mu^i}}{r^v!} \\
& \cdot \prod_{v\in V_2(\Gamma), \mathfrak{v}_1^{-1}(v)=\{e,e'\}} \left(\sqrt{-1}^{\ell(\nu^e)+\ell(\nu^{e'})} \cdot \left(\sqrt{-1}\frac{\mathfrak{f}(e)}{\mathfrak{p}(e)}\right)^{r^v} \right. \\
& \qquad \left. \cdot \sum_{(\chi^+,\chi^-,\sigma)\in J_{\chi^v,\nu^e,\nu^{e'}}} \frac{H^\bullet_{\chi^+,\nu^e,\sigma}}{r^+_{\chi^+,\sigma}!} z_\sigma (-1)^{\ell(\sigma)} \frac{H^\bullet_{\chi^-,\nu^{e'},\sigma}}{r^-_{\chi^-,\sigma}!} \right)
\end{aligned}$$

7.4. **Sum over labels.** We have

$$F^{\bullet\Gamma}_{\chi,\vec{d},\vec{\mu}}(u_1, u_2) = \frac{1}{|\operatorname{Aut}(\vec{\mu})|} \sum_{(\vec{\chi},\vec{\nu})\in G^\bullet_\chi(\Gamma,\vec{d},\vec{\mu})} I_{\vec{\chi},\vec{\nu}}(u_1, u_2).$$

Define

$$F^{\bullet\Gamma}_{\vec{d},\vec{\mu}}(\lambda; u_1, u_2) = \sum_{\chi\in 2\mathbb{Z}, \chi\leq l(\vec{\mu})} \lambda^{-\chi+l(\vec{\mu})} F^{\bullet\Gamma}_{\chi,\vec{d},\vec{\mu}}(u_1, u_2). \tag{7.4}$$

Then

$$F^{\bullet\Gamma}_{\vec{d},\vec{\mu}}(\lambda; u_1, u_2) \tag{7.5}$$

$$
\begin{aligned}
= & \sum_{|\nu^{\bar e}|=\vec d(\bar e)} \prod_{\bar e\in E(\Gamma)} (-1)^{n^e \vec d(\bar e)} z_{\nu^{\bar e}} \prod_{v\in V_3(\Gamma)} \sqrt{-1}^{\ell(\vec\nu^v)} G^\bullet_{\vec\nu^v}(\lambda;\mathbf{w}_v)\\
& \cdot \prod_{v\in V_1(\Gamma),\mathfrak{v}_1(e)=v} (-1)^{\vec d(\bar e)}\sqrt{-1}^{\ell(\nu^e)+\ell(\mu^v)}\Phi^\bullet_{\nu^e,\mu^v}\left(\sqrt{-1}\frac{\mathfrak{f}(e)}{\mathfrak{p}(e)}\lambda\right)\\
& \cdot \prod_{v\in V_2(\Gamma),\mathfrak{v}_1^{-1}(v)=\{e,e'\}} \sqrt{-1}^{\ell(\nu^e)+\ell(\nu^{e'})}\Phi^\bullet_{\nu^e,\sigma}\left(\sqrt{-1}\frac{\mathfrak{f}(e)}{\mathfrak{p}(e)}\lambda\right)(-1)^{\ell(\sigma)} z_\sigma \Phi^\bullet_{\nu^{e'},\sigma}\left(\sqrt{-1}\frac{\mathfrak{f}(e')}{\mathfrak{p}(e')}\lambda\right)
\end{aligned}
$$

where

$$
\begin{aligned}
(7.6)\quad & \sqrt{-1}^{\ell(\vec\mu)} G^\bullet_{\vec\mu}(\lambda;\mathfrak{p}(e_1),\mathfrak{p}(e_2),\mathfrak{p}(e_3))\\
= & \sqrt{-1}^{\ell(\vec\mu)} \sum_{|\nu^i|=|\mu^i|} \tilde F^\bullet_{\vec\nu}(\lambda;0)\prod_{i=1}^3 z_{\nu^i}\Phi^\bullet_{\nu^i,\mu^i}\left(\sqrt{-1}\frac{\mathfrak{p}_0(e_i)}{\mathfrak{p}(e_i)}\right)\\
= & (-1)^{\sum_{i=1}^3 \vec d(\bar e_i)} \sum_{|\nu^i|=|\mu^i|} F^\bullet_{\vec\nu}(\lambda;0)\prod_{i=1}^3 \sqrt{-1}^{\ell(\nu^i)-\ell(\mu^i)} z_{\nu^i}\Phi^\bullet_{\nu^i,\mu^i}\left(-\sqrt{-1}\frac{\mathfrak{p}_0(e_i)}{\mathfrak{p}(e_i)}\lambda\right)
\end{aligned}
$$

7.5. **Invariance.** Let Γ be a FTCY graph, and let

$$\Gamma_2=\Gamma_{V_2(\Gamma)},\quad \Gamma^2=\Gamma^{V_2(\Gamma)}.$$

Then Γ_2, Γ^2 are *regular* FTCY graphs. We call Γ_2 the *full smoothing* of Γ, and Γ^2 the full resolution of Γ. We have surjective maps

$$\pi_2=\pi_{V_2(\Gamma)}:E^o(\Gamma)\to E^o(\Gamma_2),\quad \pi^2=\pi^{V_2(\Gamma)}:V(\Gamma^2)\to V(\Gamma).$$

Definition 7.2. *Let Γ be a FTCY graph, and let Γ^2 be the full resolution of Γ,*

Let $(\vec d,\vec\mu)$ be an effective class of Γ. A splitting type *of $(\vec d,\vec\mu)$ is a map $\vec\sigma$: $V_2(\Gamma)\to\mathcal{P}$ such that $|\vec\sigma(v)|=\vec d(\bar e)$ if $\mathfrak{v}_1(e)=v$.*

Given a splitting type $\vec\sigma$ of an effective class $(\vec d,\vec\mu)$ of Γ, let $(\vec d,\vec\mu\sqcup\vec\sigma)$ denote the effective class of Γ^2 defined by $\vec d: E(\Gamma^2)=E(\Gamma)\to\mathbb{Z}_{\geq 0}$ and

$$\vec\mu\sqcup\vec\sigma(v)=\begin{cases}\vec\mu(\pi^2(v)), & \pi^2(v)\in V_1(\Gamma)\\ \vec\sigma(\pi^2(v)), & \pi^2(v)\in V_2(\Gamma)\end{cases}$$

Let $I_\Gamma(\vec d,\vec\mu)$ denote the set of all splitting types of $(\vec d,\vec\mu)$.

The following is clear from the expression (7.5).

Lemma 7.3. *Let Γ be a FTCY graph, and let $(\vec d,\vec\mu)$ be an effective class of Γ. Then*

$$F^{\bullet\Gamma}_{\vec d,\vec\mu}(\lambda;u_1,u_2)=\sum_{\sigma\in I_\Gamma(\vec d,\vec\mu)} z_{\vec\sigma}F^{\bullet\Gamma^2}_{\vec d,\vec\mu\sqcup\vec\sigma}(\lambda;u_1,u_2)$$

where

$$z_{\vec\sigma}=\prod_{v\in V_2(\Gamma)} z_{\vec\sigma(v)}.$$

By Lemma 7.3, it suffices to consider regular FTCY graphs. For a regular FTCY graph Γ, (7.5) reduces to

$$(7.7)\qquad F^{\bullet\Gamma}_{\vec d,\vec\mu}(\lambda;u_1,u_2)$$

$$= \sum_{|\nu^{\bar{e}}|=\vec{d}(\bar{e})} \prod_{\bar{e}\in E(\Gamma)} (-1)^{n^e \vec{d}(\bar{e})} z_{\nu^{\bar{e}}} \prod_{v\in V_3(\Gamma)} \sqrt{-1}^{\ell(\vec{\nu}^v)} G^\bullet_{\vec{\nu}^v}(\lambda;\mathbf{w}_v)$$
$$\cdot \prod_{v\in V_1(\Gamma),\mathfrak{v}_1(e)=v} (-1)^{\vec{d}(\bar{e})} \sqrt{-1}^{\ell(\nu^e)+\ell(\mu^v)} \Phi^\bullet_{\nu^e,\mu^v}\left(\sqrt{-1}\frac{\mathfrak{f}(e)}{\mathfrak{p}(e)}\lambda\right)$$

since $V_2(\Gamma)=\emptyset$.

Let $(\vec{d},\vec{\mu})$ be the effective class of a regular FTCY graph. Let $P(\vec{d},\vec{\mu})$ be the set of all maps $\vec{\nu}: E^o(\Gamma)\to\mathcal{P}$ such that

- $|\vec{\nu}(e)| = \vec{d}(\bar{e})$.
- $\vec{\nu}(e) = \vec{\mu}(v)$ if $\mathfrak{v}_0(e) = v\in V_1(\Gamma)$.

Note that we do not require $\vec{\nu}(e)=\vec{\nu}(-e)$. Denote $\vec{\nu}(e)$ by ν^e. Given $v\in V_3(\Gamma)$, there exist $e_1,e_2,e_3\in E(\Gamma)$, unique up to a cyclic permutation, such that $\mathfrak{v}_0^{-1}(v) = \{e_1,e_2,e_3\}$ and $\mathfrak{p}(e_1)\wedge\mathfrak{p}(e_2) = u_1\wedge u_2$. Define

$$\vec{\nu}^v = (\nu^{e_1},\nu^{e_2},\nu^{e_3}),\quad z_{\vec{\nu}^v} = z_{\nu^{e_1}} z_{\nu^{e_2}} z_{\nu^{e_3}}. \tag{7.8}$$

Note that $F^\bullet_{\vec{\nu}^v}(\lambda;\mathbf{0})$ and $z_{\vec{\nu}^v}$ are invariant under cyclic permutation of e_1,e_2,e_3, thus well-defined.

Using (7.6) and the sum formula (3.12) of double Hurwitz numbers, we can rewrite (7.7) as follows:

$$F^{\bullet\Gamma}_{\vec{d},\vec{\mu}}(\lambda;u_1,u_2) \tag{7.9}$$
$$= \sum_{\vec{\nu}\in P(\vec{d},\vec{\mu})} \prod_{v\in V_3(V)} F^\bullet_{\vec{\nu}^v}(\lambda;0) z_{\vec{\nu}(v)} \prod_{\bar{e}\in E(\Gamma)} \sqrt{-1}^{\ell(\nu^e)-\ell(\nu^{-e})} (-1)^{n^e\vec{d}(\bar{e})} \Phi^\bullet_{\nu^e,\nu^{-e}}(\sqrt{-1}n^e\lambda).$$

Note that the right hand side of (7.9) does not depend on u_1,u_2. This completes the proof of Theorem 5.7. From now on, we write $F^{\bullet\Gamma}_{\vec{d},\vec{\mu}}(\lambda)$ instead of $F^{\bullet\Gamma}_{\vec{d},\vec{\mu}}(\lambda;u_1,u_2)$. We define

$$F^{\bullet\Gamma}_{\chi,\vec{d},\vec{\mu}} = F^{\bullet\Gamma}_{\chi,\vec{d},\vec{\mu}}(u_1,u_2),$$

to be *formal relative Gromov-Witten invariants* of $\hat{Y}^{\mathrm{rel}}_\Gamma$.

7.6. **Gluing formulae.** Let $(\vec{d},\vec{\mu})$ be an effective class of a regular FTCY graph Γ. Let $T(\vec{d},\vec{\mu})$ be the set of all maps $\vec{\nu}: E(\Gamma)\to\mathcal{P}$ such that

- $|\vec{\nu}(e)| = \vec{d}(\bar{e})$.
- $\vec{\nu}(-e) = \vec{\nu}(e)^t$.

Note that we do not require $\vec{\nu}(e)=\vec{\mu}(v)$ if $\mathfrak{v}_0(e)=v\in V_1(E)$. We have

$$F^\bullet_{\vec{\mu}}(\lambda;\mathbf{0}) = \frac{(-1)^{|\mu^1|+|\mu^2|+|\mu^3|}}{\sqrt{-1}^{\ell(\vec{\mu})}} \sum_{|\nu^i|=|\mu^i|} \tilde{C}_{\vec{\nu}}(\lambda) \prod_{i=1}^3 \frac{\chi_{\nu^i}(\mu^i)}{z_{\mu^i}}, \tag{7.10}$$

where $\tilde{C}_{\vec{\nu}}(\lambda) = \tilde{C}_{\vec{\nu}}(\lambda;\mathbf{0})$. Applying (7.10) and the Burnside formula (3.10) of double Hurwitz numbers, we see that (7.9) is equivalent to the following.

Proposition 7.4. *Let Γ be a regular FTCY graph. Then*

$$\begin{aligned} F^{\bullet\Gamma}_{\vec{d},\vec{\mu}}(\lambda) &= \sum_{\vec{\nu}\in T(\vec{d},\vec{\mu})} \prod_{\bar{e}\in E(\Gamma)} (-1)^{(n^e+1)\vec{d}(\bar{e})} e^{-\sqrt{-1}\kappa_{\nu^e} n^e \lambda/2} \\ &\quad \cdot \prod_{v\in V_3(\Gamma)} \tilde{C}_{\vec{\nu}^v}(\lambda) \prod_{v\in V_1(\Gamma), v_0(e)=v} \frac{\chi_{\nu^e}(\mu^v)}{\sqrt{-1}^{\ell(\mu^v)} z_{\mu^v}} \end{aligned}$$

Note that

$$\kappa_{\nu^{-e}} n^{-e} = \kappa_{(\nu^e)^t}(-n^e) = \kappa_{\nu^e} n^e.$$

Theorem 7.5 (gluing formula)**.** *Let Γ be a FTCY graph, and let Γ_2 and Γ^2 be its full smoothing and its full resolution, respectively. Let $(\vec{d},\vec{\mu})$ be an effective class of Γ which can also be viewed as an effective class of Γ_2. Then*

$$F^{\bullet\Gamma_2}_{\vec{d},\vec{\mu}}(\lambda) = F^{\bullet\Gamma}_{\vec{d},\vec{\mu}}(\lambda) = \sum_{\vec{\sigma}\in I_\Gamma(\vec{d},\vec{\mu})} z_{\vec{\sigma}} F^{\bullet\Gamma^2}_{\vec{d}_\Gamma,\vec{\mu}\sqcup\vec{\sigma}}(\lambda). \tag{7.11}$$

Proof. By Lemma 7.3 and Proposition 7.4, it suffices to show that if $|\mu| = |\nu| = d$, then

$$\sum_{|\sigma|=d} \frac{\chi_\mu(\sigma)}{\sqrt{-1}^{\ell(\sigma)} z_\sigma} z_\sigma \frac{\chi_\nu(\sigma)}{\sqrt{-1}^{\ell(\sigma)} z_\sigma} = (-1)^d \delta_{\mu(\nu)^t},$$

which is obvious. □

7.7. **Sum over effective classes.** Given a regular FTCY graph, let $\mathrm{Eff}(\Gamma)$ denote the set of effective classes of Γ. Introduce formal Kähler parameters

$$\mathbf{t} = \{t^{\bar{e}} : \bar{e} \in E(\Gamma)\}$$

and winding parameters

$$\mathbf{p} = \{p^v = (p^v_1, p^v_2, \ldots) : v \in V_1(\Gamma)\}$$

We define the *formal relative Gromov-Witten partition function* of $\hat{Y}^{\mathrm{rel}}_\Gamma$ to be

$$Z^\Gamma_{\mathrm{rel}}(\lambda;\mathbf{t};\mathbf{p}) = \sum_{(\vec{d},\vec{\mu})\in\mathrm{Eff}(\Gamma)} F^{\bullet\Gamma}_{\vec{d},\vec{\mu}}(\lambda) e^{-\sum_{\bar{e}\in E(\Gamma)} \vec{d}(\bar{e}) t^{\bar{e}}} \prod_{v\in V_1(\Gamma)} p^v_{\mu^v} \tag{7.12}$$

where

$$p^v_\mu = p^v_{\mu_1}\cdots p^v_{\mu_{\ell(\mu)}}.$$

Let T^Γ denote the set of pairs $(\vec{\nu},\vec{\mu})$ such that

- $\vec{\nu} : E^o(\Gamma) \to \mathcal{P}$ such that $\vec{\nu}(-e) = \vec{\nu}(e)^t$.
- $\vec{\mu} : V_1(\Gamma) \to \mathcal{P}$.
- $|\vec{\nu}(e)| = |\vec{\mu}(v)|$ if $\mathfrak{v}_0(e) = v$.

Let ν^e denote $\vec{\nu}(e)$ for $e \in E^o(\Gamma)$, and let μ^v denote $\mu(v)$ for $v \in V_1(\Gamma)$. Define $\vec{\mu}^v$ by (7.8) for $v \in V_3(\Gamma)$. The following is a direct consequence of Proposition 7.4.

Corollary 7.6.

$$\begin{aligned} Z^\Gamma_{\mathrm{rel}}(\lambda;\mathbf{t};\mathbf{p}) &= \sum_{(\vec{\nu},\vec{\mu})\in T^\Gamma} \prod_{\bar{e}\in E(\Gamma)} e^{-|\nu^e| t^{\bar{e}}} (-1)^{(n^e+1)|\nu^e|} e^{-\sqrt{-1}\kappa_{\nu^e} n^e \lambda/2} \\ &\quad \cdot \prod_{v\in V_3(\Gamma)} \tilde{C}_{\vec{\nu}^v}(\lambda) \prod_{v\in V_1(\Gamma), v_0(e)=v} \frac{\chi_{\nu^e}(\mu^v)}{\sqrt{-1}^{\ell(\mu^v)} z_{\mu^v}} \end{aligned}$$

8. Combinatorial Expressions for the topological vertex

In this section, we will derive the following combinatorial expression for $\tilde{C}_{\vec{\mu}}(\lambda)$:

Theorem 8.1. *Let $\vec{\mu} \in \mathcal{P}_+^3$. Then*

$$\tilde{C}_{\vec{\mu}}(\lambda) = \tilde{\mathcal{W}}_{\vec{\mu}}(q),$$

where $q = e^{\sqrt{-1}\lambda}$, and $\tilde{\mathcal{W}}_{\vec{\mu}}(q)$ is defined as in Section 3.3.

We now outline our strategy. We use the notation introduced in Section 3.3. By Proposition 6.6,

$$\tilde{C}_{\vec{\mu}}(\lambda) = \sum_{|\nu^i|=|\mu^i|} \prod_{i=1}^{3} \chi_{\mu^i}(\nu^i) q^{-\frac{1}{2}\left(\sum_{i=1}^{3} \kappa_{\nu^i} \frac{w_{i+1}}{w_i}\right)} G^{\bullet}_{\vec{\nu}}(\lambda; \mathbf{w}),$$

where $\mathbf{w} = (w_1, w_2, w_3)$, $w_3 = -w_1 - w_2$, $w_4 = w_1$. In particular,

$$\tilde{C}_{\vec{\mu}}(\lambda) = \sum_{|\nu^i|=|\mu^i|} \prod_{i=1}^{3} \chi_{\mu^i}(\nu^i) q^{-\frac{1}{2}\kappa_{\nu^1} + \kappa_{\nu^2} + \frac{1}{4}\kappa_{\nu^3}} \cdot G^{\bullet}_{\vec{\nu}}(\lambda; 1, 1, -2). \tag{8.1}$$

In Section 8.1, we will show that the main result in [23] gives a combinatorial expression of $G^{\bullet}_{\mu,\nu,\emptyset}(\lambda; \mathbf{w})$ (Theorem 8.7). In Section 8.2, we will relate $G^{\bullet}_{\vec{\mu}}(\lambda; 1, 1, -2)$ to $G^{\bullet}_{\emptyset,\mu^1\cup\mu^2,\mu^3}(\lambda; 1, 1, -2)$. This gives the combinatorial expression $\tilde{\mathcal{W}}_{\vec{\mu}}(q)$ in Theorem 8.1. Moreover, (6.16) and Theorem 8.1 imply the following formula of three-partition Hodge integrals.

Theorem 8.2 (Formula of three-partition Hodge integrals)**.** *Let $\mathbf{w} = (w_1, w_2, w_3)$, where $w_3 = -w_1 - w_2$. Let $\vec{\mu} = (\mu^1, \mu^2, \mu^3) \in \mathcal{P}_+^3$, $w_4 = w_1$. Then*

$$G^{\bullet}_{\vec{\mu}}(\lambda; \mathbf{w}) = \sum_{|\nu^i|=|\mu^i|} \sum_{|\nu^i|=|\mu^i|} \prod_{i=1}^{3} \frac{\chi_{\nu^i}(\mu^i)}{z_{\mu^i}} q^{\frac{1}{2}\left(\sum_{i=1}^{3} \kappa_{\nu^i} \frac{w_{i+1}}{w_i}\right)} \mathcal{W}_{\vec{\nu}}(q). \tag{8.2}$$

The cyclic symmetry of $\tilde{C}_{\vec{\mu}}(\lambda)$ is obvious from definition. By Theorem 8.1 we have the following cyclic symmetry

$$\tilde{\mathcal{W}}_{\mu^1,\mu^2,\mu^3}(q) = \tilde{\mathcal{W}}_{\mu^2,\mu^3,\mu^1}(q) = \tilde{\mathcal{W}}_{\mu^3,\mu^1,\mu^2}(q)$$

which is far from obvious.

Finally, we conjecture that the combinatorial expression $\tilde{\mathcal{W}}_{\vec{\mu}}(q)$ coincides with $\mathcal{W}_{\vec{\mu}}(q)$ predicted in [1]:

Conjecture 8.3. *Let $\vec{\mu} \in \mathcal{P}_+^3$. Then*

$$\tilde{\mathcal{W}}_{\vec{\mu}}(q) = \mathcal{W}_{\vec{\mu}}(q),$$

where $q = e^{\sqrt{-1}\lambda}$, and $\mathcal{W}_{\vec{\mu}}(q)$ is defined as in Section 3.3.

We have strong evidence for Conjecture 8.3. By Theorem 8.1 and Corollary 8.8, Conjecture 8.3 holds when one of the three partitions is empty. When none of the partitions is empty, A. Klemm has checked by computer that Conjecture 8.3 holds in all the cases where

$$|\mu^i| \leq 6, \quad i = 1, 2, 3.$$

We will list some of these cases in Appendix B.

8.1. **One-partition and two-partition Hodge integrals.** We recall some notation in [22].

$$\mathcal{C}^\bullet_\mu(\lambda;\tau)=\sqrt{-1}^{|\mu|}G^\bullet_{\mu,\emptyset,\emptyset}(\lambda;1,\tau,-\tau-1). \tag{8.3}$$

$$V_\mu(q)=q^{-\kappa_\mu/4}\sqrt{-1}^{|\mu|}\mathcal{W}_\mu(q). \tag{8.4}$$

where $\mathcal{W}_\mu(q)=\mathcal{W}_{\mu,\emptyset,\emptyset}(q)$ is defined in Section 3.3. The main result of [22] is the following formula conjectured by Mariño and Vafa [25]:

Theorem 8.4.

$$\mathcal{C}^\bullet_\mu(\lambda;\tau)=\sum_{|\nu|=|\mu|}\frac{\chi(\mu)}{z_\mu}q^{\kappa_\nu(\tau+\frac{1}{2})/2}V_\nu(q) \tag{8.5}$$

Theorem 8.4 can be reformulated in our notation as follows:

Theorem 8.5 (Formula of one-partition Hodge integrals)**.** *Let* $\mathbf{w}=(w_1,w_2,w_3)$, *where* $w_3=-w_1-w_2$. *Let* $\mu\in\mathcal{P}_+$. *Then*

$$G^\bullet_{\mu,\emptyset,\emptyset}(\lambda;\mathbf{w})=\sum_{|\nu|=|\mu|}\frac{\chi_\nu(\mu)}{z_\mu}q^{\frac{1}{2}\kappa_\nu\frac{w_2}{w_1}}\mathcal{W}_{\nu,\emptyset,\emptyset}(q). \tag{8.6}$$

Let

$$G^\bullet_{\mu^+,\mu^-}(\lambda;\tau)=(-1)^{|\mu^-|-\ell(\mu^-)}G^\bullet_{\mu^+,\mu^-}(\lambda;1,\tau,-1-\tau). \tag{8.7}$$

The main result of [23] is the following formula conjectured in [31]:

Theorem 8.6. *Let* $(\mu^+,\mu^-)\in\mathcal{P}^2_+$. *Then*

$$G^\bullet_{\mu^+,\mu^-}(\lambda;\tau)=\sum_{|\nu^\pm|=|\mu^\pm|}\frac{\chi_{\nu^+}(\mu^+)}{z_{\mu^+}}\frac{\chi_{\nu^-}(\mu^-)}{z_{\mu^-}}q^{\left(\kappa_{\nu^+}\tau+\kappa_{\nu^-}\tau^{-1}\right)/2}\mathcal{W}_{\nu^+,\nu^-}(q).$$

We now reformulate Theorem 8.6 in the notation of this paper.

$$\begin{aligned}
&G^\bullet_{\mu^1,\mu^2,\emptyset}(\lambda;1,\tau,-1-\tau)\\
=\;&(-1)^{|\mu^2|-\ell(\mu^2)}\sum_{|\nu^i|=|\mu^i|}\frac{\chi_{\nu^1}(\mu^1)}{z_{\mu^1}}\frac{\chi_{\nu^2}(\mu^2)}{z_{\mu^2}}q^{\left(\kappa_{\nu^1}\tau+\kappa_{\nu^2}\tau^{-1}\right)/2}\mathcal{W}_{\nu^1,\nu^2}(q)\\
=\;&\sum_{|\nu^i|=|\mu^i|}\frac{\chi_{\nu^1}(\mu^1)}{z_{\mu^1}}\frac{\chi_{(\nu^2)^t}(\mu^2)}{z_{\mu^2}}q^{\left(\kappa_{\nu^1}\tau+\kappa_{\nu^2}\tau^{-1}\right)/2}q^{\kappa_{\nu^2}/2}\mathcal{W}_{\nu^1,(\nu^2)^t,\emptyset}(q)\\
=\;&\sum_{|\nu^i|=|\mu^i|}\frac{\chi_{\nu^1}(\mu^1)}{z_{\mu^1}}\frac{\chi_{\nu^2}(\mu^2)}{z_{\mu^2}}q^{\left(\kappa_{\nu^1}\tau+\kappa_{\nu^2}\frac{-\tau-1}{\tau}\right)/2}\mathcal{W}_{\nu^1,\nu^2,\emptyset}(q)
\end{aligned}$$

Theorem 8.6 is equivalent to the following:

Theorem 8.7 (Formula of two-partition Hodge integrals)**.** *Let* $\mathbf{w}=(w_1,w_2,w_3)$, *where* $w_3=-w_1-w_2$. *Let* $(\mu^1,\mu^2)\in\mathcal{P}^2_+$. *Then*

$$G^\bullet_{\mu^1,\mu^2,\emptyset}(\lambda;\mathbf{w})=\sum_{|\nu^i|=|\mu^i|}\sum_{|\nu^i|=|\mu^i|}\frac{\chi_{\nu^1}(\mu^1)}{z_{\mu^1}}\frac{\chi_{\nu^2}(\mu^2)}{z_{\mu^2}}q^{\frac{1}{2}\left(\kappa_{\nu^1}\frac{w_2}{w_1}+\kappa_{\nu^2}\frac{w_3}{w_2}\right)}\mathcal{W}_{\nu^1,\nu^2,\emptyset}(q). \tag{8.8}$$

Note that Theorem 8.5 corresponds the special case where $(\mu^1,\mu^2)=(\mu,\emptyset)$. Theorem 8.7 and (6.15) imply

Corollary 8.8. *Let* $\vec{\mu} = (\mu^1, \mu^2, \mu^3) \in \mathcal{P}_+^3$*, and let* $q = e^{\sqrt{-1}\lambda}$*. Then*

$$\tilde{C}_{\vec{\mu}}(\lambda) = \mathcal{W}_{\vec{\mu}}(q)$$

when one of μ^1, μ^2, μ^3 *is empty.*

8.2. **Reduction.** Recall that

$$G_{g,\vec{\mu}}(\tau) = G_{g,\vec{\mu}}(1, \tau, -\tau - 1).$$

We have

Lemma 8.9. *Let* $\vec{\mu} = (\mu^1, \mu^2, \mu^3) \in \mathcal{P}_+^3$*. Then*

(8.9)

$$G_{g,\vec{\mu}}(\lambda; 1) = (-1)^{|\mu^1| - \ell(\mu^1)} \frac{z_{\mu^1 \cup \mu^2}}{z_{\mu^1} \cdot z_{\mu^2}} G_{g,\emptyset,\mu^1\cup\mu^2,\mu^3}(\lambda; 1) + \delta_{g0} \sum_{m \geq 1} \delta_{\mu^1(m)} \delta_{\mu^2 \emptyset} \delta_{\mu^3(2m)} \frac{(-1)^{m-1}}{m}$$

Proof. Let

$$\begin{aligned} I_{g,\vec{\mu}}(w_1, w_2, w_3) &= \int_{\overline{\mathcal{M}}_{g,\ell(\vec{\mu})}} \prod_{i=1}^{3} \frac{\Lambda^\vee(w_i) w_i^{\ell(\vec{\mu})-1}}{\prod_{j=1}^{\ell(\mu^i)} (w_i(w_i - \mu_j^i \psi_{d_{\vec{\mu}}^i + j})} \\ I_{g,\vec{\mu}}(\tau) &= I_{g,\vec{\mu}}(1, \tau, -\tau - 1) \end{aligned}$$

Then

$$I_{0,\vec{\mu}}(\tau) = \frac{(\tau(-\tau-1))^{\ell(\vec{\mu})-1}}{\tau^{2\ell(\mu^2)}(-\tau-1)^{2\ell(\mu^3)}} \left(|\mu^1| + \frac{|\mu^2|}{\tau} + \frac{|\mu^3|}{-\tau-1} \right)^{\ell(\vec{\mu})-3} \tag{8.10}$$

Note that $I_{g,\vec{\mu}}(\tau)$ has a pole at $\tau = 1$ only if

$$g = 0, \quad \vec{\mu} = ((m), \emptyset, (2m)) \text{ or } (\emptyset, (m), (2m)), \tag{8.11}$$

where $m > 0$. Let

$$E_\mu(\tau) = \prod_{j=1}^{\ell(\mu)} \frac{\prod_{a=1}^{\mu_j - 1} (\tau \mu_j + a)}{(\mu_j - 1)!}. \tag{8.12}$$

Then $E_\mu(\tau)$ is a polynomial in τ of degree $|\mu| - \ell(\mu)$, and

$$E_\mu(-\tau - 1) = (-1)^{|\mu| - \ell(\mu)} E_\mu(\tau).$$

$$\begin{aligned} G_{g,\vec{\mu}}(\tau) &= \frac{(-\sqrt{-1})^{\ell(\vec{\mu})}}{|\operatorname{Aut}(\vec{\mu})|} E_{\mu^1}(\tau) E_{\mu^2}(-1 - \tau^{-1}) E_{\mu^3}\left(\frac{1}{-\tau - 1}\right) I_{g,\vec{\mu}}(\tau) \\ &= (-1)^{|\mu^1| - \ell(\mu^1)} \frac{(-\sqrt{-1})^{\ell(\vec{\mu})}}{|\operatorname{Aut}(\vec{\mu})|} E_{\mu^1}(-1 - \tau) E_{\mu^2}(-1 - \tau^{-1}) E_{\mu^3}\left(\frac{1}{-\tau - 1}\right) I_{g,\vec{\mu}}(\tau) \end{aligned}$$

$$\begin{aligned} & G_{g,\emptyset,\mu^1\cup\mu^2,\mu^3}(\tau) \\ = & \frac{(-\sqrt{-1})^{\ell(\vec{\mu})}}{|\operatorname{Aut}(\mu^1 \cup \mu^2) \times \operatorname{Aut}(\mu^3)|} E_{\mu^1}(-1 - \tau^{-1}) E_{\mu^2}(-1 - \tau^{-1}) E_{\mu^3}\left(\frac{1}{-\tau - 1}\right) I_{g,\emptyset,\mu^1\cup\mu^2,\mu^3}(\tau). \end{aligned}$$

Suppose that $(g, \vec{\mu})$ is not the exceptional case listed in (8.11). Then neither is $(g, \emptyset, \mu^1 \cup \mu^2, \mu^3)$. It is immediate from the definition that

$$I_{g,\mu^1,\mu^2,\mu^3}(1) = I_{g,\emptyset,\mu^1\cup\mu^2,\mu^3}(1),$$

so

$$G_{g,\vec{\mu}}(1)=(-1)^{|\mu^1|-\ell(\mu^1)}\frac{|\operatorname{Aut}(\mu^1\cup\mu^2)|}{|\operatorname{Aut}(\mu^1)\times\operatorname{Aut}(\mu^2)|}G_{g,\emptyset,\mu^1\cup\mu^2,\mu^3}(1) \tag{8.13}$$

For the exceptional case (8.11), we have

$$\begin{aligned}&G_{0,(m),\emptyset,(2m)}(\tau)\\ =\;&\frac{\tau}{(\tau+1)(m-1)!(2m-1)!}\prod_{a=1}^{m-1}(\tau m+a)\prod_{a=1}^{m-1}(\frac{2m}{-\tau-1}+a)\prod_{a=m+1}^{2m-1}(\frac{2m}{-\tau-1}+a)\\ &G_{0,\emptyset,(m),(2m)}(\tau)\\ =\;&\frac{-1}{(\tau+1)(m-1)!(2m-1)!}\prod_{a=1}^{m-1}(\frac{-\tau-1}{\tau}m+a)\prod_{a=1}^{m-1}(\frac{2m}{-\tau-1}+a)\prod_{a=m+1}^{2m-1}(\frac{2m}{-\tau-1}+a)\end{aligned}$$

So

$$G_{0,(m),\emptyset,(2m)}(1)=\frac{(-1)^{m-1}}{2m},\quad G_{0,\emptyset,(m),(2m)}(1)=\frac{-1}{2m}. \tag{8.14}$$

Combining the general case (8.13) and the exceptional case (8.14), we obtain (8.9). □

We have

$$G(\lambda;\mathbf{p};1)=G(\lambda;\mathbf{p};1,1,-2)=\sum_{\vec{\mu}\in\mathcal{P}_+^3}\sum_{g=0}^{\infty}\lambda^{2g-2+\ell(\vec{\mu})}G_{g,\vec{\mu}}(\lambda;1)p^1_{\mu^1}p^2_{\mu^2}p^3_{\mu^3},$$

where $\mathbf{p}=(p^1,p^2,p^3)$, $p^i=(p^i_1,p^i_2,\ldots)$, and

$$p^i_\mu=\prod_{j=1}^{\ell(\mu)}p^i_{\mu_j}.$$

By Lemma 8.9,

$$\begin{aligned}&G(\lambda;\mathbf{p};1)\\ =\;&\sum_{\vec{\mu}\in\mathcal{P}_+^3}\sum_{g=0}^{\infty}\lambda^{2g-2+\ell(\vec{\mu})}G_{g,\emptyset,\mu^1\cup\mu^2,\mu^3}(1)\frac{z_{\mu^1\cup\mu^2}}{z_{\mu^1}z_{\mu^2}}(-1)^{|\mu^1|-\ell(\mu^1)}p^1_{\mu^1}p^2_{\mu^2}p^3_{\mu^3}\\ &+\sum_{m\geq1}\frac{(-1)^{m-1}}{m}p^1_m p^3_{2m}\\ =\;&\sum_{(\mu^+,\mu^3)\in\mathcal{P}_+^2}\sum_{g=0}^{\infty}\lambda^{2g-2+\ell(\mu^+)+\ell(\mu^3)}G_{g,\emptyset,\mu^+,\mu^3}(1)\left(\sum_{\mu^1\cup\mu^2=\mu^+}\frac{z_{\mu^+}}{z_{\mu^1}z_{\mu^2}}(-1)^{|\mu^1|-\ell(\mu^1)}p^1_{\mu^1}p^2_{\mu^2}\right)p^3_{\mu^3}\\ &+\sum_{m\geq1}\frac{(-1)^{m-1}}{m}p^1_m p^3_{2m}\end{aligned}$$

Let $p^+_i=(-1)^{i-1}p^1_i+p^2_i$, and let

$$p^+_\mu=\prod_{j=1}^{\ell(\mu)}p^+_{\mu_j}.$$

It is easy to see that

$$\sum_{\mu^1\cup\mu^2=\mu^+} \frac{z_{\mu^+}}{z_{\mu^1}z_{\mu^2}}(-1)^{|\mu^1|-\ell(\mu^1)} p^1_{\mu^1}p^2_{\mu^2} = p^+_{\mu^+}. \tag{8.15}$$

So

$$G(\lambda; p^1, p^2, p^3; 1) = G(\lambda; 0, p^+, p^3; 1) + \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} p^1_m p^3_{2m} \tag{8.16}$$

Let

$$G^\bullet(\lambda; \mathbf{p}; \tau) = \exp(G(\lambda; \mathbf{p}; \tau)).$$

We have

Lemma 8.10.

$$G^\bullet(\lambda; p^1, p^2, p^3; 1) = G^\bullet(\lambda; 0, p^+, p^3; 1) \exp\left(\sum_{m\geq 1} \frac{(-1)^{m-1}}{m} p^1_m p^3_{2m}\right) \tag{8.17}$$

8.3. **Combinatorial expression.**

Lemma 8.11.
(8.18)

$$G^\bullet(\lambda; 0, p^+, p^3; 1) = \sum_{\nu^+, \nu^i, \mu^i \in \mathcal{P}} c^{\nu^+}_{(\nu^1)^t\nu^2} q^{(-2\kappa_{\nu^+} - \frac{\kappa_{\nu^3}}{2})/2} \mathcal{W}_{\nu^+,\nu^3}(q)(-1)^{|\mu^3|-l(\mu^3)} \prod_{i=1}^{3} \frac{\chi_{\nu^i}(\mu^i)}{z_{\mu^i}} p^i_{\mu^i}.$$

Proof. By Theorem 8.7,

$$G^\bullet(\lambda; 0, p^+, p^3; 1) = \sum_{\mu^\pm, \nu^\pm, \mu^3 \in \mathcal{P}} \frac{\chi_{\nu^+}(\mu^+)}{z_{\mu^+}} \frac{\chi_{\nu^3}(\mu^3)}{z_{\mu^3}} q^{(-2\kappa_{\nu^+} - \frac{\kappa_{\nu^3}}{2})/2} \mathcal{W}_{\emptyset,\nu^+,\nu^3}(q) p^+_{\mu^+} p^3_{\mu^3}.$$

Recall that

$$\mathcal{W}_{\emptyset,\nu^+,\nu^3}(q) = q^{\kappa_{\nu^3}/2}\mathcal{W}_{\nu^+,(\nu^3)^t}(q),$$

$$p^+_{\mu^+} = \sum_{\mu^1\cup\mu^2=\mu^+} \frac{z_{\mu^+}}{z_{\mu^2}z_{\mu^2}}(-1)^{|\mu^1|-\ell(\mu^1)} p^1_{\mu^1}p^2_{\mu^2},$$

so

$$\begin{aligned}
& G^\bullet(\lambda; 0, p^+, p^3; 1) \\
= & \sum_{\mu^\pm, \nu^\pm, \mu^3 \in \mathcal{P}} \frac{\chi_{\nu^+}(\mu^+)}{z_{\mu^+}} \frac{\chi_{\nu^3}(\mu^3)}{z_{\mu^3}} q^{(-2\kappa_{\nu^+} + \frac{\kappa_{\nu^3}}{2})/2} \mathcal{W}_{\nu^+,(\nu^3)^t}(q) p^+_{\mu^+} p^3_{\mu^3} \\
= & \sum_{\mu^\pm, \nu^\pm, \mu^3 \in \mathcal{P}} \frac{\chi_{\nu^+}(\mu^+)}{z_{\mu^+}} \frac{\chi_{(\nu^3)^t}(\mu^3)}{z_{\mu^3}} q^{(-2\kappa_{\nu^+} - \frac{\kappa_{\nu^3}}{2})/2} \mathcal{W}_{\nu^+,\nu^3}(q) p^3_{\mu^3} \\
& \cdot \sum_{\mu^1\cup\mu^2=\mu^+} \frac{z_{\mu^+}}{z_{\mu^1}\cdot z_{\mu^2}} (-1)^{|\mu^1|-\ell(\mu^1)} p^1_{\mu^1} p^2_{\mu^2} \\
= & \sum_{\mu^i, \nu^+, \nu^3 \in \mathcal{P}} \frac{\chi_{\nu^+}(\mu^1\cup\mu^2)}{z_{\mu^1}\cdot z_{\mu^2}} \frac{\chi_{(\nu^3)^t}(\mu^3)}{z_{\mu^3}} q^{(-2\kappa_{\nu^+} - \frac{\kappa_{\nu^3}}{2})/2} \mathcal{W}_{\nu^+,\nu^3}(q) (-1)^{|\mu^1|-\ell(\mu^1)} p^1_{\mu^1} p^2_{\mu^2} p^3_{\mu^3}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\mu^1,\mu^2,\nu^+,\nu^i\in\mathcal{P}} \frac{\chi_{\nu^+}(\mu^1\cup\mu^2)\chi_{\nu^1}(\mu^1)\chi_{\nu^2}(\mu^2)}{z_{\mu^1}\cdot z_{\mu^2}} q^{(-2\kappa_{\nu^+}-\frac{\kappa_{\nu^3}}{2})/2}\mathcal{W}_{\nu^+,\nu^3}(q)(-1)^{|\mu^1|-\ell(\mu^1)} s^1_{\nu^1}s^2_{\nu^2}s^3_{(\nu^3)^t}\\
&= \sum_{\mu^1,\mu^2,\nu^+,\nu^i\in\mathcal{P}} \frac{\chi_{\nu^+}(\mu^1\cup\mu^2)\chi_{(\nu^1)^t}(\mu^1)\chi_{\nu^2}(\mu^2)}{z_{\mu^1}\cdot z_{\mu^2}} q^{(-2\kappa_{\nu^+}-\frac{\kappa_{\nu^3}}{2})/2}\mathcal{W}_{\nu^+,\nu^3}(q) s^1_{\nu^1}s^2_{\nu^2}s^3_{(\nu^3)^t}\\
&= \sum_{\nu^+,\nu^i\in\mathcal{P}} c^{\nu^+}_{(\nu^1)^t\nu^2} q^{(-2\kappa_{\nu^+}-\frac{\kappa_{\nu^3}}{2})/2}\mathcal{W}_{\nu^+,\nu^3}(q) s^1_{\nu^1}s^2_{\nu^2}s^3_{(\nu^3)^t}\\
&= \sum_{\nu^+,\nu^i,\mu^i\in\mathcal{P}} c^{\nu^+}_{(\nu^1)^t\nu^2} q^{(-2\kappa_{\nu^+}-\frac{\kappa_{\nu^3}}{2})/2}\mathcal{W}_{\nu^+,\nu^3}(q)(-1)^{|\mu^3|-l(\mu^3)}\prod_{i=1}^3\frac{\chi_{\nu^i}(\mu^i)}{z_{\mu^i}}p^i_{\mu^i}.
\end{aligned}
$$

In the above we have used (8.15) and the following identity:

$$
c^{\mu}_{\mu^+\mu^-} = \sum_{\nu^+,\nu^-}\frac{\chi_{\mu^+}(\nu^+)\chi_{\mu^-}(\nu^-)\chi_{\mu}(\nu^+\cup\nu^-)}{z_{\nu^+}z_{\nu^-}}.
$$

□

Lemma 8.12. *We have*

$$
\exp\left(-\sum_{m\geq 1}\frac{(-1)^{m-1}}{m}p^1_m p^3_{2m}\right) = \sum_{\mu\in\mathcal{P}}\frac{(-1)^{|\mu|-\ell(\mu)}}{z_\mu}p^1_\mu p^3_\mu \tag{8.19}
$$

where 2μ is the partition $(2\mu_1, 2\mu_2, \ldots, 2\mu_{\ell(\mu)})$.

Proof. Let $(x^i_1,\ldots,x^i_n,\ldots)$ be formal variables such that

$$
p^i_m = \sum_n (x^i_n)^m.
$$

Then we have by standard series manipulations

$$
\begin{aligned}
&\exp\left(-\sum_{m\geq 1}\frac{(-1)^{m-1}}{m}p^2_m p^3_{2m}\right)\\
= &\exp\left(\sum_{m\geq 1}\frac{(-1)^{m-1}}{m}\sum_{n_1,n_3}(x^1_{n_1})^m(x^3_{n_3})^{2m}\right)\\
= &\prod_{n_1,n_3}\exp\left(\sum_{m\geq 1}\frac{(-1)^{m-1}}{m}(p^1_{n_1}(p^3_{n_3})^2)^m\right)\\
= &\prod_{n_1,n_3}(1+x^1_{n_1}(x^3_{n_3})^2).
\end{aligned}
$$

Now recall (cf. [24], p. 65, (4.1')):

$$
\prod_{i,j}(1+x_iy_j) = \sum_{\mu\in\mathcal{P}}\frac{(-1)^{|\mu|-\ell(\mu)}}{z_\mu}p_\mu(x)p_\mu(y),
$$

hence we have

$$\begin{aligned}\exp\left(\sum_{m\geq 1}\frac{(-1)^{m-1}}{m}p^1_m p^3_{2m}\right) &= \sum_{\mu\in\mathcal{P}}\frac{(-1)^{|\mu|-\ell(\mu)}}{z_\mu}p_\mu(x^1)p_\mu((x^3)^2)\\ &= \sum_{\mu\in\mathcal{P}}\frac{(-1)^{|\mu|-\ell(\mu)}}{z_\mu}p_\mu(x^1)p_{2\mu}(x^3).\end{aligned}$$

□

By Lemma 8.10, Lemma 8.11, and Lemma 8.12, we have

$$\begin{aligned}& G^\bullet(\lambda;p^1,p^2,p^3;1)\\ =\ & G^\bullet(\lambda;0,p^+,p^3;1)\exp\left(\sum_{m\geq 1}\frac{(-1)^{m-1}}{m}p^1_m p^3_{2m}\right)\\ =\ & \sum_{\nu^+,\nu^i\in\mathcal{P}} c^{\nu^+}_{(\nu^1)^t\nu^2} q^{(-2\kappa_{\nu^+}-\frac{\kappa_{\nu^3}}{2})/2}\mathcal{W}_{\nu^+,\nu^3} s^1_{\nu^1}s^2_{\nu^2}s^3_{(\nu^3)^t}\\ & \cdot\sum_{\mu\in\mathcal{P}}\frac{(-1)^{|\mu|-\ell(\mu)}}{z_\mu}p_\mu(x^1)p_{2\mu}(x^3)\\ =\ & \sum_{\nu^+,\nu^i\in\mathcal{P}} c^{\nu^+}_{(\nu^1)^t\nu^2} q^{(-2\kappa_{\nu^+}-\frac{\kappa_{\nu^3}}{2})/2}\mathcal{W}_{\nu^+,\nu^3}(q) s^1_{\nu^1}s^2_{\nu^2}s^3_{(\nu^3)^t}\\ & \cdot\sum_{\mu,\eta^1,\eta^3\in\mathcal{P}}\frac{(-1)^{|\mu|-\ell(\mu)}}{z_\mu}\chi_{\eta^1}(\mu)\chi_{\eta^3}(2\mu)s^1_{\eta^1}s^3_{\eta^3}\\ =\ & \sum_{\nu^+,\nu^1,\nu^3,\eta^1,\eta^3,\mu\in\mathcal{P}} c^{\nu^+}_{(\nu^1)^t\nu^2}c^{\rho^1}_{\eta^1\nu^1}c^{\rho^3}_{\eta^3(\nu^3)^t} q^{(-2\kappa_{\nu^+}-\frac{\kappa_{\nu^3}}{2})/2}\mathcal{W}_{\nu^+,\nu^3}(q)\frac{\chi_{(\eta^1)^t}(\mu)\chi_{\eta^3}(2\mu)}{z_\mu}s^1_{\rho^1}s^2_{\rho^2}s^3_{\rho^3}\\ =\ & \sum_{\nu^+,\nu^1,\nu^3,\eta^1,\eta^3,\mu\in\mathcal{P}} c^{\nu^+}_{(\nu^1)^t\nu^2}c^{\rho^1}_{(\eta^1)^t\nu^1}c^{\rho^3}_{\eta^3(\nu^3)^t} q^{(-2\kappa_{\nu^+}-\frac{\kappa_{\nu^3}}{2})/2}\mathcal{W}_{\nu^+,\nu^3}(q)\frac{\chi_{\eta^1}(\mu)\chi_{\eta^3}(2\mu)}{z_\mu}s^1_{\rho^1}s^2_{\rho^2}s^3_{\rho^3}.\end{aligned}$$

By Proposition 6.6,

$$\begin{aligned}G^\bullet(\lambda;\mathbf{p};1) &= \sum_{\mu^i,\nu^i\in\mathcal{P}}\tilde{C}_{\vec{\nu}}(\lambda)q^{(\kappa_{\nu^1}-2\kappa_{\nu^2}-\frac{1}{2}\kappa_{\nu^3})/2}\prod_{i=1}^3\frac{\chi_{\nu^i}(\mu^i)}{z_{\mu^i}}p^i_{\mu^i}\\ &= \sum_{\nu^i\in\mathcal{P}}\tilde{C}_{\vec{\nu}}(\lambda)q^{(\kappa_{\nu^1}-2\kappa_{\nu^2}-\frac{1}{2}\kappa_{\nu^3})/2}\prod_{i=1}^3 s_{\nu^i}(x^i).\end{aligned}$$

Hence

$$\begin{aligned}& \sum c^{\nu^+}_{(\nu^1)^t\nu^2}c^{\rho^1}_{(\eta^1)^t\nu^1}c^{\rho^3}_{\eta^3(\nu^3)^t} q^{(-2\kappa_{\nu^+}-\frac{\kappa_{\nu^3}}{2})/2}\mathcal{W}_{\nu^+,\nu^3}(q)\frac{\chi_{\eta^1}(\mu)\chi_{\eta^3}(2\mu)}{z_\mu}s^1_{\rho^1}s^2_{\rho^2}s^3_{\rho^3}\\ =\ & \sum_{\mu^i,\nu^i\in\mathcal{P}}\tilde{C}_{\vec{\nu}}(\lambda)q^{(\kappa_{\nu^1}-2\kappa_{\nu^2}-\frac{1}{2}\kappa_{\nu^3})/2}\prod_{i=1}^3 s^i_{\nu^i}.\end{aligned}$$

Therefore,

$$\tilde{C}_{\vec{\rho}}(\lambda)=\tilde{\mathcal{W}}_{\vec{\rho}}(q)$$

where

$$\tilde{\mathcal{W}}_{\vec{\rho}}(q) = q^{-(\kappa_{\rho^1}-2\kappa_{\rho^2}-\frac{1}{2}\kappa_{\rho^3})/2}\sum c^{\nu^+}_{(\nu^1)^t\rho^2}c^{\rho^1}_{(\eta^1)^t\nu^1}c^{\rho^3}_{\eta^3(\nu^3)^t}q^{(-2\kappa_{\nu^+}-\frac{\kappa_{\nu^3}}{2})/2}\mathcal{W}_{\nu^+,\nu^3}(q)\frac{\chi_{\eta^1}(\mu)\chi_{\eta^3}(2\mu)}{z_\mu}$$

This completes the proof of Theorem 8.1.

Appendix A. Proof of Theorem 5.9

We begin with our strategy in proving the invariance of the integral[1]

$$F^{\bullet\Gamma^0}_{\chi,\vec{d},\vec{\mu}}(u_1,u_2) = \frac{1}{|\operatorname{Aut}(\vec{\mu})|}\int_{[\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(\hat{Y}^{\mathrm{rel}}_{\Gamma^0},\hat{L})^T]^{\mathrm{vir}}}\frac{e^T(\mathcal{T}^{1,m})}{e^T(\mathcal{T}^{2,m})}\in\mathbb{Q}(u_1/u_2) \tag{A.1}$$

that should define the values of the topological vertex Γ^0. (Recall the precise meaning of the symbol $\int$ from Section 5.4.) For any integer χ and effective class $(\vec{d},\vec{\mu})$ of Γ^0, we first come up with a moduli space of relative stable morphisms $\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(W^{\mathrm{rel}},L)$ and a T-equivariant morphism

$$\Phi : \mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(\hat{Y}^{\mathrm{rel}}_{\Gamma^0},\hat{L}) \longrightarrow \mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(W^{\mathrm{rel}},L)$$

so that the induced map on the T-fixed loci

$$\Phi^T : \mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(\hat{Y}^{\mathrm{rel}}_{\Gamma^0},\hat{L})^T \longrightarrow \mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(W^{\mathrm{rel}},L)^T$$

is an open and closed embedding. In addition, we require that the obstruction theories of $\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(W^{\mathrm{rel}},L)$ along its fixed loci is identical to that of $\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(\hat{Y}^{\mathrm{rel}}_{\Gamma^0},\hat{L})$ via Φ^T. Because the two obstruction theories are identical, the integrals

$$\int_{[\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(\hat{Y}^{\mathrm{rel}}_{\Gamma^0},\hat{L})^T]^{\mathrm{vir}}}\frac{e^T(\mathcal{T}^{1,m})}{e^T(\mathcal{T}^{2,m})} = \int_{[\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(W^{\mathrm{rel}},L)^T_0]^{\mathrm{vir}}}\frac{e^T(\mathcal{T}^{1,m})}{e^T(\mathcal{T}^{2,m})}, \tag{A.2}$$

where $\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(W^{\mathrm{rel}},L)^T_0$ is the image of the fixed loci of $\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(\hat{Y}^{\mathrm{rel}}_{\Gamma^0},\hat{L})$. (Here by abuse of notation, we denote by $\mathcal{T}^{i,m}$ the moving parts of the obstruction complex $[\mathcal{T}^1\to\mathcal{T}^2]$ of $\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(\hat{Y}^{\mathrm{rel}}_{\Gamma^0},\hat{L})$ as well as $\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(W^{\mathrm{rel}},L)$ along their fixed loci.)

In case the integral (A.2) vanishes along the fixed loci of $\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(W^{\mathrm{rel}},L)$ other than $\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(W^{\mathrm{rel}},L)^T_0$, the right hand side of (A.2) becomes

$$\int_{[\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(W^{\mathrm{rel}},L)^T]^{\mathrm{vir}}}\frac{e^T(\mathcal{T}^{1,m})}{e^T(\mathcal{T}^{2,m})} = \deg[\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(W^{\mathrm{rel}},L)]^{\mathrm{vir}},$$

which is a topological number; hence is independent of u_1/u_2. This was how the invariance of the two partition topological vertex was proved in [23].

This time we do not have a similar vanishing result. What we will do is to devise a local contribution of $\deg[\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(W^{\mathrm{rel}},L)]^{\mathrm{vir}}$ along $\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(Y^{\mathrm{rel}}_{\Gamma^0},L)$; this local contribution is a sum of the desired term (A.1) with *some other terms*; we will then show that this *some other terms* vanish completely. This will settle the invariance of the topological vertex Γ^0 (Theorem 5.9).

We now outline in more details of our proof. To proceed, a quick review of the construction of the virtual cycles of the moduli stack is in order. For notational simplicity, in the remainder of this Appendix we will abbreviate $\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(W^{\mathrm{rel}},L)$

[1] In the previous sections, $T\cong(\mathbb{C}^*)^2$ and $T_{\mathbb{R}}\cong U(1)^2$ is its maximal compact subgroup; in this Appendix, we will only consider the compact torus $U(1)^2$, denoted T instead of $T_{\mathbb{R}}$.

to $\mathcal{M}$ and abbreviate $\mathcal{M}^{\bullet}_{\chi,\vec{d},\vec{\mu}}(Y^{\mathrm{rel}}_{\Gamma^0}, L)$ to $\mathcal{M}_0$. As shown in [3, 4, 20], the virtual cycle $[\mathcal{M}]^{\mathrm{vir}}$ is constructed by

a. identifying the perfect obstruction theory of $\mathcal{M}$;
b. picking a vector bundle (locally free sheaf) $\mathcal{E}$ on $\mathcal{M}$ so that it surjects onto the obstruction sheaf of $\mathcal{M}$ and
c. constructing an associated cone $\mathcal{C} \subset \mathcal{E}$ of pure dimension $\operatorname{rank} \mathcal{E}$.

The virtual cycle $[\mathcal{M}]^{\mathrm{vir}}$ is the image of the cycle $[\mathcal{C}] \in H_*(\mathcal{E}, \mathcal{E} - \mathcal{M})$ under the Thom isomorphism

$$\varphi_{\mathcal{E}} : H_*(\mathcal{E}, \mathcal{E} - \mathcal{M}) \longrightarrow H_{*-2r}(\mathcal{M}), \quad r = \operatorname{rank} \mathcal{E}.$$

Here as usual, we denote by $\mathcal{E}$ the total space of $\mathcal{E}$ and denote by $\mathcal{M} \subset \mathcal{E}$ its zero section that is isomorphic to $\mathcal{M}$. Also, all (co)homologies are taken with $\mathbb{Q}$ coefficient.

Following [11], we can make the above construction T-equivariant. We choose a smooth DM T-stack $\mathcal{Z}$ and a T-equivariant embedding $\iota : \mathcal{M} \to \mathcal{Z}$; we choose $\mathcal{E}$ to be T-equivariant and extend $\iota_* \mathcal{E}$ to a T-equivariant vector bundle $\mathcal{F}$ over $\mathcal{Z}$. Then the cone $\mathcal{C}$ alluded before is a T-invariant subcone of $\mathcal{F}$ that lies entirely over $\mathcal{M} \subset \mathcal{Z}$. Because $\mathcal{C} \subset \mathcal{F}$ is T-equivariant, the composite

$$T \times \mathcal{C} \xrightarrow{\mathrm{pr}_2} \mathcal{C} \longrightarrow \mathcal{F}$$

defines a T-equivariant class $[\mathcal{C}]^T \in H^T_*(\mathcal{F}, \mathcal{F} - \mathcal{Z})$; its image under the T-equivariant Thom isomorphism $\varphi_{\mathcal{F}}$ is the T-equivariant virtual moduli cycle

$$\varphi_{\mathcal{F}}\big([\mathcal{C}]^T\big) = [\mathcal{M}]^{\mathrm{vir},T} \in H^T_*(\mathcal{Z}).$$

(Indeed, this class lies in the image of $H^T_*(\mathcal{M}) \to H^T_*(\mathcal{Z})$.) Note that the equivariant homologies are $H^T_*(pt) = \mathbb{Q}[u_1, u_2]$ modules.

Next, we apply the localization theorem to the class $[\mathcal{M}]^{\mathrm{vir},T}$. Let

$$\coprod_{a \in A} \mathcal{M}_a = \mathcal{M}^T$$

be the decomposition of the T-fixed loci into connected components; let

$$\tau_a : H^T_*(\mathcal{M}_a) \otimes \mathbb{Q}[u_1, u_2]_{\mathfrak{m}} \longrightarrow H^T_*(\mathcal{M}) \otimes \mathbb{Q}[u_1, u_2]_{\mathfrak{m}}$$

be induced by the inclusion. According to [11], to each $\mathcal{M}_a$ there is a canonically defined virtual cycle $[\mathcal{M}_a]^{\mathrm{vir},T} \in H^T_*(\mathcal{M}_a)$ and a virtual T-equivariant normal bundle $\mathcal{N}^{\mathrm{vir}}_a = \mathcal{T}^{1,m} - \mathcal{T}^{2,m}$ so that, after localized at $\mathfrak{m} = (u_1, u_2)$,

$$[\mathcal{M}]^{\mathrm{vir},T}_{\mathfrak{m}} = \sum_{a \in A} \tau_{a*}\Big(\frac{[\mathcal{M}_a]^{\mathrm{vir},T}}{e^T(\mathcal{N}^{\mathrm{vir}}_a)}\Big) \in (H^T_*(\mathcal{M}) \otimes \mathbb{Q}[u_1, u_2]_{\mathfrak{m}})_0,$$

where

$$\frac{[\mathcal{M}_a]^{\mathrm{vir},T}}{e^T(\mathcal{N}^{\mathrm{vir}}_a)} = \frac{1}{e^T(\mathcal{N}^{\mathrm{vir}}_a)} \cap [\mathcal{M}_a]^{\mathrm{vir},T}$$

and $(H^T_*(\mathcal{M}) \otimes \mathbb{Q}[u_1, u_2]_{\mathfrak{m}})_0$ is the degree zero part of the graded ring $H^T_*(\mathcal{M}) \otimes \mathbb{Q}[u_1, u_2]_{\mathfrak{m}}$ (with intersection product). Let

$$[\mathcal{M}]^{\mathrm{vir},T}_{\mathfrak{m},0} = \sum_{a \in A_0} \tau_{a*}\Big(\frac{[\mathcal{M}_a]^{\mathrm{vir},T}}{e^T(\mathcal{N}^{\mathrm{vir}}_a)}\Big) \in (H^T_*(\mathcal{M}) \otimes \mathbb{Q}[u_1, u_2]_{\mathfrak{m}})_0,$$

where A_0 is the collection of those $\mathcal{M}_a$ that lies entirely in $\mathcal{M}_0$. Since the $\mathcal{M}$ we will work with has the property that either $\mathcal{M}_a \subset \mathcal{M}_0$ or $\mathcal{M}_a \cap \mathcal{M}_0 = \emptyset$, such A_0 is well-defined. In this format, the formal relative GW invariant defined in (A.1) is

$$F^{\bullet\Gamma^0}_{\chi,\vec{d},\vec{\mu}}(u_1,u_2) = \frac{1}{|\operatorname{Aut}(\vec{\mu})|}\deg_{\mathfrak{m}}[\mathcal{M}]^{\mathrm{vir},T}_{\mathfrak{m},0} \in \mathbb{Q}(u_2/u_2)$$

where

$$\deg_{\mathfrak{m}} : H^T_*(\mathcal{M})\otimes\mathbb{Q}[u_1,u_2]_{\mathfrak{m}} \longrightarrow \mathbb{Q}[u_1,u_2]_{\mathfrak{m}}$$

is defined as in Section 5.4 (with A^T_* replaced by H^T_*).

To show that the above sum is independent of the (u_1,u_2), we shall devise a way to show that this partial sum is the localization of a purely topological quantity. We pick a T-equivariant Riemannian metric on $\mathcal{Z}$; we form a T-invariant closed tubular neighborhood of $\mathcal{M}_0$:

$$\Sigma_\epsilon\mathcal{M}_0 = \{u\in\mathcal{Z} \mid \operatorname{dist}(u,\mathcal{M}_0)\le\epsilon\}\subset\mathcal{Z}.$$

For sufficiently small $\epsilon>0$, $\Sigma_\epsilon\mathcal{M}_0$ is a smooth orbifold with smooth boundary

$$\partial\Sigma_\epsilon\mathcal{M}_0 = \{u\in\mathcal{Z} \mid \operatorname{dist}(u,\mathcal{M}_0)=\epsilon\}.$$

We then close the boundary of $\Sigma_\epsilon\mathcal{M}_0$ by picking a subgroup $S^1\subset T$ and contract individual S^1 orbits $S^1\cdot z\subset\partial\Sigma_\epsilon\mathcal{M}_0$ to points $[S^1\cdot z]$. For this purpose, we pick a subgroup $S^1\subset T$ so that $\mathcal{M}^T=\mathcal{M}^{S^1}$. To stay away from the S^1-fixed points $(\partial\Sigma_\epsilon\mathcal{M}_0)^{S^1}$ in $\partial\Sigma_\epsilon\mathcal{M}_0$, we shall take $\Sigma_\epsilon\mathcal{M}_0-(\partial\Sigma_\epsilon\mathcal{M}_0)^{S^1}$ and then contract individual S^1-orbits in $\partial\Sigma_\epsilon\mathcal{M}_0-(\partial\Sigma_\epsilon\mathcal{M}_0)^{S^1}$. We denote the resulting space by $\tilde{\mathcal{Z}}$. It is a smooth orbifold; its construction depends on the choice of ϵ and $S^1\subset T$. Because $\mathcal{M}\cap(\partial\Sigma_\epsilon\mathcal{M}_0)^{S^1}=\emptyset$, the image of $\mathcal{M}\cap\Sigma_\epsilon\mathcal{M}_0$ in $\tilde{\mathcal{Z}}$, denoted by $\tilde{\mathcal{M}}$, is compact. We let $\tilde{\mathcal{Z}}_\infty$ be $\big(\partial\Sigma_\epsilon\mathcal{M}_0-(\partial\Sigma_\epsilon\mathcal{M}_0)^{S^1}\big)/S^1$ and let $\tilde{\mathcal{M}}_\infty=\tilde{\mathcal{Z}}_\infty\cap\tilde{\mathcal{M}}$.

We next define the virtual cycle of $\tilde{\mathcal{M}}$. Because $\mathcal{F}$ is a T-equivariant vector bundle, and because points in $\partial\Sigma_\epsilon\mathcal{M}_0-(\partial\Sigma_\epsilon\mathcal{M}_0)^{S^1}$ have finite stabilizers, $\mathcal{F}$ descends to a vector bundle $\tilde{\mathcal{F}}$ on $\tilde{\mathcal{Z}}$. For the same reason, the cone $\mathcal{C}$ descends to a cone $\tilde{\mathcal{C}}\subset\tilde{\mathcal{F}}$. Since $\mathcal{C}$ stay away from fibers over $(\partial\Sigma_\epsilon\mathcal{M}_0)^{S^1}$, $\tilde{\mathcal{C}}$ defines an element $[\tilde{\mathcal{C}}]\in H^T_*(\tilde{\mathcal{F}},\tilde{\mathcal{F}}-\tilde{\mathcal{Z}})$; since all data are T-equivariant, it also defines an equivariant class

$$[\tilde{\mathcal{C}}]^T\in H^T_*(\tilde{\mathcal{F}},\tilde{\mathcal{F}}-\tilde{\mathcal{Z}}).$$

Their respective images under the obvious Thom isomorphisms give the virtual and equivariant virtual cycles:

$$[\tilde{\mathcal{M}}]^{\mathrm{vir}}\in H_*(\tilde{\mathcal{Z}}) \quad\text{and}\quad [\tilde{\mathcal{M}}]^{\mathrm{vir},T}\in H^T_*(\tilde{\mathcal{Z}}).$$

We now apply the localization theorem to the cycle $[\tilde{\mathcal{M}}]^{\mathrm{vir},T}$. Let $\coprod_{b\in B}\tilde{\mathcal{Z}}_b$ be the connected components decomposition of $\tilde{\mathcal{Z}}^T$; let

$$\tilde{r}_b : H^T_*(\tilde{\mathcal{Z}})\longrightarrow H^T_*(\tilde{\mathcal{Z}},\tilde{\mathcal{Z}}-\tilde{\mathcal{Z}}_b)\longrightarrow H^T_*(\tilde{\mathcal{Z}}_b)$$

be the composites of the homomorphisms induced by the inclusion and the Thom isomorphism.

Lemma A.1. *In case $\tilde{\mathcal{Z}}_b$ is contained in $\tilde{\mathcal{Z}}_\infty$, then $\tilde{r}_b\big([\tilde{\mathcal{M}}]^{vir,T}\big)=0$.*

Proof of Theorem 5.9. We let $\tilde{\mathcal{Z}}^T_\infty$ be the union of those $\tilde{\mathcal{Z}}_b$ that are contained in $\tilde{\mathcal{Z}}_\infty$ and let $\tilde{\mathcal{Z}}^T_0$ the union of the remaining $\tilde{\mathcal{Z}}_b$. Since $\tilde{\mathcal{Z}}$ is the quotient of

$\Sigma_\epsilon\mathcal{M}_0 - (\partial\Sigma_\epsilon\mathcal{M}_0)^{S^1}$, $\tilde{\mathcal{Z}}_0^T$ is disjoint from $\tilde{\mathcal{Z}}_\infty^T$. Because of Lemma A.1, the image of $[\tilde{\mathcal{M}}]^{\mathrm{vir},T}$ under

$$H_*^T(\tilde{\mathcal{Z}}) \longrightarrow H_*^T(\tilde{\mathcal{Z}}, \tilde{\mathcal{Z}} - \tilde{\mathcal{Z}}_\infty^T) \longrightarrow H_*^T(\tilde{\mathcal{Z}}_\infty^T)$$

is zero. Hence if we let

$$\tilde{r}_0 : H_*^T(\tilde{\mathcal{Z}}_0^T) \otimes \mathbb{Q}[u_1,u_2]_{\mathfrak{m}} \longrightarrow H_*^T(\tilde{\mathcal{Z}}) \otimes \mathbb{Q}[u_1,u_2]_{\mathfrak{m}}$$

be induced by the inclusion, then $[\tilde{\mathcal{M}}]^{\mathrm{vir},T} \in \mathrm{Im}(\tilde{r}_0)$. On the other hand, $\tilde{\mathcal{Z}}$ is a DM-stack near $\tilde{\mathcal{Z}}_0^T$; by following the proof of [11] line by line we conclude that

$$[\tilde{\mathcal{M}}]^{\mathrm{vir},T}_{\mathfrak{m}} = \sum_{\tilde{\mathcal{M}}_a \subset \tilde{\mathcal{Z}}_0^T} \tilde{r}_a\Big(\frac{[\mathcal{M}_a]^{\mathrm{vir},T}}{e^T(\mathcal{N}_a^{\mathrm{vir}})}\Big),$$

where

$$\tilde{r}_a : H_*^T(\mathcal{M}_a) \otimes \mathbb{Q}[u_1,u_2]_{\mathfrak{m}} \to H_*^T(\tilde{\mathcal{Z}}) \otimes \mathbb{Q}[u_1,u_2]_{\mathfrak{m}}$$

is the obvious homomorphism. So

$$\deg_{\mathfrak{m}}[\tilde{\mathcal{M}}]^{\mathrm{vir},T}_{\mathfrak{m}} = \deg_{\mathfrak{m}}\Big(\sum_{\tilde{\mathcal{M}}_a \subset \tilde{\mathcal{Z}}_0^T} \tilde{r}_a\Big(\frac{[\mathcal{M}_a]^{\mathrm{vir},T}}{e^T(\mathcal{N}_a^{\mathrm{vir}})}\Big)\Big).$$

But the right hand side is exactly $F^{\bullet\Gamma^0}_{\chi,\vec{d},\vec{\mu}}(u_1,u_2)$, and it is independent of u_1/u_2 since

$$\deg_{\mathfrak{m}}[\tilde{\mathcal{M}}]^{\mathrm{vir},T}_{\mathfrak{m}} = \deg[\tilde{\mathcal{M}}]^{\mathrm{vir}} \in \mathbb{Q}.$$

This completes the proof of Theorem 5.9. □

The proof of Lemma A.1 will occupy the remainder of this Appendix. We shall first construct the moduli space $\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(W^{\mathrm{rel}}, L)$ and the equivariant morphism Φ^T. This will be the done in the next subsection. Afterwards, we will study fixed loci of $\mathcal{M}$ of any subtorus T_η of T that are small deformations of $\mathcal{M}_0$. We will prove a structure theorem of such loci and study the obstruction theory of such loci. The details of these will occupy the remainder of this Appendix.

A.1. **The relative Calabi-Yau manifold W^{rel} and the morphism Φ.** Our first task is to come up with a toric Calabi-Yau manifold W^{rel} as mentioned. Looking at the graph Γ^0 that we chose, the obvious choice of W is the blowing up of $\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$ along three disjoint lines

$$\ell_1 = \infty\times\mathbb{P}^1\times 0, \qquad \ell_2 = 0\times\infty\times\mathbb{P}^1 \quad\text{and}\quad \ell_3 = \mathbb{P}^1\times 0\times\infty. \tag{A.3}$$

Here we follow the convention that (z_1,z_2,z_3) is the point $([z_1,1],[z_2,1],[z_3,1])$ in $(\mathbb{P}^1)^3$. We let $D\subset W$ be the exceptional divisor and let $D_i\subset D$ be its connected component that lies over ℓ_i. Each D_i is isomorphic to $\mathbb{P}^1\times\mathbb{P}^1$. Clearly,

$$\wedge^3\Omega_W(\log D) \cong \mathcal{O}_W, \tag{A.4}$$

and hence the relative pair $W^{\mathrm{rel}} = (W,D)$ is a relative Calabi-Yau threefold. We next let C_1, C_2 and C_3 be the proper transforms of

$$\mathbb{P}^1\times 0\times 0, \qquad 0\times\mathbb{P}^1\times 0 \quad\text{and}\quad 0\times 0\times\mathbb{P}^1,$$

and let $L_1\subset D_1, L_2\subset D_2, L_3\subset D_3$ be the preimage of

$$(\infty,0,0)\in\ell_1, \qquad (0,\infty,0)\in\ell_2 \quad\text{and}\quad (0,0,\infty)\in\ell_3.$$

For later discussion, we agree that under the isomorphisms $D_i \cong \mathbb{P}^1 \times \mathbb{P}^1$ and $\ell_i \cong \mathbb{P}^1$, the tautological projection $D_i \to \ell_i$ is the first projection. Under this convention, the line $L_i \subset D_i$ is the line $0 \times \mathbb{P}^1$ and the intersection $p_i = C_i \cap D_i$ is the point $(0,0)$. (See Figure 13 for details.)

As to the torus action, we pick the obvious one on $(\mathbb{P}^1)^3$ via

$$(z_1, z_2, z_3)^{(t_1,t_2,t_3)} = (t_1 z_1, t_2 z_2, t_3 z_3), \qquad (t_1, t_2, t_3) \in (\mathbb{C}^*)^3. \tag{A.5}$$

It lifts to a $(\mathbb{C}^*)^3$-action on W that leaves D_i and L_i invariant. Within $(\mathbb{C}^*)^3$ there is a subgroup defined by $t_1 t_2 t_3 = 1$; it is isomorphic to $(\mathbb{C}^*)^2$ and is the subgroup that leaves (A.4) invariant. We let T be the maximal compact subgroup of $\{t_1 t_2 t_3 = 1\} \subset (\mathbb{C}^*)^3$. In the following, we shall view $W^{\mathrm{rel}} = (W, D)$ as a T-relative Calabi-Yau manifold.

Next we will define the moduli space $\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(W^{\mathrm{rel}}, L)$. Clearly, each C_i induces a homology class $[C_i] \in H_2(W;\mathbb{Z})$. For

$$\vec{\mu} = (\mu^1, \mu^2, \mu^3) \in \mathcal{P}^3_+,$$

we let $\vec{d}$ be the homology class

$$\vec{d} = |\mu^1|[C_1] + |\mu^2|[C_2] + |\mu^3|[C_3] \in H_2(W;\mathbb{Z}).$$

The pair $(\vec{d}, \vec{\mu})$ is an effective class of Γ^0:

$$\vec{d}(\bar{e}_i) = |\mu^i|, \quad \vec{\mu}(v_i) = \mu^i, \quad i = 1, 2, 3.$$

We then let

$$\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(W^{\mathrm{rel}}, L)$$

be the moduli of relative stable morphisms

$$u : (X; R_1, R_2, R_3) \longrightarrow (W[\mathbf{m}], D[\mathbf{m}]_1, D[\mathbf{m}]_2, D[\mathbf{m}]_3)$$

having fundamental classes $\vec{d}$, having ramification patterns μ^i along $D[\mathbf{m}]_i$, and satisfying $u(R_i) \subset L[\mathbf{m}]_i$, modulo the equivalence relation introduced in [18]. It is a proper, separated DM-stack; it has a perfect obstruction theory [17, 18], and thus admits a virtual cycle[2].

It follows from our construction that the scheme Y_{Γ^0} is the union $C_1 \cup C_2 \cup C_3$ in W and the formal scheme $\hat{Y}_{\Gamma^0}$ is the formal completion of W along Y_{Γ^0}. Further, the relative divisor $\hat{D}_0$ of $\hat{Y}_{\Gamma^0}$ is the preimage of the relative divisor $D \subset W$; the induced morphism

$$\phi : (\hat{Y}_{\Gamma^0}, \hat{D}, \hat{L}) \longrightarrow (W, D, L) \tag{A.6}$$

is T-equivariant; and the two effective classes $(\vec{d}, \vec{\mu})$ are consistent under the map φ. Therefore, it induces a T-equivariant morphism of the moduli spaces

$$\Phi : \mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(\hat{Y}^{\mathrm{rel}}_{\Gamma^0}, \hat{L}) \longrightarrow \mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(W^{\mathrm{rel}}, L), \tag{A.7}$$

which induces a morphism

$$\Phi^T : \mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(\hat{Y}^{\mathrm{rel}}_{\Gamma^0}, \hat{L})^T \longrightarrow \mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(W^{\mathrm{rel}}, L)^T$$

between their respective fixed loci.

[2] The scheme $W[\mathbf{m}]$ is defined in subsection A.2; the L_i are line in the relative divisor D_i in $W[\mathbf{m}]$.

Lemma A.2. *The morphism* Φ^T *is an open and closed embedding; the obstruction theories of* $\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(\hat{Y}^{\rm rel}_{\Gamma^0},\hat{L})$ *and* $\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(W^{\rm rel},L)$ *are identical under* Φ *along the fixed loci* $\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(\hat{Y}^{\rm rel}_{\Gamma^0},\hat{L})^T$ *and its image in* $\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(W^{\rm rel},L)$.

Proof. Let $[u]\in\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(\hat{Y}^{\rm rel}_{\Gamma^0},\hat{L})^T$ be a closed point and let $[\tilde{u}]\in\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(W^{\rm rel},L)^T$ be the image of $[u]$ under Φ. To prove the first statement it suffices to show that any deformation $[\tilde{u}_s]\in\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(W^{\rm rel},L)^T$ of $[\tilde{u}]$ lies entirely in $\operatorname{Im}\Phi$. But this follows from a straightforward analysis of the maps in $\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(W^{\rm rel},L)^T$. This settles the first part of the Lemma.

For the second part, we note that the similarly defined moduli $\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(Y^{\rm rel}_{\Gamma^0},L)$ is a closed substack of $\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(W^{\rm rel},L)$, the moduli space $\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(\hat{Y}^{\rm rel}_{\Gamma^0},\hat{L})$ is the formal completion of $\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(W^{\rm rel},L)$ along $\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(Y^{\rm rel}_{\Gamma^0},L)$. Hence the obstruction theory of $\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(\hat{Y}^{\rm rel}_{\Gamma^0},\hat{L})$ is the one induced from that of $\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(W^{\rm rel},L)$. This proves the second statement of the Lemma. □

In the remainder of this Appendix, we shall fix χ, $\vec{d}$, $\vec{\mu}$ and the relative variety $W^{\rm rel}$ once and for all. For notational simplicity, we will abbreviate $\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(W^{\rm rel},L)$, $\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(Y^{\rm rel}_{\Gamma^0},L)$ and $\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(\hat{Y}^{\rm rel}_{\Gamma^0},\hat{L})$ to $\mathcal{M}$, $\mathcal{M}_0$ and $\hat{\mathcal{M}}_0$, respectively.

A.2. **Invariant relative stable morphisms.** Let $a_1,a_2,a_3\in\mathbb{Z}$ with $a_1+a_2+a_3=0$ be three relative prime integers; $\eta=(a_1,a_2,a_3)$ defines a subgroup

$$T_\eta=\{(e^{\sqrt{-1}a_1\theta},e^{\sqrt{-1}a_2\theta},e^{\sqrt{-1}a_3\theta}\})\mid\theta\in[0,2\pi]\}\subset T.$$

Our next task is to characterize those stable relative morphisms that are invariant under $T_\eta\subset T$ and are small deformations of elements in $\mathcal{M}_0$.

To begin with, we sketch the variety W by its image under the moment map $\Upsilon: W\longrightarrow\mathbb{R}^3$ of the $(S^1)^3$-action on W. The image is a polytope that is diffeomorphic to the quotient $W/(S^1)^3$. As shown in Figure 13, all faces of this polytope represent the $(\mathbb{C}^*)^3$ invariant divisors of W; the point p_0 is the image of the point $(0,0,0)\in W$ and the line $\overline{p_0p_i}$ is the image of the curve C_i; the rectangle face containing the edge $\overline{p_iq_i}$ is the image of the relative divisor D_i.

To investigate relative stable morphisms to W, we need the expanded relative pair $(W[\mathbf{m}],D[\mathbf{m}])$, $\mathbf{m}=(m_1,m_2,m_3)$. The main part $W[\mathbf{m}]$ is the result after attaching three chains, of length m_1, m_2 and m_3 respectively, of a ruled variety Δ over $\mathbb{P}^1\times\mathbb{P}^1$ to D_1, D_2 and D_3 in W. Here Δ is the projective bundle $\mathbb{P}\big(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(0,1)\big)$ with two sections

$$D_+=\mathbb{P}\big(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}\oplus 0\big)\quad\text{and}\quad D_-=\mathbb{P}\big(0\oplus\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(0,1)\big);$$

an m-chain of Δ is the gluing of m ordered copies of Δ by identifying the D_- of one Δ to the D_+ of the next Δ via the canonical isomorphism $\mathrm{pr}: D_\pm\to\mathbb{P}^1\times\mathbb{P}^1$; the chain is attached to D_i by identifying the D_+ of the first Δ in the chain with D_i and declaring the D_- of the last Δ be $D[\mathbf{m}]_i$; the union

$$D[\mathbf{m}]=D[\mathbf{m}]_1\cup D[\mathbf{m}]_2\cup D[\mathbf{m}]_3$$

is the new relative divisor of $W[\mathbf{m}]$. Note that our construction is consistent with that the normal bundle of D_i in W has degree -1 along L_i.

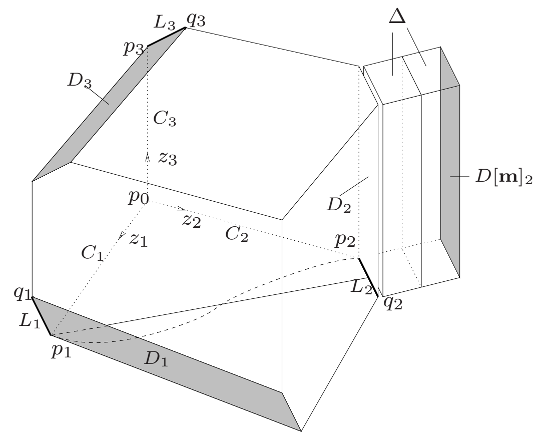


FIGURE 13. This is a sketch of the scheme $W[\mathbf{m}]$ for $\mathbf{m} = (0, 2, 0)$. The main part is the image of W under the moment map Υ; the faces are the images of T-invariant divisors; the point p_0 is the image of $(0, 0, 0)$, the z_i indicates the choice of coordinate chart of the line C_i; The D_i are the blown-up divisors with points p_i and q_i shown; the line L_i are the thickened lines in D_i. The added two solids to the left are the two Δ's attached to D_2, resulting the scheme $W[\mathbf{m}]$ with $\mathbf{m} = (0, 2, 0)$. The shaded faces are the relative divisor $D[\mathbf{m}]$ of $W[\mathbf{m}]$.

The straight diagonal line contained in the bottom face indicates the image of $\phi_{k,c}$ in case $\eta = (1, -1, 0)$; the curved line indicates the image of $\phi_{k,c}$ in the other case.

For future convenience, we denote by $\Delta[m_i]$ the chain of Δ's that is attached to D_i; we denote by $L[\mathbf{m}]_i \subset D[\mathbf{m}]_i$ the same line as $L_i \subset D_i$. The new scheme $W[\mathbf{m}]$ contains W as its main irreducible component; it also admits a stable contraction $W[\mathbf{m}] \to W$. Unless otherwise mentioned, the maps $W \to W[\mathbf{m}]$ and $W[\mathbf{m}] \to W$ are these inclusion and projection.

The pair (W, D) contains (Y_{Γ^0}, p), $p = p_1 + p_2 + p_3$, as its subpair. Accordingly, the pair $(W[\mathbf{m}], D[\mathbf{m}])$ contains a subpair $(Y_{\Gamma^0}[\mathbf{m}], p[\mathbf{m}])$ whose main part $Y_{\Gamma^0}[\mathbf{m}]$ is the preimage of Y_{Γ^0} under the contraction $W[\mathbf{m}] \to W$. The relative divisor $p[\mathbf{m}]$ is the intersection $Y_{\Gamma^0}[\mathbf{m}] \cap D[\mathbf{m}]$. It is the embedding $Y_{\Gamma^0}[\mathbf{m}] \subset W[\mathbf{m}]$ that induces the embedding $\mathcal{M}_0 \subset \mathcal{M}$.

Now let u_0 be a relative stable morphism in $\mathcal{M}_0$, considered as an element in $\mathcal{M}$; let u_s be a small deformation of u_0 in $\mathcal{M}^{T_\eta}$ that is not entirely contained in $\mathcal{M}_0$. Each u_s is a morphism from X_s to $W[\mathbf{m}]$ for some triple $\mathbf{m}$ possibly depending on s. We let $\tilde{u}_s : X_s \to W$ be the composite of u_s with the contraction $W[\mathbf{m}] \to W$; $\tilde{u}_s$ is a flat family of morphisms. Further, $\tilde{u}_s$ specializes to $\tilde{u}_0$ as s specializes to 0. Hence as sets,

$$\lim_{s \to 0} \tilde{u}_s(X_s) = \tilde{u}_0(X_0). \tag{A.8}$$

Because $\tilde{u}_s(X_s)$ are union of algebraic curves in W and $\tilde{u}_0(X_0)$ is contained in $C_1 \cup C_2 \cup C_3$, for general s the intersection $\tilde{u}_s(X_s) \cap D$ is discrete. Hence every connected component $Y \subset \tilde{u}_s^{-1}(D_i)$ must be mapped to a fiber of $\Delta[m_i]/D_i$; in particular $u_s(Y) \cap D[\mathbf{m}]_i \neq \emptyset$. Because of the requirement $u_s(Y) \cap D[\mathbf{m}]_i \subset L[\mathbf{m}]_i$

we impose on the $\mathcal{M}$, we have

$$\tilde{u}_s(X) \cap D_i \subset L_i. \tag{A.9}$$

This leads to the following definition.

Definition A.3. *We let $\mathcal{M}^{T_\eta}_{def}$ be the union of all connected components of*

$$\{(u, X) \in \mathcal{M}^{T_\eta} \mid \tilde{u}(X) \cap D \text{ is finite}\}$$

that intersect $\mathcal{M}_0$ but are not entirely contained in it.

In short, elements in $\mathcal{M}^{T_\eta}_{\rm def}$ are those u that are T_η-invariant and are small deformations of elements in $\mathcal{M}_0$. Following the discussion before Definition A.3, all u in $\mathcal{M}^{T_\eta}_{\rm def}$ satisfies (A.9). In case $a_{i+1} \neq 0$ (we agree $a_4 = a_1$), the only T_η-fixed points of L_i are p_i and q_i; hence all u in $\mathcal{M}^{T_\eta}_{\rm def}$ satisfies a strengthened version to (A.9):

$$\tilde{u}_s(X) \cap D_i \subset p_i, \qquad \text{when} \;\; a_{i+1} \neq 0. \tag{A.10}$$

Here q_i is ruled out because each connected component of $\mathcal{M}^{T_\eta}_{\rm def}$ intersects $\mathcal{M}_0$.

We now characterize elements in $\mathcal{M}^{T_\eta}_{\rm def}$. We comment that we shall reserve a_1, a_2 and a_3 for the three components of η; we always assume the three a_i's are relatively prime and that $a_1 + a_2 + a_3 = 0$. In this and the next two Subsections, we shall workout the case $a_1 > 0$ and $a_2, a_3 < 0$; the case $\eta = (1, -1, 0)$ will be considered in Subsection A.5. Now let $(u, X) \in \mathcal{M}^{T_\eta}_{\rm def}$ and let $V \subset \tilde{u}(X)$ be any irreducible component. Since u is T_η-invariant, V is T_η-invariant. Hence V must be the lift of the set

$$\bar{V} = \{(c_1 t^{a_1}, c_2 t_2^{a_2}, c_3 t_3^{a_3}) \mid t \in \mathbb{C} \cup \{\infty\} \ \} \subset (\mathbb{P}^1)^3$$

for some (c_1, c_2, c_3). In case all c_i are non-zero, then $V \cap D \not\subset \{p_1, p_2, p_3\}$, which violates the requirement (A.8); when $c_1 = 0$ but the other two are non-zero, then $V \cap D \neq p_2$, which is impossible. Similarly, in the case $c_2 = 0$ but the others are not zero, the set V contains q_1, which is impossible.

This leaves us with the only two possibilities: when only one of c_i is non-zero or $c_3 = 0$ but the other two are non-zero. In the first case we have $V = C_i$ for some i; in the later case V is the image of the map

$$\phi_{k,c} : \mathbb{P}^1 \longrightarrow W, \qquad k \in \mathbb{Z}^+, \;\; c \in \mathbb{C}^* \tag{A.11}$$

that is the lift of $\mathbb{P}^1 \to (\mathbb{P}^1)^{\times 3}$ defined by $\xi \mapsto (\xi^{ka_1}, c^{-ka_2}\xi^{ka_2}, 0)$. Clearly, $\phi_{k,c}$ is T_η-invariant. It is easy to see that these are the only T_η-equivariant maps $Y \to W$ whose images are not entirely lie in $C_1 \cup C_2 \cup C_3$ and the divisor D. This proves

Lemma A.4. *Suppose $a_1 > 0$ and a_2 and $a_3 < 0$. Then any $(u, X) \in \mathcal{M}^{T_\eta}_{def}$ has at least one irreducible component $Y \subset X$ and a pair (k, c) so that $u|_Y \cong \phi_{k,c}$.*

Here by $u|_Y \cong \phi_{k,c}$ we mean that there is an isomorphism $Y \cong \mathbb{P}^1$ so that under this isomorphism $u|_Y \equiv \phi_{k,c}$.

When c specialize to 0, the map $\phi_{k,c}$ specializes to

$$\phi_{k,0} : \mathbb{P}^1 \sqcup \mathbb{P}^1 \longrightarrow W$$

defined as follows. We endow the first copy (of $\mathbb{P}^1 \sqcup \mathbb{P}^1$) with the coordinate ξ_1 and the second copy with ξ_2; we then form the nodal curve $\mathbb{P}^1 \sqcup \mathbb{P}^1$ by identifying 0 of the first $\mathbb{P}^1$ with 0 of the second $\mathbb{P}^1$; we define $\phi_{k,0}$ to be the lift of the maps

$$\xi_1 \mapsto (\xi_1^{ka_1}, 0, 0) \quad \text{and} \quad \xi_2 \mapsto (0, \xi_1^{-ka_2}, 0).$$

Since $\xi_1 = 0$ and $\xi_2 = 0$ are both mapped to the origin in $(\mathbb{P}^1)^{\times 3}$, they glue together to form a morphism $\phi_{k,0} : \mathbb{P}^1 \sqcup \mathbb{P}^1 \to W$.

This leads to the following definition:

Definition A.5. *A deformable part of a* $(u, X) \in \mathcal{M}_{def}^{T_\eta}$ *consists of a curve* $Y \subset X$ *and an isomorphism* $u|_Y \cong \phi_{k,c}$ *for some* (k, c).

Suppose (u, X) has at least two deformable parts, say (Y_1, ϕ_{k_1,c_1}) and (Y_2, ϕ_{k_1,c_2}), then the explicit expression of $\phi_{k,c}$ ensures that Y_1 and Y_2 share no common irreducible components. Should $Y_1 \cap Y_2 \neq \emptyset$, their intersection would be a nodal point of X that could only be mapped to either D_1 or D_2 of W under u. (Note that it could not be mapped to p_0 since then both c_1 and $c_2 = 0$, and that node would be in more than two irreducible components of X.) However, the case that the node is mapped to D_1 or D_2 can also be ruled out because it violates the pre-deformable requirement of relative stable morphisms [17]. Hence Y_1 and Y_2 are disjoint. This way, we can talk about the maximal collection of deformable parts of (u, X); let it be

$$(Y_1, \phi_{k_1,c_1}), \cdots , (Y_l, \phi_{k_l,c_l}).$$

Definition A.6. *We define the deformation type of* $(u, X) \in \mathcal{M}_{def}^{T_\eta}$ *be the unordered collection* $\{k_1, \cdots , k_l\}$. *It defines a function on* $\mathcal{M}_{def}^{T_\eta}$*, called the deformation type function.*

Let (u, X) be an element in $\mathcal{M}_{\mathrm{def}}^{T_\eta}$ of type $\{k_1, \cdots , k_l\}$ as before. Intuitively, we should be able to deform u within $\mathcal{M}_{\mathrm{def}}^{T_\eta}$ by varying $u|_{Y_i}$ using $\phi_{k,t}$ to generate an $\mathbb{A}^l$-family in $\mathcal{M}_{\mathrm{def}}^{T_\eta}$. It is our next goal to make this precise.

To proceed, we need to show how to put $\phi_{k,t}$ into a family. We first blow up $\mathbb{P}^1 \times \mathbb{A}^1$ at $(0, 0)$ to form a family of curves $\mathfrak{Y}$ over $\mathbb{A}^1$. The complement of the exceptional divisor $\mathfrak{Y} - E = \mathbb{P}^1 \times \mathbb{A}^1 - (0, 0)$ comes with an induced coordinate (ξ, t). We define

$$\Phi_k|_{\mathfrak{Y}-E} : \mathfrak{Y} - E \longrightarrow W; \quad (\xi, t) \mapsto (\xi^{ka_1}, t^{-ka_2}\xi^{ka_2}, 0).$$

We claim that $\Phi_k|_{\mathfrak{Y}-E}$ extends to a $\Phi_k : \mathfrak{Y} \to W$. Indeed, if we pick a local coordinate chart near E, which is (ξ, v) with $t = \xi v$, then

$$\Phi_k|_{\mathfrak{Y}-E} : (\xi, v) \mapsto (\xi^{ka_1}, (\xi v)^{-ka_2}\xi^{ka_2}, 0) = (\xi, v) \mapsto (\xi^{ka_1}, v^{-ka_2}, 0),$$

which extends to a regular

$$\Phi_k : \mathfrak{Y} \longrightarrow W.$$

Note that for $c \in \mathbb{A}^1$, the fiber of $(\Phi_k, \mathfrak{Y})$ over c is exactly the $\phi_{k,c}$ we defined earlier. Henceforth, we will call $(\Phi_k, \mathfrak{Y})$ the standard model of the family $\phi_{k,t}$; we will use $\mathfrak{Y}_c$ to denote the fiber of $\mathfrak{Y}$ over $c \in \mathbb{A}^1$.

To deform u using the family Φ_k, we need to glue $\mathfrak{Y}$ onto the domain X. We let $\mathfrak{D}_1$ be the proper transform of $0 \times \mathbb{A}^1 \subset \mathbb{P}^1 \times \mathbb{A}^1$ and let $\mathfrak{D}_2 = \infty \times \mathbb{A}^1$ in $\mathcal{Y}$. Both $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are canonically isomorphic to $\mathbb{A}^1$ via the second projection. For $Y \subset X$, we fix an isomorphism $Y \cong \mathfrak{Y}_c$ so that $u|_Y \cong \phi_{k,c}$; we specify v_1, $v_2 \in Y$ so that $u(v_i) \in D_i$; we let X_0 be the closure of $X - Y$ in X.

We now glue $\mathfrak{Y}$ onto $X_0 \times \mathbb{A}^1$. In case both v_1 and v_2 are nodes of X, we glue $\mathfrak{Y}$ onto $X_0 \times \mathbb{A}^1$ by identifying $\mathfrak{D}_1$ with $v_1 \times \mathbb{A}^1$ and $\mathfrak{D}_2$ with $v_2 \times \mathbb{A}^1$, using the their standard isomorphisms with $\mathbb{A}^1$; in case v_1 is a marked point of X and v_2 is a node, we glue $\mathfrak{Y}$ onto $X_0 \times \mathbb{A}^1$ by identifying $\mathfrak{D}_2$ with $v_2 \times \mathbb{A}^1$ and declaring $\mathfrak{D}_1$

to be the new marked points, replacing v_1; in case v_1 is a node and v_2 is a marked points, we repeat the same procedure with the role of v_1 and v_2 and of $\mathfrak{D}_1$ and $\mathfrak{D}_2$ exchanged; finally in case both v_1 and v_2 are marked points, we simply replace $Y\times\mathbb{A}^1$ in $X\times\mathbb{A}^1$ by $\mathfrak{Y}$ while declaring that $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are the two marked points replacing v_1 and v_2. We let $\mathcal{X}\to\mathbb{A}^1$ be the resulting family.

The morphisms

$$X_0\times\mathbb{A}^1 \xrightarrow{\text{pr}} X_0 \xrightarrow{u|_{X_0}} W[\mathbf{m}] \quad\text{and}\quad \Phi_k:\mathfrak{Y}\longrightarrow W$$

glue together to form a morphism

$$\mathcal{U}:\mathcal{X}\longrightarrow W[\mathbf{m}].$$

The pair $(\mathcal{U},\mathcal{X})$ is the family in $\mathcal{M}_{\text{def}}^{T_\eta}$ that keeps $u|_{X_0}$ fixed.

More generally, we can deform u inside $\mathcal{M}_{\text{def}}^{T_\eta}$ by identifying and altering its restriction to the deformable parts of X simultaneously. This way, any $u\in\mathcal{M}_{\text{def}}^{T_\eta}$ of type $\{k_1,\cdots,k_l\}$ generates an $\mathbb{A}^l$ family of elements in $\mathcal{M}_{\text{def}}^{T_\eta}$.

A.3. **Global structure of the loci of invariant relative morphisms.** Before we move on to the next part, we shall give a complete description of all infinitesimal deformations of an element in $\mathcal{M}_{\text{def}}^{T_\eta}$. At the moment, we continue to concentrate on the case a_2 and $a_3<0$.

Let $u:X\to W[\mathbf{m}]$ be any such element and let $R\subset X$ be the divisor of the marked points of X. Unlike the case of ordinary stable morphisms, infinitesimal deformations of u may involve the smoothing of the nodal divisors of the target $W[\mathbf{m}]$. In the case studied, we are fortunate that the nodes $u^{-1}(D)\subset X$ will stay intact when u varies as T_η-invariant morphisms.

We now make this precise. We let $\hat S$ be the convex hull of the infinitesimal deformations of $[u]$ in $\mathcal{M}_{\text{def}}^{T_\eta}$; its quotient under a finite group (as a stack) is the formal completion of $\mathcal{M}_{\text{def}}^{T_\eta}$ at $[u]$. We let

$$\hat{\mathcal{U}}:\hat{\mathcal{X}}\to\hat{\mathcal{W}}\quad\text{and}\quad\hat{\mathcal{R}}\subset\hat{\mathcal{X}}$$

be the tautological family over $\hat S$ and its divisor of its marked points; its only closed fiber is $u:X\to W[\mathbf{m}]$ and $R\subset X$. We denote by $\hat{\mathcal{D}}\subset\hat{\mathcal{W}}$ the relative divisor and $\hat{\mathcal{L}}\subset\hat{\mathcal{D}}$ the flat family of three lines in $\hat{\mathcal{D}}$ associated to $L[\mathbf{m}]\subset D[\mathbf{m}]$. We then let $\hat{\mathcal{N}}$ be the closed substack of nodes of $\hat{\mathcal{X}}$. We say that a node $v\in X$ stays intact when u deforms in $\mathcal{M}_{\text{def}}^{T_\eta}$ if the stack $\hat{\mathcal{N}}$ is flat over $\hat S$ at v.

We have the following Lemma about the possible smoothings of nodes of X.

Lemma A.7. *Let $v\in X_{node}$ be a node in $u^{-1}(D)$. Then v stays intact when u deforms in $\mathcal{M}_{def}^{T_\eta}$.*

The key to the proof relies on the fact that the restrictions of the automorphisms induced by T_η on one irreducible component of X that contains v are infinite while on the other irreducible component are finite.

More precisely, since u is T_η-invariant, there are homomorphisms $h_1:T_\eta\to\operatorname{Aut}(X)$ and $h_2:T_\eta\to\operatorname{Aut}(W[\mathbf{m}])$ so that

$$\begin{array}{ccccc} X & \xrightarrow{u} & W[\mathbf{m}] & \longrightarrow & W \\ \downarrow{\scriptstyle h_1(\sigma)} & & \downarrow{\scriptstyle h_2(\sigma)} & & \downarrow{\scriptstyle\sigma} \\ X & \xrightarrow{u} & W[\mathbf{m}] & \longrightarrow & W \end{array}\qquad \forall\,\sigma\in T_\eta \tag{A.12}$$

is commutative. Now let $v \in u^{-1}(D_i)$ be a node of X that is mapped to D_i under u; let V_- be the irreducible component of X that contains v that is mapped onto C_i and let V_+ be the other irreducible component of X that contains v. Then $u(V_+)$ either is contained in $\Delta[m_i]$. Since $h_1(\mathrm{id}) = \mathrm{id}$ and that T_η is connected, $h_1(\sigma)(V_\pm) \subset V_\pm$. Hence $h_i(\sigma)$ are automorphisms of $V_\pm$ that fixed v. We let

$$T_\eta|_{V_\pm} \triangleq \{h_1(\sigma) \mid \sigma \in T_\eta\} \subset \mathrm{Aut}(V_\pm, v);$$

it is a subgroup.

Lemma A.8. *The group $T_\eta|_{V_-}$ is an infinite group while the group $T_\eta|_{V_+}$ is a finite group.*

Proof. The group $T_\eta|_{V_-}$ is infinite is obvious. Since $u(v) = p_i$ and $u(V_-) \subset W$, $u(V_-)$ must be the line C_i. Since the induced action on C_i is infinite, $T_\eta|_{V_-}$ must be infinite because u is T_η-invariant.

We now show that in case $T_\eta|_{V_+}$ is infinite, then the automorphism group of u is infinite, violating the stability requirement on u. For this, we will construct homomorphisms $h_1' : T_\eta \to \mathrm{Aut}(X)$ and $h_2' : T_\eta \to \mathrm{Aut}(W[\mathbf{m}]/W)$ that makes the diagram (A.12) commutative with h_i replaced by h_i' and the last vertical arrow replace by the identity[3].

For $\sigma \in T_\eta$, we let $h_1'(\sigma) : X \to X$ be $h_1(\sigma)$ when restricted to $u^{-1}(\Delta[m_1])$ and be the identity in its complement. The homomorphism h_2' is slightly tricky since $h_2(\sigma)$ does not commute with $W \equiv W$. In our case this does not pose any problem since the image $u(u^{-1}(\Delta[m_i]))$ is entirely contained in the fiber of $\Delta[m_i]$ over $p_i \in D_i$.

We let $\pi : \Delta[m_1] \to D_1$ be the projection and let $h_2|_{\pi^{-1}(p_1)}$ be the restriction of h_2 to the this fiber. We then extend $h_2|_{\pi^{-1}(p_i)}$ to a D_i-automorphism of $\Delta[m_1]$; namely, it is an equivariant automorphism of the pair $\Delta[m_1] \to D_1$ with T_η acting trivially on D_1. Lastly, we extend this action (on $\Delta[m_1]$) to $W[\mathbf{m}]$ by identity. Such $h_2' : T_\eta \to \mathrm{Aut}(W[\mathbf{m}]/W)$ and satisfies the required commutativity. This proves the Lemma. □

We have the following easy observation:

Lemma A.9. *Let V_- and V_+ be the two connected components of X as before. Then the node v stays intact when u deforms in $\mathcal{M}^{T_\eta}_{def}$.*

Proof. Suppose $\mathcal{N}$ is not flat over v, then v is smoothed at least of first order within $\hat{S}$. Since $\hat{\mathcal{X}}/\hat{S}$ is T_η-equivariant with T_η acting on $\hat{S}$ trivially, that v has been smoothed of first order and $T_\eta|_{V_-}$ is infinite forces $T_\eta|_{V_+}$ to be infinite as well. This violates the assumption that $T_\eta|_{V_+}$ is finite. This proves the Lemma. □

Because all nodes $u^{-1}(D) \subset X$ stay intact when u varies in $\mathcal{M}^{T_\eta}_{\mathrm{def}}$, the domain of the universal family $\hat{\mathcal{X}}$ over $\hat{S}$ will split into four parts: those that are mapped D_i under the composite of $\hat{\mathcal{U}}$ and the contraction $\hat{\mathcal{W}} \to W$, and the one that is mapped to W.

To begin with, we first divide X into connected components according to their images in $W[\mathbf{m}]$. We let $\Delta[m_i]$ be the chain of ruled varieties attached to D_i; for $i =\geq 1$, we let $X^{[i]} = u^{-1}(\Delta[m_i])$; we endow $X^{[i]}$ with the marked points $R^{[i]}$ that

[3]The automorphisms $\zeta \in \mathrm{Aut}(W[\mathbf{m}])$ that makes the right square commutative with $h_2(\sigma)$ replace by ζ and with σ replace by id are called relative automorphisms of $W[\mathbf{m}]/W$; the group of all such automorphisms is denoted by $\mathrm{Aut}(W[\mathbf{m}]/W)$.

is the union of $R\cap X^{[i]}$ with $u^{-1}(D_i)$; we let $X^{[0]}=u^{-1}(W)$ with marked points $u^{-1}(D)$. (In case $m_i=0$, we simply take $X^{[i]}=\emptyset$.) Because of the discussion so far in this subsection, the nodes $u^{-1}(D)$ of X all stay intact when u varies in $\hat S$. Therefore, there is a family of subcurves $\hat{\mathcal X}^{[i]}$ in $\hat{\mathcal X}$ that is flat over $\hat S$ and has $X^{[i]}$ as its only closed fiber. We let $\hat{\mathcal R}^{[i]}\subset\hat{\mathcal X}^{[i]}$ be the divisor of marked points extending $R^{[i]}\subset X^{[i]}$.

For each $i>0$, $u^{[i]}\triangleq u|_{X^{[i]}}$ is a relative stable morphism to Δ, relative to both D_- and D_+. Because of the condition (A.10), the family $\hat{\mathcal U}^{[i]}\triangleq\hat{\mathcal U}|_{\hat{\mathcal X}^{[i]}}$ is a family of relative stable morphisms in $\mathcal M^\bullet(\mathbb P^1,0,\infty)//\mathbb C^*$ — the moduli of relative stable morphisms to $\mathbb P^1$ relative to 0 and ∞ in $\mathbb P^1$ up to an additional equivalence: dilation by $\mathbb C^*$ on $\mathbb P^1$ fixing 0 and ∞. We let $\hat S^{[i]}$ be the convex hull to deformations of such $u^{[i]}$.

We next look at the part $X^{[0]}$; we will divide its connected components into four classes. The *class I* consists of those $V\subset X^{[0]}$ that satisfy $u|_V\cong\phi_{k,c}$ for some (k,c); these are the deformable components. We let l be the number of such components. Since each deformable component varies in a family $\mathbb A^1$, this class contributes an $\hat{\mathbb A}^l$ to $\hat S$.

The *class II* consists of those connected components V of $X^{[0]}$ so that

$$V=\mathbb P^1\sqcup\mathbb P^1,\quad u|_V:\xi_1\mapsto\xi_1^{k_V a_1}\in C_1,\ \text{and}\ \xi_2\mapsto\xi_2^{k_V|a_3|}\in C_3,\quad k_V\in\mathbb Z^+. \tag{A.13}$$

Such $V=\mathbb P^1\sqcup\mathbb P^1$ has two marked points $v_1=\{\xi_1=\infty\}$ and $v_2=\{\xi_2=\infty\}$. An easy calculation shows that there is a finite order T_η-equivariant deformation of $u|_V:\mathbb P^1\sqcup\mathbb P^1\to W$ satisfying $u(v_i)\in p_i$; the deformation space is $\operatorname{Spec}\mathbb C[t]/(t^{k_V|a_3|})$.

The *class III* consists of connected components $V\subset X^{[0]}$ that have $u^{-1}(p_0)\cap V=\{pt\}$ but are not in the *class I* and *II*. It is easy to see that they do not admit even first order deformations as T_η-equivariant maps; they are rigid.

The *class IV* consists of those $V\subset X^{[0]}$ so that $u^{-1}(p_0)\triangleq V_0$ is a curve. Since V is connected, V_0 is also connected. We endow V_0 with the marked points $R_0=\{\text{nodes of } V-\text{nodes of } V_0\}$. Then the space of T_η-equivariant deformations of $u|_V$ coincide with the deformations of the pointed curve (V_0,R_0). We let g_V be the arithmetic genus of V and let n_V be the number of nodes in V_0. The convex full $\hat{\mathcal M}_{g_V,n_V}$ of deformations of (V_0,R_0) contributes to $\hat S$.

Combined, we have proved

Lemma A.10. *The convex hull $\hat S$ of the infinitesimal deformations of $[u]$ in $\mathcal M_{def}^{T_\eta}$ is isomorphic to the formal completion of*

$$\prod_{i=1}^3\hat S^{[i]}\times\hat{\mathbb A}^l\times\Big(\prod_{V\in\ class\ II}\operatorname{Spec}\mathbb C[t]/(t^{k_V|a_3|})\Big)\times\Big(\prod_{V\in\ class\ IV}\hat{\mathcal M}_{g_V,n_V}\Big)$$

at its only closed point.

We next prove a structure theorem to the stack $\mathcal M_{\rm def}^{T_\eta}$; we shall show that each of its connected component is a trivial $\mathbb A^l$-bundle over its intersection with $\mathcal M_0$.

We let (u,X) be any element in $\mathcal M_{\rm def}^{T_\eta}$; let $Y_1,\cdots,Y_l\subset X$ be all its deformable parts so that $u|_{Y_i}\cong\phi_{k_i,c_i}$; we let $v_{i,1}$ and $v_{i,2}\in Y_i$ be the marked points so that $u(v_{i,j})=p_j$. Then according to the discussion in the previous Subsection, by varying $u|_{Y_i}$ using Φ_{k_i} we get a copy $\mathbb A^1$ in $\mathcal M_{\rm def}^{T_\eta}$; together they provide a copy $\mathbb A^l$ in

$\mathcal{M}_{\mathrm{def}}^{T_\eta}$. This is one of the fiber of the fiber bundle structure on $\mathcal{M}_{\mathrm{def}}^{T_\eta}$ we are about to construct.

To extend this $\mathbb{A}^l \subset \mathcal{M}_{\mathrm{def}}^{T_\eta}$ to nearby elements of $[u]$, we need to extend all Y_i in X to a flat family of subcurves.

Lemma A.11. *The deformation type function on $\mathcal{M}_{def}^{T_\eta}$ is locally constant.*

Proof. We pick a disk $0 \in S$ and a morphism $\psi : S \to \mathcal{M}_{\mathrm{def}}^{T_\eta}$ so that $\psi(0) = [u]$. The morphism ψ pulls back the tautological family on $\mathcal{M}_{\mathrm{def}}^{T_\eta}$ to a family $\mathcal{U} : \mathcal{X} \to \mathcal{Z}$ over S. The central fiber $\mathcal{X}_0 \equiv X$ and thus contains Y_i. We let $\mathcal{N} \subset \mathcal{X}$ be the subscheme of the nodes of all fibers of $\mathcal{X}/S$. Since $v_{i,j}$ is either a marked point or a node of X, $v_{i,j} \in \mathcal{N} \cup \mathcal{R}$. Let $\mathcal{P}_{i,j}$ be the connected component of $\mathcal{N} \cup \mathcal{R}$ that contains $v_{i,j}$. We claim that $\mathcal{P}_{i,j}$ is a section of $\mathcal{N} \cup \mathcal{R} \to S$. First, $\mathcal{P}_{i,j}$ is flat over S at $v_{i,j}$. This is true in case $v_{i,j}$ is a marked point since $\mathcal{R}$ is flat over S by definition; in case $v_{i,j}$ is a node it is true because of Lemma A.9. Therefore, $\mathcal{P}_{i,j}$ dominates over S. Then because $\mathcal{N} \cup \mathcal{R}$ is proper and unramified over S, dominating over S guarantees that $\mathcal{P}_{i,j}$ is finite and étale over S. But then since S is a disk, $\mathcal{P}_{i,j}$ must be isomorphic to S via the projection.

We now pick the desired family of curves $\mathcal{Y}_i$. In case $\mathcal{P}_{i,j}$ is one of the section of the marked points of $\mathcal{X}/S$, we do nothing; otherwise, we resolve the singularity of the fibers of $\mathcal{X}$ along $\mathcal{P}_{i,j}$. As a result, we obtain a flat family of subcurves $\mathcal{Y}_i \subset \mathcal{X}$ that contains Y_i as its central fiber. We let $\mathcal{U}_i : \mathcal{Y}_i \to \mathcal{W}$ be the restriction of $\mathcal{U}$ to $\mathcal{Y}_i$. Because $\mathcal{U}_i(Y_i) \subset W \subset W[\mathbf{m}]$, $\mathcal{U}_i(\mathcal{Y}_i) \subset W \times S \subset \mathcal{W}$ as well.

Since $\mathcal{U} : \mathcal{X} \to \mathcal{W}$ is a family of T_η-equivariant relative stable maps, $\mathcal{U}_i : \mathcal{Y}_i \to W$ is also a family of T_η-equivariant stable morphisms. Then because $\mathcal{U}_i|_{Y_i}$ is isomorphic to ϕ_{k_i,c_i}, each member of $\mathcal{U}_i$ must be an $\phi_{k_i,c}$ for some $c \in \mathbb{C}$. This proves that the deformation type of $\mathcal{U}|_{\mathcal{X}_s}$ contains that of $\mathcal{U}|_{\mathcal{X}_0}$ as a subset. Because this holds true with 0 and s exchanged, it shows that the deformation type function stay constant over S.

Finally, because any two elements in the same connected component of $\mathcal{M}_{\mathrm{def}}^{T_\eta}$ can be connected by a chain of analytic disks, the deformation type function does take same values on such component. This proves the lemma. □

We are now ready to exhibit a fiber bundle structure of any connected component of $\mathcal{M}_{\mathrm{def}}^{T_\eta}$.

Let $\mathcal{Q} \subset \mathcal{M}_{\mathrm{def}}^{T_\eta}$ be any connected component. According to the previous subsection, all elements in $\mathcal{Q}$ are of the same deformation types, say $\{k_1, \cdots, k_l\}$. In case $\mathcal{Q}$ is not entirely contained in $\mathcal{M}_0$, $l > 0$. To get the fiber structure, we need to take a finite (branched) cover of $\mathcal{Q}$, which we now construct.

Definition A.12. *We define the groupoid $\bar{\mathcal{Q}}$ over $\mathcal{Q}$ as follows. For any scheme S over $\mathcal{Q}$, we let $\bar{\mathcal{Q}}(S)$ be the collection of data $\{(\mathcal{U}, \mathcal{X}, \mathcal{W}), \rho_i, \mathcal{Y}_i, \pi_i \mid i = 1 \cdots, l\}$ of which*

(1) $\mathcal{U} : \mathcal{X} \to \mathcal{W}$ *is an object*[4] *in* $\mathcal{Q}(S)$
(2) ρ_i *are morphisms from* S *to* $\mathbb{A}^1_i$, $\mathbb{A}^1_i \cong \mathbb{A}^1$*;*
(3) $\mathcal{Y}_i$ *are flat families of subcurves in* $\mathcal{X}$ *over* S *with all marked points discarded;*
(4) $\pi_i : \mathcal{Y}_i \to \rho_i^* \mathfrak{Y}_{k_i}$ *an isomorphism over* S

[4]Here we consider $\mathcal{Q}$ as a groupoid and $\mathcal{Q}(S)$ is the collection of objects over S.

that satisfies

$$\mathcal{U}|_{\mathcal{Y}_i} \equiv \rho_i^* \Phi_{k_i} \circ \pi_i : \mathcal{Y}_i \longrightarrow W.$$

An arrow from $\{(\mathcal{U}, \mathcal{X}, \mathcal{W}), \rho_i, \mathcal{Y}_i, \pi_i\}$ *to* $\{(\mathcal{U}', \mathcal{X}'.\mathcal{W}'), \rho_i', \mathcal{Y}_i', \pi_i'\}$ *consists of an isomorphism* $h_1 : \mathcal{X} \to \mathcal{X}'$ *and an isomorphism* $h_2 : \mathcal{W} \to \mathcal{W}'$ *relative to* W *so that under these isomorphisms* $\mathcal{Y}_i = \mathcal{Y}_i'$, $\rho_i = \rho_i'$, $\pi_i = \pi_i'$ *(for all* i*) and* $\mathcal{U} = \mathcal{U}'$.

Here we use $\mathbb{A}_i^1$ to denote the target of ρ_i, which is $\mathbb{A}^1$ though, since later we need to distinguish them for different i.

Proposition A.13. *The groupoid* $\bar{\mathcal{Q}}$ *is a DM-stack; it is finite and étale over* $\mathcal{Q}$. *The morphisms* ρ_i *in each object in* $\bar{\mathcal{Q}}$ *glue to a morphism* $\bar{\rho}_i : \bar{\mathcal{Q}} \to \mathbb{A}_i^1$. *Let* $\bar{\mathcal{Q}}_0 = (\bar{\rho}_1, \cdots, \bar{\rho}_l)^{-1}(0)$. *Then there is a canonical projection* $\pi : \bar{\mathcal{Q}} \to \bar{\mathcal{Q}}_0$ *making it a* $\mathbb{C}^l$*-vector bundle over* $\bar{\mathcal{Q}}_0$. *Finally, the morphism*

$$(\pi, (\bar{\rho}_1, \cdots, \bar{\rho}_l)) : \bar{\mathcal{Q}} \longrightarrow \bar{\mathcal{Q}}_0 \times \mathbb{A}^l$$

is an isomorphism of DM-stacks.

Proof. The proof is straightforward, following Lemmas A.10 and A.11, and will be omitted. □

A.4. **The obstruction sheaves.** In this subsection, we will investigate the obstruction sheaf to deforming a $[u]$ in $\mathcal{M}_{\mathrm{def}}^{T_\eta}$ for the case a_2, $a_3 < 0$; we will follow the convention introduced in Subsection A.3.

According to [18], the obstruction sheaf $\mathcal{T}^2$ over $\hat{S}$ of the obstruction theory of $\mathcal{M}_{\mathrm{def}}^{T_\eta}$ fits into the long exact sequences

$$\longrightarrow \mathcal{E}xt^1_{\hat{\mathcal{X}}/\hat{S}}(\Omega_{\hat{\mathcal{X}}/\hat{S}}(\hat{\mathcal{R}}), \mathcal{O}_{\hat{\mathcal{X}}})_{T_\eta} \xrightarrow{\beta} \mathcal{A}^1_{T_\eta} \xrightarrow{\delta} \tilde{\mathcal{T}}^2_{T_\eta} \longrightarrow 0; \tag{A.14}$$

$$\longrightarrow \mathcal{B}^0_{T_\eta} \longrightarrow R^1\pi_*\big(\mathcal{U}^*\Omega_{\hat{\mathcal{W}}^\dagger/\hat{S}}(\log \hat{\mathcal{D}})^\vee\big)_{T_\eta} \xrightarrow{\alpha} \mathcal{A}^1_{T_\eta} \longrightarrow \mathcal{B}^1_{T_\eta} \longrightarrow 0 \tag{A.15}$$

and

$$\longrightarrow \tilde{\mathcal{T}}^1_{T_\eta} \longrightarrow \mathcal{H}_{T_\eta} \longrightarrow \mathcal{T}^2_{T_\eta} \longrightarrow \tilde{\mathcal{T}}^2_{T_\eta} \longrightarrow 0. \tag{A.16}$$

Within these sequences, $\mathcal{B}^i = \oplus_{j=1}^3 \mathcal{B}^i_j$; each summand $\mathcal{B}^i_j$ is a sheaf that associates to the smoothing of the nodes of the fibers of $\hat{\mathcal{X}}$ that are mapped under $\hat{\mathcal{U}}$ to D_i and the singular loci of $\Delta[m_i]$; the $\hat{\mathcal{W}}^\dagger$ is the scheme $\hat{\mathcal{W}}$ with the log structure defined in [18] and $\Omega_{\hat{\mathcal{W}}^\dagger}$ is the sheaf of log differentials. In our case, $\hat{\mathcal{U}}^*\Omega_{\hat{\mathcal{W}}^\dagger/\hat{S}}(\hat{\mathcal{D}}) = \tilde{\mathcal{U}}^*\Omega_W(\log D)$, where $\tilde{\mathcal{U}} : \hat{\mathcal{X}} \to W$ is the obvious induced morphism.

Without taking the T_η-invariant part, the top two exact sequences define the obstruction sheaf $\tilde{\mathcal{T}}^2$ to deforming $[u]$ in $\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(W^{\mathrm{rel}})$ — the moduli of relative stable morphisms without requiring $u(R) \subset L$. Taking the invariant part and adding the last exact sequence defines the obstruction sheaf $\mathcal{T}^2_{T_\eta}$. The sheaf $\mathcal{H}$ is the pull back of the normal line bundle to $\mathcal{L} \subset \mathcal{D}$. For the η we are interested, $\mathcal{H}_{T_\eta} = 0$; hence the last exact sequence reduces to $\mathcal{T}^2_{T_\eta} \equiv \tilde{\mathcal{T}}^2_{T_\eta}$.

In the following we shall show that the l families $\hat{\mathcal{Y}}_i \subset \hat{\mathcal{X}}$ of deformable parts of $(\hat{\mathcal{U}}, \hat{\mathcal{X}})$ each contributes to a weight zero trivial quotient sheaf of $\mathcal{T}^2_{T_\eta}$.

We begin with the sheaf

$$R^1\pi_{k*}\big(\Phi_k^*\Omega_W(\log D)^\vee\big), \tag{A.17}$$

where $\Phi_{k_i} : \mathfrak{Y} \to W$ is the family constructed before and $\pi_k : \mathfrak{Y} \to \mathbb{A}^1$ is the projection. We let Z_{12} (resp. Z_{31}) be the T-invariant divisor of W that contains

C_1 and C_2 (resp. C_1 and C_3); let $\pi_{12}: W \to Z_{12}$, $\pi_{31}: W \to Z_{31}$ and $\pi_1: W \to C_1$ be the obvious projections. By definition, $\Omega_W(\log D)^\vee|_{Z_{12}}$ has a quotient sheaf $\mathcal{N}_{Z_{12}/W} = \pi_1^*\mathcal{O}(-1)$. However, the projection π_{31} gives us a subbundle

$$\pi_{31}^*\Omega_{Z_{31}}(\log L_1)^\vee|_{Z_{12}} \longrightarrow \Omega_W(\log D)^\vee|_{Z_{12}}.$$

Note that L_1 is a divisor in Z_{31}. Because

$$\Omega_{Z_{31}}(\log L_1)^\vee|_{C_1} \cong \mathcal{N}_{C_1/Z_{31}} \oplus \Omega_{C_1}(\log p_1),$$

we obtain

$$\pi_1^*\mathcal{N}_{C_1/Z_{31}}|_{C_1} \longrightarrow \Omega_W(\log D)|_{Z_{12}} \longrightarrow \mathcal{N}_{Z_{12}/W} \equiv \pi_1^*\mathcal{N}_{C_1/Z_{31}}|_{C_1}.$$

Because their compositions is the identity homomorphism, $\Omega_W(\log D)^\vee|_{Y_{12}}$ has a direct summand $\pi_1^*\mathcal{N}_{C_1/Z_{31}}$. Consequently, $\Phi_k^*\Omega_W(\log D)$ has a direct summand $\Phi_k^*(\pi_1^*\mathcal{N}_{C_1/Z_{31}})$.

Because of our choice, the weight of dz_i is a_i; the weight of $T_0^\vee\mathfrak{Y}_c$ at 0 is $1/k_i$ and the weight of $\Phi_k^*(\pi_1^*\mathcal{N}_{C_1/Z_{31}})$ at $0 \times \mathbb{A}^1 \subset \mathfrak{Y}$ is $-a_3$. Hence, the sheaf A.17 splits to line bundles of weights

$$-a_3 - a_1 + \frac{1}{k}, -a_3 - a_1 + \frac{2}{k}, \cdots\cdots, -a_3 - \frac{1}{k}.$$

Since all a_i are integers, and $a_3 \le -1$ and $-a_3 - a_1 = a_2 \le -1$, within the above list there is exactly one that is zero. Hence

$$R^1\pi_{k*}\big(\Phi_k^*\Omega_W(\log D)^\vee\big)_{T_\eta} \cong \mathcal{O}_{\mathbb{A}^1}. \tag{A.18}$$

We now let $\rho_i: \hat{S} \to \mathbb{A}_i^1$ be so that $\hat{\mathcal{U}}|_{\hat{\mathcal{Y}}_i} \cong \rho_i^*\Phi_{k_i}$. Since $\hat{\mathcal{Y}}_i \subset \hat{\mathcal{X}}$ is a flat family of subcurves,

$$R^1\pi_*\big(\hat{\mathcal{U}}^*\Omega_{\hat{\mathcal{W}}^\dagger/\hat{S}}(\log\hat{\mathcal{D}})^\vee\big)_{T_\eta} \longrightarrow R^1\pi_*\big(\hat{\mathcal{U}}^*\Omega_{\hat{\mathcal{W}}^\dagger/\hat{S}}(\log\hat{\mathcal{D}})^\vee|_{\hat{\mathcal{Y}}_i}\big)_{T_\eta}$$

is surjective; but the last term is isomorphic to the pull back ρ_i^* of (A.18); hence we obtain a quotient sheaf

$$\varphi_i : R^1\pi_*\big(\hat{\mathcal{U}}^*\Omega_{\hat{\mathcal{W}}^\dagger/\hat{S}}(\log\hat{\mathcal{D}})^\vee\big)_{T_\eta} \longrightarrow \rho_i^*\mathcal{O}_{\mathbb{A}_i^1}. \tag{A.19}$$

In the next part, we will show that this homomorphism canonically lifts to surjective

$$\hat{\varphi}_i : \mathcal{T}_{T_\eta}^2 \longrightarrow \rho_i^*\mathcal{O}_{\mathbb{A}_i^1}. \tag{A.20}$$

The default proof is to follow the construction of the sheaves and the exact sequences in (A.14-A.16); once it is done, the required vanishing will follow immediately. However, to follow this strategy, we need to set up the notation as in [18] that itself requires a lot of efforts. Instead, we will utilize the decomposition of $\hat{S}$ to give a more conceptual argument; bypassing some straightforward but tedious checking.

We let $\hat{\mathcal{X}}^{[i]} \subset \hat{\mathcal{X}}$ be the families of subcurves derived in the previous Subsection. Since $\hat{\mathcal{U}}^{[0]}: \hat{\mathcal{X}}^{[0]} \to W$ is a family of T_η-equivariant relative stable morphisms, and since $\hat{\mathcal{U}}^{[i]}$ is a family of T_η-equivariant relative stable morphisms to Δ relative to D_- and D_+, modulo an additional equivalence induced by the $\mathbb{C}^*$ action on Δ, the obstruction sheaves $\mathcal{T}^{[i],2}$ over $\hat{S}$ to deforming T_η-equivariant maps $\hat{\mathcal{U}}^{[i]}$ fit into similar exact sequences

$$\longrightarrow \mathcal{E}xt^1_{\hat{\mathcal{X}}^{[i]}/\hat{S}}(\Omega_{\hat{\mathcal{X}}^{[i]}/\hat{S}}(\hat{\mathcal{R}}^{[i]}), \mathcal{O}_{\hat{\mathcal{X}}^{[i]}})_{T_\eta} \xrightarrow{\beta^{[i]}} \mathcal{A}^{[i],1}_{T_\eta} \xrightarrow{\delta^{[i]}} \tilde{\mathcal{T}}^{[i],2}_{T_\eta} \longrightarrow 0 \tag{A.21}$$

and

$$\longrightarrow \mathcal{B}^{[i],0}_{T_\eta} \longrightarrow R^1\pi_*\big(\hat{\mathcal{U}}^{[i]*}\Omega_{\hat{\mathcal{W}}^\dagger/\hat{S}}(\log\hat{\mathcal{D}})^\vee\big)_{T_\eta} \xrightarrow{\alpha^{[i]}} \mathcal{A}^{[i],1}_{T_\eta} \longrightarrow \mathcal{B}^{[i],1}_{T_\eta} \longrightarrow 0. \tag{A.22}$$

Now let $\hat{\mathcal{N}}_{\rm sp} \subset \hat{\mathcal{U}}^{-1}(D_i \times \hat{S})$ be any section of nodes of $\hat{\mathcal{X}}$ that separates $\hat{\mathcal{X}}^{[0]}$ and $\hat{\mathcal{X}}^{[i]}$. Because the induced T_η-automorphisms on the connected component of $\hat{\mathcal{X}}^{[0]}$ adjacent to $\hat{\mathcal{N}}_{\rm sp}$ is infinite and on $\hat{\mathcal{X}}^{[i]}$ is finite,

$$\bigoplus_{i=0}^{3} \mathcal{E}xt^1_{\hat{\mathcal{X}}^{[i]}/\hat{S}}(\Omega_{\hat{\mathcal{X}}^{[i]}/\hat{S}}(\hat{\mathcal{R}}^{[i]}), \mathcal{O}_{\hat{\mathcal{X}}^{[i]}})_{T_\eta} = \mathcal{E}xt^1_{\hat{\mathcal{X}}/\hat{S}}(\Omega_{\hat{\mathcal{X}}/\hat{S}}(\hat{\mathcal{R}}), \mathcal{O}_{\hat{\mathcal{X}}})_{T_\eta}.$$

For the similar reason, because the tangent bundle $T_{p_i}W$ has no weight 0 non-trivial T_η-invariant subspaces,

$$\bigoplus_{i=0}^{3} R^1\pi_*\big(\hat{\mathcal{U}}^{[i]*}\Omega_{\hat{\mathcal{W}}^\dagger/\hat{S}}(\log\hat{\mathcal{D}})^\vee\big)_{T_\eta} = R^1\pi_*\big(\hat{\mathcal{U}}^{*}\Omega_{\hat{\mathcal{W}}^\dagger/\hat{S}}(\log\hat{\mathcal{D}})^\vee\big)_{T_\eta}. \tag{A.23}$$

Further, if we follow the definition of the the sheaves $\mathcal{B}^i$ and $\mathcal{A}^i$, we can prove that

$$\bigoplus_{i=0}^{3} \mathcal{A}^{[i],j}_{T_\eta} = \mathcal{A}^j_{T_\eta} \quad \text{and} \quad \bigoplus_{i=0}^{3} \mathcal{B}^{[i],j}_{T_\eta} = \mathcal{B}^j_{T_\eta}; \tag{A.24}$$

that under these isomorphisms,

$$\bigoplus_{i=0}^{3} \alpha^{[i]} = \alpha, \quad \bigoplus_{i=0}^{3} \beta^{[i]} = \beta \quad \text{and} \quad \bigoplus_{i=0}^{3} \delta^{[i]} = \delta; \tag{A.25}$$

and

$$\bigoplus_{i=0}^{3} \mathcal{T}^{[i],2}_{T_\eta} = \mathcal{T}^2_{T_\eta}. \tag{A.26}$$

The exact sequences (A.14) and (A.15) become the direct sums of the exact sequences (A.21) and (A.22).

Now we come back to the weight zero quotient φ_i in (A.19). By its construction, φ_i is merely the canonical quotient homomorphism

$$R^1\pi_*\big(\hat{\mathcal{U}}^{[0]*}\Omega_{\hat{\mathcal{W}}^\dagger/\hat{S}}(\log\hat{\mathcal{D}})^\vee\big)_{T_\eta} \longrightarrow R^1\pi_*\big(\hat{\mathcal{U}}^{[0]*}\Omega_{\hat{\mathcal{W}}^\dagger/\hat{S}}(\log\hat{\mathcal{D}})^\vee|_{\mathcal{Y}_i}\big)_{T_\eta} = \rho_i^*\mathcal{O}_{\mathbb{A}^1_i}. \tag{A.27}$$

under the isomorphism (A.23). Because of (A.26), to lift φ_i to $\hat{\varphi}_i$ we only need to lift (A.27) to $\mathcal{T}^{[0],2}_{T_\eta} \to \rho_i^*\mathcal{O}_{\mathbb{A}^1_i}$.

For this, we need to look at the exact sequence (A.22) for $\hat{\mathcal{X}}^{[0]}$. Since $\hat{\mathcal{U}}^{[0]}$ is a relative stable map to (W, D) — namely no Δ's has been attached to W — the sheaf $\mathcal{B}^{[0],j} = 0$. Therefore the sequence (A.22) reduces to $\alpha^{[0]} = \mathrm{id}$. On the other hand, $\mathcal{T}^{[0],2}_{T_\eta}$ is the obstruction sheaf on $\hat{S}$ to deformations of $\hat{\mathcal{U}}^{[0]}$; since deforming one connected component is independent of the geometry of the other connected components, the exact sequence (A.21) decomposes into direct sum of individual exact sequences

$$\longrightarrow \mathcal{E}xt^1_{\hat{\mathcal{Y}}/\hat{S}}\big(\Omega_{\hat{\mathcal{Y}}/\hat{S}}(\hat{\mathcal{R}}^{[0]}), \mathcal{O}_{\hat{\mathcal{Y}}}\big)_{T_\eta} \xrightarrow{\beta^{[\hat{\mathcal{Y}}]}} R^1\pi_*\big(\hat{\mathcal{U}}^{[0]*}\Omega_{\hat{\mathcal{W}}^\dagger/\hat{S}}(\log\hat{\mathcal{D}})^\vee|_{\hat{\mathcal{Y}}}\big)_{T_\eta} \xrightarrow{\delta^{[\hat{\mathcal{Y}}]}} \mathcal{T}^{[\hat{\mathcal{Y}}],2}_{T_\eta} \longrightarrow 0, \tag{A.28}$$

one exact sequence for each connected component $\hat{\mathcal{Y}} \subset \hat{\mathcal{X}}^{[0]}$.

For $\hat{\mathcal{Y}}_i$, since it is smooth, it has expected dimension zero and has actual dimension one, the obstruction sheaf $\mathcal{T}^{[\hat{\mathcal{Y}}_i],2}_{T_\eta}$ must be a rank one locally free sheaf on $\hat{S}$.

Then because the middle term in (A.28) is $\rho_i^*\mathcal{O}_{\mathbb{A}^1_i}$, which is a rank one locally free sheaf, the arrow $\delta^{[\hat{\mathcal{Y}}]}$ must be an isomorphism while $\beta^{[\hat{\mathcal{Y}}]}=0$. Hence φ_i lifts to

$$\mathcal{T}^{[0],2}_{T_\eta}\equiv\bigoplus_{\hat{\mathcal{Y}}\subset\hat{\mathcal{X}}^{[0]}}\mathcal{T}^{[\hat{\mathcal{Y}}],2}_{T_\eta}\longrightarrow\mathcal{T}^{[\hat{\mathcal{Y}}_i],2}_{T_\eta}\equiv\rho_i^*\mathcal{O}_{\mathbb{A}^1_i},$$

and lifts to $\hat{\varphi}_i:\mathcal{T}^2_{T_\eta}\to\rho_i^*\mathcal{O}_{\mathbb{A}^1_i}$, thanks to (A.26).

The above lift works over the stack $\bar{\mathcal{Q}}$. We let $\mathcal{U}:\mathcal{X}\to\mathcal{W}$ be the tautological family over $\bar{\mathcal{Q}}$; we let $\mathcal{Y}_i$ and $\rho_i:\bar{\mathcal{Q}}\to\mathbb{A}^1_i$ be the l families of subcurves and morphisms given by Definition A.12 and Proposition A.13. Let $\mathcal{T}^2_{T_\eta}$ be the obstruction sheaf over $\bar{\mathcal{Q}}$. Then according to the discussion above, the quotient homomorphism

$$R^1\pi_*\big(\mathcal{U}^*\Omega_{\mathcal{W}^\dagger/\bar{\mathcal{Q}}}(\log\mathcal{D})^\vee\big)_{T_\eta}\longrightarrow\bigoplus_{i=1}^{l}R^1\pi_*\big(\mathcal{U}^*\Omega_{\mathcal{W}^\dagger/\bar{\mathcal{Q}}}(\log\mathcal{D})^\vee|_{\mathcal{Y}_i}\big)_{T_\eta}=\bigoplus_{i=1}^{l}\rho_i^*\mathcal{O}_{\mathbb{A}^1_i}$$

lifts to a quotient homomorphism

$$\mathcal{T}^2_{T_\eta}\longrightarrow\bigoplus_{i=1}^{l}\rho_i^*\mathcal{O}_{\mathbb{A}^1_i}. \tag{A.29}$$

In the remainder part of this Subsection, we shall apply the knowledge gained to investigate the obstruction theory to deforming $[u]$ in $\hat{S}$; we shall prove that the obstruction classes to deforming $[u]$ lies in the kernel of

$$\mathcal{T}^2_{0,T_\eta}:=\ker\{\bigoplus_{i=1}^{l}\hat{\varphi}_i:\mathcal{T}^2_{T_\eta}\longrightarrow\bigoplus_{i=1}^{l}\rho_i^*\mathcal{O}_{\mathbb{A}^1_i}\}.$$

For this purpose, a quick review of the set up of the obstruction theory is in order.

We say the vector space $T^2_{T_\eta}\triangleq\mathcal{T}^2_{T_\eta}\otimes_{\mathcal{O}_{\hat{S}}}\mathbf{k}([u])$ is the obstruction space to deforming $[u]$ in $\hat{S}$ if the following holds. Let $(A,\mathfrak{m},I)$ be a triple of an Artinian ring, its maximal ideal and an ideal of A so that $\mathfrak{m}\cdot I=0$; let $\psi_{A/I}:\operatorname{Spec}A/I\to\hat{S}$ be a morphism containing $[u]$ in its image. Then there is a *canonical* obstruction class

$$\operatorname{ob}(A,I,\psi_{A/I})\in T^2_{T_\eta}\otimes_{\mathbb{C}}I$$

whose vanishing is the necessarily and sufficient condition for extending $\psi_{A/I}$ to a $\psi_A:\operatorname{Spec}A\to\hat{S}$. The assignment

$$(A,I,\psi_{A/I})\mapsto\operatorname{ob}(A,I,\psi_{A/I})$$

is canonical in that it satisfies the obvious base change property. (See [18] for details.)

We let $T^2_{0,T_\eta}=\mathcal{T}^2_{0,T_\eta}\otimes_{\mathcal{O}_{\hat{S}}}\mathbf{k}([u])$. What we will show is that

Proposition A.14. *The obstruction class $ob(A,I,\psi_{A/I})$ lies in the subspace $I\otimes_{\mathbb{C}}T^2_{0,T_\eta}\subset T^2_{T_\eta}\otimes_{\mathbb{C}}I$.*

Proof. Let $(A,I,\psi_{A/I})$ be as before; let $\mathcal{U}_{A/I}:\mathcal{X}_{A/I}\to\mathcal{W}_{A/I}$ over $\operatorname{Spec}A/I$ be the pull back of the universal family $\hat{\mathcal{U}}$ over $\hat{S}$. The decomposition $\hat{\mathcal{X}}=\cup\hat{\mathcal{X}}^{[i]}$ divides $\mathcal{X}_{A/I}$ into four parts $\mathcal{X}^{[i]}_{A/I}$. Let $\mathcal{U}^{[i]}_{A/I}=\mathcal{U}_{A/I}|_{\mathcal{X}^{[i]}_{A/I}}$. First, in case $(\mathcal{U}_{A/I},\mathcal{X}_{A/I})$ extends to $(\mathcal{U}_A,\mathcal{X}_A)$, the decomposition $\mathcal{X}_{A/I}=\cup\mathcal{X}^{[i]}_{A/I}$ remains valid; hence extending $\mathcal{X}_{A/I}$ to $\mathcal{X}_A$ is equivalent to extending each $\mathcal{X}^{[i]}_{A/I}$ to $\mathcal{X}^{[i]}_A$. Secondly, if we can extend $\mathcal{U}^{[i]}_{A/I}$ to T_η-equivariant $\mathcal{U}^{[i]}_A$, because $\mathcal{U}^{[i]}$ has to map its marked points either to $\{p_1,p_2,p_3\}$ or to the relative divisors of the target of $\hat{\mathcal{W}}_A$, $\mathcal{U}^{[i]}_A$ glue together to a map $\mathcal{U}_A:\mathcal{X}_A\to\mathcal{W}_A$, after gluing $\mathcal{X}_A^{[i]}$ accordingly. Therefore, extending $\mathcal{U}_{A/I}$ to T_η-equivariant $\mathcal{U}_A$ over $\operatorname{Spec} A$ is equivalent to extending $\mathcal{U}_{A/I}^{[i]}$ to T_η-equivariant $\mathcal{U}_A^{[i]}$ for each i. Hence the obstruction class

$$\operatorname{ob}(A,I,\mathcal{U}_{A/I})=\sum_{i=0}^{3}\operatorname{ob}(A,I,\mathcal{U}_{A/I}^{[i]})\in\bigoplus_{i=0}^{3}\mathcal{T}_{T_\eta}^{[i]}\otimes_{\mathcal{O}_{\hat{S}}} I.$$

To complete the proof, we need to show that $\operatorname{ob}(A,I,\mathcal{U}_{A/I}^{[0]})$ lies in the kernel of

$$\mathcal{T}_{T_\eta}^{[0]}\otimes_{\mathcal{O}_{\hat{S}}} I\longrightarrow\bigoplus_{i=1}^{l}\rho_i^*\mathcal{O}_{\mathbb{A}^1_i}\otimes_{\mathcal{O}_{\hat{S}}} I. \tag{A.30}$$

But this is obvious. For extending $\mathcal{U}_{A/I}^{[0]}$ is equivalent to extending each of its connected components to over $\operatorname{Spec} A$; to each connected component $\mathcal{Y}_{A/I}$ of $\mathcal{X}_{A/I}^{[0]}$ the obstruction class $\operatorname{ob}(A,I,\mathcal{U}_{A/I}|_{\mathcal{Y}_{A/I}})$ to extending it and the map $\mathcal{U}_{A/I}|_{\mathcal{Y}_{A/I}}$ to over $\operatorname{Spec} A$ lies in $\mathcal{T}_{T_\eta}^{[\hat{\mathcal{Y}}],2}\otimes_{\mathcal{O}_{\hat{S}}} I$. Here $\hat{\mathcal{Y}}\subset\hat{\mathcal{X}}$ is the connected component that contains $\mathcal{Y}_{A/I}$. For the components $\mathcal{Y}_{A/I}$ that contains one of the deformable parts $Y_1,\cdots,Y_l$, the obstruction class $\operatorname{ob}(A,I,\mathcal{Y}_{A/I})=0$ since each $u|_{Y_i}$ varies within a smooth family. Because $\mathcal{T}_{T_\eta}^{[0],2}$ is the direct sum of $\mathcal{T}_{T_\eta}^{[\hat{\mathcal{Y}}],2}$ for all connected components $\hat{\mathcal{Y}}$ of $\hat{\mathcal{X}}^{[0]}$, the obstruction class $\operatorname{ob}(A,I,\mathcal{U}_{A/I}^{[0]})$ does lie in the kernel of (A.30). This proves the Proposition. □

A.5. **The case for $\eta=(1,-1,0)$.** We now investigate the structures of maps $[u]\in\mathcal{M}_{\text{def}}^{T_\eta}$ in case $\eta=(1,-1,0)$. Let (u,X) be any such map, let R be the marked points and let $\tilde{u}$ be the contraction $X\to W$. Because $a_3=0$, $\tilde{u}(X)$ intersects D_1 at p_1; intersects D_3 at p_3 while intersects D_2 can be any point in L_2. Thus being T_η-equivariant forces $\tilde{u}(X)$ to be a finite union of a subset of C_1, C_2, C_3 and the lifts of the sets $\{z_1z_2=c,z_3=0\}\subset(\mathbb{P}^1)^3$.

In case all irreducible components are mapped to $\cup C_i$ under $\tilde{u}$, $[u]\in\mathcal{M}_0$. For those that are not in $\mathcal{M}_0$, there bound to be some $Y\subset X$ so that $\tilde{u}(Y)$ is the lifts of $\{z_1z_2=c,z_3=0\}$. Such $u|_Y$ are realized by the morphism $\phi_{k,c}:\mathbb{P}^1\to W$ that are the lifts of

$$\xi\mapsto(c^k\xi^k,\xi^{-k},0)\in(\mathbb{P}^1)^3. \tag{A.31}$$

When c specializes to 0, the map $\phi_{k,c}$ specializes to $\phi_{k,0}:\mathbb{P}^1\sqcup\mathbb{P}^1\to W$ that is the lift of $\xi_1\mapsto(\xi_1^k,0,0)$ and $\xi_2\mapsto(0,\xi_2^{-k},0)$. Indeed, there is a family $\mathfrak{Y}\to\mathbb{A}^1$ and a morphism $\Phi_k:\mathfrak{Y}\to W$ so that its fiber over $c\in\mathbb{A}^1$ is the $\phi_{k,c}$ defined; also this is a complete list T_η-equivariant deformations of $\phi_{k,c}$. Since the argument is exactly the same as in the case studied, we shall not repeat it here.

Here comes the main difference between this and the case studied earlier. In the previous case, $\operatorname{Im}\phi_{k,c}\cap D_i=p_i$ for both $i=1$ and 2; hence we can deform each $u|_Y\cong\phi_{k,c}$ to produce an $\mathbb{A}^1$ family in $\mathcal{M}_{\text{def}}^{T_\eta}$. In the case under consideration, though $\operatorname{Im}\phi_{k,c}\cap D_1=p_1$, if we fix an embedding $\mathbb{A}^1\subset L_2$ so that $0\in\mathbb{A}^1$ is the $p_2\in L_2$, then $\operatorname{Im}\phi_{k,c}\cap D_2=c^k\in L_2$. In other words, if we deform $u|_F\cong\phi_{k,c}$, we need to move the connected component of $X^{[2]}$ that is connected to Y.

This leads to the following definition.

Definition A.15. *We say that a connected component $Y \subset X^{[0]}$ is subordinated to a connected component $E \subset X^{[2]}$ if $Y \cap E \neq \emptyset$; we say a connected component $E \subset X^{[2]}$ is deformable if every connected component of $X^{[0]}$ that is subordinate to E is of the form $\phi_{k,c}$ for some pair (k, c).*

We say u has deformation type l if it has exactly l deformable connected components in $X^{[2]}$. The deformation types define a function on $\mathcal{M}_{\mathrm{def}}^{T_\eta}$.

Lemma A.16. *The deformation type function is locally constant on $\mathcal{M}_{def}^{T_\eta}$.*

Proof. The proof is parallel to the case studied, and will be omitted. □

As in the previous case, any $[u] \in \mathcal{M}_{\mathrm{def}}^{T_\eta}$ of deformation type l generates an $\mathbb{A}^l$ in $\mathcal{M}_{\mathrm{def}}^{T_\eta}$ so that its origin lies in $\mathcal{M}_0$. Let $E_1, \cdots, E_l \subset X^{[2]}$ be the complete set of deformable parts of u; let $Y_{i,j}$, $j = 1, \cdots, n_i$ be the complete set of connected components in $X^{[0]}$ that are subordinate to E_i. By definition, each $u|_{Y_{i,j}} \cong \phi_{k_{i,j}, c_{i,j}}$. To deform u, we shall vary the $c_{i,j}$ in each $\phi_{k_{i,j}, c_{i,j}}$ and move E_i accordingly to get a new map.

In accordance, we shall divide X into three parts. We let X_0 be the union of irreducible components of X other than the E_i's and $Y_{i,j}$'s. The variation of u will remain unchanged over this part of the curve. The second part is the moving part E_i's. Recall that each $u|_{E_i}$ is a morphism to $\Delta[m_2]$. Suppose it maps to the fiber $\Delta[m_2]_c$ of $\Delta[m_2]$ over $c \in L_2 \subset D_2$. To deform u, we need to make the new map maps E_i to $\Delta[m_2]_{c'}$. Since the total space of $\Delta[m_2]$ over L_2 is a trivial $\mathbb{P}^1[m_2]$ bundle, there is a canonical way to do this. We let

$$\varphi_{c,c'} : \Delta[m_2]_c \xrightarrow{\cong} \Delta[m_2]_{c'}$$

be the isomorphism of the two fibers of $\Delta[m_2]$ over c and $c' \in L_2$ induced by the projection $\Delta[m_2] \to \mathbb{P}^1[m_2]$ that is induced by the product structure on $\Delta[m_2]$ over L_2. The third parts are those $Y_{i,j}$ that are subordinate to E_i.

We now deform the map u using the parameter space $\mathbb{A}^l$. We let K_i be the least common multiple of $(k_{i,1}, \cdots, k_{i,n_i})$; we let $e_{i,j} = K_i/k_{i,j}$. Since $Y_{i,j}$ and $Y_{i,j'}$ are connected to the same connected component $E_i \subset X^{[2]}$, $c_{i,j}^{k_{i,j}} = c_{i,j'}^{k_{i,j'}}$; we let it be c_i. For $\mathbf{t} = (t_1, \cdots, t_l) \in \mathbb{A}^l$, we define

$$u^{\mathbf{t}}|_{X_0} = u|_{X_0}, \quad u^{\mathbf{t}}|_{\tilde{Y}_i} = \varphi_{c_i, t_i^{K_i}} \quad \text{and} \quad u^{\mathbf{t}}|_{Y_{i,j}} = \phi_{k_{i,j}, t_i^{e_{i,j}}}. \tag{A.32}$$

Here in case $Y_{i,j} \cong \mathbb{P}^1$, which is the case when $c_{i,j} \neq 0$, by $u^{\mathbf{t}}|_{Y_{i,j}} = \phi_{k_{i,j},0}$ we mean that we will replace $Y_{i,j}$ by $\mathbb{P}^1 \sqcup \mathbb{P}^1$ with necessarily gluing if required; and vice versa.

The $\mathbb{A}^l$ family $u^{\mathbf{t}}$ is a family of T_η-equivariant relative stable morphisms in $\mathcal{M}_{\mathrm{def}}^{T_\eta}$; the map u^0 associated to $0 \in \mathbb{A}^l$ lies in $\mathcal{M}_0$; the induced morphism $\mathbb{A}^l \to \mathcal{M}_{\mathrm{def}}^{T_\eta}$ is an embedding up to a finite quotient.

By extending this to any connected component $\mathcal{Q}$ of $\mathcal{M}_{\mathrm{def}}^{T_\eta}$, we obtain

Proposition A.17. *Let $\mathcal{Q}$ be any connected component of $\mathcal{M}_{def}^{T_\eta}$ that is not entirely contained in $\mathcal{M}_0$. Suppose elements of $\mathcal{Q}$ has deformation type l. Then there is a stack $\bar{\mathcal{Q}}$, a finite quotient morphism $\bar{\mathcal{Q}} \to \mathcal{Q}$, a closed substack $\bar{\mathcal{Q}}_0 \subset \bar{\mathcal{Q}}$, l projections $\rho_i : \bar{\mathcal{Q}} \to \mathbb{A}_i^1$ and a projection $\pi : \bar{\mathcal{Q}} \to \bar{\mathcal{Q}}_0$ so that*

$$\big(\pi, (\rho_1, \cdots, \rho_l)\big) : \mathcal{Q} \xrightarrow{\cong} \mathcal{Q}_0 \times \mathbb{A}^l$$

is an isomorphism. Further, given a $[u] \in \mathcal{Q}$*, the fiber* $\mathbb{A}^l$ *in* $\bar{\mathcal{Q}}$ *that contains a lift of* $[u] \in \mathcal{Q}$ *is the* $\mathbb{A}^l$ *family* $\{u^{\mathbf{t}} \mid \mathbf{t} \in \mathbb{A}^l\}$*; its intersection with the zero section* $\bar{\mathcal{Q}}_0$ *is* u^0*. Finally, the intersection* $\mathcal{Q} \cap \mathcal{M}_0$ *is the image of* $\bar{\mathcal{Q}}_0$*.*

Proof. Let $\mathcal{U}: \mathcal{X} \to \mathcal{W}$ be the tautological family over $\bar{\mathcal{Q}}$. We choose $\bar{\mathcal{Q}}$ so that there are families of subcurves $\mathcal{E}_1, \cdots, \mathcal{E}_l \subset \mathcal{X}$ so that for each $z \in \bar{\mathcal{Q}}$, $\mathcal{E}_1 \cap \mathcal{X}_z, \cdots, \mathcal{E}_l \cap \mathcal{X}_z$ are exactly the l deformable parts of $\mathcal{X}_z$. Then the composite $\mathcal{E}_i \to \mathcal{W} \longrightarrow W$ factor through $L_2 \subset W$, and the resulting morphism $\mathcal{E}_i \to L_2$ factor through $\bar{\mathcal{Q}} \to L_2$. Because each $\mathcal{E}_i \cap \mathcal{X}_z$ has a $\phi_{k,c}$ connected to it, the image of $\bar{\mathcal{Q}} \to L_2$ lies in $L_2 - q_2$. We then fix an isomorphism $\mathbb{A}^1 \cong L_2 - q_2$ with 0 corresponding to p_2. This way we obtain the desired morphism

$$\rho_i : \bar{\mathcal{Q}} \longrightarrow \mathbb{A}^1_i \cong L_2 - q_2.$$

The proof of the remainder part of the Proposition is exactly the same as the case studied; we shall not repeat it here. □

The last step is to investigate the obstruction sheaf over $\mathcal{Q}$, or its lift to $\bar{\mathcal{Q}}$.

Let $\mathcal{R} \subset \mathcal{X}$ the divisor of marked points. By passing to an étale covering of $\bar{\mathcal{Q}}$, we can assume that $\mathcal{R} \to \mathcal{Q}$ is a union of sections; in other words, we can index the marked points of $[u]$ in $\bar{\mathcal{Q}}$ globally. We then pick an indexing so that for $i \le l$ the i-th section of the marked points $\mathcal{R}_i$ lies in $\mathcal{E}_i$. We let $\mathcal{U}_i : \bar{\mathcal{Q}} \to \mathcal{L}_2$ be

$$\mathcal{U}_i \triangleq \mathcal{U}|_{\mathcal{R}_i} : \mathcal{R}_i \cong \bar{\mathcal{Q}} \longrightarrow \mathcal{L}_2 \subset \mathcal{W}.$$

Since $\mathcal{L}_2 \subset \mathcal{D}_2$ is isomorphic to $L_2 \times \bar{\mathcal{Q}} \subset D_2 \times \bar{\mathcal{Q}}$ under the contraction $\mathcal{W} \to W \times \bar{\mathcal{Q}}$ and since $\mathcal{R}_i$ lies in $\mathcal{E}_i$, for $i \le l$ the morphism $\mathcal{U}_i$ is exactly the ρ_i under the isomorphism $\mathbb{A}^1_i \cong L_2 - q_2$, and $\mathcal{U}_i^* \mathcal{N}_{\mathcal{L}/\mathcal{D}}$ is canonically isomorphic to $\rho_i^* N_{L_2/D_2}$. Because D_2 is fixed by T_η, N_{L_2/D_2} is fixed as well, and hence $\rho_i^* N_{L_2/D_2}$ is a trivial line bundle on $\bar{\mathcal{Q}}$ with trivial T_η-linearization.

Because $\mathcal{H}$ is the direct sum of $\mathcal{U}_i^* \mathcal{N}_{\mathcal{L}/\mathcal{D}}$, $\oplus_{i=1}^l \rho_i^* N_{L_2/D_2}$ becomes a direct summand of $\mathcal{H}$. Because it has weight zero, it induces a canonical homomorphism

$$\oplus_{i=1}^l \rho_i^* N_{L_2/D_2} \longrightarrow \mathcal{T}^2_{T_\eta},$$

a weight zero subsheaf of $\mathcal{T}^2_{T_\eta}$.

Lemma A.18. *The homomorphism* $\oplus^n \rho_i^* N_{L_2/D_2} \to \mathcal{T}^2_{T_\eta}$ *in (A.16) is injective; thus* $\mathcal{T}^2_{T_\eta}$ *contains* $\oplus^n \rho_i^* N_{L_2/D_2}$ *as its subsheaf. Indeed, this subsheaf is canonically a direct summand of* $\mathcal{T}^2_{T_\eta}$*.*

Proof. First the first l marked points lie in the connected components of $X^{[2]}$ that are connected to the domain of at least one $\phi_{k,c}$ in W. Because all deformations of $\phi_{k,c}$ as T_η-invariant maps are $\phi_{k,c'}$, and they intersect D_2 in L_2 only; hence for these i even if we do not impose the condition $\mathcal{U}(\mathcal{R}_i) \subset \mathcal{L}_2$ the condition will be satisfied automatically. In short, the arrow $\tilde{\mathcal{T}}^1_{T_\eta} \to \mathcal{H}_{T_\eta}$ has image lies in the summand $\oplus_{i>l} \mathcal{U}_i^* \mathcal{N}_{\mathcal{L}/\mathcal{D}}$. This proves that the homomorphism $\oplus^n \rho_i^* N_{L_2/D_2} \to \mathcal{T}^2_{T_\eta}$ is injective.

We now show that this subsheaf is canonically a summand of the obstruction sheaf. The ordinary moduli of stable relative morphisms $\mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(W^{\mathrm{rel}})$ requires that the marked points be sent to the relative divisor. The moduli space $\mathcal{M} = \mathcal{M}^\bullet_{\chi,\vec{d},\vec{\mu}}(W^{\mathrm{rel}}, L)$ we worked on imposes one more restriction: the marked points be sent to $L \subset D$. The obstruction sheaves of the two moduli spaces are related by the exact sequence (A.16) because of the exact sequences

$$0 \longrightarrow N_{L_i/D_i} \longrightarrow N_{L_i/W} \longrightarrow N_{D_i/W}|_{D_i} \longrightarrow 0.$$

In our case, L_i is a $\mathbb{P}^1$ and the above exact sequence splits T-equivariantly. Hence the sheaf $\mathcal{T}^2_{T_\eta}$ splits off a factor that is the kernel of $\tilde{\mathcal{T}}^1_{T_\eta} \to \mathcal{H}_{T_\eta}$. Therefore $\oplus_{i=1}^{l} \rho_i^* N_{L_2/D_2}$, which is a summand of $\mathcal{H}_{T_\eta}$ and a subsheaf of $\mathcal{T}^2_{T_\eta}$, becomes a summand $\mathcal{T}^2_{T_\eta}$. □

We next investigate the obstruction theory of $\bar{\mathcal{Q}}$. Let $[u]$ be any closed point; then the canonical obstruction theory of $\bar{\mathcal{Q}}$ at $[u]$ takes values in

$$T^2_{T_\eta} = \mathcal{T}^2_{T_\eta} \otimes_{\mathcal{O}_{\bar{\mathcal{Q}}}} \mathbf{k}([u]).$$

We let

$$\mathcal{T}^2_{0,T_\eta} \triangleq \ker\bigl\{\mathcal{T}^2_{T_\eta} \longrightarrow \overset{l}{\underset{i=1}{\oplus}} \rho_i^* \mathcal{O}_{\mathbb{A}^1_i}\bigr\} \quad \text{and} \quad T^2_{0,T_\eta} = \mathcal{T}^2_{0,T_\eta} \otimes_{\mathcal{O}_{\bar{\mathcal{Q}}}} \mathbf{k}([u]).$$

Proposition A.19. *The obstruction to deformation of* $[u]$ *in* $\mathcal{M}^{T_\eta}_{def}$ *takes value in* T^2_{0,T_η}.

Proof. The proof is exactly the same as the proof of the first statement of Lemma A.18. □

A.6. **The proof of Lemma A.1.** Following the convention set up in the introduction of this Appendix, the class $\tilde{\tau}_b\bigl([\tilde{\mathcal{M}}]^{\mathrm{vir},T}_{\mathfrak{m}}\bigr)$ is the summand in $H^T_*(\tilde{\mathcal{Z}}_a)$ of the image of the equivariant class $[\tilde{\mathcal{C}}]^T \in H^T_*(\tilde{\mathcal{F}}, \tilde{\mathcal{F}} - \tilde{\mathcal{Z}})$ under the composites

$$H^T_*(\tilde{\mathcal{F}}, \tilde{\mathcal{F}} - \tilde{\mathcal{Z}}) \xrightarrow{\text{Thom}} H^T_*(\tilde{\mathcal{Z}}) \longrightarrow H^T_*(\tilde{\mathcal{Z}}, \tilde{\mathcal{Z}} - \tilde{\mathcal{Z}}_b) \longrightarrow H^T_*(\tilde{\mathcal{Z}}_b).$$

This is the same as the summand of the image under the composite

$$H^T_*(\tilde{\mathcal{F}}, \tilde{\mathcal{F}} - \tilde{\mathcal{Z}}) \xrightarrow{\iota_b} H^T_*(\tilde{\mathcal{F}}, \tilde{\mathcal{F}} - \tilde{\mathcal{Z}}_b) \xrightarrow{\phi_a} H^T_*(\tilde{\mathcal{Z}}_b). \tag{A.33}$$

Here ι_b is the homomorphism induced by the inclusion and ϕ_a is the equivariant Thom-isomorphism.

Now we localize the above sequence. Let a and b be indices so that $\tilde{\mathcal{M}}_a \subset \tilde{\mathcal{Z}}_b$. We let $\tilde{\mathcal{N}}_a$ be the restriction to $\tilde{\mathcal{M}}_a$ of the normal bundle to $\tilde{\mathcal{Z}}_b$ in $\tilde{\mathcal{F}}$. We then consider the normal cone to $\tilde{\mathcal{C}} \cap \tilde{\mathcal{Z}}_b$ in $\tilde{\mathcal{C}}$. It is a cone contained in the total space of $\coprod_{a \prec b} \tilde{\mathcal{N}}_a$ (we say $a \prec b$ if $\tilde{\mathcal{M}}_a \subset \tilde{\mathcal{Z}}_b$). We let $\tilde{\mathcal{C}}_a$ be its part in $\tilde{\mathcal{N}}_a$. We let

$$\tau_a : H^T_*(\tilde{\mathcal{N}}_a, \tilde{\mathcal{N}}_a - \tilde{\mathcal{M}}_a) \longrightarrow H^T_*(\tilde{\mathcal{M}}_a) \quad \text{and} \quad \zeta_a : H^T_*(\tilde{\mathcal{M}}_a) \longrightarrow H^T_*(\tilde{\mathcal{Z}}_b) \tag{A.34}$$

be the Thom homomorphism and the one induced by the inclusion. Then obviously

$$\sum_{a \prec b} \zeta_a \circ \tau_a([\tilde{\mathcal{C}}_a]^T) = \phi_b \circ \iota_b([\tilde{\mathcal{C}}]^T) = \tilde{\tau}_b\bigl([\tilde{\mathcal{M}}]^{\mathrm{vir},T}\bigr).$$

Hence to prove Lemma A.1, it suffices to show that $\tau_a([\tilde{\mathcal{C}}_a]^T) = 0$ for all $\tilde{\mathcal{M}}_a \subset \tilde{\mathcal{M}}_\infty$.

Before we prove this, we need two structure results: one is on $\tilde{\mathcal{M}}_a$ and the other is on the bundle $\tilde{\mathcal{N}}_a|_{\tilde{\mathcal{M}}_a}$. We will show that each connected component $\tilde{\mathcal{M}}_a$ of $\tilde{\mathcal{M}}^T_\infty$, up to a finite branched cover $\bar{\mathcal{M}}_a \to \tilde{\mathcal{M}}_a$, admits a T-equivariant map

$$\bar{\pi}_a : \bar{\mathcal{M}}_a \to \mathbb{P}^{n_a}_{\mathbf{w}} \tag{A.35}$$

to a weighted projective space $\mathbb{P}^{n_a}_{\mathbf{w}}$ with trivial T action; that there is a rank n_a+1 vector bundle $\mathcal{V}_a$ on $\mathbb{P}^{n_a}_{\mathbf{w}}$ with trivial T-action and a T-equivariant quotient vector bundle homomorphism

$$\bar{\mathcal{N}}_a \longrightarrow \bar{\pi}_a^* \mathcal{V}_a \tag{A.36}$$

of the lift $\bar{\mathcal{N}}_a$ to $\bar{\mathcal{M}}_a$ of $\tilde{\mathcal{N}}_a$ on $\tilde{\mathcal{M}}_a$.

We first construct $\bar{\pi}_a$. Let $\tilde{z} \in \tilde{\mathcal{M}}^T_\infty$ be any closed point. By the construction of $\tilde{\mathcal{M}}_\infty$, $\tilde{z}$ is an T_ν-orbit $[T_\nu \cdot z]$ of some $z \in \partial\Sigma_\epsilon\mathcal{M}_0 \cap \mathcal{M}$; therefore $\tilde{z}$ is fixed by T if and only if the T_ν-orbit $T_\nu \cdot z$ is identical to the T-orbit $T \cdot z$, which is possible only if $\dim_{\mathbb{R}} \operatorname{stab}_T(z) \geq 1$. Because $\tilde{z} \in \partial\Sigma_\epsilon\mathcal{M}_0$, it is not in $\mathcal{M}^T$; hence there must be a subgroup $T_\eta \subset T$ so that $\tilde{z} \in \mathcal{M}^{T_\eta}$. Finally, because ϵ is sufficiently small, $z \in \mathcal{M}^{T_\eta}_{\text{def}}$. This shows that

$$\tilde{\mathcal{M}}^T_\infty = \coprod_{T_\eta \subset T} \Big(\mathcal{M}^{T_\eta}_{\text{def}} \cap \partial\Sigma_\epsilon\mathcal{M}_0 \Big) / T_\nu.$$

We now analyze its individual connected components. Before we move on, we remark that we only need to consider the case $\eta = (a_1, a_2, a_3)$, for $a_1 > 0$ and $a_2, a_3 < 0$, and the case $\eta = (1, -1, 0)$. Indeed, since the symmetry of $(\mathbb{P}^1)^3$ defined by $(z_1, z_2, z_3) \mapsto (z_2, z_3, z_1)$ lifts to a symmetry of W, any statement that holds true for the $\eta = (a_1, a_2, a_3)$ holds true for $\eta' = (a_2, a_3, a_1)$. Consequently, we only need to work with those η so that $|a_1| \geq |a_2|$ and $|a_3|$. Then because $T_\eta = T_{-\eta}$, we can assume further that $a_1 > 0$. Hence either a_2 and $a_3 < 0$ or one of them is zero. For former is the *case one*; in the later case, by applying the S_3 symmetry we can reduce it to the case $\eta = (1, -1, 0)$.

We fix a $T_\eta \subset T$ belongs to the two classes just mentioned. We let $\mathcal{Q}_a$ be a connected component of $\mathcal{M}^{T_\eta}_{\text{def}}$ associated to $\tilde{\mathcal{M}}_a$. According to Proposition A.13 and A.17, after a finite branched covering $\bar{\mathcal{Q}} \to \mathcal{Q}$, $\bar{\mathcal{Q}}$ is isomorphic to $\mathcal{Q}_0 \times \mathbb{A}^l$ for some integer $l > 0$; the T_ν-action on $\bar{\mathcal{Q}}$ is the product of the action on $\bar{\mathcal{Q}}_0$ induced by that on $\mathcal{M}_0$ and the action

$$(u_1, \cdots, u_l)^\sigma = (\sigma^{w_1} u_1, \cdots, \sigma^{w_l} u_l) \in \mathbb{A}^l \tag{A.37}$$

for some $\mathbf{w} = (w_1, \cdots, w_l)$. Hence if we let $P_a : \bar{\mathcal{Q}} \to \mathbb{A}^l$ be the projection, which is $(\rho_1, \cdots, \rho_l)$ by our convention, and if we endow $\mathbb{A}^l$ with the T_ν-action (A.37), then $\bar{\mathcal{Q}} \to \mathbb{A}^l$ is T_ν-equivariant.

We now pick an T_ν-invariant Riemannian metric on $\mathbb{A}^l$; we let $S^{2l-1}_\epsilon \subset \mathbb{A}^l$ be the ϵ-sphere under this metric. $P_a^{-1}(S^{2l-1}_\epsilon)$. Without lose of generality, we can assume that the metric on $\mathbb{A}^l$ and on $\mathcal{Z}$ are chosen so that $P_a^{-1}(S^{2l-1}_\epsilon) \subset \bar{\mathcal{Q}}$ is the preimage of $\mathcal{Q} \cap \partial\Sigma_\epsilon\mathcal{M}_0$ in $\bar{\mathcal{Q}}$. Hence P_a induces an T_ν-equivariant map

$$P_a^{-1}(S^{2l-1}_\epsilon) \longrightarrow S^{2l-1}_\epsilon \tag{A.38}$$

and thus induces a map between their quotients

$$\bar{\pi}_a : \bar{\mathcal{M}}_a \triangleq P_a^{-1}(S^{2l-1}_\epsilon)/T_\nu \longrightarrow S^{2l-1}_\epsilon / T_\nu = \mathbb{P}^{n_a}_{\mathbf{w}}.$$

(Here we use the subscript $\mathbf{w}$ to indicate the weighted and the superscript n_a to denote the dimension of weighted projective space; to be precise, we shall view the weighted projective spaces as DM-stack. Since the specific weight is irrelevant to our study, we shall not keep track of it in our study.)

We next construct the quotient bundle (A.36). Let $\mathcal{Q} \subset \mathcal{M}^{T_\eta}_{\text{def}}$ be the connected component that gives rise to the fixed loci $\tilde{\mathcal{M}}_a$; let $\bar{\mathcal{Q}} \to \mathcal{Q}$ be the finite branched covering alluded to in Propositions A.13 and A.17; let $\mathcal{T}^2_{T_\eta}$ be the obstruction sheaf on $\bar{\mathcal{Q}}$. By (A.29) and Lemma A.18, there is a canonical quotient sheaf homomorphism

$$\mathcal{T}^2_{T_\eta} \longrightarrow \bigoplus_{i=1}^{l} \rho_i^* \mathcal{O}_{\mathbb{A}^1_i}, \tag{A.39}$$

both with trivial T_η-actions.

A direct check shows that to each i there is a T_ν-linearization on $\mathcal{O}_{\mathbb{A}^1_i}$ so that the above homomorphism is T_ν-equivariant. Because $T_\nu \cdot T_\eta = T$, the adopted T_ν-linearization and the trivial T_η-linearization on $\mathcal{O}_{\mathbb{A}^1_i}$ makes (A.39) T-equivariant.

Since the obstruction sheaf $\mathcal{T}^2$ on $\mathcal{M}$ is a T-equivariant quotient sheaf of $\mathcal{F}|_{\mathcal{M}}$, restricting to $\bar{\mathcal{Q}}$ and then composing with (A.39) give us a T-equivariant quotient sheaf

$$\mathcal{F}|_{\bar{\mathcal{Q}}} \longrightarrow \bigoplus_{i=1}^{l} \rho_i^* \mathcal{O}_{\mathbb{A}^1_i}. \tag{A.40}$$

Here $\mathcal{F}|_{\bar{\mathcal{Q}}}$ is the lift of $\mathcal{F}|_{\mathcal{Q}}$ to $\bar{\mathcal{Q}}$. Their descents to $\bar{\mathcal{M}}_a$ then give rise to a quotient homomorphism

$$\bar{\mathcal{F}}|_{\bar{\mathcal{M}}_a} \longrightarrow \bar{\pi}_a^* \mathcal{V}_a.$$

Here $\mathcal{V}_a$ is the descent (or the T_ν-quotient) of $\oplus_{i=1}^l \mathcal{O}_{\mathbb{A}^1_i}|_{S^{2l-1}}$ — a vector bundle on $\mathbb{P}^{n_a}_{\mathbf{w}}$ with trivial T-action; $\bar{\mathcal{F}}|_{\bar{\mathcal{M}}_a}$ is the lift of $\tilde{\mathcal{F}}|_{\tilde{\mathcal{M}}_a}$ to $\bar{\mathcal{M}}_a$.

Finally, since the normal bundle $\tilde{\mathcal{N}}_a$ has $\tilde{\mathcal{F}}|_{\tilde{\mathcal{M}}_a}$ as its quotient sheaf, its lift to $\bar{\mathcal{M}}_a$, denoted by $\bar{\mathcal{N}}_a$, has $\bar{\pi}_a^* \mathcal{V}_a$ as its T-equivariant quotient. This is the homomorphism we aimed at.

We next prove the following Lemma.

Lemma A.20. *Let $\tilde{\mathcal{C}}_a$ be the part in $\tilde{\mathcal{N}}_a$ of the normal cone to $\tilde{\mathcal{C}} \cap \tilde{\mathcal{Z}}_a$ in $\tilde{\mathcal{C}}$; let $\bar{\mathcal{C}}_a \subset \bar{\mathcal{N}}_a$ be the canonical lift of $\tilde{\mathcal{C}}_a$ under $\bar{\mathcal{N}}_a \to \tilde{\mathcal{N}}_a$; and let $\bar{\mathcal{N}}_{a,0}$ be the kernel of $\bar{\mathcal{N}}_a \to \bar{\pi}_a^* \mathcal{V}_a$. Then there is a T-equivariant cone $\bar{\mathcal{B}}_a \subset \bar{\mathcal{N}}_a$ so that $\bar{\mathcal{C}}_a$ is T-equivariant rationally equivalent to $\bar{\mathcal{B}}_a$.*

The desired vanishing $\tau_a([\tilde{\mathcal{C}}_a]^T) = 0$ follows immediately from this Lemma. Indeed, let

$$\bar{\tau}_a : H^T_*(\bar{\mathcal{N}}_a, \bar{\mathcal{N}}_a - \bar{\mathcal{M}}_a) \longrightarrow H^T_*(\bar{\mathcal{M}}_a)$$

and

$$\bar{\tau}_{a,0} : H^T_*(\bar{\mathcal{N}}_{a,0}, \bar{\mathcal{N}}_{a,0} - \bar{\mathcal{M}}_a) \longrightarrow H^T_*(\bar{\mathcal{M}}_a)$$

be the T-equivariant Thom homomorphisms and let $\pi : \bar{\mathcal{M}}_a \to \tilde{\mathcal{M}}_a$ be the projection. Then

$$\deg \pi \cdot \tau_a([\tilde{\mathcal{C}}_a]^T) = \bar{\tau}_a([\bar{\mathcal{B}}_a]^T) = e^T(\bar{\pi}_a^* \mathcal{V}_a) \cdot \bar{\tau}_{a,0}([\bar{\mathcal{B}}_a]^T) = \bar{\pi}_a^* e^T(\mathcal{V}_a) \cdot \bar{\tau}_{a,0}([\bar{\mathcal{B}}_a]^T) = 0.$$

Here the first equality follows from the projection formula and the rational equivalence of the Lemma; the second follows from the fact that $\bar{\mathcal{B}}_a \subset \bar{\mathcal{N}}_{a,0}$; the last follows from that $\mathcal{V}_a$ is T-trivial and its rank is bigger than $\dim \mathbb{P}^{n_a}_{\mathbf{w}}$.

Proof of Lemma A.20. We let $\tilde{\mathcal{Z}}_b$ be the connected component of $\tilde{\mathcal{Z}}^T_\infty$ that contains $\tilde{\mathcal{M}}_a$; let $\mathcal{Z}_b$ be the connected component of $\mathcal{Z}^{T_\eta}$ so that $(\mathcal{Z}^{T_\eta} \cap \partial\Sigma_\epsilon \mathcal{M}_0)/T_\nu$ is $\tilde{\mathcal{Z}}_b$; and let $\mathcal{Q}$ as before be the connected component of $\mathcal{M}^{\mathbb{C}^*_\eta}$ that gives rise to $\tilde{\mathcal{M}}_a$. We next let $\mathcal{N}_b$ be the normal bundle to $\mathcal{Z}_b$ in $\mathcal{Z}$; it admits a canonical T-linearization. Since $\partial\Sigma_\epsilon \mathcal{M}_0$ intersects $\mathcal{Z}_b$ transversally along $\mathcal{Q} \cap \partial\Sigma_\epsilon \mathcal{M}_0$, $\mathcal{N}_b|_{\mathcal{Q} \cap \partial\Sigma_\epsilon \mathcal{M}_0}$ is identical to the restriction to $\mathcal{Q} \cap \partial\Sigma_\epsilon \mathcal{M}_0$ of the normal bundle to $\mathcal{Z}_b \cap \partial\Sigma_\epsilon \mathcal{M}_0$ in $\partial\Sigma_\epsilon \mathcal{M}_0$.

Hence the descent of $\mathcal{N}_b|_{\mathcal{Q}\cap\partial\Sigma_\epsilon\mathcal{M}_0}$ to $\tilde{\mathcal{M}}_a$, which we denote by $\tilde{\mathcal{N}}_0$, is canonically a subbundle of $\tilde{\mathcal{N}}_a$ — the restriction to $\tilde{\mathcal{M}}_a$ of the normal bundle to $\tilde{\mathcal{Z}}_b$ in $\tilde{\mathcal{Z}}$. Further, the quotient $\tilde{\mathcal{L}}_a \triangleq \tilde{\mathcal{N}}_a/\tilde{\mathcal{N}}_0$ is an $\mathbb{R}^2$-bundle on $\tilde{\mathcal{M}}_a$ that admits a canonical T-linearization that makes the following exact sequence T-equivariant:

$$0 \longrightarrow \tilde{\mathcal{N}}_0 \longrightarrow \tilde{\mathcal{N}}_a \longrightarrow \tilde{\mathcal{L}}_a \longrightarrow 0. \tag{A.41}$$

For later application, we now exhibit explicitly the T-linearization on $\tilde{\mathcal{L}}_a$ and show that the above exact sequence splits canonically. For this we need to use the complexified subgroup $\mathbb{C}^*_\nu$ of $(\mathbb{C}^*)^2$. Namely, $\mathbb{C}^*_\nu \subset (\mathbb{C}^*)^2$ is the algebraic subgroup that contains the T_ν as its maximal compact subgroup. For any $\sigma \in T_\nu$, we let $\sigma\mathbb{R}^+$ be the obvious real subgroup in $\mathbb{C}^*_\nu$. For any $z \in \mathcal{Q}$ the tangent of the orbit $\sigma\mathbb{R}^+ \cdot z$ at σz defines an element

$$n_\sigma(z) \triangleq \frac{d}{dt}\big(\sigma t \cdot z\big)|_{t=1} \in T_{\sigma z}\mathcal{Z}.$$

Since $\mathbb{C}^*_\nu$ acts on $\mathcal{Z}$ algebraically, the vector $n_\sigma(z)$ is never zero. Descending to $\tilde{\mathcal{M}}_a$, these vectors $\{n_\sigma(z) \mid \sigma \in T_\nu\}$ are loci of an T_ν orbit of the fiber $\tilde{\mathcal{L}}_a|_{[z]}$ at $[z] \in \tilde{\mathcal{M}}_a$. Here $[z]$ is the point in $\tilde{\mathcal{M}}_a$ that is the T_ν-orbit $T_\nu \cdot z$ of z. The vectors also span an $\mathbb{R}^2$-subspace in $\tilde{\mathcal{N}}_a|_{[z]}$ and generate the $\mathbb{R}^2$ normal vector space $\tilde{\mathcal{L}}_a|_{[z]}$. Therefore, it induces a splitting of the exact sequence (A.41) at $[z]$. Because this splitting is canonical, it extends to a splitting of the exact sequence along $\tilde{\mathcal{M}}_a$. In the following we will fix this splitting and write $\tilde{\mathcal{N}}_a \equiv \tilde{\mathcal{N}}_0 \oplus \tilde{\mathcal{L}}_a$. We remark that this splitting is induced by the $\mathbb{C}^*_\nu$-action on $\mathcal{Z}$; it does not depend on the choice of the Riemannian metric.

We now consider the pair $(\mathcal{Q}, \tilde{\mathcal{M}}_a)$. Following its construction, $\tilde{\mathcal{M}}_a$ is the quotient of $\mathcal{Q} \cap \partial\Sigma_\epsilon\mathcal{M}_0$ by T_ν. If we replace $\mathcal{Q}$ by its covering $\bar{\mathcal{Q}}$ studied before, we obtain $\bar{\mathcal{M}}_a$ after quotient out $\bar{\mathcal{Q}} \cap P_a^{-1}(S_\epsilon^{2l-1})$ by T_ν. (See (A.38) for notations.) Parallel to that of symplectic reduction, $\tilde{\mathcal{M}}_a$ and $\bar{\mathcal{M}}_a$ are canonically isomorphic to the quotient of $\mathcal{Q}_+ \triangleq \mathcal{Q} - \mathcal{M}_0$ by $\mathbb{C}^*_\nu$ and of $\bar{\mathcal{Q}}_+ \triangleq \bar{\mathcal{Q}} - \bar{\mathcal{Q}}_0$ by $\mathbb{C}^*_\nu$.

As to the cone $\tilde{\mathcal{C}}_a$, the part of the normal cone to $\tilde{\mathcal{C}} \times_{\tilde{\mathcal{Z}}} \tilde{\mathcal{Z}}_b$ in $\tilde{\mathcal{C}}$ that lies over $\tilde{\mathcal{M}}_a$, there is a similar quotient description which we now describe. We let $\mathcal{C}_a$ be the part of the normal cone to $\mathcal{C} \times_{\mathcal{Z}} \mathcal{Z}_b$ in $\mathcal{C}$ that lies above $\mathcal{Q}$; we let $\mathcal{C}_{a+}$ be $\mathcal{C}_a \times_{\mathcal{Q}} \mathcal{Q}_+$. Obviously, $\mathcal{C}_{a+}$ is a $\mathbb{C}^*_\nu$-equivariant subcone of the vector bundle $\mathcal{N}_{a+} \triangleq \mathcal{N}_a|_{\mathcal{Q}_+}$. Because $\mathbb{C}^*_\nu$ acts on $\mathcal{Q}_+$ with finite stabilizers everywhere, the quotient $\mathcal{Q}_+/\mathbb{C}^*_\nu$ is a DM-stack and is canonically isomorphic to $\tilde{\mathcal{M}}_a$; the quotient $\mathcal{N}_{a+}/\mathbb{C}^*_\nu$ is canonically isomorphic to $\tilde{\mathcal{N}}_0$; the quotient $\mathcal{C}_{a+}/\mathbb{C}^*_\nu$ is a subcone of $\tilde{\mathcal{N}}_0$ and its direct product with $\tilde{\mathcal{L}}_a$ over $\tilde{\mathcal{M}}_a$ is the cone $\tilde{\mathcal{C}}_a$; that is

$$\tilde{\mathcal{C}}_a = \mathcal{C}_{a+}/\mathbb{C}^*_\nu \times_{\tilde{\mathcal{M}}_a} \tilde{\mathcal{L}}_a \subset \tilde{\mathcal{N}}_a \equiv \tilde{\mathcal{N}}_0 \oplus \tilde{\mathcal{L}}_a. \tag{A.42}$$

Our next step is to construct a $\mathbb{C}^*_\eta \times \mathbb{C}^*_\nu$-equivariant subcone $\mathcal{B}_a \subset \mathcal{N}_a$ whose quotient by $\mathbb{C}^*_\nu$ will give us the cycle $\tilde{\mathcal{B}}_a$; we shall construct also a $\mathbb{C}^*_\eta \times \mathbb{C}^*_\nu$-equivariant rational equivalence between $\mathcal{C}_a$ and $\mathcal{B}_a$ whose descent to $\tilde{\mathcal{M}}_a$ will give the required T-equivariant rational equivalence relation.

The cone $\mathcal{B}_a$ and the equivalence relation was already constructed in [11]. We consider the moduli space $\mathcal{M}$ and the open substack $\mathcal{Q} \subset \mathcal{M}^{\mathbb{C}^*_\eta}$. Following the proof in [11, Section 3], we can construct a $(\mathbb{C}^*)^2$-invariant cone cycle $\mathcal{B}_a \subset \mathcal{N}_a$ so that it is rational equivalent to $\mathcal{C}_a$ and its intersection with the $\mathbb{C}^*_\eta$ fixed part of $\mathcal{N}_a$, namely $\mathcal{B}_a \cap \mathcal{N}_a^{\mathbb{C}^*_\eta}$, is the cone cycle constructed in [20, 4] that defines the virtual cycle of the $\mathbb{C}^*_\eta$ fixed part $[\mathcal{Q}]^{\mathbb{C}^*_\eta}$;

We now let $\bar{\mathcal{N}}'_{a+}$ be the lift of $\mathcal{N}_a|_{\mathcal{Q}_+}$ to $\bar{\mathcal{Q}}_+$; let $\mathcal{F}|_{\bar{\mathcal{Q}}_+}$ be the lift of $\mathcal{F}|_{\mathcal{Q}_+}$. Since $\mathcal{F}|_{\mathcal{Q}_+}$ is canonically a quotient bundle of $\mathcal{N}'_{a+}$, $\mathcal{F}|_{\bar{\mathcal{Q}}_+}$ is a quotient bundle of $\bar{\mathcal{N}}'_{a+}$. Then the homomorphism (A.40) induces a quotient homomorphism

$$\bar{\mathcal{N}}'_{a+} \longrightarrow \bigoplus_{i=1}^{l} \rho_i^* \mathcal{O}_{\mathbb{A}^1_i}|_{\bar{\mathcal{Q}}_+}.$$

We let $\bar{\mathcal{N}}'_{a,0+}$ be the kernel of this homomorphism.

Because of Lemma A.14 and A.19, the lifting $\bar{\mathcal{B}}'_a \subset \bar{\mathcal{N}}'_{a+}$ of $\mathcal{B}_a \subset \mathcal{N}_a$ satisfies

$$\bar{\mathcal{B}}'_a \cap \bar{\mathcal{N}}'^{\mathbb{C}^*_\eta}_{a+} \subset \bar{\mathcal{N}}'^{\mathbb{C}^*_\eta}_{a,0+}.$$

Therefore, $\bar{\mathcal{B}}'_a \subset \bar{\mathcal{N}}'_{a,0+}$.

Now let $\bar{\mathcal{C}}'_a$ be the lift of $\mathcal{C}_a$ to $\bar{\mathcal{Q}}_+$, which is a subcone of $\bar{\mathcal{N}}'_{a+}$. Because the whole construction is $\mathbb{C}^*_\eta \times \mathbb{C}^*_\zeta$ equivariant, the cycle $\bar{\mathcal{B}}'_a$ and the rational equivalence of $\bar{\mathcal{C}}'_a$ and $\bar{\mathcal{B}}'_a$ are $\mathbb{C}^*_\eta \times \mathbb{C}^*_\zeta$-equivariant. Then because $\mathbb{C}^*_\nu$ acts on $\bar{\mathcal{Q}}_+$ with finite stabilizers, we can take the quotient $\bar{\mathcal{B}}'_a/\mathbb{C}^*_\nu$ and the similar quotient of the rational equivalence. The former lies in $\bar{\mathcal{N}}_0$ and the later is a $\mathbb{C}^*_\eta$-equivariant rational equivalence between $\bar{\mathcal{B}}'_a/\mathbb{C}^*_\nu$ and $\bar{\mathcal{C}}'_a/\mathbb{C}^*_\nu$. Then because of (A.42), if we define

$$\bar{\mathcal{B}}_a = \bar{\mathcal{B}}'_a/\mathbb{C}^*_\nu \times_{\bar{\mathcal{M}}_a} \bar{\mathcal{L}}_a \subset \tilde{\mathcal{N}}_a \equiv \tilde{\mathcal{N}}_0 \oplus \tilde{\mathcal{L}}_a,$$

then the direct produce of $\bar{\mathcal{L}}_a$ with the quotient of the rational relation by $\mathbb{C}^*_\nu$ defines the desired rational equivalence between $\bar{\mathcal{B}}_a$ and $\bar{\mathcal{C}}_a$. This completes the proof of the Lemma. □

Appendix B. Examples of Conjecture 8.3

Conjecture 8.3. *Let $\vec{\mu} \in \mathcal{P}^3_+$. Then*

$$\tilde{\mathcal{W}}_{\vec{\mu}}(q) = \mathcal{W}_{\vec{\mu}}(q),$$

where $q = e^{\sqrt{-1}\lambda}$, and $\mathcal{W}_{\vec{\mu}}(q)$ is defined as in Section 3.3.

We have seen in Section 8 that Conjecture 8.3 holds when one of the three partitions is empty. When none of the partitions is empty, A. Klemm has checked by computer that Conjecture 8.3 holds in all the cases where

$$|\mu^i| \leq 6, \quad i = 1, 2, 3.$$

We list some of these cases here.

$$\tilde{\mathcal{W}}_{(1),(1),(1)}(q) = \mathcal{W}_{(1),(1),(1)}(q) = \frac{q^4 - q^3 + q^2 - q + 1}{q^{1/2}(q-1)^3}$$

$$\tilde{\mathcal{W}}_{(1),(1),(2)}(q) = \mathcal{W}_{(1),(1),(2)}(q) = \frac{q^6 - q^5 + q^3 - q + 1}{(q^2-1)(q-1)^3}$$

$$\tilde{\mathcal{W}}_{(1),(1),(1,1)}(q) = \mathcal{W}_{(1),(1),(1,1)}(q) = \frac{q^6 - q^5 + q^3 - q + 1}{q(q^2-1)(q-1)^3}$$

$$\tilde{\mathcal{W}}_{(1),(1),(3)}(q) = \mathcal{W}_{(1),(1),(3)}(q) = \frac{q^{3/2}(q^8 - q^7 + q^4 - q + 1)}{(q^3-1)(q^2-1)(q-1)^3}$$

$$\tilde{\mathcal{W}}_{(1),(1),(2,1)}(q) = \mathcal{W}_{(1),(1),(2,1)}(q) = \frac{q^8 - 2q^7 + 3q^6 - 3q^5 + 3q^4 - 3q^3 + 3q^2 - 2q + 1}{q^{1/2}(q^3-1)(q-1)^4}$$

$$\tilde{\mathcal{W}}_{(1),(1),(1,1,1)}(q) = \mathcal{W}_{(1),(1),(1,1,1)}(q) = \frac{q^8 - q^7 + q^4 - q + 1}{q^{3/2}(q^3-1)(q^2-1)(q-1)^3}$$

$$\tilde{\mathcal{W}}_{(1),(2),(2)}(q) = \mathcal{W}_{(1),(2),(2)}(q) = \frac{q^{1/2}(q^8 - q^7 + q^5 - q^4 + q^3 - q + 1)}{(q^2-1)^2(q-1)^3}$$

$$\tilde{\mathcal{W}}_{(1),(1,1),(2)}(q) = \mathcal{W}_{(1),(1,1),(2)}(q) = \frac{q^9 - q^8 + q^6 - q^5 + 2q^3 - q^2 - q + 1}{q^{3/2}(q^2-1)^2(q-1)^3}$$

$$\tilde{\mathcal{W}}_{(1),(2),(1,1)}(q) = \mathcal{W}_{(1),(2),(1,1)}(q) = \frac{q^9 - q^8 - q^7 + 2q^6 - q^4 + q^3 - q + 1}{q^{1/2}(q^2-1)^2(q-1)^3}$$

$$\tilde{\mathcal{W}}_{(1),(1,1),(1,1)}(q) = \mathcal{W}_{(1),(1,1),(1,1)}(q) = \frac{q^8 - q^7 + q^5 - q^4 + q^3 - q + 1}{q^{3/2}(q^2-1)^2(q-1)^3}$$

$$\tilde{\mathcal{W}}_{(1),(1),(4)}(q) = \mathcal{W}_{(1),(1),(4)}(q) = \frac{q^4(q^{10} - q^9 + q^5 - q + 1)}{(q^4-1)(q^3-1)(q^2-1)(q-1)^3}$$

$$\tilde{\mathcal{W}}_{(1),(1),(3,1)}(q) = \mathcal{W}_{(1),(1),(3,1)}(q) = \frac{q(q^{10} - 2q^9 + 2q^8 - 2q^6 + 3q^5 - 2q^4 + 2q^2 - 2q + 1)}{(q^4-1)(q^2-1)(q-1)^4}$$

$$\tilde{\mathcal{W}}_{(1),(1),(2,2)}(q) = \mathcal{W}_{(1),(1),(2,2)}(q) = \frac{q(q^8 - 2q^6 + q^5 + q^4 + q^3 - 2q^2 + 1)}{(q^3-1)(q^2-1)^2(q-1)^3}$$

$$\tilde{\mathcal{W}}_{(1),(1),(2,1,1)}(q) = \mathcal{W}_{(1),(1),(2,1,1)}(q) = \frac{q^{10} - 2q^9 + 2q^8 - 2q^6 + 3q^5 - 2q^4 + 2q^2 - 2q + 1}{q(q^4-1)(q^2-1)(q-1)^4}$$

$$\tilde{\mathcal{W}}_{(1),(1),(1,1,1,1)}(q) = \mathcal{W}_{(1),(1),(1,1,1,1)}(q) = \frac{q^{10} - q^9 + q^5 - q + 1}{q^2(q^4-1)(q^3-1)(q^2-1)(q-1)^3}$$

$$\tilde{\mathcal{W}}_{(1),(2),(3)}(q) = \mathcal{W}_{(1),(2),(3)}(q) = \frac{q^2(q^{10} - q^9 + q^6 - q^4 + q^3 - q + 1)}{(q^3-1)(q^2-1)^2(q-1)^3}$$

$$\tilde{\mathcal{W}}_{(1),(3),(2)}(q) = \mathcal{W}_{(1),(3),(2)}(q) = \frac{q^2(q^{10} - q^9 + q^7 - q^6 + q^4 - q + 1)}{(q^3-1)(q^2-1)^2(q-1)^3}$$

$$\tilde{\mathcal{W}}_{(1),(2),(2,1)}(q) = \mathcal{W}_{(1),(2),(2,1)}(q) = \frac{q^{11} - 2q^{10} + 2q^9 - q^8 + q^7 - q^6 + q^4 - q + 1}{(q^3-1)(q^2-1)(q-1)^4}$$

$$\tilde{\mathcal{W}}_{(1),(2,1),(2)}(q) = \mathcal{W}_{(1),(2,1),(2)}(q) = \frac{q^{11} - q^{10} + q^7 - q^5 + q^4 - q^3 + 2q^2 - 2q + 1}{q(q^3-1)(q^2-1)(q-1)^4}$$

$$\tilde{\mathcal{W}}_{(1),(2),(1,1,1)}(q) = \mathcal{W}_{(1),(2),(1,1,1)}(q) = \frac{q^{12} - q^{11} - q^{10} + q^9 + q^8 - q^6 + q^4 - q + 1}{q(q^3-1)(q^2-1)^2(q-1)^3}$$

$$\tilde{\mathcal{W}}_{(1),(1,1,1),(2)}(q) = \mathcal{W}_{(1),(1,1,1),(2)}(q) = \frac{q^{12} - q^{11} + q^8 - q^6 + q^4 + q^3 - q^2 - q + 1}{q^3(q^3-1)(q^2-1)^2(q-1)^3}$$

$$\tilde{\mathcal{W}}_{(1),(1,1),(3)}(q) = \mathcal{W}_{(1),(1,1),(3)}(q) = \frac{q^{12} - q^{11} + q^8 - q^6 + q^4 + q^3 - q^2 - q + 1}{q(q^3-1)(q^2-1)^2(q-1)^3}$$

$$\tilde{\mathcal{W}}_{(1),(3),(1,1)}(q) = \mathcal{W}_{(1),(3),(1,1)}(q) = \frac{q(q^{12} - q^{11} - q^{10} + q^9 + q^8 - q^6 + q^4 - q + 1)}{(q^3-1)(q^2-1)^2(q-1)^3}$$

$$\tilde{\mathcal{W}}_{(1),(1,1),(2,1)}(q) = \mathcal{W}_{(1),(1,1),(2,1)}(q) = \frac{q^{11} - q^{10} + q^7 - q^5 + q^4 - q^3 + 2q^2 - 2q + 1}{q^2(q^3-1)(q^2-1)(q-1)^4}$$

$$\tilde{\mathcal{W}}_{(1),(2,1),(1,1)}(q) = \mathcal{W}_{(1),(2,1),(2,1)}(q) = \frac{q^{11} - 2q^{10} + 2q^9 - q^8 + q^7 - q^6 + q^4 - q + 1}{q(q^3-1)(q^2-1)(q-1)^4}$$

$$\tilde{\mathcal{W}}_{(1),(1,1),(1,1,1)}(q) = \mathcal{W}_{(1),(1,1),(1,1,1)}(q) = \frac{q^{10} - q^9 + q^7 - q^6 + q^4 - q + 1}{q^2(q^3-1)(q^2-1)^2(q-1)^3}$$

$$\tilde{\mathcal{W}}_{(1),(1,1,1),(1,1)}(q) = \mathcal{W}_{(1),(1,1,1),(1,1)}(q) = \frac{q^{10} - q^9 + q^6 - q^4 + q^3 - q + 1}{q^2(q^3-1)(q^2-1)^2(q-1)^3}$$

$$\tilde{\mathcal{W}}_{(2),(2),(2)}(q) = \mathcal{W}_{(2),(2),(2)}(q) = \frac{q(q^{10} - 3q^8 + 3q^7 + 2q^6 - 5q^5 + 2q^4 + 3q^3 - 3q^2 + 1)}{(q^2-1)^3(q-1)^3}$$

$$\tilde{\mathcal{W}}_{(2),(2),(1,1)}(q) = \mathcal{W}_{(2),(2),(1,1)}(q) = \frac{q^{12} - q^{11} - q^{10} + 2q^9 - q^7 + q^6 - q^5 + 2q^3 - q^2 - q + 1}{q(q^2-1)^3(q-1)^3}$$

$$\tilde{\mathcal{W}}_{(2),(1,1),(1,1)}(q) = \mathcal{W}_{(2),(1,1),(1,1)}(q) = \frac{q^{12} - q^{11} - q^{10} + 2q^9 - q^7 + q^6 - q^5 + 2q^3 - q^2 - q + 1}{q^2(q^2-1)^3(q-1)^3}$$

$$\tilde{\mathcal{W}}_{(1,1),(1,1),(1,1)}(q) = \mathcal{W}_{(1,1),(1,1),(1,1)}(q) = \frac{q^{10} - 3q^8 + 3q^7 + 2q^6 - 5q^5 + 2q^4 + 3q^3 - 3q^2 + 1}{q^2(q^2-1)^3(q-1)^3}$$

$$\begin{aligned} & \mathcal{W}_{(1),(2),(3,1)}(q) = \tilde{\mathcal{W}}_{(1),(2),(3,1)}(q) \\ = \ & \frac{q^{3/2}(q^{13} - 2q^{12} + q^{11} + 2q^{10} - 3q^9 + 2q^8 - 2q^6 + 2q^5 - q + 1)}{(q^4-1)(q^2-1)^2(q-1)^4} \end{aligned}$$

$$\begin{aligned} & \mathcal{W}_{(1,1),(2,1),(3)}(q) = \tilde{\mathcal{W}}_{(1,1),(2,1),(3)}(q) \\ = \ & (q^{19} - q^{18} - q^{17} + q^{16} + q^{15} - q^{13} + q^{11} - q^{10} + q^8 + q^7 - q^6 - 2q^5 \\ & +2q^4 + q^2 - 2q + 1) \cdot \left(q^2(q^3-1)^2(q^2-1)^2(q-1)^4\right)^{-1} \end{aligned}$$

$$\begin{aligned} & \mathcal{W}_{(2),(2),(2,1,1,1)}(q) = \tilde{\mathcal{W}}_{(2),(2),(2,1,1,1)}(q) \\ = \ & (q^{22} - q^{21} - 2q^{20} + 3q^{19} + q^{18} - 3q^{17} + 3q^{15} - q^{14} - 2q^{13} + q^{12} \\ & +q^{11} + q^{10} - 2q^9 - q^8 + 3q^7 - 3q^5 + q^4 + 3q^3 - 2q^2 - q + 1) \\ & \cdot \left(q^{7/2}(q^5-1)(q^3-1)(q^2-1)^3(q-1)^4\right)^{-1} \end{aligned}$$

$$\begin{aligned} & \mathcal{W}_{(1),(2,2),(3,2)}(q) = \tilde{\mathcal{W}}_{(1),(2,2),(3,2)}(q) \\ = \ & (q^{23} - 2q^{22} + q^{21} + q^{20} - q^{19} + q^{18} - 2q^{17} + q^{16} + q^{15} + q^{13} - 3q^{12} \\ & +q^{10} + 2q^9 + q^8 - 2q^7 - 2q^6 + 2q^4 + 2q^3 - 2q^2 - q + 1) \\ & \cdot \left(q(q^4-1)(q^3-1)^2(q^2-1)^3(q-1)^4\right)^{-1} \end{aligned}$$

$$\begin{aligned} & \mathcal{W}_{(3),(3),(2,2,1)}(q) = \tilde{\mathcal{W}}_{(3),(3),(2,2,1)}(q) \\ = \ & (q^{28} - q^{27} - q^{26} + 2q^{24} + 2q^{23} - 3q^{22} - 3q^{21} + 2q^{20} + 3q^{19} + 2q^{18} - 4q^{17} \\ & -3q^{16} + 2q^{15} + 3q^{14} + 2q^{13} - 3q^{12} - 4q^{11} + 2q^{10} + 3q^9 + 2q^8 - 3q^7 - 3q^6 \\ & +2q^5 + 2q^4 - q^2 - q + 1) \cdot \left(q^{1/2}(q^4-1)(q^3-1)^3(q^2-1)^3(q-1)^4\right)^{-1} \end{aligned}$$

$$\begin{aligned} & \mathcal{W}_{(2,1),(4),(3,2)}(q) = \tilde{\mathcal{W}}_{(2,1),(4),(3,2)}(q) \\ = \ & q^2(q^{31} - 2q^{30} + 2q^{29} - 2q^{28} + q^{27} + q^{26} - 2q^{25} + 4q^{24} - 5q^{23} + 4q^{22} - 4q^{21} \\ & +2q^{20} + 2q^{19} - 3q^{18} + 5q^{17} - 7q^{16} + 5q^{15} - 2q^{14} + 2q^{12} - 4q^{11} + 5q^{10} - 3q^6 \\ & +4q^5 - q^4 + q^3 - q^2 - q + 1) \cdot \left((q^4-1)^2(q^3-1)^3(q^2-1)^2(q-1)^5\right)^{-1} \end{aligned}$$

$$
\begin{aligned}
& \mathcal{W}_{(1,1,1,1),(2,2),(2,2,1)}(q) = \tilde{\mathcal{W}}_{(1,1,1,1),(2,2),(2,2,1)}(q) \\
= \; & (q^{35} - q^{34} - 2q^{33} + 2q^{32} + 2q^{31} - 3q^{29} - 2q^{28} + 3q^{27} + 2q^{26} + 2q^{25} - 5q^{24} \\
& -3q^{23} + 3q^{22} + 4q^{21} + 4q^{20} - 9q^{19} - 3q^{18} + 3q^{17} + 4q^{16} + 6q^{15} - 7q^{14} - 3q^{13} \\
& -2q^{12} + 3q^{11} + 7q^{10} - 4q^{9} - q^{8} - 2q^{7} + 3q^{5} - q^{4} - q^{2} - q + 1) \\
& \cdot \Big(q^{13/2}(q^4-1)^2(q^3-1)^3(q^2-1)^4(q-1)^4 \Big)^{-1}
\end{aligned}
$$

$$
\begin{aligned}
& \mathcal{W}_{(2),(3),(3,2,2,1,1)}(q) = \tilde{\mathcal{W}}_{(2),(3),(3,2,2,1,1)}(q) \\
= \; & (q^{37} - q^{36} - 3q^{35} + 5q^{34} - 5q^{32} + 6q^{31} - 3q^{30} - 5q^{29} + 10q^{28} - 4q^{27} - 5q^{26} \\
& +8q^{25} - 5q^{24} - q^{23} + 6q^{22} - 5q^{21} + q^{20} + q^{19} - 2q^{18} + 3q^{17} - 2q^{16} + 3q^{14} - 5q^{13} \\
& +3q^{12} + 3q^{11} - 7q^{10} + 4q^{9} + 2q^{8} - 5q^{7} + 5q^{6} - q^{5} - 4q^{4} + 4q^{3} - 2q + 1) \\
& \cdot \big(q(q^7-1)(q^5-1)(q^4-1)^2(q^3-1)(q^2-1)^4(q-1)^5 \big)^{-1}
\end{aligned}
$$

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