Local Langlands correspondence for GL(2)

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Preliminary Notes

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These pages contain a preliminary version of lectures to be delivered at the International Summer School on p-adic arithmetic geometry, at Zhejiang University, Hangzhou, China in August 2004.

The notes are arranged in three parts. In the first, we give a complete account of the irreducible smooth (complex) representations of $\mathrm{GL}_2(F)$, for a non-Archimedean local field F. In the second, we define L-functions and local constants for such representations, using the approach of Godement and Jacquet. We show that an irreducible representation π of $\mathrm{GL}_2(F)$ is completely determined by the local constants $\varepsilon(\chi\pi, s, \psi)$, as χ ranges over the characters of $F^\times$. In the third part, we recall the local constant of Langlands-Deligne, and discuss the two-dimensional representations of the Weil group of F. We give an explicit construction for the Langlands correspondence when the residual characteristic p of F is odd, but have to take a more indirect approach when $p = 2$.

The notes are not complete: we have left many gaps, some quite large. The main lacunæ cover application of global methods to parts of the construction in Part III.

It has to be emphasized that these are preliminary notes: there are gaps and, doubtless errors. In parts, we have adopted a novel approach which might require reconsideration at a later date.

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I: Representation Theory

In this first part, we study the smooth representation theory of the general linear group $G = \mathrm{GL}_2(F)$ of a non-Archimedean local field F, working always with complex representations. Our ultimate aim is to classify the irreducible smooth representations of G.

1. Finite linear groups

By way of an introductory exercise, we consider the (complex) representation theory of the finite group $G = \mathrm{GL}_2(\boldsymbol{k})$, where $\boldsymbol{k} = \mathbb{F}_q$ is the finite field of q elements.

1.1. The group G has several particularly important (conjugacy classes of) subgroups of geometric significance:

$$B = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \right\}, \quad N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right\}, \quad T = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \right\};$$

B is a *Borel subgroup,* N is the *unipotent radical* of B, and T is a *split maximal torus.* We have $N \lhd B$, $B = T \ltimes N$, and we can identify T with B/N.

The double coset space $B\backslash G/B$ has two elements, namely B and BwB where, for example, $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

We can give a complete account of the conjugacy classes in G. For each irreducible polynomial $f(X) = X^2 + aX + b \in \boldsymbol{k}[X]$, there is a unique conjugacy class of elements of G with characteristic polynomial $f(X)$. Thus there are $(q^2 - q)/2$ such *elliptic* conjugacy classes. The other conjugacy classes have representatives

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \qquad \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix},$$

with a, b ranging over $\boldsymbol{k}^\times$. There are $(q-1) + (q-1)(q-2)/2$ classes of the first kind and $q-1$ of the second

It follows that G has precisely $q^2 - 1$ conjugacy classes and hence $q^2 - 1$ distinct isomorphism classes of irreducible representations.

1.2. We can account for $q-1$ of these representations straightaway: we have the $q-1$ one-dimensional representations $\chi \circ \det$, where χ ranges over the characters of $\boldsymbol{k}^\times$.

The group N is isomorphic to the additive group of $\boldsymbol{k}$. This has q characters, the trivial one plus $q-1$ non-trivial ones. If we fix a non-trivial character ψ of $\boldsymbol{k}$, the function $\psi_a : x \mapsto \psi(ax)$, $x \in \boldsymbol{k}$, ranges over the characters of $\boldsymbol{k}$ as a ranges over $\boldsymbol{k}$. Under the natural action of T (or B) on N, the characters of N thus fall into two conjugacy classes, consisting respectively of the trivial character and all of the non-trivial ones.

We consider next the irreducible representations of G which contain the trivial character of N. Let χ_1, χ_2 be characters of $\boldsymbol{k}^\times$. We form the character

$$\chi = \chi_1 \otimes \chi_2 : \left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right) \longmapsto \chi_1(a)\chi_2(b)$$

of T: we can equally regard this as a character of B trivial on N. An irreducible representation of G containing the trivial character of N is thus an irreducible component of an induced representation $\mathrm{Ind}_B^G\,\chi$, for some character χ of B of this form.

In the same situation, we can form the character

$$\chi^w : \left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right) \longmapsto \chi_2(a)\chi_1(b)$$

of T. We abuse notation and view this as a character of B trivial on N.

Proposition. *Let χ, μ be characters of T, viewed as characters of B which are trivial on N.*

(1) *The space $\mathrm{Hom}_G\big(\mathrm{Ind}_B^G(\chi), \mathrm{Ind}_B^G(\mu)\big)$ is trivial unless $\chi = \mu$ or $\chi = \mu^w$.*

(2) *The spaces $\mathrm{Hom}_G\big(\mathrm{Ind}_B^G(\chi), \mathrm{Ind}_B^G(\chi)\big)$, $\mathrm{Hom}_G\big(\mathrm{Ind}_B^G(\chi), \mathrm{Ind}_B^G(\chi^w)\big)$ have the same dimension. This dimension is 2 if $\chi = \chi^w$, and is 1 otherwise.*

Proof. We use the canonical isomorphism of Frobenius Reciprocity

$$\begin{aligned}\mathrm{Hom}_G\big(\mathrm{Ind}_B^G(\chi), \mathrm{Ind}_B^G(\mu)\big) &\cong \mathrm{Hom}_B\big(\mathrm{r}_B^G\,\mathrm{Ind}_B^G(\chi), \mu\big) \\ &\subset \mathrm{Hom}_N\big(\mathrm{r}_N^G\,\mathrm{Ind}_B^G(\chi), 1_N\big),\end{aligned}$$

where r_B^G, r_N^G denotes the functor of restriction of representations from G to B, N, and 1_N is the trivial character of N. We use the restriction-induction formula of Mackey to get

$$\mathrm{r}_B^G\,\mathrm{Ind}_B^G(\chi) \cong \sum_{y \in B\backslash G/B} \mathrm{Ind}_{B\cap B^y}^B\big(\mathrm{r}_{B\cap B^y}^{B^y}(\chi^y)\big), \tag{1.2.1}$$

where χ^y denotes the character $b \mapsto \chi(yby^{-1})$ of the group $B^y = y^{-1}By$. As remarked in 1.1, we only have to consider the cases $y = 1$ and $y = w$. The term in (1.2.1) corresponding to $y = 1$ is just χ. The restriction of the other term to N is the regular representation of N. The trivial character of N occurs in this term, therefore, with multiplicity one, as the restriction of the character χ^w of B. All assertions of the proposition are now obvious. $\square$

We deduce:

Corollary 1. *The representation* $\operatorname{Ind}_B^G \chi$ *is irreducible if and only if* $\chi \neq \chi^w$. *If* $\chi = \chi^w$, *the representation* $\operatorname{Ind}_B^G \chi$ *has length* 2, *the irreducible components being distinct.*

As an example, consider the trivial character 1_B of B. Clearly, the trivial character 1_G occurs in $\operatorname{Ind}_B^G 1_B$, so

$$\operatorname{Ind}_B^G 1_B = 1_G \oplus \mathrm{St}_G,$$

for a unique irreducible representation St_G, of dimension q, called the *Steinberg representation.*

The representations $\operatorname{Ind}_B^G \chi$, for characters χ of T, are called the *principal series.* The terminology is often extended to include their irreducible components, that is, all irreducible representations of G containing the trivial character of N. Altogether, these account for $(q-1)(q-2)/2 + 2q-2 = (q+q^2)/2 - 1$ of the irreducible representations of G.

1.3. An irreducible representation of G not containing the trivial character of N is called *cuspidal.* Any such representation σ must contain some, and hence every, non-trivial character of N. The cuspidal representations cannot be constructed directly by induction.

Let $\boldsymbol{l}/\boldsymbol{k}$ be a quadratic field extension. Choosing a $\boldsymbol{k}$-basis of $\boldsymbol{l}$ identifies $\boldsymbol{l}$ with the vector space $\boldsymbol{k} \oplus \boldsymbol{k}$ and G with $\operatorname{Aut}_{\boldsymbol{k}}(\boldsymbol{l})$. The natural action of $\boldsymbol{l}^\times$ on $\boldsymbol{l}$ thus gives an embedding of $\boldsymbol{l}^\times$ in G, the G-conjugacy class of which is uniquely determined. We henceforward regard $\boldsymbol{l}^\times$ as a subgroup E of G. Note that any elliptic element of G is conjugate to an element of E.

The non-trivial $\boldsymbol{k}$-automorphism of the field $\boldsymbol{l}$ is $x \mapsto x^q$. A character θ of $\boldsymbol{l}^\times$ is called *regular* if $\theta^q \neq \theta$.

Let Z denote the group of matrices $\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)$, $a \in \boldsymbol{k}^\times$. Let θ be a regular character of $\boldsymbol{l}^\times$ and ψ a non-trivial character of N. We form a character θ_ψ of ZN by

$$\theta_\psi : \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right) u \longmapsto \theta(a)\psi(u), \quad a \in \boldsymbol{k}^\times,\ u \in N.$$

We observe that the representation $\operatorname{Ind}_{ZN}^G \theta_\psi$ is, up to equivalence, independent of the choice of ψ.

Theorem. *Let* θ *be a regular character of* $E = \boldsymbol{l}^\times$ *and* ψ *a non-trivial character of* N.

(1) *The virtual representation*

$$\pi_\theta = \operatorname{Ind}_{ZN}^G \theta_\psi - \operatorname{Ind}_E^G \theta$$

is an irreducible cuspidal representation of G, *of dimension* $q-1$.

(2) *Let θ_1, θ_2 be regular characters of E; then $\pi_{\theta_1} \cong \pi_{\theta_2}$ if and only if $\theta_2 = \theta_1$ or θ_1^q.*

(3) *Every irreducible cuspidal representation of G is of the form π_θ, for some regular character θ of E.*

Proof. We observe that part (3) follows from parts (1) and (2) on counting up the dimensions. One may prove the first two parts directly, by computing characters. One finds:

$$(*)\qquad \begin{aligned} \operatorname{tr}\pi_\theta(z) &= (q-1)\,\theta(z), \quad z \in Z; \\ \operatorname{tr}\pi_\theta(zu) &= -\theta(z), \quad z \in Z,\ 1 \neq u \in N; \\ \operatorname{tr}\pi_\theta(y) &= -(\theta(y)+\theta^q(y)), \quad y \in E \smallsetminus Z. \end{aligned}$$

The character value $\operatorname{tr}\pi_\theta(g)$ is zero if g is not conjugate to an element of $ZN \cup E$. The character table $(*)$ gives part (2) straightaway. One checks that

$$\frac{1}{|G|}\sum_{g\in G} |\operatorname{tr}\pi_\theta(g)|^2 = 1,$$

so π_θ is an irreducible representation of G. Finally,

$$\sum_{u\in N} \operatorname{tr}\pi_\theta(u) = 0,$$

so $\pi_\theta \mid N$ does not contain the trivial character of N, and π_θ is therefore cuspidal. $\square$

Exercise. *Let ψ be a non-trivial character of N, and consider the representation*

$$\Gamma = \operatorname{Ind}_N^G \psi.$$

Let σ be an irreducible representation of G. Show that:

(1) *If $\sigma = \phi \circ \det$, for a character ϕ of $\boldsymbol{k}^\times$, then σ does not occur in Γ.*

(2) *Otherwise, σ occurs in Γ with multiplicity one.*

2. Linear groups over p-adic fields

We fix a prime number p and consider a finite field extension $F/\mathbb{Q}_p$ of the p-adic number field $\mathbb{Q}_p$. We let $\mathfrak{o}$ denote the discrete valuation ring in F and $\mathfrak{p}$ the maximal ideal of $\mathfrak{o}$. (We use notation like $\mathfrak{o}_F$, $\mathfrak{p}_F$ when it is necessary to emphasize the field F.) The field $\boldsymbol{k} = \mathfrak{o}/\mathfrak{p}$ is then finite, with q elements say, and characteristic p.

2.1. Let ϖ be a prime element of F, meaning $\varpi\mathfrak{o} = \mathfrak{p}$. We denote by $x \mapsto \|x\|$ the absolute value on F for which $\|\varpi\| = q^{-1}$. This puts a metric on F, with respect to which F is complete; the metric defines a topology on F, which is locally compact and totally disconnected. The fractional ideals

$$\mathfrak{p}^n = \{x \in F : \|x\| \leqslant q^{-n}\}, \quad n \in \mathbb{Z},$$

are compact open subgroups of F, providing a fundamental system of open neighbourhoods of 0. Likewise, the subgroups $U_F^n = 1+\mathfrak{p}^n$, $n \geqslant 1$, of $F^\times$ are compact open and form a fundamental system of open neighbourhoods of 1. We also write $U_F = \mathfrak{o}^\times$.

2.2. Let V be a finite-dimensional F-vector space. Choosing a basis identifies V with F^n, $n = \dim V$, and F^n carries the product topology. The corresponding topology on V is independent of the choice of basis, and makes V into a topological F-vector space. In exactly the same way, $A = \mathrm{End}_F(V)$ becomes a topological F-algebra which is again locally compact and totally disconnected. The group $G = \mathrm{Aut}_F(V)$ is an open subset of A and, in the induced topology, the operation of inversion of group elements is continuous. Thus G is a locally compact, totally disconnected topological group. If we choose a basis of V and so identify G with $\mathrm{GL}_n(F)$, the compact open subgroups

$$\begin{aligned} K_0 &= \mathrm{GL}_n(\mathfrak{o}), \\ K_d &= 1 + \mathfrak{p}^d \mathrm{M}_n(\mathfrak{o}), \quad d \geqslant 1, \end{aligned}$$

provide a fundamental system of open neighbourhoods of 1.

We recall that a *lattice in V* is a compact open subgroup of V; it is an $\mathfrak{o}$-lattice if it is also an $\mathfrak{o}$-module. One knows that an $\mathfrak{o}$-lattice is the same as the $\mathfrak{o}$-span of an F-basis of V: in particular, any $\mathfrak{o}$-lattice in V is a free $\mathfrak{o}$-module of rank n.

Exercise.

(1) *Let K be a compact subgroup of G. Show that G stabilizes some $\mathfrak{o}$-lattice in V.*
(2) *Deduce that G possesses maximal compact subgroups, and that these are the conjugates of $\mathrm{GL}_n(\mathfrak{o})$.*

2.3. We specialize to the case of $G = \mathrm{GL}_2(F)$. As in §1, we have the following subgroups of G:

$$B = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \right\} \quad N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right\}, \quad T = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \right\}.$$

These are closed subgroups of G; we again have $B = TN$ and we may identify the topological groups $T \cong B/N$.

The elements 1 and $w = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ again provide representatives for the double coset space $B\backslash G/B$, that is

$$G = B \cup BwB.$$

2.4. In this case, the group $G = \mathrm{GL}_2(F)$ admits useful decompositions besides the *Bruhat decomposition* of (2.3). Set $K = \mathrm{GL}_2(\mathfrak{o})$. We then have the *Iwasawa decomposition*

$$G = BK$$

and the related *Cartan decomposition*

$$G = \dot{\bigcup_{\substack{a,b \in \mathbb{Z}, \\ a \leqslant b}}} K \left(\begin{smallmatrix} \varpi^a & 0 \\ 0 & \varpi^b \end{smallmatrix}\right) K.$$

2.5. The standard Iwahori subgroup of G is

$$I = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) : a, c \in U,\ b \in \mathfrak{o},\ c \in \mathfrak{p} \right\}.$$

Write $N' = wNw^{-1}$ for the group of lower triangular unipotent matrices in G. The group I then admits an *Iwahori decomposition*

$$I = I \cap N' \cdot I \cap T \cdot I \cap N.$$

With $K = \mathrm{GL}_2(\mathfrak{o})$ as before, we have a decomposition of Bruhat type:

$$K = I \cup IwI,$$

giving

$$G = BI \cup BwI = B(I \cap N') \cup Bw(I \cap N).$$

These decompositions are particularly useful when examining the asymptotics of various functions on the group G.

Remark 1. Let L be an $\mathfrak{o}$-lattice in the vector space $V = F^2$. The group $K = \mathrm{Aut}_{\mathfrak{o}}(L)$ is then a maximal compact subgroup of G, conjugate to $\mathrm{GL}_2(\mathfrak{o})$. Let L' be an $\mathfrak{o}$-lattice such that

$$\varpi L \subsetneq L' \subsetneq L.$$

The group $I = \mathrm{Aut}_{\mathfrak{o}}(L) \cap \mathrm{Aut}_{\mathfrak{o}}(L')$ is then an Iwahori subgroup of G, conjugate to the standard one above, which is given by $L = \mathfrak{o} \oplus \mathfrak{o}$ and $L' = \mathfrak{o} \oplus \mathfrak{p}$.

Remark 2. Practically everything we do in the next few chapters applies, without change, if we replace F by a non-Archimedean local field of positive characteristic, that is, a field $\boldsymbol{k}((t))$ of formal Laurent series in one variable over a finite field $\boldsymbol{k}$.

3. Smooth representations of locally profinite groups

We now work more generally. Let G be a topological group which is locally compact and totally disconnected: this simply means that the compact open subgroups of G provide a fundamental system of open neighbourhoods of the identity in G. One also calls such groups "locally profinite". We describe the appropriate notion of representation for such groups. Throughout, we have in mind the example of $G = \mathrm{GL}_2(F)$ described above.

3.1. Let G be a locally profinite group, and let (π, V) be a representation of G. Thus V is a complex vector space (usually of infinite dimension) and π is a group homomorphism $G \to \mathrm{Aut}_{\mathbb{C}}(V)$. It is called *smooth* if, for every $v \in V$, there is a compact open subgroup $K = K_v$ of G such that $\pi(x)v = v$, for all $x \in K$. Equivalently, if V^K is the space of $\pi(K)$-fixed vectors in V, we have

$$V = \bigcup_K V^K,$$

where K ranges over the compact open subgroups of G.

A smooth representation (π, V) is called *admissible* if the space V^K is finite-dimensional, for each compact open subgroup K of G.

If (π, V) is a smooth representation of G, then any G-stable subspace of G provides a further smooth representation of G. One says that (π, V) is *irreducible* if V has no proper, G-stable subspace.

For smooth representations (π_i, V_i) of G, the set $\mathrm{Hom}_G(\pi_1, \pi_2)$ is just the space of linear maps $f : V_1 \to V_2$ commuting with the G-actions: $f \circ \pi_1(g) = \pi_2(g) \circ f$, $g \in G$. With this definition, the class of smooth representations of G forms a category $\mathrm{Rep}(G)$.

Example 1. A one-dimensional smooth representation of G is called a *character* of G. Thus a character of G is a homomorphism $G \to \mathbb{C}^\times$ with open kernel.

Example 2. Suppose that G is *compact;* thus G is profinite in the usual sense:

(1) Any open neigbourhood of the identity contains an open *normal* subgroup.
(2) An irreducible smooth representation (π, V) of G is then of finite dimension, since any non-zero $v \in V$ is fixed by an open normal subgroup K, and the translates $\pi(g)v$, $g \in G/K$, span V.
(3) A smooth representation of G is expressible as a direct sum of irreducible representations, such a decomposition being unique up to permutation of the factors.

3.2. Let $\mathcal{H}(G)$ be the space of functions $f : G \to \mathbb{C}$ which are locally constant and of compact support. The group G acts on $\mathcal{H}(G)$ by both left translation λ

and right translation ρ:

$$\begin{aligned} \lambda_g f &: x \longmapsto f(g^{-1}x), \\ \rho_g f &: x \longmapsto f(xg), \end{aligned} \qquad x, g \in G,\ f \in \mathcal{H}(G).$$

Both of these representations of G are smooth.

Since G is locally compact, it admits a left-invariant Haar integral. In our case, it is enough to think of this as a non-zero linear functional $I : \mathcal{H}(G) \to \mathbb{C}$ such that

$$\begin{aligned} I(\lambda_g f) &= I(f), \\ f \geqslant 0 &\Longrightarrow I(f) \geqslant 0, \end{aligned} \quad g \in G,\ f \in \mathcal{H}(G).$$

These conditions determine I up to a positive constant factor. One customarily uses a notation like

$$I(f) = \int_G f(x)\, d\mu(x),$$

thinking of μ as the measure taking a compact open set S to the integral $\mu(S)$ of its characteristic function.

Likewise, G admits a right-invariant Haar integral. Unless otherwise stated, we assume that G is *unimodular,* meaning that any left Haar integral is also a right Haar integral.

Let us fix a Haar measure μ on G. For $f_1, f_2 \in \mathcal{H}(G)$, we define

$$f_1 * f_2(g) = \int_G f_1(x) f_2(x^{-1}g)\, d\mu(x).$$

One checks easily that $f_1 * f_2 \in \mathcal{H}(G)$, and that the binary operation $*$, called *convolution,* is associative.

Exercises. *Let $f \in \mathcal{H}(G)$. Show:*

(1) *There is a compact open subgroup K of G such that $f(xgy) = f(g)$, for $g \in G$, $x, y \in K$.*
(2) *With K as in (1), the function f is a finite linear combination of characteristic functions of double cosets KgK.*
(3) *Let K be a compact open subgroup of G; let $e_K \in \mathcal{H}(G)$ be the function which takes the value $\mu(K)^{-1}$ on K and is zero elsewhere. Show that $e_K * e_K = e_K$, and that $e_K * \mathcal{H}(G)$ is the space of functions in $\mathcal{H}(G)$ fixed under left translation by K.*
(4) *The space $e_K * \mathcal{H}(G) * e_K$ is a sub-algebra of $\mathcal{H}(G)$, with unit element e_K.*

3.3. Let (π, V) be a smooth representation of G. We can extend π to an algebra homomorphism $\pi : \mathcal{H}(G) \to \mathrm{End}_{\mathbb{C}}(V)$ by setting

$$\pi(f)v = \int_G f(g)\pi(g)v \, d\mu(g), \quad f \in \mathcal{H}(G), \ v \in V.$$

Arguing formally, one sees easily that $\pi(f_1 * f_2) = \pi(f_1) \circ \pi(f_2)$, $f_i \in \mathcal{H}(G)$.

The integral actually reduces to a finite sum: the key point to observe is that, if K is a compact open subgroup of G and $v \in V^K$, then $\pi(e_K)v = v$. One can recover the action of G from the action of $\mathcal{H}(G)$, as follows. For $g \in G$ and $v \in V$, choose a compact open subgroup K such that $v \in V^K$ and let f be the characteristic function of the coset gK. We then have

$$\pi(g)v = \mu(K)^{-1}\, \pi(f)v.$$

It is particularly useful to note the following. The space V^K has a unique K-complement $V(K)$ in V: $V = V^K \oplus V(K)$. One can recognize this decomposition in terms of the action of $\pi(e_K)$:

$$\begin{aligned} V^K &= \{v \in V : \pi(e_K)v = v\}, \\ V(K) &= \{v \in V : \pi(e_K)v = 0\}. \end{aligned}$$

Let M be an $\mathcal{H}(G)$-module. One says that M is smooth if

$$M = \bigcup_K e_K M,$$

where K ranges over the compact open subgroups of G. The process just outlined gives an equivalence of $\mathrm{Rep}(G)$ with the category $\mathcal{H}(G)$-Mod of smooth $\mathcal{H}(G)$-modules. Thus the *Hecke algebra* $\mathcal{H}(G)$ plays the role of the group algebra in the smooth representation theory of the group G.

Exercises. *Let K be a compact open subgroup of G, and let (π, V) be a smooth representation of G.*

1. *Show that $V^K = \pi(e_K)V$ is a module over the algebra $e_K * \mathcal{H}(G) * e_K$, on which $\pi(e_K)$ acts as the identity.*
2. *Suppose that (π, V) is irreducible. Show that V^K is either zero or a simple $e_K * \mathcal{H}(G) * e_K$-module.*
3. *Write $\mathcal{H}(G, K) = e_K * \mathcal{H}(G) * e_K$ and let M be a simple $\mathcal{H}(G, K)$-module. Let G act on $\mathcal{V} = \mathcal{H}(G) \otimes_{\mathcal{H}(G,K)} M$ by left translation on the first factor. Show that*
   (a) *$\mathcal{V}$ has a unique maximal G-subspace $\mathcal{W}$ such that $\mathcal{W} \cap e_K \otimes M = 0$;*

(b) *$V_M = \mathcal{V}/\mathcal{W}$ is an irreducible smooth representation of G such that $V_M^K \cong M$ as $\mathcal{H}(G,K)$-module;*
(c) *the process $M \mapsto V_M$ gives a bijection between the set of isomorphism classes of simple $\mathcal{H}(G,K)$-modules and the set of equivalence classes of irreducible smooth representations (π, V) of G such that $V^K \neq 0$.*

(4) *Deduce that if (π_i, V_i) are irreducible smooth representations of G with $V_i^K \neq 0$, $i = 1, 2$, then $\pi_1 \cong \pi_2$ if and only if $V_1^K \cong V_2^K$ over $\mathcal{H}(G,K)$.*

Comment. Exercises 3(c) and 4 do not extend to reducible representations.

3.4. Let (π, V) be a smooth representation of G. We set $V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$; thus we have the canonical evaluation pairing

$$\begin{aligned} V^* \times V &\longrightarrow \mathbb{C}, \\ (v^*, v) &\longmapsto \langle v^*, v \rangle. \end{aligned}$$

We define a representation π^* of G on V^* by

$$\langle \pi^*(g)v^*, v \rangle = \langle v^*, \pi(g^{-1})v \rangle.$$

The representation π^* is not necessarily smooth; we accordingly put

$$V^\vee = \bigcup_K (V^*)^K,$$

with K ranging over the compact open subgroups of G, and we let $\check{\pi}$ be the natural action of G on $V^\vee$. Thus $(\check{\pi}, V^\vee)$ provides a smooth representation of G, called the *contragredient,* or *smooth dual,* of (π, V).

Observe that the space $(V^\vee)^K$ is naturally identified with the linear dual $\mathrm{Hom}_{\mathbb{C}}(V^K, \mathbb{C})$. We write $V = V^K \oplus V(K)$, where $V(K)$ is the space of $v \in V$ annihilated by $\pi(e_K)$; then $(V^\vee)^K$ is the space of linear forms on V which vanish on $V(K)$.

In all cases we have a canonical G-injection $V \to (V^\vee)^\vee$. This is not usually an isomorphism:

Exercise. *Let (π, V) be a smooth representation of G. The following are equivalent:*

(1) *(π, V) is admissible;*
(2) *$(\check{\pi}, V^\vee)$ is admissible;*
(3) *the canonical map $V \to (V^\vee)^\vee$ is an isomorphism.*

In particular, the smooth dual of an irreducible admissible representation is irreducible.

3.5. We now impose a technical hypothesis: we assume that the set G/K is *countable,* for some (equivalently any) compact open subgroup K of G. (This holds, for example, if $G = \mathrm{GL}_2(F)$, as in §2.) The $\mathbb{C}$-algebra $e_K * \mathcal{H}(G) * e_K$ then has countable dimension. The algebra $\mathcal{H}(G)$ is the union of an ascending sequence of sub-algebras $e_{K_i} * \mathcal{H}(G) * e_{K_i}$, so the dimension of $\mathcal{H}(G)$ is also countable. This has the following consequence:

Lemma. *If (π, V) is an irreducible smooth representation of G, then the dimension of V is countable.*

The main reason for introducing the countability hypothesis is:

Schur's Lemma. *If (π, V) is an irreducible smooth representation of G, then* $\mathrm{End}_G(V) = \mathbb{C}$.

Proof. Let $\phi \in \mathrm{End}_G(V)$, $\phi \neq 0$. The image and the kernel of ϕ are both G-subspaces of V, so ϕ is bijective and invertible. Thus $\mathrm{End}_G(V)$ is a complex division algebra. If we fix $v \in V$, $v \neq 0$, an element $\phi \in \mathrm{End}_G(V)$ is determined uniquely by the value $\phi(v)$. We deduce that $\mathrm{End}_G(V)$ has countable dimension. However, any $\phi \in \mathrm{End}_G(V)$, $\phi \notin \mathbb{C}$, is transcendental over $\mathbb{C}$, and generates a field $\mathbb{C}(\phi) \subset \mathrm{End}_G(V)$; the $\mathbb{C}$-dimension of $\mathbb{C}(\phi)$ is then uncountable, which is impossible. $\square$

Corollary 1. *The centre Z of G acts on V via a character $\omega_\pi : Z \to \mathbb{C}^\times$.*

One calls ω_π the *central character* of π.

Corollary 2. *If G is abelian, any irreducible smooth representation of G is one-dimensional.*

Exercise. *Suppose that G/Z is compact. Show that any irreducible smooth representation of G is finite-dimensional.*

Remark. The converse of Schur's Lemma is *false:* see 4.9.2 below for an example.

4. Induced representations

We continue, for the moment, in the general situation of §3, and introduce a basic construction. We then specialize to the case $G = \mathrm{GL}_2(F)$.

4.1. Let H be a closed subgroup of G. If (π, V) is a smooth representation of G, the restriction of π to H provides a smooth representation of H.

In the opposite direction, let (σ, W) be a smooth representation of H, and let V_∞ denote the space of functions

$$\begin{aligned} f : G &\longrightarrow W, \\ f(hg) &= \sigma(h) f(g), \end{aligned} \qquad h \in H,\ g \in G. \tag{4.1.1}$$

The group G then acts on V_∞ by right translation: for $g \in G$ and $f \in V_\infty$, gf is the function $x \mapsto f(xg)$. The representation of G provided by V_∞ is not necessarily smooth. We therefore define

$$V = \bigcup_K (V_\infty)^K,$$

with K ranging over the compact open subgroups of G. The set V is a G-subspace of V_∞, and it provides a smooth representation of G. We usually denote it by $\mathrm{Ind}_H^G \sigma$, and call it the representation of G which is *smoothly induced* by σ. In particular, the process $\sigma \mapsto \mathrm{Ind}_H^G \sigma$ is an exact, additive functor $\mathrm{Rep}(H) \to \mathrm{Rep}(G)$. It is transitive with respect to the closed subgroup H.

The key point is the following. The space V underlying $\mathrm{Ind}_H^G \sigma$ carries a canonical H-homomorphism

$$\begin{aligned} a_\sigma : V &\longrightarrow W, \\ f &\longmapsto f(1). \end{aligned}$$

One sees readily:

Frobenius Reciprocity. *Let (π, V) be a smooth representation of G. The map $\phi \mapsto a_\sigma \circ \phi$ induces an isomorphism*

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G(\sigma)) \xrightarrow{\approx} \mathrm{Hom}_H(\pi, \sigma).$$

Remark. It is helpful to note that, as a consequence, the kernel of the map a_σ contains no non-zero G-subspace of $\mathrm{Ind}_H^G \sigma$.

There is a variation on this construction; one replaces V by its G-subspace V_c of right G-smooth functions satisfying (4.1.1) and which are *compactly supported modulo H*. This yields a representation c-$\mathrm{Ind}_H^G \sigma$. If the quotient space $H\backslash G$ is compact, there is no distinction between Ind_H^G and c-Ind_H^G. In general, however, these representations are different, and they often have rather different properties.

4.2. We now return to the situation $G = \mathrm{GL}_2(F)$, and use the notation of §2. We construct representations of G using the technique of induction introduced above, following the example of (1.2).

Let χ be a character of T; therefore χ is of the form

$$\chi : \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \longmapsto \chi_1(a)\chi_2(b),$$

for characters χ_1, χ_2 of $F^\times$. We write $\chi = \chi_1 \otimes \chi_2$.

We view χ as an irreducible smooth representation of B which is trivial on N, and form the induced representation $\mathrm{Ind}_B^G \chi$ of G. The Iwasawa decomposition $G = BK$, $K = \mathrm{GL}_2(\mathfrak{o})$, implies that $B\backslash G$ is compact, so $\mathrm{Ind}_B^G \chi = c\text{-}\mathrm{Ind}_B^G \chi$.

Let (π, V) be a smooth representation of G. Let $V(N)$ be the subspace of V spanned by all vectors $v - \pi(n)v$, $v \in V$, $n \in N$. Thus $V_N = V/V(N)$ is the maximal quotient of V on which N acts trivially. The space V_N provides a smooth representation of $T = B/N$, which we denote (π_N, V_N). One calls (π_N, V_N) the *Jacquet module* of (π, V) at N. The process $\pi \mapsto \pi_N$ gives an exact, additive functor $\mathrm{Rep}(G) \to \mathrm{Rep}(T)$.

Exercise. *Let $v \in V$; then $v \in V(N)$ if and only if there exists a compact open subgroup N_0 of N such that*

$$\int_{N_0} \pi(n)v\, dn = 0.$$

We have

$$\mathrm{Hom}_B(\pi, \chi) = \mathrm{Hom}_T(\pi_N, \chi),$$

so Frobenius Reciprocity implies

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_B^G(\chi)) \cong \mathrm{Hom}_T(\pi_N, \chi).$$

More generally:

(4.2.1) *Let π be a smooth representation of G and σ a smooth representation of T. Frobenius Reciprocity induces a canonical isomorphism*

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_B^G(\sigma)) \cong \mathrm{Hom}_T(\pi_N, \sigma),$$

which is functorial in both arguments.

Remark. If we take $\pi = \mathrm{Ind}\,\sigma$, this isomorphism gives a canonical T-homomorphism

$$a_\sigma : (\mathrm{Ind}\,\sigma)_N \longrightarrow \sigma$$

corresponding to the identity map on $\mathrm{Ind}\,\sigma$. It is induced the canonical map a_σ of (4.1).

Parallel to the situation of §1, we have a fundamental division of the irreducible representations of the group $G = \mathrm{GL}_2(F)$ into two classes. The characterization of the first of these is given by:

Proposition. *Let (π, V) be an irreducible smooth representation of G. The following are equivalent:*

(1) *The Jacquet module V_N is non-zero.*
(2) *The representation π is isomorphic to a G-subspace of a representation $\mathrm{Ind}_B^G \chi$, for some character χ of T.*

Proof. The implication (2) $\Rightarrow$ (1) follows from Remark 4.1. To prove (1) $\Rightarrow$ (2), we have to show that the Jacquet module V_N admits an irreducible T- (or B-) quotient. To do this, it is enough to show that V_N is finitely generated over B.

Choose $v \in V$, $V \neq 0$. Since V is irreducible over G, any element of V is a finite linear combination of translates $\pi(g)v$ of v, for various $g \in G$. Write $K = \mathrm{GL}_2(\mathfrak{o})$. The vector v is fixed by a subgroup K' of K of finite index; let $\{v_1, v_2, \dots, v_r\}$ be the distinct elements of the form $\pi(k)v$, $k \in K$. In particular, $r \leqslant (K{:}K')$. Since $G = BK$, the elements $v_1, \dots, v_r$ generate V over B, and their images generate V_N over T. $\square$

An irreducible smooth representation (π, V) of G is called *cuspidal* if V_N is zero. In the literature, cuspidal representations are usually called *supercuspidal* or *absolutely cuspidal.*

Exercise. *Let* (π, V) *be an irreducible smooth representation* (π, V) *with a non-trivial* $\pi(N)$*-fixed vector. Show that* $\pi = \phi \circ \det$*, for some character* ϕ *of* $F^\times$*.*

Hint. If $v \in V$ is fixed by N, it is fixed by the subgroup H of G generated by N and some open subgroup K. Show that H contains $\mathrm{SL}_2(F)$.

4.3. An important fact concerns the structure of the Jacquet module $(\mathrm{Ind}_B^G\,\chi)_N$ of an induced representation. Here, it will be useful to give a more general result.

Let μ_N be a Haar measure on N and $t \in T$. The measure $X \mapsto \mu_N(t^{-1}Xt)$ is then another Haar measure, of the form $\delta_B(t)\mu_N$, for some $\delta_B(t) > 0$. Thus

$$\int_N f(txt^{-1})\,d\mu_N(x) = \delta_B(t)\int_N f(x)\,d\mu_N(x).$$

Explicitly,

$$\delta_B \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} = \|t_2/t_1\|.$$

Thus δ_B is a character of T taking positive real values.

Bernstein-Zelevinsky Lemma. *Let* σ *be a smooth representation of* T *and set* $\pi = \mathrm{Ind}_B^G\,\sigma$*. There is an exact sequence*

$$0 \to \sigma^w \otimes \delta_B^{-1} \longrightarrow \pi_N \xrightarrow{\ a_\sigma\ } \sigma \to 0.$$

We give the proof later, in (4.10).

4.4. Let (π, V) be an irreducible smooth representation of G, and let $(\check{\pi}, V^\vee)$ be its smooth dual. A *coefficient* of π is a function on G which is a finite linear combination of functions

$$\gamma_{\check{v}\otimes v} : g \longmapsto \langle \check{v}, \pi(g)v\rangle,$$

for $v \in V$, $\check{v} \in V^\vee$. Observe that if f is a coefficient of π and ω_π is the central character of π, then

$$f(zg) = \omega_\pi(z)\, f(g), \quad g \in G,\ z \in Z.$$

We say that (π, V) is γ-*cuspidal* if every coefficient of π is compactly supported modulo Z. (This is a temporary and idiosyncratic terminology, not used elsewhere.)

Remark. Let (π, V) be irreducible and admissible, and suppose that π has a non-zero coefficient which is compactly supported modulo Z. Since the dual $(\check{\pi}, V^\vee)$ is also irreducible, it follows that every coefficient of π is compactly supported modulo Z and hence that π is γ-cuspidal.

The following pair of fundamental results are proved together:

Theorem 1. *Let (π, V) be an irreducible smooth representation of G; then (π, V) is admissible.*

Theorem 2. *Let (π, V) be an irreducible smooth representation of G; then (π, V) is cuspidal if and only if it is γ-cuspidal.*

There are several stages to the proofs. We start with an elementary (and very general) result.

We fix a character χ of Z and consider the space $\mathcal{H}_\chi(G)$ of smooth functions $f : G \to \mathbb{C}$ which satisfy

$$f(zg) = \chi(z)^{-1} f(g), \quad z \in Z,\ g \in G,$$

and which are compactly supported modulo Z. Using a Haar measure $\mu_{G/Z}$ on the unimodular, locally profinite group G/Z, we define convolution on $\mathcal{H}_\chi(G)$, thereby making it into an associative $\mathbb{C}$-algebra of countable dimension. Let (π, V) be a smooth representation of G *admitting* χ *as central character,* meaning that $\pi(zg) = \chi(z)\pi(g)$, $z \in Z$, $g \in G$. As in the standard case (3.2), we extend π to a representation of $\mathcal{H}_\chi(G)$ on V.

Let K be a compact open subgroup of G such that $K \cap Z \subset \operatorname{Ker}\chi$. Let $e_K \in \mathcal{H}_\chi(G)$ be the function zero outside KZ and taking the value $\mu_{G/Z}(KZ/Z)^{-1}$ on K. Then e_K is idempotent and $\mathcal{H}_\chi(G, K) = e_K * \mathcal{H}_\chi(G) * e_K$ is the sub-algebra of $\mathcal{H}_\chi(G)$ consisting of the functions invariant under K on both sides. Let (π, V) be some smooth representation of G, admitting χ as central character. The natural action of $\mathcal{H}_\chi(G)$ on V makes V^K into an $\mathcal{H}_\chi(G, K)$-module on which e_K acts as the identity.

Lemma. *Let (π, V) be a smooth representation of G admitting χ as central character. The following are equivalent:*

(1) *(π, V) is irreducible;*
(2) *for every compact open subgroup K of G with $K \cap Z \subset \operatorname{Ker}\chi$, the space V^K is either zero or a simple $\mathcal{H}_\chi(G, K)$-module.*

Proof. Choose K so that $V^K \neq 0$. If $\mathcal{V}$ is a proper $\mathcal{H}_\chi(G,K)$-submodule of V^K, then $W = \pi(\mathcal{H}_\chi(G))\mathcal{V}$ is a G-subspace of V and $\pi(e_K)W = \mathcal{V}$. The lemma follows immediately. $\square$

4.5. We now prove Theorem 1 in the case where the irreducible smooth representation (π, V) is γ-*cuspidal.* Let $\chi = \omega_\pi$ be the central character of π, and let K range over the compact open subgroups of G such that $V^K \neq 0$ and $K \cap Z \subset \operatorname{Ker}\chi$. Suppose, for a contradiction, that there exists K for which V^K has infinite dimension. The dimension of V^K is countably infinite, so that of $(V^\vee)^K = \operatorname{Hom}_{\mathbb{C}}(V^K, \mathbb{C})$ is uncountably infinite. However, for $\check{v} \in (V^\vee)^K$ and $v \in V^K$, the function $g \mapsto \langle \check{v}, \pi(g^{-1})v\rangle$ lies in $\mathcal{H}_\chi(G,K)$. Assuming $v \neq 0$, we have $V^K = \pi(\mathcal{H}_\chi(G,K))v$, since V^K is $\mathcal{H}_\chi(G,K)$-simple. It follows that $\check{v} \mapsto \gamma_{\check{v}\otimes v}$ gives an injective linear map $(V^\vee)^K \to \mathcal{H}_\chi(G,K)$. Since $\mathcal{H}_\chi(G,K)$ has countable dimension, this is impossible. $\square$

The fact that the coefficients of π lie in $\mathcal{H}_\chi(G,K)$ has a further interesting consequence, which we shall need later in the proof:

Casselman's projectivity theorem. *Let (π, V) be irreducible and γ-cuspidal. Let (σ, W) be a smooth representation of G satisfying $\sigma(zg) = \omega_\pi(z)\sigma(g)$, for $g \in G$ and $z \in Z$. Any surjective G-homomorphism $f : W \to V$ splits.*

We give the proof in (4.11).

4.6. We now prove one implication in Theorem 2:

Proposition. *Let (π, V) be an irreducible smooth representation of G such that $V_N = 0$. Then π is γ-cuspidal.*

Proof. We define

$$\alpha(t) = t_1/t_2, \qquad t = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \in T,$$

and, for $0 < c \leqslant 1$, we set

$$T^-(c) = \{t \in T : \|\alpha(t)\| < c\}, \quad T^- = \bigcup_{c \leqslant 1} T^-(c).$$

Thus, if $g \in G$ is written as a matrix (g_{ij}), we have

$$\|(tgt^{-1})_{12}\| < c\|g_{12}\|, \quad t \in T^-(c).$$

Moreover, the set $T^- \smallsetminus T^-(c)$ is compact modulo Z when $c < 1$.

Lemma. *Let $v \in V$, $\check{v} \in V^\vee$. There exists c, $0 < c < 1$, such that*

$$\operatorname{supp}(\gamma_{\check{v}\otimes v}) \cap T^-(c) = \emptyset.$$

In particular, $\operatorname{supp}(\gamma_{\check{v}\otimes v}) \cap T^-$ *is compact modulo Z.*

Proof. We choose a compact open subgroup N_1 of N which fixes $\check{v}$. Since $V_N = 0$, there is a compact open subgroup N_2 of N such that

$$\int_{N_2} \pi(x)v \, dx = 0.$$

We then have

$$\int_{N_0} \pi(x)v \, dx = 0,$$

for any compact open subgroup N_0 of N containing N_2. However, there exists $c > 0$ such that $tN_2t^{-1} \subset N_1$ for all $t \in T^-(c)$. For such t we have (for certain positive constants κ_1, κ_2)

$$\begin{aligned} \langle \check{v}, \pi(t)v\rangle &= \kappa_1 \int_{N_1} \langle \check{\pi}(x^{-1})\check{v}, \pi(t)v\rangle \, dx \\ &= \kappa_1 \int_{N_1} \langle \check{\pi}(t^{-1})\check{v}, \pi(t^{-1}xt)v\rangle \, dx \\ &= \kappa_2 \int_{t^{-1}N_1t} \langle \check{\pi}(t^{-1})\check{v}, \pi(x)v\rangle \, dx \\ &= 0, \end{aligned}$$

since $t^{-1}N_1t \supset N_2$. $\square$

Returning to the proof of the proposition, we fix a non-zero coefficient $f = \gamma_{\check{v}\otimes v}$ of π. We write $K = \mathrm{GL}_2(\mathfrak{o})$ and recall that $G = KT^-K$. We let K' be an open subgroup of K fixing both $\check{v}$ and v, and we let $k_1, k_2, \ldots, k_r$ be a set of coset representatives for K/K'. Thus, if $g \in G$, we have $t \in T^-$ such that

$$KgK = KtK = \bigcup_{i,j} K'k_i^{-1}tk_jK'.$$

It follows that

$$\operatorname{supp} f \subset \bigcup_{i,j} k_i^{-1}(\operatorname{supp} f_{ij} \cap T^-)k_j,$$

where f_{ij} denotes the coefficient function $x \mapsto f(k_ixk_j^{-1})$. This set is compact modulo Z, by the lemma. $\square$

4.7. We now complete the proof of Theorem 1. We take an irreducible smooth representation (π, V) of G, which is not γ-cuspidal: thus (4.6) $V_N \neq 0$. We have to show that π is admissible. By Proposition 4.2, π is equivalent to a sub-representation of $\mathrm{Ind}_B^G \chi$, for some character χ of T. It is enough, therefore, to prove that $\mathrm{Ind}\,\chi$ is admissible.

We fix a compact open subgroup K of G; we may assume that $K \subset K_0 = \mathrm{GL}_2(\mathfrak{o})$. The space $X(K)$ of K-fixed points in $\mathrm{Ind}\,\chi$ consists of the functions $f : G \to \mathbb{C}$ satisfying

$$f(bgk) = \chi(b)f(g), \quad b \in B,\ g \in G,\ k \in K. \tag{4.7.1}$$

We have $G = BK_0$, so the set $B\backslash G/K$ is finite, and each double coset BgK supports, at most, a one-dimensional space of functions satisfying (4.7.1). Thus $X(K)$ is finite-dimensional, as required.

This completes the proof of Theorem 1. $\square$

4.8. We now complete the proof of Theorem 2. We are given an irreducible smooth representation (π, V) of G such that $V_N \neq 0$. We have to show that π is not γ-cuspidal. We can assume that π is a sub-representation of $\mathrm{Ind}_B^G \chi$, for a character χ of B trivial on N. We suppose, for a contradiction, that π is γ-cuspidal.

We first remark that π is then a direct summand of $\mathrm{Ind}\,\chi$. For, the inclusion $\pi \hookrightarrow \mathrm{Ind}\,\chi$ gives a G-surjection $(\mathrm{Ind}\,\chi)^\vee \twoheadrightarrow \check{\pi}$. By the projectivity theorem, there is a sub-representation τ of $(\mathrm{Ind}\,\chi)^\vee$ such that $(\mathrm{Ind}\,\chi)^\vee = \check{\pi} \oplus \tau$. Since both π and $\mathrm{Ind}\,\chi$ are admissible, we may dualize a second time to get $\mathrm{Ind}\,\chi = \pi \oplus \check{\tau}$.

The lemma of Bernstein-Zelevinsky implies that $(\mathrm{Ind}\,\chi)_N$ has dimension 2, and π_N is a non-zero direct summand of this.

Set $\omega = \chi \mid Z$, and consider the representation $\varOmega = c\text{-}\mathrm{Ind}_Z^T \omega$. There is an obvious surjection $\varOmega \to \chi$, but $\varOmega$ has no finite-dimensional T-subspace. We view $\varOmega$ as a smooth representation of B trivial on N and form the representation $\mathrm{Ind}_B^G \varOmega$. The Bernstein-Zelevinsky Lemma implies that $(\mathrm{Ind}\,\varOmega)_N$ has no finite-dimensional T-subspace.

The B-surjection $\varOmega \to \chi$ induces a G-surjection $\mathrm{Ind}\,\varOmega \to \mathrm{Ind}\,\chi$ and hence a G-surjection $\varPhi : \mathrm{Ind}\,\varOmega \to \pi$. Since π is γ-cuspidal, $\varPhi$ must split: there is a G-homomorphism $\varTheta : \pi \to \mathrm{Ind}\,\varOmega$ such that $\varPhi \circ \varTheta$ is the identity. On the other hand, $\varPhi$ induces a surjection $\varPhi_N : (\mathrm{Ind}\,\varOmega)_N \to \pi_N$ which is split by $\varTheta_N : \pi_N \to (\mathrm{Ind}\,\varOmega)_N$. This is impossible. $\square$

We have completed the proofs of Theorems 1 and 2. We can re-assemble the arguments to get:

Corollary. *Let χ be a character of T and let (π, V) be a sub-quotient of $\mathrm{Ind}_B^G \chi$.*

(1) *The Jacquet V_N is non-zero, and*
(2) *the representation $\mathrm{Ind}\,\chi$ has G-composition length at most 2.*

Proof. If V_N were zero, π would be γ-cuspidal and, arguing as in the preceding paragraph, π would be isomorphic to a G-subspace of $\operatorname{Ind}\chi$. Its Jacquet module π_N would therefore be non-zero (*cf.* Remark 4.1). It follows that the G-composition length of $\operatorname{Ind}\chi$ is at most the T-composition length of $(\operatorname{Ind}\chi)_N$ which, by the Bernstein-Zelevinsky Lemma, is 2. $\square$

4.9. To describe the principal series representations $\operatorname{Ind}_B^G\chi$ completely, we have to determine which of them are irreducible, and describe the composition series of those which are not.

It will be useful to have a new notation. If π is a smooth representation of G and ϕ is a character of $F^\times$, the representation $g \mapsto \phi(\det g)\pi(g)$ is again smooth. We denote it $\phi\pi$. We call $\phi\pi$ the *twist* of π by ϕ. If σ is a smooth representation of T, we can similarly define $\phi\sigma$, and we have the relation

$$\operatorname{Ind}_B^G(\phi\sigma) \cong \phi\cdot\operatorname{Ind}_B^G\sigma.$$

We start with some examples.

(4.9.1) Let 1_T be the trivial character of T. The space $\operatorname{Ind}1_T$ contains the constant function $g\mapsto 1$, and this spans a G-subspace, of dimension 1, equivalent to the trivial representation 1_G of G. The quotient $\operatorname{Ind}1_T/1_G$ is therefore irreducible; it is called the *Steinberg representation* St_G of G.

The space of left B-invariant functions on G admits no positive, right G-invariant integral; it follows that the exact sequence

$$0\to 1_G \longrightarrow \operatorname{Ind}1_T \longrightarrow \mathrm{St}_G \to 0 \qquad (\dagger)$$

does not split over G. (One can equally deduce this from Corollary 4.8 and Remark 4.1.)

Remark. The non-splitting of the exact sequence (†) implies that $\operatorname{End}_G(\operatorname{Ind}1_T)$ has dimension 1, while $\operatorname{Ind}1_T$ is not irreducible. Thus, unlike in the complex representation theory of finite groups, the converse of Schur's Lemma does not hold here.

(4.9.2) Consider now the representation $\operatorname{Ind}_B^G\delta_B^{-1}$. The space of functions $f : G\to\mathbb{C}$ satisfying $f(bg) = \delta_B^{-1}(b)f(g)$, for $g\in G$, $b\in B$, does admit a right G-invariant, positive integral. Let us denote this by

$$f \longmapsto \int_{B\backslash G} f(g)\,d\dot g.$$

The map $f\mapsto \int f(g)\,d\dot g$ then identifies 1_G with a G-quotient of $\operatorname{Ind}\delta_B^{-1}$. Let us temporarily denote the kernel of this map by St'_G, so that we have a G-exact sequence

$$0\to \mathrm{St}'_G \longrightarrow \operatorname{Ind}\delta_B^{-1} \longrightarrow 1_G \to 0.$$

We note that

$$\left(\mathrm{St}'_G\right)_N = \delta_B^{-1} = (\mathrm{St}_G)_N\,.$$

Remark. We recall that $G = BK$, $K = \mathrm{GL}_2(\mathfrak{o})$. For a function f on G, satisfying $f(bg) = \delta_B^{-1}(b)f(g)$, $b \in B$, $g \in G$, we have

$$\int_{B\backslash G} f(g)\, d\dot{g} = \int_K f(k)\, dk,$$

for a Haar measure dk on K.

(4.9.3) In the next section, we will prove:

Theorem 3. *Let $\chi = \chi_1 \otimes \chi_2$ be a character of T, not of the form $\phi \circ \det$ or $\delta_B \otimes \phi \circ \det$, for a character ϕ of $F^\times$. Then $\mathrm{Ind}_B^G\, \chi$ is irreducible, and we have*

$$\mathrm{Ind}_B^G\, \chi \cong \mathrm{Ind}_B^G\, \chi^w \delta_B^{-1}.$$

Exercises. *Assume Theorem 3.*

(1) *Show that the isomorphism of Theorem 3 is the only isomorphism between principal series representations, that is, if χ and ξ are characters of T, $\chi \neq \xi$, then $\mathrm{Ind}_B^G\, \xi \cong \mathrm{Ind}_B^G\, \xi$ if and only χ satisfies the conditions of the theorem and $\xi = \chi^w \delta_B^{-1}$.*
(2) *Show that $\mathrm{St}'_G \cong \mathrm{St}_G$.*

Remark. It is often convenient to use a modified, or "normalized" version of the induction functor Ind_B^G: if σ is a smooth representation of T, one defines

$$\boldsymbol{\iota}_B^G\, \sigma = \mathrm{Ind}_B^G\, \delta_B^{-\frac{1}{2}}\, \sigma.$$

This has the property $\boldsymbol{\iota}_B^G(\check{\sigma}) \cong (\boldsymbol{\iota}_B^G \sigma)^\vee$, and gives rise to more convenient combinatorics.

4.10. We now prove the lemma of Bernstein-Zelevinsky. We are given a smooth representation (σ, W) of T, which we view as a representation of B trivial on N. We set $(\pi, V) = \mathrm{Ind}_B^G\, \sigma$: thus V is the space of G-smooth functions $f : G \to W$ such that $f(bg) = \sigma(b)f(g)$, $b \in B$, $g \in G$. The canonical map $a_\sigma : V \to W$ is then restriction of functions to B. Set $V' = \mathrm{Ker}\, a_\sigma$. Thus V' provides a smooth representation of B and we get an exact sequence

$$0 \to V'_N \longrightarrow V_N \longrightarrow W \to 0,$$

where V'_N is the maximal quotient of the B-space V' on which N acts trivially.

We recall the Bruhat decomposition $G = B \cup BwB$, where $w = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. A function $f \in V$ thus lies in V' if and only if $\mathrm{supp}\, f \subset BwB = BwN$. More precisely:

Lemma. *Let $f \in V$; then $f \in V'$ if and only if there is a compact open subgroup N_0 of N (depending on f) such that* $\operatorname{supp} f \subset BwN_0$.

Proof. Let I be the standard Iwahori subgroup of G (as in (2.5)); write $I_N = I \cap N$, $I_{N'} = I \cap N'$. We use the decomposition

$$G = BwI_N \cup BI_{N'}.$$

A function $f \in V$ lies in V' if and only if $f(1) = 0$. Since f is G-smooth, such a function f vanishes on a set $B\mathcal{N}$, for some closed neighbourhood $\mathcal{N}$ of 1 in $I_{N'}$. The identity

$$\left(\begin{smallmatrix} 1 & 0 \\ x & 1 \end{smallmatrix}\right) \in Bw \left(\begin{smallmatrix} 1 & x^{-1} \\ 0 & 1 \end{smallmatrix}\right), \quad x \neq 0,$$

implies that $\operatorname{supp} f \subset BwN_0$, for some compact open subgroup N_0 of N, as required. □

Let $f \in V'$; in view of the lemma, we can define a function $f_N : T \to W$ by

$$f_N(x) = \int_N f(xwn)\, dn, \quad x \in T.$$

The kernel of the map $f \mapsto f_N$ is clearly the same as the kernel of $V' \to V'_N$. Taking $t \in T$ and $f \in V'$, we have

$$\begin{aligned}(tf)_N(x) &= \int_N f(xwnt)\, dn \\ &= \delta_B(t^{-1}) \int_N f(xt^w wn)\, dn,\end{aligned}$$

and the result follows. □

4.11. We prove Casselman's projectivity theorem. We start with an irreducible smooth representation (π, V) of G and write $\chi = \omega_\pi$. The first step in the proof is:

Schur's first orthogonality relation. *Let $d\dot{g}$ be a Haar measure on G/Z. There is a constant $d(\pi) > 0$, depending on $d\dot{g}$, such that*

$$\int_{G/Z} \langle \check{\pi}(g)\check{v}_1, v_1\rangle\langle \check{v}_2, \pi(g)v_2\rangle\, d\dot{g} = d(\pi)^{-1} \langle \check{v}_1, v_2\rangle\, \langle \check{v}_2, v_1\rangle,$$

for all $v_i \in V$ and $\check{v}_i \in V^\vee$.

Sketch of proof. Since π is γ-cuspidal, the integral converges. If we fix, say, $\check{v}_1$ and v_2, the integral determines a G-invariant pairing $V^\vee \times V \to \mathbb{C}$, which must therefore be of the form $(\check{v}_2, v_1) \mapsto c(\check{v}_1, v_2)\langle \check{v}_1, v_2\rangle$, for a constant $c(\check{v}_1, v_2)$. By the same argument $c(\check{v}_1, v_2) = c_\pi \langle \check{v}_1, v_2\rangle$, for an absolute constant c_π.

It remains only to prove that $c_\pi > 0$. To do this, we can replace π by a twist and assume that $|\chi| = 1$. The space V then admits a positive definite, G-invariant Hermitian form h, constructed as follows. One chooses a nonzero element $\check{v} \in V^\vee$ and sets

$$h(v_1, v_2) = \int_{G/Z} \langle \check{v}, \pi(g)v_1\rangle \overline{\langle \check{v}, \pi(g)v_2\rangle}\, d\dot{g}.$$

There is then a complex anti-linear G-isomorphism $\Theta : (\pi, V) \to (\check{\pi}, V^\vee)$ such that $h(v_1, v_2) = \langle \Theta v_1, v_2\rangle$. It is enough to prove that

$$\int_{G/Z} h(\pi(g)v_1, v_2)\, h(v_3, \pi(g)v_4)\, d\dot{g} = b_\pi\, h(v_1, v_4)\, h(v_3, v_2),$$

where $b_\pi > 0$. One goes through the same argument; the positivity of b_π follows on taking $v_1 = v_2 = v_3 = v_4$. $\square$

The positive constant $d(\pi)$ is called the *formal degree* of π, relative to the Haar measure $d\dot{g}$.

Let $\mathrm{Rep}_\chi(G)$ be the category of smooth representations of G which admit χ as central character. We have to show that (π, V) is projective in this category. Arguing as in (3.3), the category $\mathrm{Rep}_\chi(G)$ is equivalent to the category of smooth $\mathcal{H}_\chi(G)$-modules. This reduces us to showing that V is equivalent to a direct summand of the G-representation $\mathcal{H}_\chi(G)$, with G acting by left translation.

We fix a non-zero vector $\check{v}_0 \in V^\vee$. The function $\phi_v : g \mapsto \langle \check{\pi}(g)\check{v}_0, v\rangle$ lies in $\mathcal{H}_\chi(G)$ and the map

$$\begin{aligned} \Phi : V &\longrightarrow \mathcal{H}_\chi(G), \\ v &\longmapsto \phi_v, \end{aligned}$$

is a non-zero, therefore injective, G-homomorphism. Let us now fix $v_0 \in V$ such that $d(\pi)^{-1}\langle \check{v}_0, v_0\rangle = 1$. We have a G-surjection $\Pi : \mathcal{H}_\chi(G) \to V$ given by $f \mapsto \pi(f)v_0$. The composite map $\Pi \circ \Phi$ is given by

$$w \longmapsto \pi(\phi_w)v_0 = \int_{G/Z} \pi(g)\phi_w(g)v_0\, d\dot{g}.$$

For $\check{w} \in V^\vee$, this gives

$$\begin{aligned} \langle \check{w}, \Pi\Phi(w)\rangle &= \int_{G/Z} \langle \check{\pi}(g^{-1})\check{w}, v_0\rangle \langle \check{\pi}(g)\check{v}_0, w\rangle\, d\dot{g} \\ &= \langle \check{w}, w\rangle, \end{aligned}$$

whence $\Pi\Phi(w) = w$, as required. $\square$

5. Strata

We introduce a finer method of analyzing the smooth representations of G, that of *restriction to compact open subgroups.* This will enable us, in the following sections, to complete our discussion of principal series representations and to classify the cuspidal ones.

5.1. We start with some basic facts: these hold in the completely general context of §3.

Let K be a compact open subgroup of G and let (π, V) be a smooth representation of G. If we view V as a representation of K, it is *semisimple:* it is a direct sum of irreducible representations of K. If ρ is an irreducible smooth representation of K, we denote by V^ρ the sum of all irreducible K-subspaces of V which are equivalent to ρ. One calls V^ρ *the ρ-isotypic component* of (π, V), and one says that π *contains* ρ if $V^\rho \neq 0$. We have

$$V = \bigoplus_{\rho \in \widehat{K}} V^\rho,$$

where ρ ranges over the set $\widehat{K}$ of equivalence classes of irreducible smooth representations of K.

The occurrence of representations ρ in a given irreducible representation of G is not entirely random. For $i = 1, 2$, let K_i be a compact open subgroup of G and ρ_i an irreducible smooth representation of K_i. We say that an element $g \in G$ *intertwines* ρ_1 *with* ρ_2 if

$$\mathrm{Hom}_{K_1^g \cap K_2}(\rho_1^g, \rho_2) \neq 0,$$

where ρ_1^g denotes the representation $x \mapsto \rho_1(gxg^{-1})$ of the group $K_1^g = g^{-1}K_1 g$. Note that this property of the element g depends only on the double coset $K_1 g K_2$. Also, the relation of interwining is *symmetric,* but it is not transitive.

Lemma. *Let (π, V) be an irreducible smooth representation of G containing both ρ_1 and ρ_2. There then exists $g \in G$ which intertwines ρ_1 with ρ_2.*

Proof. Let e_2 denote the K_2-projection $V \to V^{\rho_2}$. Since π is irreducible and $V^{\rho_1} \neq 0$, the spaces $\pi(g)V^{\rho_1}$, $g \in G$, span V. We can therefore choose $g \in G$ such that $e_2 \circ \pi(g)$ induces a non-zero map $V^{\rho_1} \to V^{\rho_2}$: this is the required element g. □

Exercise. *(In this Exercise, G can be any locally profinite group.) Let K be a compact open subgroup of G and ρ an irreducible smooth representation of K. Define a function $e_\rho \in \mathcal{H}(G)$ to vanish outside K and*

$$e_\rho(x) = \frac{\dim \rho}{\mu_G(K)} \,\mathrm{tr}\, \rho(x^{-1}), \quad x \in K.$$

Show that e_ρ *is idempotent and, for a smooth representation* (π, V) *of* G*, we have* $V^\rho = \pi(e_\rho)V$*. Show that the analogues of Exercises 3.2 hold. Show further that, if an element* $g \in G$ *supports a function* $f \in e_\rho * \mathcal{H}(G) * e_\rho$*, then* g *intertwines* ρ*.*

Hint. Let K' be the kernel of ρ. Thus the space $V^{K'}$ of K'-fixed vectors provides a representation of the finite group K/K', and $V^\rho \subset V^{K'}$. This reduces the problem to one for finite groups.

5.2. We need to introduce a special family of compact open subgroups of $G = \mathrm{GL}_2(F)$. These are groups of units of certain rings, which we now define.

Temporarily write $V = F^{(2)}$, so that $G = \mathrm{Aut}_F(V)$ and $A = \mathrm{End}_F(V)$. An $\mathfrak{o}$*-lattice chain in* V is a non-empty set $\mathfrak{L}$ of $\mathfrak{o}$-lattices in V, linearly ordered under inclusion and stable under multiplication by $F^\times$. Thus we can number the elements of $\mathfrak{L}$:

$$\mathfrak{L} = \{L_i : i \in \mathbb{Z}\}, \quad L_i \supsetneq L_{i+1}.$$

Stability under translation by $F^\times$ says that there exists a positive integer $e = e_{\mathfrak{L}}$ such that

$$xL_i = L_{i+ev_F(x)},$$

for all $i \in \mathbb{Z}$ and $x \in F^\times$.

We define

$$\mathfrak{A} = \mathfrak{A}_{\mathfrak{L}} = \bigcap_{i\in\mathbb{Z}} \mathrm{End}_{\mathfrak{o}}(L_i) = \bigcap_{0\leqslant i\leqslant e-1} \mathrm{End}_{\mathfrak{o}}(L_i).$$

This is a ring and also an $\mathfrak{o}$-lattice in A: such is called an $\mathfrak{o}$-order in $\mathfrak{A}$. It is called the *hereditary* $\mathfrak{o}$*-order* defined by the lattice chain $\mathfrak{L}$. The lattices $L_i \in \mathfrak{L}$ are $\mathfrak{A}$-lattices; they are the only $\mathfrak{A}$-lattices in V, so we can recover $\mathfrak{L}$ from $\mathfrak{A}$. We therefore work with $\mathfrak{A}$, rather than $\mathfrak{L}$, and set $e_{\mathfrak{A}} = e_{\mathfrak{L}}$.

Examples.

(1) In $A = \mathrm{M}_2(F)$, there are only two conjugacy classes of hereditary $\mathfrak{o}$-orders. These are represented by:

$$\mathfrak{M} = \mathrm{M}_2(\mathfrak{o}), \qquad (e_{\mathfrak{M}} = 1),$$

$$\mathfrak{J} = \begin{pmatrix} \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{o} \end{pmatrix}, \quad (e_{\mathfrak{J}} = 2).$$

(2) Let E/F be a quadratic field extension, contained in A. Thus $F^{(2)}$ is an E-vector space of dimension 1. If x is some non-zero element of $F^{(2)}$, we have $F^{(2)} = Ex$. The set $\mathfrak{L} = \{\mathfrak{p}_E^i x : i \in \mathbb{Z}\}$ is a lattice chain, defined independent of the choice of x. It is the unique lattice chain in $F^{(2)}$ stable under translation by $E^\times$. It follows that $\mathfrak{A} = \mathfrak{A}_{\mathfrak{L}}$ is the unique hereditary $\mathfrak{o}$-order in A which is stable under conjugation by $E^\times$. It satisfies $e_{\mathfrak{A}} = e(E|F)$.

Let $\mathfrak{A} = \mathfrak{A}_{\mathfrak{L}}$ be a hereditary order in $A = \mathrm{M}_2(F)$, and let $\mathfrak{P} = \mathfrak{P}_{\mathfrak{A}}$ be the Jacobson radical of $\mathfrak{A}$. In the examples above,

$$\mathfrak{P}_{\mathfrak{M}} = \mathfrak{p}\mathfrak{M}, \quad \mathfrak{P}_{\mathfrak{J}} = \begin{pmatrix} \mathfrak{p} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{p} \end{pmatrix}.$$

In all cases, there is an element $\Pi \in \mathfrak{A}$ such that $\mathfrak{P}_{\mathfrak{A}} = \Pi\mathfrak{A} = \mathfrak{A}\Pi$. We define

$$\mathfrak{P}^n = \Pi^n\mathfrak{A} = \mathfrak{A}\Pi^n, \quad n \in \mathbb{Z}.$$

One calls such an element Π a *prime element of* $\mathfrak{A}$. In fact, if $\mathfrak{L} = \{L_i\}_{i\in\mathbb{Z}}$, then

$$\mathfrak{P}^n = \bigcap_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathfrak{o}}(L_i, L_{i+n}).$$

We put $U_{\mathfrak{A}} = U_{\mathfrak{A}}^0 = \mathfrak{A}^\times$ and, for $n \geqslant 1$, $U_{\mathfrak{A}}^n = 1+\mathfrak{P}^n$. The groups $U_{\mathfrak{A}}^n$ are open normal subgroups of $U_{\mathfrak{A}}$, and, if $2m \geqslant n > m \geqslant 1$, the map $x \mapsto 1+x$ induces an isomorphism $\mathfrak{P}^m/\mathfrak{P}^n \cong U_{\mathfrak{A}}^m/U_{\mathfrak{A}}^n$.

The groups $U_{\mathfrak{A}}^j$, $j \geqslant 0$, are compact open subgroups of G. We shall also need the group

$$\mathcal{K}_{\mathfrak{A}} = \{g \in G : g\mathfrak{A}g^{-1} = \mathfrak{A}\} = N_G(U_{\mathfrak{A}}).$$

One can equally define $\mathcal{K}_{\mathfrak{A}}$ as the automorphism group of the associated lattice chain $\mathfrak{L}$:

$$\mathcal{K}_{\mathfrak{A}} = \{x \in G : x\mathfrak{L} = \mathfrak{L}\}.$$

Either way, $\mathcal{K}_{\mathfrak{A}}$ is the group generated by $U_{\mathfrak{A}}$ and a prime element Π of $\mathfrak{A}$. It is open, it contains the centre Z of G, and $\mathcal{K}_{\mathfrak{A}}/Z$ is compact. Moreover, $\mathcal{K}_{\mathfrak{A}}$ normalizes all of the groups $U_{\mathfrak{A}}^j$, $j \geqslant 0$.

5.3. Let $\widehat{F}$ denote the Pontrjagin dual of F: this is the group of characters of F with the compact open topology. There is a non-canonical isomorphism $F \to \widehat{F}$ (of topological groups) as follows. Let $\psi \in \widehat{F}$, $\psi \neq 1$. The isomorphism is then $a \mapsto a\psi$, where $a\psi(x) = \psi(xa)$.

It will be useful to fix a character $\psi_F \in \widehat{F}$ *of level one.* This means that the largest fractional ideal of $\mathfrak{o}$ contained in $\mathrm{Ker}\,\psi$ is $\mathfrak{p}$. With this convention, we have

$$\{x \in F : \psi(x\mathfrak{p}^m) = 1\} = \mathfrak{p}^{1-m}, \quad m \in \mathbb{Z}.$$

There are similar structures in $A = \mathrm{M}_2(F)$. Define $\psi_A \in \widehat{A}$ by

$$\psi_A(x) = \psi_F(\mathrm{tr_A}\, x),$$

where tr_A denotes the trace map $A \to F$. Again we get an isomorphism $A \to \widehat{A}$ by $a \mapsto a\psi_A$.

Let L be some $\mathfrak{o}$-lattice in A; we define

$$L^* = \{x \in A : \psi_A(xy) = 1, y \in L\}.$$

Then L^* is an $\mathfrak{o}$-lattice, and we can identify L^* with the group of characters of A which are trivial on L.

Let $\mathfrak{A}$ be a hereditary order in A with radical $\mathfrak{P}$. For $n \in \mathbb{Z}$, we then have

$$\left(\mathfrak{P}^n\right)^* = \mathfrak{P}^{1-n}.$$

Therefore:

Lemma. *Let m, n be integers such that $2m+1 \geqslant n > m \geqslant 0$. We have an isomorphism*

$$\begin{aligned} \mathfrak{P}^{-n}/\mathfrak{P}^{-m} &\longrightarrow \left(U_{\mathfrak{A}}^{m+1}/U_{\mathfrak{A}}^{n+1}\right)^{\widehat{}}, \\ a+\mathfrak{P}^{-m} &\longmapsto \psi_{A,a}, \end{aligned}$$

where $\psi_{A,a}$ is the character $x \mapsto \psi_A(a(x-1))$.

5.4. Let (π, V) be an irreducible smooth representation of G. We define the *normalized level* $l(\pi)$ of π to be the minimum value $n/e_{\mathfrak{A}}$, where n ranges over the non-negative integers and $\mathfrak{A}$ over the hereditary orders in A, subject to the condition that π contains the trivial character of $U_{\mathfrak{A}}^{n+1}$. Then clearly $2l(\pi)$ is a non-negative integer.

We use the notation $\mathfrak{M}$, $\mathfrak{I}$ of the examples in 5.2.

Proposition. *Let (π, V) be an irreducible smooth representation of G such that $l(\pi) = 0$. Exactly one of the following holds:*

(1) *π contains an irreducible representation ρ of $U_{\mathfrak{M}}$, trivial on $U_{\mathfrak{M}}^1$ and inflated from an irreducible cuspidal representation $\tilde{\rho}$ of the group $U_{\mathfrak{M}}/U_{\mathfrak{M}}^1 \cong \mathrm{GL}_2(\boldsymbol{k})$.*

(2) *π contains the trivial character of $U_{\mathfrak{I}}^1$.*

Proof. Since $U_{\mathfrak{M}}^1 \subset U_{\mathfrak{I}}^1$, the representation π surely contains the trivial character of $U_{\mathfrak{M}}^1$. It therefore contains the inflation ρ to $U_{\mathfrak{M}}$ of an irreducible representation $\tilde{\rho}$ of $\mathrm{GL}_2(\boldsymbol{k})$. If $\tilde{\rho}$ is cuspidal, we are in the first case, otherwise in the second. To prove that the two cannot occur together, we interpolate a useful general lemma.

Lemma. *For $i = 1, 2$, let $\tilde{\rho}_i$ be an irreducible representation of $\mathrm{GL}_2(\boldsymbol{k})$, and let ρ_i denote the inflation of $\tilde{\rho}_i$ to a representation of $U_{\mathfrak{M}} = \mathrm{GL}_2(\mathfrak{o})$. Suppose that $\tilde{\rho}_1$ is cuspidal.*

(1) *The representations ρ_i intertwine in G if and only if $\tilde{\rho}_1 \cong \tilde{\rho}_2$.*

(2) *An element $g \in G$ intertwines ρ_1 if and only if $g \in ZU_{\mathfrak{M}}$*

Proof. Let $g \in G$ intertwine the ρ_i. It is only the coset $U_{\mathfrak{M}} gZU_{\mathfrak{M}}$ which intervenes, so we can take g of the form

$$g = \begin{pmatrix} \varpi^a & 0 \\ 0 & 1 \end{pmatrix},$$

for some $a \geqslant 0$. If $a = 0$, we have $g = 1$ and there is nothing to do. We therefore assume $a \geqslant 1$. The group $(U^1_{\mathfrak{M}})^g \cap U_{\mathfrak{M}}$ contains the group

$$N_0 = \begin{pmatrix} 1 & \mathfrak{o} \\ 0 & 1 \end{pmatrix}$$

on which ρ_2^g is trivial. Since $\tilde{\rho}_1$ is cuspidal, ρ_1 does not contain the trivial character of N_0, so g cannot intertwine the ρ_i. All assertions now follow. $\square$

It follows that, in the Proposition, ρ cannot intertwine with the trivial character of $U^1_{\mathfrak{J}}$. $\square$

5.5. To deal with positive level representations, we introduce some new terminology. A *stratum* in A is a triple $(\mathfrak{A}, n, a)$, where $\mathfrak{A}$ is a hereditary order in A (with radical $\mathfrak{P}$, say), $n \geqslant 1$ is an integer and $a \in \mathfrak{P}^{-n}$. We say that strata $(\mathfrak{A}, n, a)$, $(\mathfrak{A}, n, a')$ are equivalent if $a \equiv a' \pmod{\mathfrak{P}^{1-n}}$: thus the equivalence class of the stratum $(\mathfrak{A}, n, a)$ is the same as the character $\psi_{A,a}$ of $U^n_{\mathfrak{A}}/U^{n+1}_{\mathfrak{A}}$.

Exercise. *Let $(\mathfrak{A}_i, n_i, a_i)$, $i = 1, 2$, be strata in A, let $\mathfrak{P}_i = \operatorname{rad}\mathfrak{A}_i$, and let $g \in G$. The following are equivalent:*

(1) *The element g intertwines the character ψ_{A,a_1} of $U^{n_1}_{\mathfrak{A}_1}$ with the character ψ_{A,a_2} of $U^{n_2}_{\mathfrak{A}_2}$.*
(2) *The intersection $g^{-1}(a_1 + \mathfrak{P}_1^{1-n_1})g \cap (a_2 + \mathfrak{P}_2^{1-n_2})$ is non-empty.*

Lemma. *Let $(\mathfrak{A}, n, a)$ be a stratum in A. The following are equivalent:*

(1) *The coset $a + \mathfrak{P}^{1-n}$ contains a (possibly zero) nilpotent element of A.*
(2) *There is an integer $r \geqslant 1$ such that $a^r \in \mathfrak{P}^{1-rn}$.*

Proof. Given the explicit form of $\mathfrak{A}$ (5.2), this is immediate. $\square$

If $a + \mathfrak{P}^{1-n}$ contains no nilpotent, we say that the stratum $(\mathfrak{A}, n, a)$ is *fundamental.*

Let (π, V) be an irreducible smooth representation of G. We say that π *contains the stratum* $(\mathfrak{A}, n, a)$ if it contains the character $\psi_{A,a}$ of $U^n_{\mathfrak{A}}$. Observe that, if this is the case, then $n/e_{\mathfrak{A}} \geqslant l(\pi)$ by definition.

The main result here is:

Theorem. *Let π be an irreducible smooth representation of G and let $(\mathfrak{A}, n, a)$ be a stratum in A, contained in π. The following are equivalent:*

(1) *$(\mathfrak{A}, n, a)$ is fundamental;*
(2) *$l(\pi) = n/e_{\mathfrak{A}}$.*

In particular, if $l(\pi) \neq 0$ then π contains a fundamental stratum.

Proof. The key steps are as follows. First, suppose we have a non-fundamental stratum $[\mathfrak{A}, n, a]$. Let $\mathfrak{P}$ be the radical of $\mathfrak{A}$. There is then a hereditary order $\mathfrak{A}_1$, with radical $\mathfrak{P}_1$, and an integer n_1, such that

$$a+\mathfrak{P}^{1-n} \subset \mathfrak{P}_1^{-n_1}, \quad \text{and} \quad n_1/e(\mathfrak{A}_1) < n/e(\mathfrak{A}).$$

(This is easily proved "by hand", for GL_2.) If π contains the stratum $[\mathfrak{A}, n, a]$, then it also contains the trivial character of $U_{\mathfrak{A}_1}^{n_1+1}$.

For the second step, one has to show that if we have two fundamental strata $[\mathfrak{A}_i, n_i, a_i]$, such that the characters ψ_{a_i} (of $U_{\mathfrak{A}_i}^{n_i}$) intertwine, then $n_1/e(\mathfrak{A}_1) = n_2/e(\mathfrak{A}_2)$. $\square$

5.6. Given that there are, in effect, only two hereditary orders in A, it is not hard to describe the fundamental strata completely, up to G-conjugacy.

Proposition 1.

(1) *Let n be odd; a stratum $(\mathfrak{I}, n, a)$ is then fundamental if and only if $a\mathfrak{I} = \mathfrak{P}_{\mathfrak{I}}^{-n}$ or, equivalently, $a \in \Pi^{-n}U_{\mathfrak{I}}$ for a prime element Π of $\mathfrak{I}$.*

(2) *Let π be an irreducible smooth representation of G with $l(\pi) > 0$. If $l(\pi) = n/2 \notin \mathbb{Z}$, then π contains a fundamental stratum $(\mathfrak{I}, n, a)$.*

Proof. The first assertion is a straightforward calculation. Theorem 5.5 says that π contains a fundamental stratum; since $l(\pi) \notin \mathbb{Z}$, it must be of the form $(\mathfrak{I}, n, a)$. $\square$

We define a *ramified simple stratum* to be a fundamental stratum $(\mathfrak{A}, n, \alpha)$ in which $e_{\mathfrak{A}} = 2$ and n is odd.

Lemma. *If $0 < l(\pi) = n \in \mathbb{Z}$, then π contains a fundamental stratum of the form $(\mathfrak{M}, n, \alpha)$.*

Proof. This follows from 5.5 and the observation that

$$U_{\mathfrak{M}}^{n+1} \subset U_{\mathfrak{I}}^{2n+1} \subset U_{\mathfrak{I}}^{2n} \subset U_{\mathfrak{M}}^{n}. \qquad \square$$

For this reason, there is no need to consider strata $(\mathfrak{I}, n, a)$ when n is even. Consider a stratum $(\mathfrak{M}, n, \alpha)$. We may write $\alpha = \varpi^{-n}\alpha_0$, for some $\alpha_0 \in \mathfrak{M}$. Let $f_\alpha(t) \in \mathfrak{o}[t]$ be the characteristic polynomial of α_0, and let $\tilde{f}_\alpha(t) \in \boldsymbol{k}[t]$ be its reduction modulo $\mathfrak{p}$: thus $\tilde{f}_\alpha(t)$ is the characteristic polynomial of the image of α_0 in $\mathrm{M}_2(\boldsymbol{k})$. The stratum $(\mathfrak{M}, n, \alpha)$ is then fundamental if and only if $\tilde{f}_\alpha(t) \neq t^2$.

Definition. *Let* $(\mathfrak{M}, n, \alpha)$ *be a fundamental stratum. We say that* $(\mathfrak{M}, n, \alpha)$ *is* unramified simple *(resp.* split, *resp.* essentially scalar*) if* $\tilde{f}_\alpha(t)$ *is irreducible in* $\boldsymbol{k}[t]$ *(resp. has distinct roots in* $\boldsymbol{k}$*, resp. has a repeated root in* $\boldsymbol{k}^\times$*).*

A stratum is called *simple* if it is either ramified or unramified simple.

Remark. Temporarily write $\mathcal{S}_1, \dots, \mathcal{S}_4$ for the set of strata which are, respectively, ramified simple, unramified simple, split, essentially scalar. An element of $\mathcal{S}_i$ does not intertwine with any element of $\mathcal{S}_j$ unless $i = j$.

Thus we have a preliminary classification of the irreducible smooth representations π of G with $l(\pi) > 0$, according to which sort of fundamental stratum occurs in π. One case is essentially uninteresting:

Proposition 2. *Let* π *be an irreducible smooth representation of* G *with* $0 < l(\pi) = n \in \mathbb{Z}$*. Let* $(\mathfrak{M}, n, \alpha)$ *be a fundamental stratum appearing in* π*. The following are equivalent:*

(1) *The stratum* $(\mathfrak{M}, n, \alpha)$ *is essentially scalar.*
(2) *There is a character* χ *of* $F^\times$ *such that* $l(\chi\pi) < l(\pi)$*.*

Proof. Suppose that $(\mathfrak{M}, n, \alpha)$ is essentially scalar. Write

$$(\star) \qquad \alpha \equiv \varpi^{-n} \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \pmod{\mathfrak{p}^{1-n}\mathfrak{M}},$$

for a prime element ϖ of F and $a \in U_F$, $b \in \mathfrak{o}$. Let χ be a character of $F^\times$, of level n, such that $\chi \mid U_F^n = \psi_{F,-\varpi^{-n}a}$. The representation $\chi\pi$ then contains the stratum $(\mathfrak{M}, n, \beta)$, with

$$\beta \equiv \varpi^{-n} \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \pmod{\mathfrak{p}^{1-n}\mathfrak{M}},$$

which is not fundamental. Thus $l(\chi\pi) < n$.

For the converse, we change notation and take for π an irreducible smooth representation of G with $l(\pi) < n$. Let ϕ be a character of $F^\times$, of level n, with $\phi \mid U_F^n = \psi_{F,\varpi^{-n}a}$, $a \in U_F$. The representation $\phi\pi$ then contains a stratum of the form $(\star)$. $\square$

In particular:

Corollary. *Let* π *be an irreducible smooth representation of* G *such that* $0 < l(\pi) \leqslant l(\chi\pi)$*, for every character* χ *of* $F^\times$*. Then* π *contains a fundamental stratum which is either split or simple.*

Remark. In the case where π is not of minimal normalized level, relative to twisting by characters of $F^\times$, one can deduce very little from the fundamental strata appearing in π. In the other cases, however, one can garner considerably more information.

6. Strata and the principal series

We now use the preliminary classification via strata to complete our description of the principal series representations of G, begun in §4.

6.1 Theorem. *Let (π, V) be an irreducible smooth representation of G. Suppose that either*

(a) *$l(\pi) = 0$ and π contains a character ξ of $U_{\mathfrak{J}}/U_{\mathfrak{J}}^1$ which does not factor through the determinant $\det_G$, or*

(b) *$l(\pi) > 0$ and π contains a split fundamental stratum $(\mathfrak{M}, n, a)$.*

The Jacquet module (π_N, V_N) then contains the character $\xi \mid U_{\mathfrak{J}} \cap T$ (in case (a)*) or $\psi_{A,a} \mid U_{\mathfrak{M}}^n \cap T$ (in case* (b)*). In particular, $V_N \neq 0$ and π is principal series.*

Proof. The two cases are very similar, so we deal only with the second. Let us write $\psi_{A,a} = \xi$. It is enough to show that the space V^ξ has non-zero image in V_N. Suppose, for a contradiction, that $V^\xi \subset V(N)$. Thus, for each $v \in V^\xi$, there is a compact open subgroup $N(v)$ of N such that

$$\int_{N(v)} \pi(u)v\, du = 0.$$

We write

$$N_j = \begin{pmatrix} 1 & \mathfrak{p}^j \\ 0 & 1 \end{pmatrix}.$$

The space V^ξ is finite-dimensional, so there exists $j \in \mathbb{Z}$ such that

$$\int_{N_j} \pi(u)v\, du = 0,$$

for all $v \in V^\xi$. We choose j maximal for this property. There exists $v_1 \in V^\xi$ such that

$$\int_{N_{j+1}} \pi(u)v_1\, du \neq 0.$$

We set $t = \left(\begin{smallmatrix} \varpi & 0 \\ 0 & 1 \end{smallmatrix}\right)$, and consider the vector $v_2 = \pi(t^{-1})v_1$. We have (for a positive constant c)

$$\begin{aligned}
\int_{N_j} \pi(u)v_2\, du &= \int_{N_j} \pi(ut^{-1})v_1\, du \\
&= \pi(t^{-1}) \int_{N_j} \pi(tut^{-1})v_1\, du \\
&= c\,\pi(t^{-1}) \int_{N_{j+1}} \pi(u)v_1\, du \neq 0,
\end{aligned}$$

by the definition of j. We have

$$\pi(y)v_2 = \xi(y)v_2, \quad y \in Y = U_{\mathfrak{M}}^n \cap t^{-1}U_{\mathfrak{M}}^n t.$$

Any irreducible representation of $U_{\mathfrak{M}}^n$ containing $\xi \mid Y$ is one-dimensional; let ϕ range over the set Φ characters of $U_{\mathfrak{M}}^n$ such that $\phi \mid Y = \xi \mid Y$. Therefore $v_2 = \sum_\phi v_\phi$, for certain vectors $v_\phi \in V^\phi$. There exists $\phi \in \Phi$ such that

$$\int_{N_j} \pi(u)v_\phi \, du \neq 0.$$

However, the splitness hypothesis implies that $\phi^x = \xi$, for some $x \in N_0$. Thus $v_3 = \pi(x)v_\phi$ lies in V^ξ and

$$\int_{N_j} \pi(u)v_3 \, du \neq 0,$$

contrary to hypothesis. □

Corollary. *Let $\chi = \chi_1 \otimes \chi_2$ be a character of T such that χ_1/χ_2 is ramified. Then $\mathrm{Ind}_B^G \chi$ is irreducible.*

Proof. Suppose that χ_1/χ_2 is not *tamely* ramified: the other case is similar, so we omit the argument. Further, twisting by χ_2^{-1}, we may as well assume that χ_2 is trivial and χ_1 is wildly ramified.

Let $n \geqslant 1$ be the level of χ_1; there exists $a \in \mathfrak{p}^{-n} \smallsetminus \mathfrak{p}^{1-n}$ such that $\chi_1 \mid U_F^n = \psi_{F,a}$. Let $(\pi, V) = \mathrm{Ind}\,\chi$; then π contains the character $\psi_{A,\alpha}$ of $U_{\mathfrak{M}}^n$, where

$$\alpha = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}.$$

That is, π contains the split fundamental stratum $(\mathfrak{M}, n, \alpha)$. By the theorem, π_N contains the character $\psi_{A,\alpha} \mid T \cap U_{\mathfrak{M}}^n$. However, π also contains the character $\psi_{A,\alpha}^w = \psi_{A,\beta}$ of $U_{\mathfrak{M}}^n$, where

$$w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \beta \equiv \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} \pmod{\mathfrak{P}_{\mathfrak{M}}^{1-n}},$$

so π_N contains $\psi_{A,\beta} \mid T \cap U_{\mathfrak{M}}^n$. The characters $\psi_{A,\alpha} \mid T \cap U_{\mathfrak{M}}^n$, $\psi_{A,\beta} \mid T \cap U_{\mathfrak{M}}^n$, are different, so $\dim V_N = 2$ and π is irreducible. □

6.2. Thus we have dealt with all principal series representations $\mathrm{Ind}_B^G(\chi_1 \otimes \chi_2)$, except those with χ_1/χ_2 unramified. To treat this case, we may as well assume that the χ_i are both unramified. Writing $I = U_{\mathfrak{J}}$ for the standard Iwahori subgroup of G, we have immediately

Lemma. *Let* $(\pi, V) = \operatorname{Ind}_B^G \chi$, *where* χ *is unramified. The space* V^I *has dimension* 2.

These representations with Iwahori fixed vector have some unusual properties.

Proposition. *Let* (π, V) *be an irreducible smooth representation of* G. *The canonical map* $V \to V_N$ *induces an isomorphism* $V^I \cong (V_N)^{I \cap T}$.

Proof. The canonical map $V \to V_N$ surely induces a map $V^I \to (V_N)^{I\cap T}$. We will only prove that it is injective: the proof of surjectivity follows the same course as earlier arguments. The proof of injectivity relies on the following fact:

Lemma. *Let* $f \in \mathcal{H}(G, I)$ *be supported on one double coset* IgI; *then* f *is invertible in* $\mathcal{H}(G, I)$.

The proof of the proposition then follows much the same course as that of Theorem 6.1. We choose j maximal for the property that

$$\int_{N_j} \pi(u)v\,du = 0$$

for all $v \in V^I$. We choose $v_1 \in V^I$ such that

$$\int_{N_{j+1}} \pi(u)v_1\,du \neq 0,$$

and consider the vector $v_2 = \pi(f)v_1$, where $f \in \mathcal{H}(G, I)$ has support $Iz^{-1}I$, $z = \left(\begin{smallmatrix} \varpi & 0 \\ 0 & 1 \end{smallmatrix}\right)$. We have $v_2 \in V^I$ and, since f is invertible, $v_2 \neq 0$. However, a simple calculation shows

$$\int_{N_j} \pi(u)v_2\,du = 0,$$

contrary to hypothesis. $\square$

Consequently:

Corollary 1. *Let* (π, V) *be an irreducible smooth representation of* G. *The following are equivalent:*

(1) π *contains an Iwahori fixed vector;*
(2) π *is a composition factor of* $\operatorname{Ind}_B^G \chi$, *for some unramified character* χ *of* T.

Set $(\pi, V) = \operatorname{Ind}\chi$, where χ is unramified, and consider the $\mathcal{H}(G, I)$-module V^I. If π is irreducible, V^I is a simple $\mathcal{H}(G, I)$-module. Otherwise, each composition factor of π has a one-dimensional Jacquet module on which $I \cap T$ acts trivially: thus each composition factor has an I-fixed vector. In other words, π is irreducible if and only if V^I is a simple $\mathcal{H}(G,I)$-module.

The structure of the algebra $\mathcal{H}(G,I)$ is known: it is an affine Hecke algebra. It has just two families of simple modules of dimension 1: these correspond respectively to twists, by unramified characters of $F^\times$, of the trivial representation 1_G and of the Steinberg representation St_G. All other simple modules have dimension 2. Thus:

Corollary 2. *Let χ be a character of T; then $\mathrm{Ind}_B^G \chi$ is reducible if and only if $\chi = \phi 1_T$ or $\phi\delta_B^{-1}$, for some character ϕ of $F^\times$.*

Let us restate this in terms of the normalized induction functor

$$\boldsymbol{\iota}_B^G : \chi \longmapsto \mathrm{Ind}\,\chi\delta_B^{-\frac{1}{2}}.$$

Corollary 3. *Let $\chi = \chi_1 \otimes \chi_2$ be a character of T. The representation $\boldsymbol{\iota}_B^G \chi$ is irreducible except when $\chi_1\chi_2^{-1}(x) = \|x\|^{\pm 1}$. In the exceptional cases, $\boldsymbol{\iota}_B^G \chi$ has composition length 2.*

In other words, up to twisting by characters of $F^\times$, all the reducible representations $\mathrm{Ind}_B^G \chi$ have already been described in 4.9.

6.3. The operation of twisting irreducible representations by characters of $F^\times$ is transparent. Thus, when classifying the irreducible representations π, it is enough to treat the cases where $l(\pi) \leqslant l(\chi\pi)$ for every character χ of $F^\times$: we say that such a representation π has *minimal level.* We have proved:

Corollary. *Let π be an irreducible smooth representation of G, of minimal level. The following are equivalent:*

(1) *The representation π is cuspidal.*
(2) *Either*
    (a) *$l(\pi) = 0$ and π contains a representation of $U_{\mathfrak{M}} = \mathrm{GL}_2(\mathfrak{o})$ inflated from an irreducible cuspidal representation of $\mathrm{GL}_2(\boldsymbol{k})$, or else*
    (b) *$l(\pi) > 0$ and π contains a simple stratum.*

This is the starting point of our description of the irreducible cuspidal representations in the next chapter.

7. Cuspidal representations

We can now give a complete description of the irreducible cuspidal representations of $\mathrm{GL}_2(F)$.

7.1. We start with a general discussion so, for the moment, G can be any unimodular, locally profinite group. Let H be an open subgroup of G and (ρ, W) a smooth representation of H. We form the compactly induced representation $c\text{-}\mathrm{Ind}_H^G\,\rho$. This representation, we recall, is realized as G acting by right translation on the space of functions $f : G \to W$ which are compactly supported modulo H and satisfy $f(hg) = \rho(h)f(g)$, for $g \in G$ and $h \in H$. (Observe that any such function on G is automatically smooth.)

There is a canonical H-homomorphism $b_\rho : W \to c\text{-}\mathrm{Ind}\,\rho$: for $w \in W$, the function $b_\rho(w)$ is supported on H and $b_\rho(w) : h \mapsto \rho(h)w$, $h \in H$. On then has:

Frobenius Reciprocity. *Let ρ be a smooth representation of H and π a smooth representation of G. Composition with b_ρ induces a canonical isomorphism*

$$\mathrm{Hom}_G(c\text{-}\mathrm{Ind}(\rho), \pi) \cong \mathrm{Hom}_H(\rho, \pi).$$

Continuing in the general case, the group H is unimodular and so the space $H\backslash G$ admits a positive G-invariant measure $d\dot{g}$.

Lemma. *Let (ρ, W) be a smooth representation of H. The pairing*

$$\mathrm{Ind}\,\check{\rho} \times c\text{-}\mathrm{Ind}\,\rho \longrightarrow \mathbb{C},$$
$$(a, b) \longmapsto \int_{H\backslash G} \langle a(g), b(g)\rangle_W \, d\dot{g},$$

where $\langle\ ,\ \rangle_W$ is the canonical pairing $W^\vee \times W \to \mathbb{C}$, induces a G-isomorphism

$$\mathrm{Ind}\,\check{\rho} \cong \big(c\text{-}\mathrm{Ind}\,\rho\big)^\vee.$$

Proof. This is straightforward. $\square$

Exercises. *Let K be an open subgroup of G, containing and compact modulo the centre Z of G. Let (ρ, W) be an irreducible smooth representation of K. Let $\mathcal{H}(G, \rho)$ be the space of functions $f : G \to \mathrm{End}_\mathbb{C}(W)$ which are compactly supported modulo Z and satisfy $f(k_1 g k_2) = \rho(k_1)f(g)\rho(k_2)$, $k_i \in K$, $g \in G$.*

(1) *Show that $\mathcal{H}(G, \rho)$ is an algebra with 1 under convolution relative to a Haar measure on G/Z.*
(2) *Fix $g \in G$; show that the space of functions $f \in \mathcal{H}(G, \rho)$ supported on KgK is canonically isomorphic to $\mathrm{Hom}_{K^g \cap K}(\rho^g, \rho)$.*
(3) *Produce a canonical algebra isomorphism*

$$\mathcal{H}(G, \rho) \cong \mathrm{End}_G(c\text{-}\mathrm{Ind}\,\rho) = \mathrm{Hom}_K(\rho, c\text{-}\mathrm{Ind}\,\rho).$$

Warning. The algebra $\mathcal{H}(G, \rho)$ defined above is often deemed by many authors (including us) to be $\mathcal{H}(G, \check{\rho})$.

We return to the specific case $G = \mathrm{GL}_2(F)$, and give the result we shall need:

Proposition. *Let H be an open subgroup of G, containing and compact modulo Z. Let ρ be an irreducible smooth representation of H and suppose that an element $g \in G$ intertwines ρ if and only if $g \in H$. The compactly induced representation $c\text{-}\mathrm{Ind}_H^G\,\rho$ is then irreducible and cuspidal.*

Proof. Since (ρ, W) is irreducible, the space W has finite dimension. The G-space $c\text{-}\mathrm{Ind}\,\rho$ is generated by the image $b_\rho(W)$, which is contained in the isotypic space $c\text{-}\mathrm{Ind}(\rho)^\rho$. We may therefore choose a finite set $\{f_1, f_2, \dots, f_r\}$ of elements of $(c\text{-}\mathrm{Ind}\,\rho)^\rho$ which generates $c\text{-}\mathrm{Ind}\,\rho$ over G and is minimal for this property. Let X be a G-subspace of $c\text{-}\mathrm{Ind}\,\rho$, maximal for the property of not containing f_1. The quotient $(\pi, V) = c\text{-}\mathrm{Ind}\,\rho/X$ is then irreducible and contains ρ.

We now use the intertwining property of ρ. The technical consequence of this (see the exercise above) is that the isotypic space $(c\text{-}\mathrm{Ind}\,\rho)^\rho$ is just $b_\rho(W)$: the representation ρ occurs in $c\text{-}\mathrm{Ind}\,\rho$ with multiplicity one. It follows that the space X does not contain ρ.

Consider the isotypic space $(\mathrm{Ind}\,\check\rho)^{\rho^\vee}$; this is canonically identified with the linear dual of $(c\text{-}\mathrm{Ind}\,\rho)^\rho \cong W$, so it is finite-dimensional and consists of functions supported on H. Moreover, $(\mathrm{Ind}\,\check\rho)^{\rho^\vee}$ is contained in the canonical image of $V^\vee$ in $\mathrm{Ind}\,\check\rho$. So, if we take $\check v \in (\mathrm{Ind}\,\check\rho)^{\rho^\vee}$ and $v \in V^\rho$, the pair $(\check v, v)$ defines a coefficient of (π, V) which is supported in H: in particular, it is compactly supported mod H. We deduce that (π, V) is cuspidal and, by Casselman's Projectivity Theorem, we have $c\text{-}\mathrm{Ind}\,\rho = X \oplus V$. Thus X is a quotient of $c\text{-}\mathrm{Ind}\,\rho$, but it does not contain ρ. Therefore $\mathrm{Hom}_H(\rho, X) = 0$ and Frobenius Reciprocity implies $X = 0$. It follows that $c\text{-}\mathrm{Ind}\,\rho = \pi$, so $c\text{-}\mathrm{Ind}\,\rho$ is irreducible and cuspidal, as required. $\square$

7.2. We introduce some particularly important representations of open, compact mod centre subgroups of G.

Definition. *A* cuspidal inducing datum *in G is a pair $(\mathfrak{A}, \varXi)$, where $\mathfrak{A}$ is a hereditary order in A and $\varXi$ is an irreducible smooth representation of $\mathcal{K}_\mathfrak{A}$, of one of the following kinds:*

1. *$\mathfrak{A} \cong \mathfrak{M}$ and $\varXi \mid U_\mathfrak{A}$ is the inflation of an irreducible cuspidal representation of $U_\mathfrak{A}/U_\mathfrak{A}^1 \cong \mathrm{GL}_2(\boldsymbol{k})$;*
2. *there is a simple stratum $(\mathfrak{A}, n, \alpha)$, $n \geqslant 1$, such that the restriction of $\varXi$ to $U_\mathfrak{A}^n$ contains the character ψ_α;*
3. *$\varXi = \varXi_0 \otimes \chi \circ \det$, where $(\mathfrak{A}, \varXi_0)$ is a cuspidal inducing datum of one of the first two kinds and χ is a character of $F^\times$.*

In (2), we have abbreviated $\psi_\alpha = \psi_{A,\alpha}$.

If $(\mathfrak{A}, \varXi)$ is a cuspidal inducing datum, its *level* is the least integer $n \geqslant 0$ such that $\varXi$ is trivial on $U_\mathfrak{A}^{n+1}$, and its *normalized level,* denoted $l(\varXi)$, is $e_\mathfrak{A}^{-1}$ times its level. The data of minimal normalized level, relative to twisting by characters of $F^\times$, are those defined in (1) and (2).

Observe that, in (3), if χ has level m, then $l(\Xi_0 \otimes \chi\circ\det) = \max(m, l(\Xi_0))$, so there is some, insignificant, overlap between the three kinds.

In this language, Corollary 6.3 gives:

Proposition. *An irreducible smooth representation π of G is cuspidal if and only if it contains a cuspidal inducing datum.*

We now complete the picture.

Classification theorem.

(1) *Let $(\mathfrak{A}, \Xi)$ be a cuspidal inducing datum in G. The representation*

$$\pi_\Xi = c\text{-}\mathrm{Ind}_{\mathcal{K}_\mathfrak{A}}^G\, \Xi$$

is an irreducible cuspidal representation of G, satisfying $l(\pi_\Xi) = l(\Xi)$.

(2) *Let π be an irreducible cuspidal representation of G. There exists a cuspidal inducing datum $(\mathfrak{A}, \Xi)$, unique up to G-conjugacy, such that $\pi \cong \pi_\Xi$.*

7.3. We deal first with the level zero case.

Theorem. *The process*

$$(\mathfrak{A}, \Xi) \longmapsto \pi_\Xi = c\text{-}\mathrm{Ind}_{\mathcal{K}_\mathfrak{A}}^G\, \Xi$$

induces a bijection between the set of G-conjugacy classes of cuspidal inducing data $(\mathfrak{A}, \Xi)$ of level zero and the set of equivalence classes of irreducible cuspidal representations of G of level zero.

Proof. Initially, let $(\mathfrak{A}, \Xi)$ be a datum of level zero: we may as well take $\mathfrak{A} = \mathfrak{M}$. Set $\xi = \Xi \mid U_\mathfrak{M}$. By Lemma 5.4, an element $g \in G$ intertwines ξ if and only if $g \in \mathcal{K}_\mathfrak{M} = ZU_\mathfrak{M}$. Proposition 7.1 implies that π_Ξ is an irreducible cuspidal representation of G. It contains Ξ and hence the trivial character of $U_\mathfrak{M}^1$: thus $l(\pi_\Xi) = 0$.

Conversely, let π be an irreducible cuspidal representation of G of level zero. Then (Corollary 6.3) π contains a representation ξ of $U_\mathfrak{M}$ inflated from an irreducible cuspidal representation of $\mathrm{GL}_2(\boldsymbol{k})$. It surely contains, therefore, an irreducible representation Ξ of $ZU_\mathfrak{M}$ with $\Xi \mid U_\mathfrak{M} = \xi$. The pair $(\mathfrak{M}, \Xi)$ is a cuspidal inducing datum of level zero and $\pi \cong \pi_\Xi$.

Suppose that π contains another datum $(\mathfrak{M}, \Xi')$. Setting $\xi' = \Xi' \mid U_\mathfrak{M}$, the representations ξ, ξ' must intertwine in G. By Lemma 5.4 again, $\xi \cong \xi'$. Thus Ξ, Ξ' are extensions to $ZU_\mathfrak{M}$ of the same representation ξ. They are intertwined by some $g \in G$ which necessarily intertwines ξ. Thus $g \in ZU_\mathfrak{M}$, whence $\Xi = \Xi'$, as required. $\square$

7.4. To proceed further, we have to analyse the structure of cuspidal inducing data of positive level. We need a preliminary digression.

Definition. *Let $\alpha \in G \setminus Z$; we say that α is* minimal over F *if $E = F[\alpha]$ is a field and, setting $n = -v_E(\alpha)$, we have:*

(1) *E/F is ramified and n is odd, or*
(2) *E/F is unramified and, if ϖ is a prime element of F, then $\mathfrak{o}_E = \mathfrak{o}[\varpi^n \alpha]$.*

Proposition.

(1) *Let $(\mathfrak{A}, n, \alpha)$ be a simple stratum; then α is minimal over F and $F[\alpha]^\times \subset \mathcal{K}_\mathfrak{A}$. Moreover, every $\alpha' \in \alpha + \mathfrak{P}^{1-n} = \alpha U^1_\mathfrak{A}$ is minimal over F, where $\mathfrak{P} = \operatorname{rad} \mathfrak{A}$.*
(2) *Let α be minimal over F. There exists a unique hereditary order $\mathfrak{A}$ in A such that $\alpha \in \mathcal{K}_\mathfrak{A}$. Moreover, if $\mathfrak{P} = \operatorname{rad} \mathfrak{A}$ and $\alpha\mathfrak{A} = \mathfrak{P}^{-n}$, the triple $(\mathfrak{A}, n, \alpha)$ is a simple stratum.*

Proof. In (1), suppose first that $(\mathfrak{A}, n, \alpha)$ is a ramified simple stratum. Thus we may take $\mathfrak{A} = \mathfrak{I}$, $n = 2m+1$ is odd and, by definition, $\alpha\mathfrak{I} = \mathfrak{P}^{-n}$, where $\mathfrak{P} = \operatorname{rad} \mathfrak{I}$. Let ϖ be a prime of F and consider $\alpha_0 = \varpi^{m+1}\alpha$. This satisfies $\alpha_0 \mathfrak{I} = \mathfrak{P}$ and its minimal polynomial over F is an Eisenstein polynomial. Thus $E = F[\alpha_0] = F[\alpha]$ is a ramified quadratic field extension of F and $-n = v_E(\alpha)$, so α is minimal over F. Moreover, $\mathfrak{o}_E = \mathfrak{o}[\alpha_0] \subset \mathfrak{I}$, whence every $\mathfrak{I}$-lattice in $F^{(2)}$ is an $\mathfrak{o}_E$-lattice. It follows that the lattice chain $\mathcal{L}$ associated with $\mathfrak{I}$ is the chain of all $\mathfrak{o}_E$-lattices. Thus $E^\times$ acts on $\mathcal{L}$ as a group of automorphisms and $E^\times \subset \mathcal{K}_\mathfrak{I}$, as required. Finally, any $\alpha' \in \alpha + \mathfrak{P}^{1-n} = \alpha U^1_\mathfrak{I}$ shares all the properties of α which we have used.

We now treat (1) in the case where $(\mathfrak{A}, n, \alpha)$ is an unramified simple stratum. Thus we take $\mathfrak{A} = \mathfrak{M}$ and $\alpha = \varpi^{-n}\alpha_0$, for some $\alpha_0 \in U_\mathfrak{M}$. By definition, the characteristic polynomial of α_0 over F is irreducible when reduced modulo $\mathfrak{p}$. Thus $E = F[\alpha] = F[\alpha_0]$ is an unramified quadratic field extension of F and $\mathfrak{o}_E = \mathfrak{o}[\alpha_0]$. The proof is finished as before.

In part (2), let $E = F[\alpha]$. As in 5.2 Example 2, there is a unique hereditary $\mathfrak{o}$-order $\mathfrak{A}_E$ in A which is stable under conjugation by $E^\times$. Let $\mathfrak{A}$ be a hereditary $\mathfrak{o}$-order with $\alpha \in \mathcal{K}_\mathfrak{A}$. We argue as before to show that $\mathfrak{o}_E \subset \mathfrak{A}$ and so every $\mathfrak{A}$-lattice is an $\mathfrak{o}_E$-lattice. If E/F is ramified, $v_F(\det \alpha)$ is odd and $\alpha \in \mathcal{K}_\mathfrak{A}$, so $e_\mathfrak{A} = 2$. If E/F is unramified, $\mathfrak{o}_E/\mathfrak{p}_E$ is a subring of $\mathfrak{A}/\mathfrak{P}$, whence $e_\mathfrak{A} = 1$. In both cases therefore, the set of $\mathfrak{A}$-lattices is the full set of $\mathfrak{o}_E$-lattices in $F^{(2)}$, whence $\mathfrak{A} = \mathfrak{A}_E$. The remaining assertions are immediate. $\square$

Remark. In particular, if $(\mathfrak{A}, n, \alpha)$ is a simple stratum and $\alpha' \in \alpha U^1_\mathfrak{A}$, then $F[\alpha']/F$ is a quadratic field extension, ramified if and only if $F[\alpha]/F$ is ramified. However, the fields $F[\alpha]$, $F[\alpha']$ need not be F-isomorphic.

7.5. We fix a simple stratum $(\mathfrak{A}, n, \alpha)$, and consider the irreducible representations Ξ of $\mathcal{K}_{\mathfrak{A}}$ such that $\Xi \mid U_{\mathfrak{A}}^n$ contains the character ψ_α. Temporarily set $m = [n/2]$: this is the least integer such that $2m+1 \geqslant n$. The group $U_{\mathfrak{A}}^{m+1}/U_{\mathfrak{A}}^{n+1}$ is abelian, so Ξ contains some character of $U_{\mathfrak{A}}^{m+1}$ extending ψ_α. This is of the form $\psi_{\alpha'}$, for some $\alpha' \in \alpha U_{\mathfrak{A}}^1$. We save notation by choosing $\alpha = \alpha'$: this determines α modulo $\mathfrak{P}^{-[n/2]}$. The key technical point is the following:

Proposition 1.

(1) *Let $(\mathfrak{A}, n, \alpha)$ be a simple stratum and set $E = F[\alpha]$. An element $g \in G$ intertwines the character ψ_α of $U_{\mathfrak{A}}^{[n/2]+1}$ if and only if $g \in E^\times U_{\mathfrak{A}}^{[(n+1)/2]}$.*

(2) *For $i = 1, 2$, let $(\mathfrak{A}, n, \alpha_i)$ be a simple stratum in A. The characters ψ_{α_i} of $U_{\mathfrak{A}}^{[n/2]+1}$ intertwine in G if and only if they are conjugate by an element of $U_{\mathfrak{A}}$.*

We shall give no proof, since the techniques required take us too far afield.

In (1), note that $[n/2]+1 \neq [(n+1)/2]$ if and only if n is even, which can happen only in the unramified case. Also, the set

$$J_\alpha = E^\times U_{\mathfrak{A}}^{[(n+1)/2]}$$

is a subgroup of $\mathcal{K}_{\mathfrak{A}}$ and it *normalizes* the character ψ_α of $U_{\mathfrak{A}}^{[n/2]+1}$. Further, J_α is open in G, and J_α/Z is compact.

We therefore consider the irreducible representations of J_α which contain ψ_α on $U_{\mathfrak{A}}^{[n/2]+1}$: any such representation, we note, becomes a multiple of ψ_α when restricted to $U_{\mathfrak{A}}^{[n/2]+1}$.

We define compact, open, normal subgroups of J_α:

$$H_\alpha^1 = U_E^1 . U_{\mathfrak{A}}^{[n/2]+1}, \quad J_\alpha^k = J_\alpha \cap U_{\mathfrak{A}}^k,$$

for $k = 0, 1$. Since U_E^1 normalizes ψ_α, there exists a character θ of H_α^1 extending ψ_α. Here we have:

Lemma 1. *Let θ be a character of H_α^1 such that $\theta \mid U_{\mathfrak{A}}^{[n/2]+1} = \psi_\alpha$.*

(1) *There is a unique irreducible representation η_θ of J_α^1 such that $\eta_\theta \mid H_\alpha^1$ contains θ. Indeed, $\eta_\theta \mid H_\alpha^1$ is a multiple of θ.*

(2) *An element $g \in G$ intertwines η_θ if and only if $g \in J_\alpha$.*

Again we give no proof. However, the group J_α^0/J_α^1 is cyclic, isomorphic to the group of roots of unity in E of order prime to p. Therefore:

Lemma 2.

(1) *The representation η_θ of Lemma 1 admits extension to an irreducible representation λ of J_α^0.*

(2) *The representation λ is stable under conjugation by J_α.*

(3) *If λ_1, λ_2 are irreducible representations of J_α^0 which extend η_θ, then they intertwine in G if and only if they are equivalent.*

Pairs (J_α^0, λ), constructed in this manner from a simple stratum $(\mathfrak{A}, n, \alpha)$, are called *simple types* in G.

Finally, the group J_α / J_α^0 is infinite cyclic, so we have:

Lemma 3.

(1) *The representation η_θ admits extension to an irreducible representation Λ of J_α.*

(2) *The representation Λ is intertwined by $g \in G$ if and only if $g \in J_\alpha$.*

(3) *Let Λ_1, Λ_2 be irreducible representations of J_α such that $\Lambda_1 \mid J_\alpha^1 = \Lambda_2 \mid J_\alpha^1 = \eta_\theta$. They intertwine in G if and only if they are equivalent.*

Pairs (J_α, Λ), constructed in this manner are called *extended simple types* in G. Note that we can recover $\mathfrak{A}$ from (J_α, Λ): it is the unique hereditary order such that $\mathcal{K}_\mathfrak{A} \supset J_\alpha$ (*cf.* 7.4). More formally:

Definition. *An* extended simple type *in G is a triple $(\mathfrak{A}, J, \Lambda)$, where $\mathfrak{A}$ is a hereditary order in A, J is an open subgroup of $\mathcal{K}_\mathfrak{A}$ and Λ is an irreducible smooth representation of J, of one of the following kinds:*

(1) *$\mathfrak{A} \cong \mathfrak{M}$, $J = \mathcal{K}_\mathfrak{A}$ and $\lambda = \Lambda \mid U_\mathfrak{A}$ is the inflation of an irreducible cuspidal representation of $U_\mathfrak{A}/U_\mathfrak{A}^1 \cong \mathrm{GL}_2(\boldsymbol{k})$;*

(2) *there is a simple stratum $(\mathfrak{A}, n, \alpha)$ such that $J = J_\alpha = F[\alpha]^\times U_\mathfrak{A}^{[(n+1)/2]}$, and Λ is an irreducible smooth representation of J such that $\Lambda \mid U_\mathfrak{A}^{[n/2]+1}$ is a multiple of ψ_α;*

(3) *there is a triple $(\mathfrak{A}, J, \Lambda_0)$, of one of the first two kinds, such that $\Lambda \cong \Lambda_0 \otimes \chi \circ \det$, for some some character χ of $F^\times$.*

A simple type *in G is a pair (J^0, λ), for some extended simple type $(\mathfrak{A}, J, \Lambda)$, with $J^0 = J \cap U_\mathfrak{A}$ and $\lambda = \Lambda \mid J^0$.*

Observe that, in all cases, the representation $\lambda = \Lambda \mid J^0$ is irreducible.

We now summarize the preceding discussion:

Proposition 2.

(1) *Let $(\mathfrak{A}, J, \Lambda)$ be an extended simple type in G. An element $g \in G$ intertwines Λ if and only if $g \in J$.*

(2) *For $i = 1, 2$, let $(\mathfrak{A}_i, J_i, \Lambda_i)$ be an extended simple type, and assume that the Λ_i intertwine in G. There then exists $g \in G$ such that $\mathfrak{A}_2 = g^{-1}\mathfrak{A}_1 g$, $J_2 = g^{-1} J_1 g$ and $\Lambda_2 \cong \Lambda_1^g$.*

Proof. Part (1) follows from the definition and Proposition 1(2). In (2), if the Λ_i intertwine, they have the same normalized level. Twisting with a character of $F^\times$ changes nothing, so we can assume that both the Λ_i have minimal normalized level. In light of 7.3, we can assume this level is positive. Thus $\mathfrak{A}_1 \cong \mathfrak{A}_2$. Applying a preliminary conjugation, we can assume $\mathfrak{A}_2 = \mathfrak{A}_1$. The result now follows from Proposition 1(2), Lemma 1(1) and Lemma 3(3). $\square$

We now prove the Classification Theorem.

Lemma. *If* $(\mathfrak{A}, \Xi)$ *is a cuspidal inducing datum, there is an extended simple type* $(\mathfrak{A}, J, \Lambda)$ *such that* $\Xi = \mathrm{Ind}_J^{\mathcal{K}_\mathfrak{A}} \Lambda$.

Proof. We write $\Xi = \Xi_0 \otimes \chi \circ \det$, where $(\mathfrak{A}, \Xi_0)$ is a cuspidal inducing datum of minimal normalized level. It is enough to prove the lemma for Ξ_0. In other words, we can assume that Ξ has minimal normalized level. If this level is zero, there is nothing to prove. We assume otherwise, and set $l(\Xi) = n/e_\mathfrak{A}$. The restriction of Ξ to $U_\mathfrak{A}^{[n/2]+1}$ contains a character ψ_α such that $(\mathfrak{A}, n, \alpha)$ is a simple stratum. The required extended simple type is then the natural representation of $F[\alpha]^\times U_\mathfrak{A}^{[(n+1)/2]}$ on the ψ_α-isotypic space in Ξ. $\square$

If $(\mathfrak{A}, \Xi)$ is a cuspidal inducing datum, the representation

$$\pi = c\text{-}\mathrm{Ind}_{\mathcal{K}_\mathfrak{A}}^G \Xi = c\text{-}\mathrm{Ind}_J^G \Lambda$$

is irreducible and cuspidal, by Proposition 2(1) and 7.1. On the other hand, if π is an irreducible cuspidal representation of G, there is a cuspidal inducing datum $(\mathfrak{A}, \Xi)$ such that π contains Ξ (7.2). If Ξ has level zero, there is nothing more to do (7.3). Otherwise, we write $\Xi = \mathrm{Ind}_J^{\mathcal{K}_\mathfrak{A}} \Lambda$, for some extended simple type $(\mathfrak{A}, J, \Lambda)$. By Proposition 2(2), this is the only extended simple type occurring in π, up to G-conjugacy. Thus $(\mathfrak{A}, \Xi)$ is uniquely determined by π, up to G-conjugacy. $\square$

II: Functional equations

In this part, we define certain analytic invariants, an L-function and a "local constant", attached to an irreducible smooth representation of $G = \mathrm{GL}_2(F)$.

8. Tate's functional equation

We again start with a brief discussion of a classical situation, as an illustrative example. We recall the notions of L-function and local constant for characters of $F^\times$, as a prelude to the discussion of the corresponding objects for representations of $\mathrm{GL}_2(F)$.

8.1. Let ψ be some non-trivial character of F (as group under addition). For $a \in F^\times$, the function $a\psi : x \mapsto \psi(xa)$ is also a non-trivial character of F. Indeed, the map $a \mapsto a\psi$ gives a topological isomorphism of F with its group $\widehat{F}$ of characters.

Let $\mathcal{S}(F)$ denote the set of locally constant functions $F \to \mathbb{C}$ of compact support. Thus, for $f \in \mathcal{S}(F)$, there are integers $n < m$ such that f vanishes outside $\mathfrak{p}^n$ and is constant on cosets of $\mathfrak{p}^m$.

Let ψ be a non-trivial character of F and dx some Haar measure on F. For $f \in \mathcal{S}(F)$, we define the Fourier transform $\hat{f}$ of f by

$$\hat{f}(x) = \int_F f(y)\psi(xy)\,dy.$$

Then $\hat{f} \in \mathcal{S}(A)$, but the definition depends on both ψ and dx. However, for fixed ψ, there is a unique choice of Haar measure dx for which the Fourier inversion formula holds:

$$\hat{\hat{f}}(x) = f(-x), \quad f \in \mathcal{S}(F),\ x \in F.$$

This is the *self-dual* Haar measure for the character ψ. If we temporarily denote this measure by $d_\psi x$, we have the relation

$$d_{a\psi}x = \|a\|^{\frac{1}{2}}\,d_\psi x,$$

where $a \in F^\times$.

8.2. Let χ be a quasicharacter of $F^\times$, let s denote a complex variable, and let $\Phi \in \mathcal{S}(F)$. We tentatively define

$$(*) \qquad \zeta(\Phi,\chi,s) = \int_{F^\times} \Phi(x)\chi(x)\|x\|^s\,d^*x,$$

where d^*x is a Haar measure on $F^\times$, the choice of which will be immaterial.

Proposition.

(1) *The integral* $(*)$ *represents a rational function in* q^{-s}. *There exists* $s_0 = s_0(\chi) \in \mathbb{R}$ *such that the integral* $(*)$ *converges absolutely and uniformly in vertical strips in the region* $\operatorname{Re} s > s_0$, *for all* $\Phi \in \mathcal{S}(F)$. *In the case* $|\chi| = 1$, *we may take* $s_0 = 0$.

(2) *The space* $\mathcal{Z}(\chi)$ *of functions* $\zeta(\Phi, \chi, s)$ *is a fractional ideal of* $\mathbb{C}[q^s, q^{-s}]$ *in the field* $\mathbb{C}(q^{-s})$ *of rational functions in* q^{-s}. *There is a unique polynomial* $P_\chi(t) \in \mathbb{C}[t]$ *such that* $P_\chi(0) = 1$ *and*

$$\mathcal{Z}(\chi) = P_\chi(q^{-s})^{-1}\, \mathbb{C}[q^s, q^{-s}].$$

Proof. We start with some formal observations. First, if $g\Phi$ denotes the function $x \mapsto \Phi(g^{-1}x)$, we have

$$\zeta(g\Phi, \chi, s) = \chi(g)\|g\|^s\, \zeta(\Phi, \chi, s).$$

Thus $\mathcal{Z}(\chi)$ is a module over the ring $\mathbb{C}[q^s, q^{-s}]$. Next, suppose that $\Phi(0) = 0$. Thus Φ is compactly supported on $F^\times$; its support is a finite union of compact open sets, on each of which both χ and the absolute value are constant. Thus $\zeta(\Phi, \chi, s) \in \mathbb{C}[q^s, q^{-s}]$. It is trivial to check that we can achieve $\zeta(\Phi, \chi, s) = 1$. Thus $\mathcal{Z}(\chi)$ is generated, as $\mathbb{C}[q^s, q^{-s}]$-module, by 1 and $\zeta(\Phi, \chi, s)$, for one Φ not vanishing at 0.

For $m \in \mathbb{Z}$, let Φ_m denote the characteristic function of the fractional ideal $\mathfrak{p}^m$. Setting $I_m = \zeta(\Phi_m, \chi, s)$, we have

$$I_m = I_{m+1} + \int_{\varpi^m U_F} \chi(x)\|x\|^s\, d^*x,$$

while also

$$I_m = \chi(\varpi)^{-1} q^s I_{m+1}.$$

The integral over $\varpi^m U_F$ is of the form cq^{-ms}, for some constant c. Moreover, $c = 0$ if χ is ramified. We so get an expression for I_{m+1} as a rational function. The absolute convergence case is that where $\chi = 1$ and $s \in \mathbb{R}$, so the assertions concerning convergence are proved. $\square$

We define

$$L(\chi, s) = P_\chi(q^{-s})^{-1}.$$

The calculations in the preceding proof give:

$$L(\chi, s) = \begin{cases} (1 - \chi(\varpi)q^{-s})^{-1} & \textit{if } \chi \textit{ is unramified,} \\ 1 & \textit{otherwise.} \end{cases} \tag{8.2.1}$$

8.3. We fix a non-trivial character ψ of F and form the Fourier transform $\Phi \mapsto \hat{\Phi}$ on $\mathcal{S}(F)$ relative to the self-dual Haar measure for ψ.

Functional equation. *There is a unique rational function $\gamma(\chi, s, \psi) \in \mathbb{C}(q^{-s})$ such that*

$$\zeta(\hat{\Phi}, \chi^{-1}, 1-s) = \gamma(\chi, s, \psi)\, \zeta(\Phi, \chi, s),$$

for all $\Phi \in \mathcal{S}(F)$. We have

$$\gamma(\chi, s, \psi) = \varepsilon(\chi, s, \psi)\, \frac{L(\chi^{-1}, 1-s)}{L(\chi, s)},$$

where $\varepsilon(\chi, s, \psi)$ is of the form aq^{bs}, for some $a \in \mathbb{C}^\times$ and $b \in \mathbb{Z}$.

Proof. The first assertion amounts to $\zeta(\hat{\Phi}, \chi^{-1}, 1-s)/\zeta(\Phi, \chi, s)$ being independent of Φ. We therefore have to prove

$$(\dagger) \qquad \zeta(\Phi, \chi, s)\, \zeta(\hat{\Psi}, \chi^{-1}, 1-s) = \zeta(\hat{\Phi}, \chi^{-1}, 1-s)\, \zeta(\Psi, \chi, s),$$

for all $\Phi, \Psi \in \mathcal{S}(F)$.

Introducing a translation on s, we can assume that $|\chi| = 1$; both sides of the relation $(\dagger)$ can therefore be expressed as double integrals, convergent in the region $0 < \operatorname{Re} s < 1$. In the following calculation, dx denotes the Haar measure on F which is self-dual for ψ, and d^*x is the measure $\|x\|^{-1}dx$ on $F^\times$. We have

$$\begin{aligned}\zeta(\Phi, \chi, s)\, \zeta(\hat{\Psi}, \chi^{-1}, 1-s) &= \iint \Phi(x)\hat{\Psi}(y)\chi(xy^{-1})\|xy^{-1}\|\|y\|\, d^*x\, d^*y \\ &= \iint \Phi(x)\hat{\Psi}(xy)\chi(y^{-1})\|y\|^{-s}\|xy\|\, d^*x\, d^*y \\ &= \int_{F^\times} \chi(y^{-1})\|y\|^{1-s} \int_F \Phi(x)\hat{\Psi}(xy)\, dx\, d^*y.\end{aligned}$$

The inner integral here can be written as

$$\begin{aligned}\int_F \Phi(x)\hat{\Psi}(xy)\, dx &= \iint \Phi(x)\Psi(z)\psi_A(xyz)\, dx\, dz \\ &= \|y\|^{-1} \int_F \hat{\Phi}(z)\Psi(y^{-1}z)\, dz,\end{aligned}$$

using the substitution $z \mapsto y^{-1}z$. Therefore

$$\begin{aligned}&\zeta(\Phi, \chi, s)\, \zeta(\hat{\Psi}, \chi^{-1}, 1-s) \\ &\quad= \iint \hat{\Phi}(z)\Psi(y^{-1}z)\chi(y^{-1})\|y\|^{-s}\|z\|\, d^*y\, d^*z \\ &\quad= \iint \hat{\Phi}(z)\Psi(yz)\chi(y)\|y\|^{s}\|z\|\, d^*y\, d^*z \qquad (\text{using } y \mapsto y^{-1}) \\ &\quad= \iint \hat{\Phi}(z)\Psi(y)\chi(z^{-1}y)\|y\|^{s}\|z\|^{1-s}\, d^*y\, d^*z \qquad (\text{using } y \mapsto z^{-1}y) \\ &\quad= \zeta(\hat{\Phi}, \chi^{-1}, 1-s)\, \zeta(\Psi, \chi, s),\end{aligned}$$

as required.

Finally, we note that

$$\varepsilon(\chi, s, \psi) = \frac{\zeta(\hat{\Phi}, \chi^{-1}, 1-s)/L(\chi^{-1}, 1-s)}{\zeta(\Phi, \chi, s)/L(\chi, s)},$$

whence it satisfies

$$\varepsilon(\chi, s, \psi)\, \varepsilon(\chi^{-1}, 1-s, \psi) = \chi(-1).$$

We can choose Φ so that $\zeta(\Phi, s, \psi)/L(\chi, s) = 1$, so $\varepsilon(\chi, s, \psi) \in \mathbb{C}[q^s, q^{-s}]$; the same applies to $\varepsilon(\chi^{-1}, 1-s, \psi)$, whence $\varepsilon(\chi, s, \psi)$ is an invertible element of $\mathbb{C}[q^s, q^{-s}]$ and so of the desired form. $\square$

8.4. It will be very useful to have a table of values for the *local constant* $\varepsilon(\chi, s, \psi)$. The dependence on ψ is transparent:

Lemma. *Let $a \in F^\times$; we then have*

$$\varepsilon(\chi, s, a\psi) = \chi(a)\, \|a\|^{s-\frac{1}{2}}\, \varepsilon(\chi, s, \psi).$$

For consistency with our other conventions, we now assume that ψ *has level one.* The self-dual Haar measure for ψ then satisfies

$$\int_{\mathfrak{o}} dx = q^{1/2}.$$

It will be convenient to choose d^*x such that

$$\int_{U_F} d^*x = 1.$$

(8.4.1) We consider first the case where χ is *unramified.* If Φ denotes the characteristic function of $\mathfrak{o}$, we get $\zeta(\Phi, \chi, s) = L(\chi, s)$. On the other side, $\hat{\Phi}(x) = q^{1/2}\Phi(\varpi^{-1}x)$, so

$$\begin{aligned}\zeta(\hat{\Phi}, \chi^{-1}, s) &= q^{1/2} \int_{F^\times} \Phi(\varpi^{-1}x)\chi(x)^{-1}\|x\|^s\, d^*x \\ &= q^{1/2-s}\chi(\varpi)^{-1} \int_{F^\times} \Phi(x)\chi(x)^{-1}\|x\|^s\, d^*x \\ &= q^{1/2-s}\chi(\varpi)^{-1}\, L(\chi^{-1}, s).\end{aligned}$$

It follows that

$$\varepsilon(\chi, s, \psi) = \frac{\zeta(\hat{\Phi}, \chi^{-1}, 1-s)}{L(\chi^{-1}, 1-s)} = q^{s-\frac{1}{2}}\chi(\varpi)^{-1}.$$

(8.4.2) We turn to the case where χ is *tamely ramified,* but not unramified. We take for $\varPhi$ the characteristic function of U_F^1. This gives us $\zeta(\varPhi, \chi, s) = (q-1)^{-1}$. On the other side, the support of $\hat{\varPhi}$ is $\mathfrak{o}$ and

$$\hat{\varPhi}(y) = q^{-1/2}\psi(y), \quad y \in \mathfrak{o}.$$

Thus

$$\zeta(\hat{\varPhi}, \chi^{-1}, s) = q^{-1/2} \int_{\mathfrak{o}\cap F^\times} \chi(x)^{-1}\psi(x)\|x\|^s \, d^*x.$$

We decompose the range of integration as the union of sets $\mathfrak{p}^i \smallsetminus \mathfrak{p}^{i+1} = \varpi^i U_F$, $i \geqslant 0$. The additive character ψ is trivial on $\mathfrak{p}^i$, for $i \geqslant 1$, while χ is non-trivial on U_F. The integral therefore vanishes on such sets, giving us

$$\zeta(\hat{\varPhi}, \chi^{-1}, s) = q^{-1/2} \int_{U_F} \chi(x)^{-1}\psi(x) \, d^*x.$$

The integrand is constant on cosets of U_F^1, so we obtain

$$\varepsilon(\chi, s, \psi) = q^{-1/2} \sum_{x\in U_F/U_F^1} \chi(x)^{-1}\psi(x).$$

We remark that $\chi \mid U_F$ is the inflation of a non-trivial character $\tilde{\chi}$ of $\boldsymbol{k}^\times$, while $\psi \mid \mathfrak{o}$ is the inflation of a non-trivial character $\tilde{\psi}$ of $\boldsymbol{k}$. The sum in the expression for the local constant is then just the classical Gauss sum

$$\tau(\tilde{\chi}^{-1}, \tilde{\psi}) = \sum_{x\in \boldsymbol{k}^\times} \tilde{\chi}(x)^{-1}\tilde{\psi}(x).$$

(8.4.3) We now assume that χ has level $n \geqslant 1$. We let $\varPhi$ be the characteristic function of U_F^{n+1}, giving $\xi(\varPhi, \chi, s) = (U_F : U_F^{n+1})^{-1}$. The Fourier transform $\hat{\varPhi}$ has support $\mathfrak{p}^{-n}$ and

$$\hat{\varPhi}(y) = q^{\frac{1}{2}-n-1}\psi(y), \quad y \in \mathfrak{p}^{-n}.$$

Thus

$$\zeta(\hat{\varPhi}, \chi^{-1}, s) = q^{\frac{1}{2}-n-1} \int_{\mathfrak{p}^{-n}\smallsetminus\{0\}} \psi(y)\chi(y)^{-1}\|y\|^s \, d^*y.$$

We again decompose the range of integration into "shells" $\mathfrak{p}^i \smallsetminus \mathfrak{p}^{i+1}$, $i \geqslant -n$. We know that the integral must reduce to a monomial in q^s, so only one of these shells can contribute. The first of these, corresponding to $i = -n$, gives (for any $\alpha \in F$ with $v_F(\alpha) = -n$)

$$q^{ns+\frac{1}{2}-n-1} \int_{U_F} \psi(\alpha u)\chi^{-1}(\alpha u) \, d^*u.$$

The integrand is constant on cosets of U_F^{n+1} and so reduces to a sum over U_F/U_F^{n+1}. An elementary calculation shows that the sum is non-zero, whence

$$\varepsilon(\chi, s, \psi) = q^{n(\frac{1}{2}-s)} \sum_{x \in U_F/U_F^{n+1}} \chi(\alpha x)^{-1}\psi(\alpha x)/q^{(n+1)/2}.$$

We remark that the denominator $q^{(n+1)/2}$ is the absolute value of the unnormalized trigonometric sum

$$\sum_{x \in U_F/U_F^{n+1}} \chi(x)^{-1}\psi(\alpha x).$$

9. Functional equation for GL(2)

We now review the functional equation of Godement-Jacquet, which generalizes the ideas of §8 to representations of $G = \mathrm{GL}_2(F)$.

9.1. Let $\mathcal{S}(A)$ be the space of locally constant, compactly supported functions $A \to \mathbb{C}$. If ψ denotes a non-trivial character of F, the function $\psi_A = \psi \circ \mathrm{tr}_A$ on $A = \mathrm{M}_2(F)$ is a non-trivial character of A, relative to which we can define Fourier transform

$$\hat{\Phi}(x) = \int_A \Phi(y)\, \psi_A(xy)\, dy,$$

for a Haar measure dy on A. There is a unique self-dual measure for which the Fourier inversion formula

$$\hat{\hat{\Phi}}(x) = \Phi(-x), \quad \Phi \in \mathcal{S}(A),$$

holds: henceforward, we use only this one to define the Fourier transform.

9.2. Let (π, V) be an irreducible smooth representation of G; we denote by $\mathcal{C}(\pi)$ the space of coefficients of π. Thus, if $f \in \mathcal{C}(\pi)$, the function $f^\vee : g \mapsto f(g^{-1})$ lies in $\mathcal{C}(\check{\pi})$ and the map $f \mapsto f^\vee$ gives a linear isomorphism $\mathcal{C}(\pi) \to \mathcal{C}(\check{\pi})$.

We fix a Haar measure d^*x on G and, as before, s denotes a complex variable. We define

$$(*) \qquad \zeta(\Phi, f, s) = \int_G \Phi(x)\, f(x)\, \|\det x\|^s\, d^*x, \quad \Phi \in \mathcal{S}(A),\ f \in \mathcal{C}(\pi).$$

The main result closely parallels the discussion in §8:

Theorem.

(1) *The integral $(*)$ represents a rational function in q^{-s}. There exists $s_0 = s_0(\pi)$ such that, for $\mathrm{Re}\, s > s_0$, $f \in \mathcal{C}(\pi)$, $\Phi \in \mathcal{S}(A)$, the integral $(*)$ converges absolutely and uniformly in vertical strips.*

(2) *The space*

$$\mathcal{Z}(\pi) = \{\zeta(\Phi, f, s) : \Phi \in \mathcal{S}(A), f \in \mathcal{C}(\pi)\}$$

is a finitely generated $\mathbb{C}[q^s, q^{-s}]$*-submodule of* $\mathbb{C}(q^{-s})$*. There is a unique polynomial* $P_\pi(t) \in \mathbb{C}[t]$ *such that* $P_\pi(0) = 1$ *and*

$$\mathcal{Z}(\pi) = P_\pi(q^{-s})^{-1}\, \mathbb{C}[q^s, q^{-s}].$$

(3) *There is a unique rational function* $\gamma(\pi, s, \psi)$ *such that*

$$\zeta(\hat{\Phi}, f^\vee, \tfrac{3}{2}-s) = \gamma(\pi, s, \psi)\, \zeta(\Phi, f, \tfrac{1}{2}+s),$$

for all $\Phi \in \mathcal{S}(A)$ *and* $f \in \mathcal{C}(\pi)$*.*

(4) *Defining* $L(\pi, s) = P_\pi(q^{-s})^{-1}$*, we have*

$$\gamma(\pi, s, \psi) = \varepsilon(\pi, s, \psi)\, \frac{L(\check{\pi}, 1-s)}{L(\pi, s)}\,,$$

where $\varepsilon(\pi, s, \psi) = aq^{bs}$*, for some* $a \in \mathbb{C}^\times$*,* $b \in \mathbb{Z}$*.*

9.3. We start with some general remarks. For $f \in \mathcal{C}(\pi)$ and $g_1, g_2 \in G$, the function

$$g_1 f g_2 : x \longmapsto f(g_2 x g_1)$$

also lies in $\mathcal{C}(\pi)$. If we fix $f \neq 0$, the functions $g_1 f g_2$, $g_i \in G$, then span $\mathcal{C}(\pi)$, since π is irreducible. We use a similar notation for the action of $G \times G$ on $\mathcal{S}(A)$. We have

$$\zeta(\Phi, g_1 f g_2, s) = \|\det g_1 g_2\|^{-s}\, \zeta(g_1^{-1}\Phi g_2^{-1}, f, s).$$

The space $\mathcal{Z}(\pi)$ is therefore spanned by the functions $\zeta(\Phi, f, s)$, as Φ ranges over $\mathcal{S}(A)$ and any fixed $f \in \mathcal{C}(\pi)$, $f \neq 0$. We may therefore choose f conveniently.

Remark. In much the same way, one can extract the dependence of $\varepsilon(\pi, s, \psi)$ on the additive character ψ: if $a \in F^\times$, we have

$$\varepsilon(\pi, s, a\psi) = \omega_\pi(a)\, \|a\|^{2s-1}\, \varepsilon(\pi, s, \psi).$$

9.4. Until further notice, we assume that π is *cuspidal.* Thus there is a cuspidal inducing datum $(\mathfrak{A}, \xi)$ in G such that $\pi = c\text{-}\mathrm{Ind}_K^G\, \Xi$, where we abbreviate $K = \mathcal{K}_{\mathfrak{A}}$. We set $\xi = \Xi \mid U_{\mathfrak{A}}$. The representation ξ is irreducible and $V^\xi = V^\Xi$; similarly for duals.

We choose non-zero elements $v \in V^\xi$ and $\check{v} \in (V^\vee)^{\xi^\vee}$; the coefficient $f = \gamma_{\check{v}\otimes v}$ is not identically zero. It satisfies $e_{\xi^\vee} * f * e_{\xi^\vee} = f$, and so has support contained in K.

Let Π be a prime element of $\mathfrak{A}$. Thus $K = \bigcup_{r\in\mathbb{Z}} \Pi^r U_{\mathfrak{A}}$. If $x \in A$, $x \notin G$, $x \neq 0$, then x has an open neighbourhood in A not meeting K: the only "point of accumulation" of the sets $\Pi^r J^0$ in A is 0. We conclude:

Lemma. *Let $\varPhi \in \mathcal{S}(A)$, and suppose $\varPhi(0) = 0$. The support of $\varPhi$ then meets only finitely many cosets $\varPi^r U_{\mathfrak{A}}$, and we have $\zeta(\varPhi, f, s) \in \mathbb{C}[q^s, q^{-s}]$.*

One shows easily that $1 \in \mathcal{Z}(\pi)$; the space $\mathcal{Z}(\pi)$ is therefore spanned by $\mathbb{C}[q^s, q^{-s}]$ and $\zeta(\varPhi, f, s)$, for any one function $\varPhi \in \mathcal{S}(A)$ with $\varPhi(0) \neq 0$. It is convenient to choose $\varPhi$ to be the characteristic function of $\mathfrak{A}$. The function $\varPhi$ is then invariant under the action of $U_{\mathfrak{A}}$ on either side. We have $e_{\xi^\vee} * f * e_{\xi^\vee} = f$, so

$$\zeta(\varPhi, f, s) = \zeta(e_\xi * \varPhi * e_\xi, f, s) = 0,$$

since the function $e_\xi * \varPhi * e_\xi$ is zero. Therefore:

Proposition. *If π is cuspidal, then $\mathcal{Z}(\pi) = \mathbb{C}[q^s, q^{-s}]$ and $L(\pi, s) = 1$.*

9.5. Let $\mathcal{H}_0(G)$ be the space of functions $\varPhi \in \mathcal{H}(G)$ such that $\hat{\varPhi} \in \mathcal{H}(G)$. For $\varPhi \in \mathcal{H}(G)$, we may form the operator-valued zeta-integral

$$\zeta(\varPhi, \pi, s) = \int_G \varPhi(g)\, \pi(g)\, \|\det g\|^s\, d^*g \in \mathrm{End}_{\mathbb{C}}(V),$$

which is, in fact, a finite sum. It satisfies

$$\big\langle \check{v}, \zeta(\varPhi, \pi, s)v \big\rangle = \zeta(\varPhi, f, s),$$

where $f = \gamma_{\check{v}\otimes v}$.

If $\varPhi \in \mathcal{H}_0(G)$, we may also form

$$\zeta(\hat{\varPhi}, \check{\pi}, s) = \int_G \hat{\varPhi}(g)\, \check{\pi}(g)\, \|\det g\|^s\, d^*g \in \mathrm{End}_{\mathbb{C}}(V^\vee),$$

and its transpose

$${}^t\zeta(\hat{\varPhi}, \check{\pi}, s) = \int_G \hat{\varPhi}(g)\, \pi(g^{-1})\, \|\det g\|^s\, d^*g \in \mathrm{End}_{\mathbb{C}}(V).$$

The first step is to prove

$$(\dagger) \qquad \zeta(\varPhi, \pi, s)\, {}^t\zeta(\hat{\varPsi}, \check{\pi}, 2-s) = \zeta(\varPsi, \pi, s)\, {}^t\zeta(\hat{\varPhi}, \check{\pi}, 2-s),$$

for all $\varPhi, \varPsi \in \mathcal{H}_0(G)$.

The two sides of ($\dagger$) can be expressed as absolutely convergent double integrals. The left hand side can be written

$$\begin{aligned}\zeta(\varPhi, \pi, s)\, {}^t\zeta(\hat{\varPsi}, \check{\pi}, 2-s) &= \iint \varPhi(x)\, \hat{\varPsi}(y)\, \pi(xy^{-1}) \|\det y\|^2\, \|\det xy^{-1}\|^s\, d^*x\, d^*y \\ &= \int_G \pi(x) \|\det x\|^s \int_G \varPhi(xy)\, \hat{\varPsi}(y)\, \|\det y\|^2\, d^*y\, d^*x,\end{aligned}$$

using the obvious substitution. The procedure is independent of the choice of Haar measure d^*g on G. It will be convenient to take $d^*g = \|\det g\|^{-2}dg$, where dg is the self dual Haar measure on A. The inner integral in the last expression can be written as

$$\begin{aligned}\int_G \Phi(xy)\,\hat{\Psi}(y)\|\det y\|^2\,d^*y &= \int_A \Phi(xy)\,\hat{\Psi}(y)\,dy\\ &= \iint \Phi(xy)\,\Psi(z)\,\psi_A(zy)\,dz\,dy\\ &= \|\det x\|^{-2}\iint \Phi(y)\,\Psi(z)\,\psi_A(zx^{-1}y)\,dy\,dz\\ &= \|\det x\|^{-2}\int_A \hat{\Phi}(zx^{-1})\,\Psi(z)\,dz.\end{aligned}$$

Substituting back, we get

$$\begin{aligned}\zeta(\Phi,\pi,s)\,{}^t\zeta(\hat{\Psi},\check{\pi},2-s) &= \iint \hat{\Phi}(zx^{-1})\,\Psi(z)\,\pi(x)\|\det x\|^{s-2}\,\|\det z\|^2\,d^*x\,d^*z\\ &= \iint \hat{\Phi}(x^{-1})\,\Psi(z)\,\pi(xz)\|\det xz\|^{s-2}\,\|\det z\|^2\,d^*x\,d^*z\\ &= \iint \hat{\Phi}(x)\,\Psi(z)\,\pi(x^{-1}z)\|\det x\|^{2-s}\,\|\det z\|^s\,d^*x\,d^*z\\ &= {}^t\zeta(\hat{\Phi},\check{\pi},2-s)\,\zeta(\Psi,\pi,s),\end{aligned}$$

as required for (†).

We now take $\Phi = e_\xi$; this gives $\zeta(\Phi,\pi,s) = \pi(e_\xi)$, which acts on V^ξ as the identity. Since $e_\xi * \Phi * e_\xi = \Phi$, we have $\hat{\Phi} = e_{\xi^\vee} * \hat{\Phi} * e_{\xi^\vee}$, and so $\hat{\Phi}$ lies in $\mathcal{H}(G)$. We may therefore *define*

$$\gamma(\pi, s-\tfrac{1}{2}, \psi) = {}^t\zeta(\hat{\Phi},\check{\pi},2-s) \in \mathrm{End}_{\mathbb{C}}(V^\xi).$$

This operator is invariant under conjugation by $\xi(U_{\mathfrak{A}})$ and, because V^ξ is irreducible over $U_{\mathfrak{A}}$, it is a scalar. Thus $\gamma(\pi,s,\psi)$ is a scalar, and lies in $\mathbb{C}[q^s,q^{-s}]$. Hence, for any $\Psi \in \mathcal{H}_0(G)$, we have

$$(\ddagger)\qquad {}^t\zeta(\hat{\Psi},\check{\pi},2-s) = \gamma(\pi,s-\tfrac{1}{2},\psi)\,\zeta(\Psi,\pi,s).$$

We now fix a non-zero coefficient $f = \check{v}\otimes v \in (V^\xi)^\vee \otimes V^\xi$. Then, for any $\Psi \in \mathcal{S}(A)$,

$$\zeta(\Psi,f,s) = \zeta(\Psi',f,s) = \langle \check{v}, \zeta(\Psi',\pi,s)v\rangle,$$

where $\Psi' = e_\xi * \Psi * e_\xi \in \mathcal{H}_0(G)$. We have $\widehat{\Psi'} = e_{\xi^\vee} * \hat{\Psi} * e_{\xi^\vee} = \hat{\Psi}'$ say, so, on the other side,

$$\zeta(\hat{\Psi},f^\vee,s) = \langle \zeta(\hat{\Psi}',\check{\pi},s)\check{v}, v\rangle = \langle \check{v}, {}^t\zeta(\widehat{\Psi'},\check{\pi},s)v\rangle.$$

Invoking (‡) for the functions Ψ' and $\widehat{\Psi'}$, we get

$$\zeta(\hat{\Psi}, f^\vee, 2-s) = \gamma(\pi, s-\tfrac{1}{2}, \psi)\, \zeta(\Psi, f, s).$$

This holds for all $\Psi \in \mathcal{S}(A)$, and our fixed f. It therefore holds for all Ψ and all coefficients f. Finally,

$$\begin{aligned}\zeta(\hat{\hat{\Psi}}, f, s) &= \gamma(\check{\pi}, \tfrac{3}{2}-s, \psi)\, \zeta(\hat{\Psi}, f^\vee, 2-s)\\ &= \gamma(\check{\pi}, \tfrac{3}{2}-s, \psi)\, \gamma(\pi, s-\tfrac{1}{2}, \psi)\, \zeta(\Psi, f, s),\end{aligned}$$

whence (using Fourier inversion)

$$\gamma(\check{\pi}, 1-s, \psi)\, \gamma(\pi, s, \psi) = \omega_\pi(-1).$$

This equation implies that $\gamma(\pi, s, \psi)$ is an invertible element of $\mathbb{C}[q^s, q^{-s}]$, and so of the required form. $\square$

9.6. We give only a very brief discussion of non-cuspidal representations π. It will be more convenient to work with normalized induction, so we regard π as a component, or composition factor, of a representation $\boldsymbol{\iota}_B^G\,\chi$, where $\chi = \chi_1 \otimes \chi_2$ is a character of T. We set $(\xi, W) = \boldsymbol{\iota}_B^G\,\chi$. The representation ξ need not be irreducible, but it is admissible and it admits a central character, namely $\chi \mid Z$.

The smooth dual $\check{\xi}$ of ξ is equivalent to $\boldsymbol{\iota}_B^G\,\check{\chi}$; the assertions of Theorem 9.2 at least make sense for ξ. Indeed:

Proposition. *The assertions of Theorem 9.2 hold for ξ. Moreover,*

$$\mathcal{Z}(\xi) = \mathcal{Z}(\chi_1)\, \mathcal{Z}(\chi_2), \quad \gamma(\xi, s, \psi) = \gamma(\chi_1, s, \psi)\, \gamma(\chi_2, s, \psi).$$

It follows that

$$L(\xi, s) = L(\chi_1, s)\, L(\chi_2, s), \quad \varepsilon(\pi, s, \psi) = \varepsilon(\chi_1, s, \psi)\, \varepsilon(\chi_2, s, \psi).$$

The argument is quite elaborate, and we omit the details. We note one immediate consequence:

Corollary. *Let $\chi = \chi_1 \otimes \chi_2$ be a character of T, and suppose that the representation $\pi = \boldsymbol{\iota}_B^G\,\chi$ is irreducible. Then:*

$$\begin{aligned}L(\pi, s) &= L(\chi_1, s)\, L(\chi_2, s),\\ \varepsilon(\pi, s, \psi) &= \varepsilon(\chi_1, s, \psi)\, \varepsilon(\chi_2, s, \psi).\end{aligned}$$

9.7. We turn to the case where ξ is not irreducible. If π is a component of ξ, we surely have $\mathcal{Z}(\pi) \subset \mathcal{Z}(\xi)$ and $\mathcal{Z}(\check{\pi}) \subset \mathcal{Z}(\check{\xi})$. The functional equation now implies $\gamma(\pi, s, \psi) = \gamma(\xi, s, \psi)$, and similarly for the duals. Thus we have polynomials $P(t)$, $P^\vee(t)$, with constant term 1, such that

$$L(\pi, s) = P(q^{-s})L(\xi, s), \quad L(\check{\pi}, s) = P^\vee(q^{-s})L(\check{\xi}, s).$$

The equality of γ-functions implies

$$\varepsilon(\pi, s, \psi) = \varepsilon(\xi, s, \psi)\, P(q^{-s})/P^\vee(q^{-1+s}).$$

The ε-functions are both monomial. Thus $\alpha \mapsto \alpha^{-1}q$ gives a bijection between the roots α of $P(t)$ and those of $P^\vee(t)$.

The reducibility of $\xi = \boldsymbol{\iota}_B^G\,\chi$ is equivalent to $\chi_1\chi_2^{-1}(x) = \|x\|^{\pm 1}$, that is, $\chi = \phi\delta_B^{\pm\frac{1}{2}}$, for some character ϕ of $F^\times$. If ϕ is ramified, the L-functions are trivial. We may as well assume, therefore, that ϕ is trivial. An elementary calculation shows that the L-function $L(1_G, s)$ is quadratic, and there is nothing more to do.

It remains only to consider the Steinberg representation St_G. Taking $\xi = \boldsymbol{\iota}_B^G\,\delta_B^{-\frac{1}{2}}$, we have

$$\gamma(\xi, s, \psi) = q^s\,\frac{(1 - q^{-\frac{1}{2}-s})(1 - q^{\frac{1}{2}-s})}{(1 - q^{s-\frac{1}{2}})(1 - q^{s-\frac{3}{2}})}.$$

The combinatorics only allow the possibilities $L(\mathrm{St}_G, s) = L(\xi, s)$ or (the correct one):

$$L(\mathrm{St}_G, s) = (1 - q^{-\frac{1}{2}-s})^{-1}, \quad \varepsilon(\mathrm{St}_G, s, \psi) = -q^{1/2}.$$

10. Cuspidal local constants

We give an explicit formula for the local constant $\varepsilon(\pi, s, \psi)$, when π is an irreducible cuspidal representation of $G = \mathrm{GL}_2(F)$.

10.1. Let $(\mathfrak{A}, \Xi)$ be a cuspidal inducing datum in G, $\Xi : \mathcal{K}_{\mathfrak{A}} \to \mathrm{Aut}_{\mathbb{C}}(W)$, say. As always, ψ denotes a character of F of level one. Let n be the level of Ξ, that is, the least integer $\geqslant 0$ such that Ξ is trivial on $U_{\mathfrak{A}}^{n+1}$. We define an element $\mathrm{T}(\Xi, \psi)$ of $\mathrm{End}_{\mathbb{C}}(W)$ by

$$(\star) \qquad \mathrm{T}(\Xi, \psi) = \sum_{x \in U_{\mathfrak{A}}/U_{\mathfrak{A}}^{n+1}} \Xi(cx)\,\psi_A(cx),$$

where $c \in \mathcal{K}_{\mathfrak{A}}$ satisfies $c\mathfrak{A} = \mathfrak{P}^{-n}$, $\mathfrak{P} = \mathrm{rad}\,\mathfrak{A}$. For fixed c, the function $x \mapsto \Xi(cx)\psi_A(cx)$ is invariant under right translation by $U_{\mathfrak{A}}^{n+1}$, so the definition $(\star)$

is independent of the choice of coset representatives. It follows that it is also independent of the choice of c, and then that $\mathrm{T}(\Xi,\psi)$ commutes with all $\Xi(y)$, $y \in \mathcal{K}_{\mathfrak{A}}$. Thus $\mathrm{T}(\Xi,\psi)$ is a scalar operator: there exists $\tau(\Xi,\psi) \in \mathbb{C}$ such that

$$(\dagger)\qquad \mathrm{T}(\Xi,\psi) = \tau(\Xi,\psi)\, 1_W.$$

(The number $\tau(\Xi,\psi)$ is the "non-abelian congruence Gauss sum" of Ξ.)

Proposition. *If $(\mathfrak{A},\Xi)$ is a cuspidal inducing datum in G, the quantity $\tau(\Xi,\psi)$ is non-zero. More precisely, if ω_Ξ is the central character of Ξ, we have*

$$\tau(\Xi,\psi)\,\tau(\check{\Xi},\psi) = \omega_\Xi(-1)\,|\mathfrak{A}/\mathfrak{P}^{n+1}|.$$

Proof. Consider first the case where Ξ has level zero. Thus $\mathfrak{A} = \mathfrak{M}$ and $\xi = \Xi \mid U_{\mathfrak{M}}$ is the inflation of an irreducible cuspidal representation $\tilde{\xi}$ of $\mathrm{GL}_2(\boldsymbol{k})$. In the formula $(\star)$, we can take $c = 1$. The character $\psi \mid \mathfrak{o}$ is the inflation of a non-trivial character of $\tilde{\psi}$ of $\boldsymbol{k}$, and $\psi_A \mid \mathfrak{M}$ is the inflation of the character $\tilde{\psi}_A = \tilde{\psi} \circ \mathrm{tr}$ of $\mathfrak{M}/\mathfrak{p}\mathfrak{M} = \mathrm{M}_2(\boldsymbol{k})$. This gives us

$$\tau(\Xi,\psi) = \frac{1}{\dim \Xi} \sum_{x \in \mathrm{GL}_2(\boldsymbol{k})} \mathrm{tr}\,\tilde{\xi}(x)\,\tilde{\psi}_A(x).$$

In the notation of §1, we have $\tilde{\xi} = \pi_\theta$, for a regular character θ of $\boldsymbol{l}^\times$, where $\boldsymbol{l}/\boldsymbol{k}$ is the quadratic field extension of $\boldsymbol{k}$. We use the character table for π_θ given in §1 to calculate the Gauss sum directly. One finds that

$$\tau(\Xi,\psi) = -q \sum_{x \in \boldsymbol{l}^\times} \theta(x)\tilde{\psi}_{\boldsymbol{l}}(x),$$

where $\tilde{\psi}_{\boldsymbol{l}} = \tilde{\psi} \circ \mathrm{Tr}_{\boldsymbol{l}/\boldsymbol{k}}$. The problem thus reduces to a standard one for classical Gauss sums.

We defer treatment of the positive level case to 10.2 below.

10.2. There is a useful variation on this theme. We write $\Xi = \mathrm{Ind}\,\Lambda$, for some extended simple type (J,Λ). Write $\lambda = \Lambda \mid J^0$, and let Y be the representation space of Λ. The operator $\mathrm{T}(\Xi,\psi)$ is a scalar matrix with eigenvalue $\tau(\Xi,\psi)$, whence

$$\Xi(e_\lambda) \circ \mathrm{T}(\Xi,\psi) \circ \Xi(e_\lambda) = \tau(\Xi,\psi)\,\Xi(e_\lambda).$$

However, we may take $c \in J$ above, and then $\Xi(e_\lambda) \circ \Xi(cx) \circ \Xi(e_\lambda) = \Lambda(cx)$ if $x \in J$, and is zero otherwise. We deduce that

$$(\ddagger)\qquad \mathrm{T}(\Lambda,\psi) = \sum_{x \in J^0/U_{\mathfrak{A}}^{n+1}} \Lambda(cx)\,\psi_A(cx) = \tau(\Xi,\psi)\,1_Y.$$

Proposition. *Let $(\mathfrak{A}, \Xi)$ be a cuspidal inducing datum, of level $n \geqslant 1$, induced by an extended simple type (J, Λ). Let $c \in J$ be such that $\Lambda \mid U_{\mathfrak{A}}^{[n/2]+1}$ is a multiple of ψ_{-c}. The Gauss sum $\tau(\Xi, \psi)$ is then the unique eigenvalue of the scalar operator*

$$k \sum_x \Lambda(cx)\, \psi_A(cx),$$

where x ranges over $U_{\mathfrak{A}}^{[(n+1)/2]}/U_{\mathfrak{A}}^{[n/2]+1}$ and $k = (U_{\mathfrak{A}}^{[n/2]+1} : U_{\mathfrak{A}}^{n+1})$.

Proof. In the expression (‡), write $x = t(1+y)$, where t ranges over $J^0/U_{\mathfrak{A}}^{[n/2]+1}$ and y over $\mathfrak{P}^{[n/2]+1}/\mathfrak{P}^{n+1}$. Only the terms $t \in U_{\mathfrak{A}}^{[(n+1)/2]}$ contribute. $\square$

We now prove Proposition 10.1 when the level n of $(\mathfrak{A}, \Xi)$ is positive. Using the notation of the last proposition, we have

$$\tau(\check{\Xi}, \psi)\, 1 = k \sum_x \Lambda(-c^{-1}x^{-1})\, \psi_A(-cx).$$

Thus $\tau(\check{\Xi}, \psi)\, \tau(\Xi, \psi)$ is the unique eigenvalue of the operator

$$k^2 \sum_{x,y} \Lambda(-x^{-1}y)\, \psi_A(c(y-x)).$$

Replacing y by xy, this reduces to

$$k^2 \sum_y \Lambda(-y) \sum_x \psi_A(cx(y-1)).$$

The inner sum vanishes unless $y \equiv 1 \pmod{\mathfrak{P}^{[n/2]+1}}$, that is, unless $y = 1$. The result now follows. $\square$

10.3. The point of these definitions is:

Theorem. *Let $(\mathfrak{A}, \Xi)$ be a cuspidal inducing datum in G, of level $n \geqslant 0$, and write $(\pi, V) = \pi_\Xi$. Set $\mathfrak{P} = \operatorname{rad}\mathfrak{A}$. Then*

$$\varepsilon(\pi, s, \psi) = |\mathfrak{P}^{-n}/\mathfrak{A}|^{(\frac{1}{2}-s)/2}\, \tau(\check{\Xi}, \psi)/|\mathfrak{A}/\mathfrak{P}^{(n+1)}|^{\frac{1}{2}}.$$

Proof. Fix a Haar measure d^*g on G. We temporarily write $K = U_{\mathfrak{A}}^{n+1}$, $\mathcal{K} = \mathcal{K}_{\mathfrak{A}}$, and put $\Phi = e_K$. Thus $\zeta(\Phi, \pi, s) = \pi(e_K)$. Let $\xi = \Xi \mid U_{\mathfrak{A}}$. Then

$$\pi(e_\xi) \circ \zeta(\Phi, \pi, s) \circ \pi(e_\xi) = \zeta(e_{\xi^\vee} * \Phi * e_{\xi^\vee}, \pi, s) = \pi(e_\xi).$$

We abbreviate $\varPsi = e_{\xi^\vee} * \varPhi * e_{\xi^\vee} = e_{\xi^\vee}$. The functional equation gives

$$\begin{aligned}\varepsilon(\pi, s-\tfrac{1}{2}, \psi)\, \pi(e_{\xi^\vee}) &= {}^t\zeta(\hat{\varPsi}, \check{\pi}, 2-s) \\ &= \int_{\mathfrak{K}} \hat{\varPsi}(x)\, {}^t\varXi^\vee(x)\, \|\det x\|^{2-s}\, d^*x \\ &= \int_{\mathfrak{K}} \hat{\varPhi}(x)\, {}^t\varXi^\vee(x)\, \|\det x\|^{2-s}\, d^*x.\end{aligned}$$

We calculate $\hat{\varPhi}$. The self-dual Haar measure on A is the one for which

$$\int_{\mathfrak{A}} dx = (\mathfrak{A} : \mathfrak{P})^{1/2}.$$

The support of $\hat{\varPhi}$ is $\mathfrak{P}^{-n}$ and

$$\hat{\varPhi}(x) = \mu(\mathfrak{P}^{n+1})\, \psi_A(x), \quad x \in \mathfrak{P}^{-n}.$$

We calculate:

$$\begin{aligned}&\int_{\mathfrak{K}} \hat{\varPhi}(x)\, {}^t\varXi^\vee(x)\, \|\det x\|^{2-s}\, d^*x \\ &\qquad = \mu(\mathfrak{P}^{n+1}) \int_{\mathfrak{P}^{-n}\cap\mathfrak{K}} \psi_A(x)\, {}^t\varXi^\vee(x) \|\det x\|^{2-s}\, d^*x.\end{aligned}$$

The range of integration is effectively $\bigcup_{r\geqslant -n} \varPi^r K$, where $\varPi$ is prime in $\mathfrak{A}$. We know that $\varepsilon(\pi, s, \psi)$ is a monomial, so only one of the "shells" $\varPi^r K$ contributes. The case $r = -n$ gives a non-zero integral, by Proposition 10.1, and the result follows. $\square$

11. Local multiplicity one theorem

We prove a fundamental theorem, on which everything depends. It is the analogue, for representations of $\mathrm{GL}_2(F)$, of the famous theorem of Hecke on classical modular forms which can claim to be the starting point of the entire theory.

11.1. As always, ψ denotes a character of F of level one. We prove:

Multiplicity one theorem. *Let π_1, π_2 be irreducible smooth representations of $G = \mathrm{GL}_2(F)$. Suppose that*

$$L(\chi\pi_1, s) = L(\chi\pi_2, s) \quad \text{and} \quad \varepsilon(\chi\pi_1, s, \psi) = \varepsilon(\chi\pi_2, s, \psi),$$

for all characters χ of $F^\times$. We then have $\pi_1 \cong \pi_2$.

We recall that $\chi\pi$ denotes the representation $g \mapsto \chi(\det g)\, \pi(g)$. The first case is rather easy:

Proposition.

(1) *Let π be an irreducible smooth representation of G; then π is principal series if and only if there exists a character χ of $F^\times$ such that $L(\chi\pi, s) \neq 1$.*

(2) *The theorem holds if one of the π_i is principal series.*

This follows on examining the calculations of L-functions in §9. Observe that the ε-relation is required by the presence of the twists of the Steinberg representation.

11.2. We are therefore reduced to the case in which the π_i are both *cuspidal.*

Proposition. *Let π_1, π_2 be irreducible cuspidal representations of G such that*

$$\varepsilon(\chi\pi_1, s, \psi) = \varepsilon(\chi\pi_2, s, \psi), \tag{†}$$

for every character χ of $F^\times$. We then have $\omega_{\pi_1} = \omega_{\pi_2}$.

Remark. In light of this result and Remark 9.3, if the relation (†) holds for one choice of ψ, then it holds for all.

Proof of Proposition. If $l(\pi_i)$ denotes the normalized level of π_i, we have

$$\varepsilon(\pi_i, s, \psi) = q^{2l(\pi_i)(\frac{1}{2}-s)}\, \varepsilon(\pi_i, \tfrac{1}{2}, \psi).$$

In particular, we have $l(\pi_1) = l(\pi_2)$. We may therefore assume that the π_i are each of minimal normalized level. It follows that there is a hereditary order $\mathfrak{A}$ such that π_i contains a cuspidal inducing datum of the form $(\mathfrak{A}, \varXi_i)$, $i = 1, 2$.

The data $\varXi_i$ then have the same level, n say. Let m be a positive integer. If χ is a character of $F^\times$ of level m, then $\chi \circ \det$, as character of $\mathcal{K}_{\mathfrak{A}}$, has level em where $e = e_{\mathfrak{A}}$. We choose m sufficiently large to ensure that $[(em{+}1)/2] \geqslant n{+}1$, and take $c \in F$ with $v_F(c) = -m$. We choose χ, of level m, such that $\chi \mid U_F^{[m/2]+1} = \psi_{F,-c}$.

We let $(J_i, \varLambda_i)$ be an extended simple type which induces $\varXi_i$, and apply (10.2), (10.3): there is a positive constant k, independent of i (it depends only on $\mathfrak{A}$ and n) such that $\varepsilon(\chi\pi, \frac{1}{2}, \psi)$ is the unique eigenvalue of the scalar operator

$$\begin{aligned} k \sum_x \varLambda_i^\vee(cx)\, \chi(\det cx)^{-1}\, \psi_A(cx) \\ = k\, \omega_{\pi_i}(c)^{-1} \dim \varLambda_i \sum_x \chi(\det cx)^{-1}\, \psi_A(cx), \end{aligned}$$

where x ranges over $U_{\mathfrak{A}}^{[em+1)/2]}/U_{\mathfrak{A}}^{[em/2]+1}$. The equality of levels implies that $\dim \varLambda_1 = \dim \varLambda_2$, so we conclude $\omega_{\pi_1}(c) = \omega_{\pi_2}(c)$. This holds for all $c \in F^\times$ with $v_F(c)$ less than some constant, whence $\omega_{\pi_1} = \omega_{\pi_2}$, as required. $\square$

11.3. We continue with the same notation and hypotheses. We write $\mathcal{K} = \mathcal{K}_{\mathfrak{A}}$ and we let $\mathcal{K}(n) = \mathcal{K} \cap (\mathfrak{P}^{-n} \smallsetminus \mathfrak{P}^{1-n})$. Thus, if Π is a prime element of $\mathfrak{A}$, we have $\mathcal{K}(n) = \Pi^{-n} U_{\mathfrak{A}}$.

We consider the set $\mathcal{K}(n)/U_{\mathfrak{A}}^{n+1}$, modulo the conjugation action of $\mathcal{K}$: we denote this set of conjugacy classes by $\operatorname{Ad}\mathcal{K}\backslash\mathcal{K}(n)/U_{\mathfrak{A}}^{n+1}$.

Proposition. *Let $x, y \in \mathcal{K}(n)$ and suppose that x is minimal over F. The cosets $xU_{\mathfrak{A}}^{n+1}$, $yU_{\mathfrak{A}}^{n+1}$ are then conjugate under $\mathcal{K}$ if and only if*

$$\begin{aligned} \det x \equiv \det y \pmod{U_F^{[n/e]+1}}, &\quad \textit{and} \\ \operatorname{tr}_A(x) \equiv \operatorname{tr}_A(y) \pmod{\mathfrak{p}}. & \end{aligned}$$

Proof. We first note that

$$\det U_{\mathfrak{A}} = U_F, \quad \det U_{\mathfrak{A}}^{n+1} = U_{\mathfrak{A}}^{[n/e]+1}, \quad \operatorname{tr}_A(\mathfrak{P}) = \mathfrak{p}.$$

This gives the necessity of the conditions. The converse is a simple exercise with the rational canonical form. □

If m is an integer, we denote by $F^\times(m)$ the set of $x \in F^\times$ with $v_F(x) = -m$. The lemma now implies:

Proposition. *The map*

$$\begin{aligned} \operatorname{Ad}\mathcal{K}\backslash\mathcal{K}(n)/U_{\mathfrak{A}}^{n+1} &\longrightarrow F^\times(2n/e)/U_F^{[n/e]+1} \times \mathfrak{p}^{1+[(-1-n)/e]}/\mathfrak{p}, \\ xU_{\mathfrak{A}}^{n+1} &\longmapsto \big(\det x . U_F^{[n/e]+1}, \operatorname{tr}_A(x) + \mathfrak{P}\big), \end{aligned}$$

is injective on the set of classes of minimal elements. In the case $e = 2$, it is bijective.

It remains only to remark that, when the order $\mathfrak{A}$ is minimal, the integer n is odd and every element of $\mathcal{K}(n)$ is minimal over F.

11.4. For a character χ of $F^\times$ and a non-trivial character μ of F, we define

$$(\ddagger) \qquad \Upsilon_i(\chi, \mu) = \int_{\mathcal{K}(n)} \operatorname{tr} \Xi_i^\vee(x)\, \chi^{-1}(\det x)\, \mu(\operatorname{tr}_A(x))\, dx,$$

for some Haar measure dx on G.

Lemma. *Suppose that χ has level $\leqslant [n/e]+1$; then*

(1) *If μ has level $\neq 1$, then $\Upsilon_i(\chi, \mu) = 0$ for all χ.*
(2) *Suppose μ has level 1. There is a constant $k > 0$, independent of i, such that*
$$\Upsilon_i(\chi, \mu) = k\, \varepsilon(\chi\pi_i, \tfrac{1}{2}, \mu).$$

In particular, we have

$$\Upsilon_1(\chi, \mu) = \Upsilon_2(\chi, \mu),$$

for all χ.

Proof. This is a reformulation of the arguments of (10.3). $\square$

Remark. The condition on χ in the lemma is equivalent to the level of $\chi\Xi_i$ being at most that of Ξ_i.

11.5. We wish to invert the relation (‡) between $\varepsilon(\chi\pi_i, \frac{1}{2}, \mu)$ and Ξ_i. We prove:

Theorem. *For every minimal element* $x \in \mathcal{K}(n)$, *we have*

$$\operatorname{tr} \Xi_1^\vee(x) = \operatorname{tr} \Xi_2^\vee(x).$$

Proof. We fix $\beta \in \mathcal{K}(n)$, so that $\mathcal{K}(n) = \beta U_{\mathfrak{A}}$. We let Γ be the group of characters of $U_F/U_F^{[n/2]+1}$ and Ψ the group of characters of $\mathfrak{p}^{1+[(-1-n)/e]}/\mathfrak{p}$. For $\chi \in \Gamma$, we let χ_0 be a character of $F^\times$, extending χ, such that $\chi_0(\det \beta) = 1$. Thus we have a function

$$\begin{aligned} \Upsilon_i^0 : \Gamma \times \Psi &\longrightarrow \mathbb{C}, \\ (\chi, \mu) \longmapsto \int_{U_{\mathfrak{A}}} &\operatorname{tr} \Xi_i^\vee(\beta u)\, \chi^{-1}(\det u)\, \mu(\operatorname{tr}_A(\beta u))\, du \\ &= \Upsilon_i(\chi_0, \mu). \end{aligned}$$

This has Fourier transform

$$\widehat{\Upsilon}_i^0 : U_F/U_F^{[n/e]+1} \times \mathfrak{p}^{1+[(-1-n)/e]}/\mathfrak{p} \longrightarrow \mathbb{C},$$

given by

$$\widehat{\Upsilon}_i^0(a, b) = k \sum_{\substack{\chi \in \Gamma, \\ \mu \in \Psi}} \Upsilon_i^0(\chi, \mu)\, \bar{\chi}(a)\, \bar{\mu}(b),$$

for a positive constant k depending only on choices of measures.

We have to relate $\widehat{\Upsilon}_i^0(a, b)$ to $\operatorname{tr} \Xi_i^\vee(x)$, for an element $x \in \mathcal{K}(n)$ which satisfies $\det(\beta^{-1}x) = a$ and $\operatorname{tr}_A(x) = b$. For $(a, b) \in U_F/U_F^{[n/e]+1} \times \mathfrak{p}^{1+[(-1-n)/e]}/\mathfrak{p}$, we define a function

$$\xi_i(a, b) = \sum_x \operatorname{tr} \Xi_i^\vee(x),$$

where x ranges over those elements of $\mathcal{K}\backslash\mathcal{K}(n)/U_{\mathfrak{A}}^{n+1}$ for which $\det(\beta^{-1}x) = a$ and $\operatorname{tr}_A(x) = b$. We then have

$$\widehat{\Upsilon}_i^0(a, b) = \xi_i(a, b), \quad i = 1, 2.$$

The conjugacy classes of minimal elements in $\mathcal{K}(n)/U_{\mathfrak{A}}^{n+1}$ all have the same volume, and this volume depends only on n and $e_{\mathfrak{A}}$. We deduce that there is a positive constant k, depending only on n and $e_{\mathfrak{A}}$, such that

$$\widehat{\Upsilon}_i^0\big(\det(\beta^{-1}x), \operatorname{tr}_A(x)\big) = k \operatorname{tr} \Xi_i^\vee(x),$$

for every minimal element x of $\mathcal{K}(n)$. The theorem now follows. $\square$

11.6. To complete the proof of the Multiplicity One Theorem, we have to show that the Ξ_i are equivalent, given that the characters of their duals agree on minimal elements in $\mathcal{K}(n)$. The representations Ξ_i agree on Z, by (11.2); the minimal elements in $\mathcal{K}$ form the set $Z\mathcal{K}(n)$, so the characters $\operatorname{tr}(\Xi_i^\vee)$ agree on all minimal elements in $\mathcal{K}$.

Suppose first that $n = 0$. Thus $\Xi_i \mid U_{\mathfrak{A}}$ is the inflation of an irreducible cuspidal representation $\tilde{\xi}_i$ of $\mathrm{GL}_2(\boldsymbol{k})$. The character relation reduces to

$$\operatorname{tr} \tilde{\xi}_1^\vee(\zeta) = \operatorname{tr} \tilde{\xi}_2^\vee(\zeta),$$

for every *elliptic* element ζ of $\mathrm{GL}_2(\boldsymbol{k})$. We deduce (*cf.* (1.3)) that $\tilde{\xi}_1 \cong \tilde{\xi}_2$, whence $\Xi_1 \cong \Xi_2$.

We may therefore assume that there is a simple stratum $(\mathfrak{A}, n, \alpha)$ such that Ξ_1 contains the character ψ_α of $U_{\mathfrak{A}}^{[n/2]+1}$. Set $E = F[\alpha]$ and write, as before, $J_\alpha = E^\times U_{\mathfrak{A}}^{[(n+1)/2]}$, $H_\alpha^1 = U_E^1 U_{\mathfrak{A}}^{[n/2]+1}$. Then Ξ_1 is induced by an extended simple type (J_α, Λ_1). The restriction of Λ_1 to H_α^1 is a multiple of a character θ, and the integral

$$\int_{H_\alpha^1} \operatorname{tr} \Xi_1(\alpha u)\, \theta(u)^{-1}\, du$$

is a positive constant times $\operatorname{tr} \Lambda_1(\alpha)$.

Lemma. *We may choose α so that* $\operatorname{tr} \Lambda_1(\alpha) \neq 0$.

Hint of proof. There is nothing to do if $\dim \Lambda_1 = 1$, as holds, in particular, when E/F is ramified. Otherwise, this is an elementary property of the representation η_θ of (7.5). $\square$

We have $\operatorname{tr} \Xi_2(\alpha u) = \operatorname{tr} \Xi_1(\alpha u)$, $u \in H_\alpha^1$, so Ξ_2 also contains θ. Thus each Ξ_i contains an extended simple type (J_α, Λ_i), with the Λ_i agreeing on H_α^1, indeed on J_α^1 (*cf.* 7.5 Lemma 1).

If E/F is ramified, we are done: the Λ_i agree on $J_\alpha^0 = U_F J_\alpha^1$, since they have the same central character; they differ at most by an unramified quadratic twist, which is trivial by the character relation.

We therefore assume that E/F is unramified; in this case, we know that the Λ_i agree on ZJ_α^1. We also have the relation

$$\operatorname{tr} \Lambda_1(\zeta) = \operatorname{tr} \Lambda_2(\zeta),$$

for every F-minimal root of unity $\zeta \in E$. This is enough to imply $\Lambda_1 \cong \Lambda_2$ and $\Xi_1 \cong \Xi_2$.

This completes the proof of the Multiplicity One Theorem. $\square$

Remark. Suppose that the π_i have minimal normalized level. We have shown that $\pi_1 \cong \pi_2$ only using that $\varepsilon(\chi\pi_1, s, \psi) = \varepsilon(\chi\pi_2, s, \psi)$ when χ has level which is either very large or rather small (as in Lemma 11.4).

Afterthoughts

We mention briefly some further ideas. These are mainly in the gap between our treatment and the classical approach of Jacquet and Langlands.

A. Characters.

A.1. Let (π, V) be an irreducible smooth representation of G and let $f \in \mathcal{H}(G)$. There exists a compact open subgroup K of G such that $e_K * f * e_K = f$. This condition translates into $\pi(e_K) \circ \pi(f) \circ \pi(e_K) = \pi(f)$. Thus $\pi(f)$ is effectively an endomorphism of the finite-dimensional space V^K, killing the K-complement of V^K in V. In particular, the trace $\operatorname{tr}\pi(f)$ of $\pi(f) \in \operatorname{End}_{\mathbb{C}}(V^K)$ is defined, and is independent of the choice of K fixing f.

Consider the linear map

$$\begin{aligned} \Theta_\pi : \mathcal{H}(G) &\longrightarrow \mathbb{C}, \\ f &\longmapsto \operatorname{tr}\pi(f). \end{aligned}$$

This is a *distribution* on G which is *invariant,* in the sense that

$$\Theta_\pi(f^g) = \Theta_\pi(f), \quad f \in \mathcal{H}(G),\ g \in G,$$

where f^g denotes the function $x \mapsto f(gxg^{-1})$. One calls Θ_π the *character distribution* of π.

Exercise. *Let π_i, $1 \leqslant i \leqslant r$, be distinct irreducible smooth representations of G and set $\Theta_i = \Theta_{\pi_i}$. Show that the set $\{\Theta_i\}_{1\leqslant i\leqslant r}$ of linear forms on $\mathcal{H}(G)$ is linearly independent.*

Hint. Use 3.3 Exercise (4) to reduce to an analogous problem for simple modules over a finite-dimensional semisimple $\mathbb{C}$-algebra.

An irreducible smooth representation π of G is therefore determined, up to equivalence, by its character distribution Θ_π.

For $f \in \mathcal{H}(G)$, let $f^\vee \in \mathcal{H}(G)$ denote the function $g \mapsto f(g^{-1})$. We have $\operatorname{tr}\pi(f) = \operatorname{tr}\check{\pi}(f^\vee)$, so

$$\Theta_{\pi^\vee} = \Theta_\pi^\vee : f \longmapsto \Theta_\pi(f^\vee).$$

A.2. An element $g \in G$ is *regular semisimple* if its characteristic polynomial has simple roots in a splitting field. We denote the set of such elements by G_{reg}: this is an open dense subset of G.

Again let π be an irreducible smooth representation of G. One knows that there exists a function θ_π on G, locally constant on G_{reg}, such that

$$\Theta_\pi(f) = \int_G \theta_\pi(g)\, f(g)\, dg, \quad f \in \mathcal{H}(G).$$

The function θ_π satisfies $\theta_\pi(gxg^{-1}) = \theta_\pi(x)$, and it determines π up to isomorphism. One sometimes call θ_π the character of π.

As an application of these simple observations, we have:

Duality Theorem. *Let π be an irreducible smooth representation of G. We have $\check{\pi} \cong \omega_\pi^{-1}\pi$.*

Proof. We have $\theta_{\pi^\vee} = \theta_\pi^\vee$. Let $w = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. For $g \in G$, the matrix ${}^t(wgw^{-1})$ is $\det g \cdot g^{-1}$, and any matrix is conjugate to its transpose. Therefore, for $g \in G_{\mathrm{reg}}$, we have

$$\theta_\pi(g) = \theta_\pi(\det g \cdot g^{-1}) = \omega_\pi(\det g)\, \theta_{\pi^\vee}(g).$$

The representations $\check{\pi}$ and $\omega_\pi^{-1}\pi$ therefore have the same character, and the result follows. $\square$

B. Whittaker models.

B.1. Let B be a Borel subgroup of G. Thus B is the stabilizer of a configuration $0 \subset V \subset F^{(2)}$, where V is an F-vector space of dimension one. Thus B is G-conjugate to the standard Borel subgroup of upper triangular matrices in G. Let N be the unipotent radical of G. Thus $N \cong F$, and we can use this isomorphism to transfer characters of F to characters of N.

Let μ be some non-trivial smooth character of N, and consider the representation $\mathrm{Ind}_N^G \mu$. The elements of this space are the right G-smooth functions $f : G \to \mathbb{C}$ such that $f(ng) = \mu(n)f(g)$, $n \in N$, $g \in G$. We observe that the choice of pair (N, μ) is immaterial: any two such pairs are G-conjugate. The basic result here is:

Theorem. *Let π be an irreducible smooth representation of G; then*

$$\dim \mathrm{Hom}_G(\pi, \mathrm{Ind}\, \mu) \leqslant 1.$$

Indeed, $\mathrm{Hom}_G(\pi, \mathrm{Ind}\,\mu)$ *is trivial if and only if* $\pi = \chi \circ \det$, *for some character* χ *of* $F^\times$.

Put another way, if π is an irreducible representation of G of infinite dimension, there is a unique G-subspace $\mathcal{W}_\pi$ of $\mathrm{Ind}\,\mu$ such that $\mathcal{W}_\pi \cong \pi$. The space $\mathcal{W}_\pi$ is called the *Whittaker model* of π (relative to (N, μ)) and its elements are the Whittaker functions on π.

B.2. We sketch a proof of the theorem, starting with the case where (π, V) is *cuspidal.* If χ is a character of $F^\times$, it is easy to see that $(\mathrm{Ind}\,\mu)\otimes\chi\circ\det \cong \mathrm{Ind}\,\mu$, so we may pretend that π has minimal normalized level. Thus either π is of level zero, or induced by an extended simple type (J, Λ) defined via simple stratum $(\mathfrak{A}, n, \alpha)$.

Let N be the unipotent radical of some Borel subgroup of G, and let μ be a character of $N \cap J$ occurring in $\Lambda \mid J \cap N$. Since N is abelian, μ extends to a *non-trivial* character of N, which we also denote by μ. Let W denote the representation space of Λ. The presence of μ in $\Lambda \mid J \cap N$ implies there is a non-trivial linear form $\ell : W \to \mathbb{C}$ such that $\ell(\Lambda(u)w) = \mu(u)\ell(w)$, for $u \in N \cap J$ and $w \in W$. We now identify (π, V) with $c\text{-}\mathrm{Ind}_J^G\,\Lambda$, in its standard realization as a space of W-valued functions on G. We define a linear form ℓ_G on $c\text{-}\mathrm{Ind}\,\Lambda$ by

$$\ell_G(f) = \int_N \bar{\mu}(u)\,\ell(f(u))\,du,$$

where du is the Haar measure on N giving $J \cap N$ mass 1. The function f has compact support on N, so the integral converges to a linear form which restricts to ℓ on the canonical image of W in $c\text{-}\mathrm{Ind}\,\Lambda$. For $u \in N$, we have $\ell_G(\pi(n)f) = \mu(n)\ell_G(f)$, so ℓ_G may be viewed as a non-zero N-homomorphism $\pi \to \mu$. The space

$$\mathrm{Hom}_M(\pi, \mu) \cong \mathrm{Hom}_G(\pi, \mathrm{Ind}\,\mu)$$

is therefore non-zero.

To prove uniqueness, we proceed rather differently. Using the same notation as before, we consider the space $\mathcal{H}(G; \Lambda, \mu)$ of functions $\phi : G \to \mathrm{Hom}(W, \mathbb{C})$ with the property

$$\phi(ugj) = \mu(u)\,\phi(g) \circ \Lambda(j), \quad u \in N,\ g \in G,\ j \in J.$$

For $f \in c\text{-}\mathrm{Ind}\,\Lambda$ and $\phi \in \mathcal{H}(G; \Lambda, \mu)$, we may form the function $\phi * f$ given by

$$\phi * f(g) = \int_{G/Z} \phi(gx^{-1}) f(x)\,d\dot{x},$$

for a Haar measure $d\dot{x}$ on G/Z. Since f is compactly supported modulo Z, the integral converges to an element of $\mathrm{Ind}\,\mu$. One may show that this process induces a isomorphism

$$\mathcal{H}(G; \Lambda, \mu) \cong \mathrm{Hom}_G(c\text{-}\mathrm{Ind}\,\Lambda, \mathrm{Ind}\,\mu).$$

We therefore have to show that $\mathcal{H}(G; \Lambda, \mu)$ has dimension one. If π has level zero, this follows, without trouble, from Exercise 1.3. We assume, therefore, that we are in the positive level case and (J, Λ) is constructed from a simple stratum $(\mathfrak{A}, n, \alpha)$. We set $E = F[\alpha]$. The group $E^\times$ acts transitively on $F^{(2)} \smallsetminus \{0\}$, whence it acts transitively (by conjugation) on the set of Borel subgroups of G. Thus, for any B, we have $G = BE^\times$ and $G = NTJ$, in our usual notation.

We now "choose coordinates" suitably. We take a basis $\{v, \alpha v\}$ of $F^{(2)}$ to re-identify G with $\mathrm{GL}_2(F)$, choosing $v \in L_0 \smallsetminus L_1$, where $\{L_i\}_{i\in\mathbb{Z}}$ is the lattice chain for $\mathfrak{A}$. (If $\mathfrak{A}$ is in standard form, this amounts to conjugating $\mathfrak{A}$ by a diagonal matrix.) The function ψ_α then determines a non-trivial character μ_α of N which, when restricted to $N \cap J = N \cap U_{\mathfrak{A}}^{[(n+1)/2]}$, occurs in Λ *with multiplicity one.* (This depends on properties of the representation $\eta = \Lambda \mid U_{\mathfrak{A}}^{[(n+1)/2]}$.) We merely have to check that $t \in T$ stabilizes this character if and only if $t \in T \cap J$: this is an explicit matrix calculation.

B.3. We now deal with the principal series case. We show that, for a character χ of T, we invariably have

$$\dim \mathrm{Hom}_G(\mathrm{Ind}_B^G(\chi), \mathrm{Ind}_N^G(\mu)) = 1.$$

Given our understanding of the principal series, and the trivial observation that a one-dimensional representation $\chi \circ \det$ certainly cannot be embedded in $\mathrm{Ind}\,\mu$, this gives the result.

We start with a general observation. Let (π, V), (ρ, W) be smooth representations of G. The space of G-invariant pairings $V \times W \to \mathbb{C}$ is then canonically isomorphic to $\mathrm{Hom}_G(\pi, \check{\rho})$ and to $\mathrm{Hom}_G(\rho, \check{\pi})$. Thus we have a canonical isomorphism

$$\mathrm{Hom}_G(\pi, \check{\rho}) \cong \mathrm{Hom}_G(\rho, \check{\pi}).$$

Applying this to our present context, we have $\mathrm{Ind}\,\mu \cong (c\text{-}\mathrm{Ind}_N^G\,\bar{\mu})^\vee$, while $\mathrm{Ind}_B^G\,\chi = (\mathrm{Ind}\,\chi^{-1}\delta_B^{-1})^\vee$. Thus we consider the space

$$\mathrm{Hom}_G(c\text{-}\mathrm{Ind}\,\bar{\mu}, \mathrm{Ind}\,\chi) = \mathrm{Hom}_T((c\text{-}\mathrm{Ind}\,\bar{\mu})_N, \chi).$$

It is not hard to show that the Jacquet module $(c\text{-}\mathrm{Ind}\,\bar{\mu})_N$ is isomorphic to $\mathcal{H}(T)$, and then that $\mathrm{Hom}_T(\mathcal{H}(T), \chi)$ is one-dimensional.

III: Galois representations and the Langlands correspondence

We start by examining finite-dimensional representations of Galois groups, defining their L-functions and local constants. This will enable us to specify the Langlands correspondence for two-dimensional representations.

12. The Langlands-Deligne local constant

We first have to develop the ideas of Tate's functional equation (§8) in a different direction.

12.1. We choose, once for all, a separable algebraic closure F^{sep}/F of F and put $\Omega_F = \mathrm{Gal}(F^{\mathrm{sep}}/F)$. Inside F^{sep}, we have the maximal *unramified* extension F^{nr}/F and the maximal *tamely ramified* extension F^{tr}/F of F. We let $\mathcal{I}_F = \mathrm{Gal}(F^{\mathrm{sep}}/F^{\mathrm{nr}})$ and $\mathcal{P}_F = \mathrm{Gal}(F^{\mathrm{sep}}/F^{\mathrm{tr}})$. Thus $\mathcal{P}_F$ is a pro p-group, while the quotients $\Omega_F/\mathcal{I}_F$ and $\mathcal{I}_F/\mathcal{P}_F$ are pro-cyclic:

$$\Omega_F/\mathcal{I}_F \cong \varprojlim_{n\geqslant 1} \mathbb{Z}/n\mathbb{Z} = \widehat{\mathbb{Z}} \cong \prod_{\ell \text{ prime}} \mathbb{Z}_\ell,$$

$$\mathcal{I}_F/\mathcal{P}_F \cong \prod_{\ell\neq p} \mathbb{Z}_\ell.$$

Let $\mathfrak{o}_{\mathrm{nr}}$ denote the integral closure of $\mathfrak{o}_F$ in F^{nr}; then $\mathfrak{p}_{\mathrm{nr}} = \mathfrak{p}\mathfrak{o}_{\mathrm{nr}}$ is the maximal ideal of $\mathfrak{o}_{\mathrm{nr}}$ and the residue field $\overline{\boldsymbol{k}}$ is an algebraic closure of $\boldsymbol{k}$. An element $\sigma \in \Omega_F$ is called an *arithmetic Frobenius* if σ acts on $\overline{\boldsymbol{k}}$ as the automorphism $x \mapsto x^q$. The inverse of an arithmetic Frobenius is called a *geometric Frobenius.*

12.2. Let σ be a Frobenius (of either sort). The image of σ in $\Omega_F/\mathcal{I}_F$ then generates an infinite cyclic group, which is the canonical image of $\mathbb{Z}$ in $\widehat{\mathbb{Z}}$. We let ${}_a\mathcal{W}_F$ denote the inverse image in Ω_F of this infinite cyclic group. Thus ${}_a\mathcal{W}_F$ is a dense subgroup of Ω_F. As *abstract* group, the Weil group $\mathcal{W}_F$ of F^{sep}/F is the same as ${}_a\mathcal{W}_F$, but it has a different topology: the subgroup $\mathcal{I}_F$ carries the topology inherited from Ω_F, but is deemed to be open in $\mathcal{W}_F$. Thus we have an exact sequence of topological groups

$$1 \to \mathcal{I}_F \longrightarrow \mathcal{W}_F \longrightarrow \mathbb{Z} \to 0,$$

in which $\mathcal{I}_F$ is open and $\mathbb{Z}$ is discrete. In particular, a subgroup H of $\mathcal{W}_F$ is open if and only if $H \cap \mathcal{I}_F$ is open in $\mathcal{I}_F$.

The identification map $\mathcal{W}_F \to {}_a\mathcal{W}_F$ is a continuous bijection but, if we view ${}_a\mathcal{W}_F$ as a topological subspace of Ω_F, the inverse map is not continuous.

12.3. Let E/F be a finite field extension, with $E \subset F^{\text{sep}}$. Thus the Galois group $\Omega_E = \text{Gal}(F^{\text{sep}}/E)$ is an open subgroup of Ω_F. We can then form the abstract group $\Omega_E \cap {}_a\mathcal{W}_F$, and take its inverse image in $\mathcal{W}_F$. The result is an open subgroup of $\mathcal{W}_F$, canonically isomorphic (and homeomorphic) to $\mathcal{W}_E$. In this way, we identify $\mathcal{W}_E$ with an open subgroup of $\mathcal{W}_F$ of finite index. The canonical map $\mathcal{W}_F/\mathcal{W}_E \to \Omega_F/\Omega_E$ is a bijection.

In the opposite direction, let $\mathcal{V}$ be an open subgroup of $\mathcal{W}_F$ of finite index. The closure $\overline{\mathcal{V}}$ of $\mathcal{V}$ in Ω_F is an open subgroup of finite index, hence of the form $\overline{\mathcal{V}} = \Omega_E$, for some finite extension E/F. Moreover, $\mathcal{V} = \Omega_E \cap \mathcal{W}_F = \mathcal{W}_E$. We have proved:

Proposition. *The map $E \mapsto \Omega_E \cap \mathcal{W}_F = \mathcal{W}_E$ is a bijection between the set of finite field extensions E/F inside F^{sep} and the set of open subgroups of $\mathcal{W}_F$ of finite index. Moreover, the canonical map $\mathcal{W}_F/\mathcal{W}_E \to \Omega_F/\Omega_E$ is bijective.*

Remark. The group we have denoted $\mathcal{W}_F$ should more properly be $\mathcal{W}_{F^{\text{sep}}/F}$, as it depends on the choice of F^{sep}. If we take an abstract field extension E/F, we have to choose a separable algebraic closure E^{sep}/E to form $\mathcal{W}_{E^{\text{sep}}/E}$. The choice of an F-isomorphism $E^{\text{sep}} \cong F^{\text{sep}}$ induces a map $\mathcal{W}_{E^{\text{sep}}/E} \to \mathcal{W}_{F^{\text{sep}}/F}$, identifying $\mathcal{W}_{E^{\text{sep}}/E}$ with an open subgroup of $\mathcal{W}_{F^{\text{sep}}/F}$, but this map is only determined up to an inner automorphism of $\mathcal{W}_{F^{\text{sep}}/F}$. However, since we shall only be concerned with the representation theory of $\mathcal{W}_F$, this nicety is irrelevant.

12.4. The groups Ω_F, $\mathcal{W}_F$ are closely related, but their representation theories are significantly different. A smooth representation of Ω_F is *semisimple,* in that it is a direct sum of irreducible smooth representations. On the other hand, among the smooth representations of $\mathcal{W}_F$ are all the representations of its quotient $\mathbb{Z}$: these are not all semisimple.

If, however, ρ is a smooth representation of Ω_F, we can compose ρ with the canonical map $\mathcal{W}_F \to \Omega_F$ to get a smooth representation ρ', say, of $\mathcal{W}_F$. We may think of ρ' as the restriction of ρ to $\mathcal{W}_F$. Since the image of $\mathcal{W}_F$ is dense in Ω_F, the representation ρ' is irreducible if and only if ρ is irreducible.

Let χ be a smooth character of Ω_F or $\mathcal{W}_F$. We say that χ is *unramified* if it is trivial on $\mathcal{I}_F$. If ρ is a finite-dimensional smooth representation of $\mathcal{W}_F$, the map $x \mapsto \det \rho(x)$ is a character of $\mathcal{W}_F$.

Proposition. *Let ρ be an irreducible smooth representation of $\mathcal{W}_F$. The following conditions are equivalent:*

(1) *There exists a smooth representation $\tilde{\rho}$ of Ω_F such that $\rho = \tilde{\rho} \mid \mathcal{W}_F$.*
(2) *The character $\det \rho$ has finite order.*

In all cases, there exists an unramified character χ of $\mathcal{W}_F$ such that $\det(\chi \otimes \rho) = \chi^{\dim \rho} \det(\rho)$ has finite order.

Proof. Let us start with a special case:

Lemma. *Let ξ be a character of $\mathcal{W}_F$. There exists a character $\tilde{\xi}$ of Ω_F such that $\tilde{\xi} \mid \mathcal{W}_F = \xi$ if and only if ξ has finite order.*

Proof. Since Ω_F is compact and totally disconnected, all of its characters have finite order and one implication is therefore obvious. Conversely, suppose ξ has finite order. The kernel of ξ is therefore an open subgroup of $\mathcal{W}_F$ of finite index, that is, $\operatorname{Ker}\xi = \mathcal{W}_E$, for some finite Galois extension E/F. The isomorphism $\mathcal{W}_F/\mathcal{W}_E \cong \Omega_F/\Omega_E$ gives the result. $\square$

Let ρ be an irreducible smooth representation of $\mathcal{W}_F$. The restriction of ρ to $\mathcal{I}_F$ is a finite direct sum $\tau_1 \oplus \cdots \oplus \tau_r$ of irreducible smooth representations τ_i of $\mathcal{I}_F$ permuted transitively by $\mathcal{W}_F$. Choose one of these, τ, say, and let $\mathcal{W}_F(\tau)$ denote the $\mathcal{W}_F$-stabilizer of the isomorphism class of τ: we observe that $\mathcal{W}_F(\tau)$ is an open subgroup of $\mathcal{W}_F$ of finite index, so $\mathcal{W}_F(\tau) = \mathcal{W}_E$, for some finite, unramified extension E/F. The Ω_F-stabilizer of the isomorphism class of τ is then Ω_E.

Now let (θ, W) be an irreducible representation of Ω_F containing τ. The isotypic space W^τ then carries a representation θ_τ of Ω_E determined by θ, and θ is induced by θ_τ. Thus, as χ ranges over the unramified characters of Ω_F, then $\chi \otimes \theta$ ranges over the irreducible representations of Ω_F containing τ. Likewise, $\xi \otimes \theta \mid \mathcal{W}_F$ ranges over the irreducible representations of $\mathcal{W}_F$ as ξ ranges over the unramified characters of $\mathcal{W}_F$. All assertions now follow from the lemma. $\square$

12.5. The motivation for the introduction of the Weil group is first provided by local class field theory. Let $\mathcal{W}_F^{\mathrm{ab}}$ denote the quotient of $\mathcal{W}_F$ by the closure of its commutator subgroup. There is then a canonical isomorphism of topological groups

$$\boldsymbol{a}_F : \mathcal{W}_F^{\mathrm{ab}} \xrightarrow{\approx} F^\times.$$

In fact there are two such canonical isomorphisms; we choose the one for which prime elements of F correspond to *geometric* Frobenius elements of $\mathcal{W}_F$.

In particular, we can use this isomorphism to identify the groups of characters of the (locally compact, totally disconnected) topological groups $\mathcal{W}_F$, $F^\times$.

12.6. Let $\mathsf{K}_0\big[\mathcal{W}_F\big]$ denote the *representation ring* of $\mathcal{W}_F$. Thus $\mathsf{K}_0\big[\mathcal{W}_F\big]$ is a free abelian group with basis $[\rho]$, where ρ ranges over the isomorphism classes of irreducible smooth representations of $\mathcal{W}_F$. If ρ is a smooth, semisimple representation of $\mathcal{W}_F$ of finite dimension, we have $\rho \cong \rho_1 \oplus \cdots \oplus \rho_r$, for uniquely determined irreducible representations ρ_j. We define $[\rho] = [\rho_1] + \cdots + [\rho_r] \in \mathsf{K}_0\big[\mathcal{W}_F\big]$. This identifies the set of positive elements of $\mathsf{K}_0\big[\mathcal{W}_F\big]$ (relative to the canonical basis) with the set of isomorphism classes of finite-dimensional, smooth, semisimple representations of $\mathcal{W}_F$. We clearly have $[\sigma_1 \oplus \sigma_2] = [\sigma_1] + [\sigma_2]$. One gives $\mathsf{K}_0\big[\mathcal{W}_F\big]$ the structure of commutative ring with 1 by defining $[\sigma_1][\sigma_2] = [\sigma_1 \otimes \sigma_2]$.

We refer to the elements of $\mathsf{K}_0\big[\mathcal{W}_F\big]$ as *virtual representations* of $\mathcal{W}_F$. We have a surjective ring homomorphism

$$\deg : \mathsf{K}_0\big[\mathcal{W}_F\big] \longrightarrow \mathbb{Z},$$

given by $\deg[\rho] = \dim \rho$ when ρ is irreducible.

12.7. We consider the set $R(F)$ of isomorphism classes of pairs $(E/F, \rho)$, where E/F is a finite field extension contained in F^{sep}, and $\rho \in \mathsf{K}_0\big[\mathcal{W}_E\big]$.

Let $R_1(F)$ denote the set of isomorphism classes of pairs $(E/F, \chi)$, where E/F is a finite sub-extension of F^{sep}/F and χ is a character of $E^\times$. We may use class field theory to identify $R_1(F)$ with a subset of $R(F)$.

If E/F is a finite extension in F^{sep}, we view $\mathcal{W}_E$ as an open subgroup of $\mathcal{W}_F$ of finite index. The operation of induction of group representations induces an additive map

$$\mathrm{Ind}_{E/F} : \mathsf{K}_0\big[\mathcal{W}_E\big] \longrightarrow \mathsf{K}_0\big[\mathcal{W}_F\big].$$

Remark. This induction functor can be thought of as either abstract induction or smooth induction: the two constructions give the same result since $\mathcal{W}_E$ is open of finite index in $\mathcal{W}_F$.

Definition. *Let $\mathcal{A}$ be an abelian group and $\mathfrak{f} : R_1(F) \to \mathcal{A}$ a function. We say that $\mathfrak{f}$ is* extendable *if there is a function $\mathfrak{F} : R(F) \to \mathcal{A}$, such that $\mathfrak{F} \mid R_1(F) = \mathfrak{f}$, and satisfying*

(1) *$\mathfrak{F}(E, \rho_1{+}\rho_2) = \mathfrak{F}(E, \rho_1)\, \mathfrak{F}(E, \rho_2)$, for $(E, \rho_i) \in R(F)$;*
(2) *if $(E, \rho) \in R(F)$ has $\deg \rho = 0$ and if $E \supset E' \supset F$, then*

$$\mathfrak{F}(E, \rho) = \mathfrak{F}(E', \mathrm{Ind}_{E/E'}\, \rho).$$

Variation. One says that $\mathfrak{f}$ is *strongly extendable* if condition (2) holds without the restriction $\deg \rho = 0$.

Proposition. *In the context of the preceding definition, there exists at most one extension $\mathfrak{F}$ of $\mathfrak{f}$, satisfying the conditions* (1) *and* (2).

Proof. Let $\mathfrak{F}_1$, $\mathfrak{F}_2$ be extensions of $\mathfrak{f}$. We take $(E/F, \rho) \in R(F)$ such that ρ is an irreducible representation of $\mathcal{W}_E$, and show that $\mathfrak{F}_1(E/F, \rho) = \mathfrak{F}_2(E/F, \rho)$: this will be enough.

Suppose, in the first instance, that $\det \rho$ has finite order: we may therefore regard ρ as a representation of Ω_E. For each i, we have

$$\mathfrak{F}_i(E/F, \rho) = \mathfrak{F}_i(E/F, \rho{-}m[1_E])\, \mathfrak{f}(E/F, [1_E])^m, \quad i = 1, 2,$$

where $m = \dim \rho$ and $[1_E]$ denotes the trivial representation $\Omega_E \to \{1\} \subset \mathbb{C}^\times$. The Brauer induction theorem for representations of finite groups gives

$$\rho - m[1_E] = \sum_i m_i \operatorname{Ind}_{E_i/E}(\chi_i - [1_{E_i}]),$$

for various integers m_i and pairs $(E_i/E, \chi_i) \in R_1(E)$. Thus

$$\begin{aligned} \mathfrak{F}_1(E/F, \rho - m[1_E]) &= \prod_i \mathfrak{f}(E_i/F, \chi_i)^{m_i}\, \mathfrak{f}(E_i/F, [1_{E_i}])^{-m_i} \\ &= \mathfrak{F}_2(E/F, \rho - m[1_E]). \end{aligned}$$

In the general case, we can write $\rho = \phi \otimes \theta$, where ϕ is an unramified character of $\mathcal{W}_E$ and $\det \theta$ has finite order. Indeed, we can take $\phi = \xi_E = \xi \mid \mathcal{W}_E$, for some unramified character ξ of $\mathcal{W}_F$. We then have

$$\mathfrak{F}_i(E/F, \rho) = \mathfrak{F}_i(E/F, \rho - m[\xi_E])\, \mathfrak{f}(E/F, [\chi_E])^m, \quad i = 1, 2.$$

Applying the Brauer induction theorem to θ, we have

$$\rho - m[\xi_E] = \sum_i m_i \operatorname{Ind}_{E_i/E}(\xi_{E_i}\chi_i - [\xi_{E_i}]),$$

and the argument goes as before. $\square$

12.8. We give some examples, incorporating significant results from class field theory. The proofs are easily accessible in the literature, but require rather different techniques from those we have been using.

Example 1. *Let $c \in F^\times$ and define*

$$\mathfrak{d}(E/F, \chi) = \chi(c), \quad (E/F, \chi) \in R_1(F).$$

The function $\mathfrak{d}$ is then extendable (but not strongly extendable). The extension $\mathfrak{D}$ is given by

$$\mathfrak{D} : (E/F, \rho) \longmapsto \det \rho(c).$$

Example 2. *Define*

$$\mathfrak{l}(E/F, \chi) = L(\chi, s), \quad (E/F, \chi) \in R_1(F).$$

The function $\mathfrak{l}$ is then strongly extendable.

The extension here is denoted $(E/F, \rho) \mapsto L(\rho, s)$, and $L(\rho, s)$ is the *Artin L-function* of the (virtual) representation ρ. It is given as follows. By additivity, we can assume that (ρ, V) is a representation of $\mathcal{W}_E$. The representation ρ then gives a representation $\rho_{\mathcal{I}}$ on the space $V^{\mathcal{I}_E}$ of $\rho(\mathcal{I}_E)$-fixed vectors in V. We take a geometric Frobenius σ_E in $\mathcal{W}_E$, and we define

$$L(\rho, s) = \det\left(1 - q_E^{-s}\rho_{\mathcal{I}}(\sigma_E)\right)^{-1},$$

where $q_E = |\boldsymbol{k}_E|$ is the size of the residue field of E.

For our next example, we need to fix a non-trivial character ψ of F. If E/F is a finite separable extension, we set $\psi_E = \psi \circ \mathrm{Tr}_{E/F}$. As in §8, we can attach to $(E/F, \chi) \in R_1(F)$ the local constant $\varepsilon(\chi, s, \psi_E)$, formed relative to the ψ_E-self dual measure on E.

Example 3. *The function $(E/F, \chi) \mapsto \varepsilon(\chi, s, \psi_E)$, $(E/F, \chi) \in R_1(\chi)$, is extendable.*

Comment. The proof of the existence of an extension is non-trivial, but requires special methods for which we shall have no further use. Since it is readily accessible in the literature, we omit it.

The extension, again denoted $(E/F, \rho) \mapsto \varepsilon(\rho, s, \psi_E)$, is called the *Langlands-Deligne local constant.* If, for example, ρ is a semisimple, finite-dimensional representation of $\mathcal{W}_F$, we have

$$\varepsilon(\rho, s, \psi) = q^{n(\rho,\psi)(\frac{1}{2}-s)}\,\varepsilon(\rho, \tfrac{1}{2}, \psi),$$

for an integer $n(\rho, \psi)$. The other factor $\varepsilon(\rho, \frac{1}{2}, \psi)$ can be thought of as a "normalized Galois Gauss sum" (and has often been called such).

We exhibit explicitly the induction relationship:

Proposition. *Let E/F be a finite separable extension, and let ρ be a smooth, semisimple representation of $\mathcal{W}_E$ of dimension m. Let ψ be a non-trivial character of F and set $\psi_E = \psi \circ \mathrm{Tr}_{E/F}$. We then have*

$$\frac{\varepsilon(\mathrm{Ind}_{E/F}\,\rho, s, \psi)}{\varepsilon(\mathrm{Ind}_{E/F}\,1_E, s, \psi)^m} = \frac{\varepsilon(\rho, s, \psi_E)}{\varepsilon(1_E, s, \psi_E)^m},$$

where 1_E denotes the trivial representation of $\mathcal{W}_E$.

It will be convenient to write $\mathrm{R}_{E/F} = \mathrm{Ind}_{E/F}\,1_E$. This "regular representation" $\mathrm{R}_{E/F}$ usually has a rather simple structure and the local constant $\varepsilon(\mathrm{R}_{E/F}, s, \psi)$ can be computed from the tables in §8. The dependence of local constants on the complex variable s is generally quite easy to determine, so it is the normalized Gauss sum factor which requires the most careful examination. For such purposes, it is convenient to have another notation:

$$\lambda_{E/F}(\psi) = \varepsilon(\mathrm{R}_{E/F}, \tfrac{1}{2}, \psi)\,/\varepsilon(1_E, \tfrac{1}{2}, \psi_E)\ .$$

This is often called the "Langlands constant" for the extension E/F.

Example 4. *The function* $(E/F, \chi) \mapsto \chi \mid F^\times$ *is extendable. The extension is given by* $(E/F, \rho) \mapsto \det \rho \mid F^\times$.

If we have $(E/F, \rho) \in R(F)$, and $\tau = \mathrm{Ind}_{E/F}\, \rho$, we have the relation

$$\det \tau = \delta_{E/F}^{\dim \rho} \otimes \det \rho \mid F^\times,$$

where $\delta_{E/F}$ denotes the "discriminant character" $\det \mathrm{R}_{E/F}$.

The dependence of the local constant $\varepsilon(\rho, s, \psi)$ on the additive character ψ is likewise easy to determine:

$$\varepsilon(\rho, s, a\psi) = \det \rho(a)\, \|a\|^{(s-\frac{1}{2})\dim \rho}\, \varepsilon(\rho, s, \psi), \quad a \in F^\times,$$

(*cf.* Remark 9.3, Lemma 8.4). In particular, the integer $n(\rho, \psi)$ depends only on ρ and *the level of* ψ. In the case where ψ has level one — we usually impose this condition — we simply write $n(\rho, \psi) = n(\rho)$.

13. The Langlands correspondence

We come to the main result.

13.1. It will be convenient to have some notation. Let $\mathcal{G}_2^0(F)$ denote the set of equivalence classes of irreducible, two-dimensional, smooth representations of the Weil group $\mathcal{W}_F$. Let $\mathcal{A}_2^0(F)$ be the set of equivalence classes of irreducible cuspidal representations of the group $G = \mathrm{GL}_2(F)$.

Main Theorem. *There is a unique bijection*

$$\begin{aligned} \mathcal{G}_2^0(F) &\longrightarrow \mathcal{A}_2^0(F), \\ \rho &\longmapsto \pi_F(\rho), \end{aligned}$$

with the following property: if ψ *is a non-trivial character of* F, *then*

$$(*) \qquad \varepsilon(\chi \otimes \rho, s, \psi) = \varepsilon(\chi \pi_F(\rho), s, \psi),$$

for all $\rho \in \mathcal{G}_2^0(F)$ *and all characters* χ *of* $F^\times$.

This bijection has the further properties:

(1) *If* $\rho \in \mathcal{G}_2^0(F)$ *and if* χ *is a character of* $\mathcal{W}_F$ *(or* $F^\times$*), then*

$$\pi_F(\chi \otimes \rho) = \chi\, \pi_F(\rho).$$

(2) *If* $\rho \in \mathcal{G}_2^0(F)$ *and* $\pi = \pi_F(\rho) \in \mathcal{A}_2^0(F)$, *then*

$$\omega_\pi = \det \rho \quad \text{and} \quad \pi_F(\check{\rho}) = \check{\pi}.$$

Remarks.

(1) The Multiplicity One Theorem (11.1) implies that there can be at most one map $\pi_F : \mathcal{G}_2^0(F) \to \mathcal{A}_2^0(F)$ which satisfies $(*)$. The problem, therefore, is to show that such a map exists, and that it is bijective. Observe that the existence of an injective map would imply that a representation $\rho \in \mathcal{G}_2^0(F)$ is determined, up to issomorphissm, by the function $\chi \mapsto \varepsilon(\chi \otimes \rho, s, \psi)$.

(2) Granting the main assertion of the theorem, property (1) follows from (11.1).

(3) We also showed in §11 that, for $\pi \in \mathcal{A}_2^0(F)$, one can recover the central character ω_π from the function $\chi \mapsto \varepsilon(\chi\pi, \frac{1}{2}, \psi)$, allowing χ to range over characters of $F^\times$ of sufficiently large level. A similar result allows one to recover $\det \rho$ from the function $\chi \mapsto \varepsilon(\chi \otimes \rho, \frac{1}{2}, \psi)$, and the first property in (2) follows.

(4) This further implies that the assertions of the theorem are independent of the choice of ψ: we can therefore work throughout with a fixed choice of ψ.

(5) The other assertion in (2) follows from the functional equation for $\varepsilon = \gamma$ (9.5).

13.2. The set of equivalence classes of irreducible, two-dimensional representations of the Galois group Ω_F corresponds, via the Langlands correspondence of the Main Theorem, to the set of $\pi \in \mathcal{A}_2^0(F)$ for which ω_π has finite order. Just as in class field theory, one introduces the Weil group in order to account for all the cuspidal representations.

13.3. Let ρ be a reducible, *semisimple* representation of $\mathcal{W}_F$ of dimension 2. Thus $\rho = \chi_1 \oplus \chi_2$, for characters χ_i of $F^\times$. One can extend the Langlands correspondence to such representations ρ by setting $\pi_F(\rho) = \pi = \boldsymbol{\iota}_B^G(\chi_1 \otimes \chi_2)$, *provided π is irreducible.* If this condition is satisfied, one gets the relations

$$(*) \qquad L(\chi \otimes \rho, s) = L(\chi\pi, s), \qquad \varepsilon(\chi \otimes \rho, s, \psi) = \varepsilon(\chi\pi, s, \psi),$$

for all characters χ of $F^\times$.

13.4. To account for the reducible principal series representations, one has to enlarge the Weil group further and define the *Weil-Deligne group* $\mathcal{W}'_F$. This is a certain extension of $\mathcal{W}_F$ by the additive group of F (although there are other ways of defining it). One considers the class of "Φ-semisimple representations" of $\mathcal{W}_F$, and one can specify these without actually defining $\mathcal{W}'_F$ itelf. *A Φ-semisimple representation of $\mathcal{W}'_F$* (or *a Deligne representation of $\mathcal{W}_F$*) is a triple

(σ, V, N), where

(1) (σ, V) is a finite-dimensional, smooth, semisimple representation of $\mathcal{W}_F$, and

(2) $N \in \mathrm{End}_{\mathbb{C}}(V)$ is nilpotent, satisfying $wNw^{-1} = \|w\|N$, $w \in \mathcal{W}_F$.

Observe that N is zero if σ is irreducible. If $\dim \sigma = 2$ and $N \neq 0$, the composition factors of σ are χ_1, χ_2, for characters χ_i of $\mathcal{W}_F$ such that $\chi_1(w)/\chi_2(w) = \|w\|^{\pm 1}$ (*cf.* (6.2) Corollary 3).

One defines L-functions and local constants for such representations. Let $\mathcal{G}_2(F)$ denote the set of equivalence classes of these representations of $\mathcal{W}'_F$, and let $\mathcal{A}_2(F)$ be the set of equivalence classes of irreducible smooth representations of $\mathrm{GL}_2(F)$. The Langlands correspondence then extends uniquely to a bijection of $\mathcal{G}_2(F)$ with $\mathcal{A}_2(F)$, satisfying the analogue of $(*)$.

From a narrowly local point of view, this process is purely formal, so we say nothing more of it. In a global context, it is of considerable significance.

14. Two-dimensional Galois representations

In preparation for the discussion of the Langlands correspondence, we give some results concerning representations of $\mathcal{W}_F$ and their local constants.

14.1. Let $\rho \in \mathcal{G}_2^0(F)$: thus ρ is an irreducible, smooth representation of $\mathcal{W}_F$ of dimension 2. We define $T(\rho)$ to be the group of characters χ of $\mathcal{W}_F$ such that $\chi \otimes \rho \cong \rho$. We say that ρ is *primitive* if $T(\rho)$ is trivial.

Suppose, on the other hand, that $\chi \in T(\rho)$, $\chi \neq 1$. Comparing determinants, we see that $\chi^2 = 1$, and so $\mathrm{Ker}\,\chi = \mathcal{W}_E$, for some separable quadratic extension E/F. Elementary Clifford theory now implies that there is a character ξ of $\mathcal{W}_E$ such that $\rho \cong \mathrm{Ind}_{E/F}\,\xi$. Moreover, ρ determines the orbit of χ under the natural action of $\mathcal{W}_F/\mathcal{W}_E = \mathrm{Gal}(E/F)$. (In practice, it is usually more convenient to think of χ, ξ as characters of $F^\times$, $E^\times$ respectively, via class field theory.)

Proposition. *The group $T(\rho)$ is an elementary abelian 2-group of order dividing 4.*

Proof. Suppose $|T(\rho| \neq 1$, and let $\kappa \in T(\rho)$ be non-trivial. Let E/F be the quadratic extension such that $\mathcal{W}_E = \mathrm{Ker}\,\kappa$. Choose a character ξ of $\mathcal{W}_E$ such that $\rho \cong \mathrm{Ind}_{E/F}\,\xi$. Let $\chi \in T(\rho)$, $\chi \neq 1, \kappa$. The equation $\rho \cong \chi \otimes \rho$ implies $\rho \cong \mathrm{Ind}_{E/F}(\chi_E \otimes \xi)$, where $\chi_E = \chi \circ \mathrm{N}_{E/F}$. It follows that $\chi_E \otimes \xi = \xi$ or ξ^σ, where $\sigma \in \mathrm{Gal}(E/F)$, $\sigma \neq 1$. By hypothesis, $\chi_E \neq 1$, so $\chi_E \otimes \xi = \xi^\sigma$, or $\chi_E = \xi^\sigma/\xi$. The only character η of $F^\times$ satisfying $\eta_E = \chi_E$, $\eta \neq \chi$, is $\kappa\chi$, and the result follows. $\square$

We say that ρ is *simply imprimitive* if $|T(\rho)| = 2$, *triply imprimitive* if $|T(\rho)| = 4$.

14.2. Let $\rho \in \mathcal{G}_2^0(F)$; we say that ρ is *unramified* if there exists an unramified character $\chi \in T(\rho)$, $\chi \neq 1$. In this case, we have:

Proposition. *Let $\rho \in \mathcal{G}_2^0(F)$ be unramified. Let E/F be the unramified quadratic extension (in F^{sep}). There is a character ξ of $\mathcal{W}_E$ such that $\rho = \mathrm{Ind}_{E/F}\,\xi$. The* $\mathrm{Gal}(E/F)$*-orbit of the character ξ is uniquely determined by ρ.*

We call the representation ρ *totally ramified* if $T(\rho)$ contains only the trivial unramified character. (In this language, therefore, primitive representations are totally ramified.)

14.3. There is another easy case, that in which $p \neq 2$. (We recall that p is the characteristic of the residue field $\boldsymbol{k}$ of F.)

Proposition. *Suppose that $p \neq 2$; let $\rho \in \mathcal{G}_2^0(F)$ be totally ramified. Then:*

(1) *The representation ρ is imprimitive: there exists a ramified quadratic extension E/F and a character ξ of $S\,\mathcal{W}_E$ such that $\rho \cong \mathrm{Ind}_{E/F}\,\xi$.*
(2) *The pair $(E/F, \xi)$ is uniquely determined by ρ, up to F-isomorphism. In particular, ρ is simply imprimitive.*

Proof. Since ρ is irreducible and has dimension $2 \neq p$, the restriction of ρ to the pro p-group $\mathcal{P}_F$ is a sum of two characters χ, χ' say, which are conjugate under $\mathcal{W}_F$. The group $\mathcal{I}_F/\mathcal{P}_F$ is pro-cyclic, so χ extends to a character of its $\mathcal{I}_F$-stabilizer and, since $\mathcal{W}_F/\mathcal{I}_F$ is pro-cyclic, this extension also extends to its $\mathcal{W}_F$-stabilizer. Thus $\chi \neq \chi'$. The $\mathcal{W}_F$-stabilizer of χ is open in $\mathcal{W}_F$ and of index two: it is therefore of the form $\mathcal{W}_E$, for some separable quadratic extension E/F. This acts on the χ-isotypic space as a character ξ. We then have $\rho = \mathrm{Ind}_{E/F}\,\xi$. The uniqueness statement is immediate. □

Remark. We have shown that, when $p \neq 2$, all irreducible two-dimensional representations of $\mathcal{W}_F$ are imprimitive. There do exist triply imprimitive representations in the case $p \neq 2$, but such representations are invariably unramified.

14.4. We now consider the case $p = 2$. The structure of ramified quadratic extensions is rather more complicated, and this is reflected in the structure of representations of $\mathcal{W}_F$.

Let E/F be ramified quadratic, and let ϖ be a prime element of E. Thus ϖ is a root of a polynomial $X^2{+}aX{+}b$, in which $\upsilon_F(b) = 1$ and $\upsilon_F(a) \geqslant 1$. (We allow the possibility $\upsilon_F(a) = \infty$, i.e., $a = 0$.) The discriminant of the field extension E/F is then $\mathfrak{d}_{E/F} = \mathfrak{p}^{d+1}$, where $d{+}1 = \min\{2\upsilon_F(a), 1{+}\upsilon_F(4)\}$. In particular, $d \geqslant 1$ and d is odd except in the case $d = \upsilon_F(4)$.

Let $\varphi = \varphi_{E/F}$ be the *inverse Herbrand function* for the extension E/F. Thus

φ is the function of positive integers given by

$$\varphi(j) = \begin{cases} j & \text{if } 1 \leqslant j \leqslant d, \\ 2j-d & \text{if } j \geqslant d. \end{cases}$$

The basic properties of the extension E/F are:

Lemma.

(1) *Let $k \in \mathbb{Z}$; then $\mathrm{Tr}_{E/F}(\mathfrak{p}_E^k) = \mathfrak{p}_F^l$, where $l = [(k+d+1)/2]$.*
(2) *Suppose $j \geqslant 1$, $j \neq d$; the field norm $\mathrm{N}_{E/F}$ induces an isomorphism $U_E^{\varphi(j)}/U_E^{1+\varphi(j)} \cong U_F^j/U_F^{1+j}$.*
(3) *The norm map $\mathrm{N}_{E/F}$ induces a map $U_E^d/U_E^{1+d} \to U_F^d/U_F^{1+d}$ with kernel and cokernel of order 2.*

Let ξ be a character of $E^\times$; the representation $\rho = \mathrm{Ind}_{E/F}\,\xi$ is then irreducible if and only if $\xi \neq \xi^\sigma$, where σ generates $\mathrm{Gal}(E/F)$. Since the kernel of $\mathrm{N}_{E/F}$ is the group of elements x^σ/x, $x \in E^\times$, this condition is equivalent to ξ not factoring through $\mathrm{N}_{E/F}$.

Let $l(\xi)$ denote the level of ξ. If χ is a character of $F^\times$, we put $\chi_E = \chi \circ \mathrm{N}_{E/F}$. We say that ξ is *minimal over F* if $l(\xi) \leqslant l(\chi_E\xi)$, for all characters χ of $F^\times$.

Proposition. *Let ξ be a character of $E^\times$.*

(1) *If ξ is minimal over F, then $l(\xi) \geqslant d$.*
(2) *Suppose that $l(\xi) \geqslant 1+d$; then ξ is minimal over F if and only if $l(\xi) \not\equiv d \pmod 2$.*
(3) *If ξ is minimal over F, then $\rho = \mathrm{Ind}_{E/F}\,\xi$ is irreducible. The representation ρ is unramified if $l(\xi) = d$ and totally ramified otherwise.*

14.5. In this same situation, one can give some account of the multiply imprimitive representations. (For the definition of the integer $n(\rho, \psi)$, see the end of §12.)

Lemma. *Let ξ be a character of $E^\times$ such that $\mathrm{Ind}_{E/F}\,\xi = \rho$ is irreducible, and let ψ be of level one; then $n(\rho, \psi) = l(\xi)+d$.*

Proof. The character $\psi_E = \psi \circ \mathrm{Tr}_{E/F}$ of E has level $1-d$. We therefore have

$$\varepsilon(\xi, s, \psi_E) = q^{(l(\xi)+d)(\frac{1}{2}-s)}\varepsilon(\xi, \tfrac{1}{2}, \psi_E),$$

and the lemma follows from the induction properties of the Langlands-Deligne local constant. □

Proposition. *Let ξ be minimal over F and put $n = l(\xi)+d$. Suppose that*

$$d < l(\xi) < 2d,$$

or, equivalently, $2d < n < 3d$. Then:

(1) *The representation $\rho = \mathrm{Ind}_{E/F}\,\xi$ is irreducible and triply imprimitive.*
(2) *The group $T(\rho)$ contains a character χ of level d', such that $n > 3d'$.*

The proof is an exercise in the properties of the norm (14.4 Lemma). In the case $n = 3d$, the representation may be simply or triply imprimitive. In the latter case, all non-trivial elements of $T(\rho)$ have level d. By a similar method:

Complement. *Suppose that ξ is minimal over F and $n = d+l(\xi) > 3d$. Suppose that $\rho = \mathrm{Ind}_{E/F}\,\xi$ is triply imprimitive. There is then a character $\chi \in T(\rho)$ of level d' such that $2d' < n < 3d'$.*

Any simply imprimitive representation $\rho \in \mathcal{G}_2^0(F)$ therefore satisfies $n \geqslant 3d$.

14.6. We briefly consider the *primitive* representations $\rho \in \mathcal{G}_2^0(F)$: we have noted that these can only exist in the case $p = 2$.

We say that ρ is *minimal over F* if $n(\chi\otimes\rho,\mu) \leqslant n(\rho,\mu)$, for all characters χ of $F^\times$ and some (equivalently any) non-trivial character μ of F.

Continuing in the general situation, let $\rho \in \mathcal{G}_2^0(F)$ and consider the image $\mathbb{G}$ of $\rho(\mathcal{W}_F)$ in the projective linear group $\mathrm{PGL}_2(\mathbb{C}) = \mathrm{GL}_2(\mathbb{C})/\mathbb{C}^\times$. The group $\mathbb{G}$ is *finite.*

Lemma 1. *Let $\mathbb{G}$ be a finite subgroup of $\mathrm{PGL}_2(\mathbb{C})$; then $\mathbb{G}$ is cyclic, dihedral, or one of the permutation groups A_4, S_4, A_5.*

This is effectively the determination of the Euclidean solids. In our situation, $\mathbb{G}$ is soluble, so the case $\mathbb{G} = A_5$ cannot arise. If $\mathbb{G}$ were cyclic, the representation ρ would be reducible. If $\mathbb{G}$ were dihedral, ρ would be an irreducible representation of a cyclic central extension of a dihedral group, and hence imprimitive. Thus:

Lemma 2. *The representation ρ is primitive if and only if $\mathbb{G} = A_4$ or S_4.*

Now we can appeal to the representation theory of cyclic central extensions of A_4 and S_4 to get:

Proposition. *Let $\rho \in \mathcal{G}_2^0(F)$ be primitive, and let $\mathbb{G}$ be the image of $\rho(\mathcal{W}_F)$ in $\mathrm{PGL}_2(\mathbb{C})$. There exists a cubic extension K/F, unique up to F-isomorphism, such that $\rho_K = \rho \mid \mathcal{W}_K$ is irreducible and imprimitive. Moreover:*

(1) *If $\mathbb{G} = A_4$, then ρ_K is triply imprimitive and K/F is cyclic.*
(2) *If $\mathbb{G} = S_4$, then ρ_K is simply imprimitive and and K/F is not cyclic. If L/K is unramified quadratic (so that L/F is the normal closure of K/F), then ρ_L is irreducible and triply imprimitive.*

Remark. Since ρ is primitive, the restriction ρ_K is irreducible for any tamely ramified extension K/F.

The situation becomes clearer if we assume that ρ is primitive and minimal over F. If L/F is a tamely ramified Galois extension such that ρ_L is triply imprimitive, the group $\mathrm{Gal}(L/F)$ acts transitively on the non-trivial elements of $T(\rho_L)$. They must all, therefore, have the same level. We deduce:

Complement. *Let $n_L = n(\rho_L, \psi_L)$; then $n_L \equiv 0 \pmod 3$ and every non-trivial element of $T(\rho_L)$ has level $n_L/3$.*

15. Multiplicity-one for imprimitive representations

For *imprimitive* two-dimensional representations of $\mathcal{W}_F$, there is an analogue of the multiplicity-one theorem (§11) for representations of $\mathrm{GL}_2(F)$.

15.1. Let E/F be a separable quadratic extension, and put $\mathrm{Gal}(E/F) = \{1, \sigma\}$.

Theorem. *For $i = 1, 2$, let ξ_i be a character of $E^\times$ and suppose that the representation $\rho_i = \mathrm{Ind}_{E/F}\, \xi_i$ is irreducible. Suppose also that*

$$(\star) \qquad \varepsilon(\chi \otimes \rho_1, s, \psi) = \varepsilon(\chi \otimes \rho_2, s, \psi),$$

for every character χ of $F^\times$. We then have $\rho_1 \cong \rho_2$ and consequently $\{\xi_1, \xi_1^\sigma\} = \{\xi_2, \xi_2^\sigma\}$.

15.2. The proof follows a course parallel to that of (11.1). First, we can assume that the ρ_i are both minimal over F, and that $n(\rho_1) = n(\rho_2)$.

Twisting with characters χ of large level, we conclude as in §11 that the characters $\det \rho_i$ of $F^\times$ are the same, and that moreover the hypothesis connecting the ρ_i is independent of the choice of character ψ of F.

The main step in the proof is:

Proposition. *Suppose that the ρ_i have minimal level, the same determinant, and that $(\star)$ holds for all characters χ of $F^\times$ such that $n(\chi \otimes \rho_i) \leqslant n(\rho_i)$. We then have $\rho_1 \cong \rho_2$.*

Let us abbreviate $\chi_E = \chi \circ \mathrm{N}_{E/F}$, $\psi_E = \psi \circ \mathrm{Tr}_{E/F}$, for characters χ, ψ of $F^\times$, F respectively. In this notation, our hypothesis amounts to

$$\varepsilon(\chi_E\, \xi_1, s, \psi_E) = \varepsilon(\chi_E\, \xi_2, s, \psi_E).$$

The equality of determinants and levels implies that

$$(\dagger) \qquad \varepsilon(\chi_E\, \xi_1, s, \mu_E) = \varepsilon(\chi_E\, \xi_2, s, \mu_E),$$

for all non-trivial characters μ of F.

15.3. We economize on notation by treating a single irreducible representation $\rho = \mathrm{Ind}_{E/F}\,\xi$, minimal over F. We have to show that we can recover ρ from $\det\rho$ and the function $(\chi,\mu) \mapsto \varepsilon(\chi\otimes\rho, \frac{1}{2}, \mu)$, where χ ranges over the characters of $F^\times$ such that $n(\rho) \geqslant n(\chi\otimes\rho)$ and μ over the characters of F of level one.

We need to examine the condition $n(\chi\otimes\rho) \leqslant n(\rho)$. For this, we first suppose that E/F is *totally ramified,* and put $\mathfrak{d}_{E/F} = \mathfrak{p}_F^{d+1}$. Thus $d \geqslant 0$, the case $d = 0$ occurring exactly when E/F is tamely ramified. We let $\varphi_{E/F}$ be the inverse Herbrand function for E/F, as in §14. (We note that, when E/F is tamely ramified, $\varphi_{E/F}(j) = 2j$, for all $j \geqslant 1$.)

Lemma. *Set $n = n(\rho)$. If E/F is totally ramified, then:*

1. *The level of ξ is $n-d$; in particular, $n \geqslant 2d$.*
2. *If χ is a character of $F^\times$ of level j, we have $n(\chi\otimes\rho) \leqslant n$ if and only if $\varphi_{E/F}(j) \leqslant n-d$, that is, if and only if $j \leqslant [n/2]$.*
3. *If μ has level one, then μ_E has level $1-d$ while if μ has level $\leqslant 0$, μ_E has level $\leqslant -d$.*

If E/F is unramified, then ξ has level n and $\chi_E\xi$ has level $\leqslant n$ if and only if the level of χ is $\leqslant n$.

Thus, in all cases,

$$n(\chi\otimes\rho) \leqslant n(\rho) \quad\Longleftrightarrow\quad \chi \text{ has level} \leqslant [n/e(E|F)].$$

This property depends only on the restriction of χ to U_F. We denote by Γ the group of characters χ of U_F satisfying the condition.

Let $E(n)$ be the set of $x \in E^\times$ with $v_E(x) = -n$, and consider the function

$$\Upsilon_\xi(\chi,\mu) = \int_{E(n)} \xi^{-1}(x)\,\chi_E^{-1}(x)\,\mu_E(x)\,d^*x, \tag{‡}$$

where χ ranges over all characters of $F^\times$ of level $\leqslant [n/e(E|F)]$ and μ over all characters of F of level $\leqslant 1$. Also, d^*x is some Haar measure on $E^\times$. The function $\Upsilon_\xi(\chi,\mu)$ is zero whenever μ has level $\leqslant 0$, while otherwise there is a positive constant, depending only on n, E/F and choices of measure, for which

$$\Upsilon_\xi(\chi,\mu) = k\,\varepsilon(\chi_E\xi, \tfrac{1}{2}, \mu_E).$$

We have, from the induction property of the Langlands-Deligne local constant, the relation

$$\varepsilon(\chi_E\xi^\sigma, \tfrac{1}{2}, \mu_E) = \varepsilon(\chi_E\xi, \tfrac{1}{2}, \psi_E),$$

so $\Upsilon_\xi(\chi,\mu) = \Upsilon_{\xi^\sigma}(\chi,\mu)$ is expressible as an integral over $\mathrm{Gal}(E/F)$-conjugacy classes.

As a function of $\mu \in (F/\mathfrak{p}_F)\widehat{\ }$, Υ_ξ depends only on the restriction of μ to traces of elements of $E(n)$. We have $\mathrm{Tr}_{E/F}(\mathfrak{p}_E^{-n}) = \mathfrak{p}_F^t$, where

$$t = \begin{cases} -n & \text{if } E/F \text{ is unramified,} \\ [(d{+}1{-}n)/2] & \text{otherwise.} \end{cases}$$

In other words, we may think of μ as an element of $(F/\mathfrak{p}_F)\widehat{\ }/(F/\mathfrak{p}_F^t)\widehat{\ } = (\mathfrak{p}_F^t/\mathfrak{p}_F)\widehat{\ }$.

Also, we fix an element $\alpha \in E$ such that $E(n) = \alpha U_E$. If E/F is ramified, n is odd and α is minimal over F. If E/F is unramified, we can take $\alpha = \varpi_F^{-n/2}$, where ϖ_F is a prime element of F.

For a character $\chi \in \Gamma$, we extend χ to a character χ^α of $F^\times$ such that $\chi_E(\alpha) = 1$. We then put

$$\Upsilon_\xi^\alpha(\chi, \mu) = \Upsilon_\xi(\chi^\alpha, \mu).$$

Thus Υ_ξ^α is a function on $\Gamma \times (\mathfrak{p}_F^t/\mathfrak{p}_F)\widehat{\ }$. Its Fourier transform $\widehat{\Upsilon}_\xi^\alpha$ is the function on $U_F/U_F^{[n/e]+1} \times \mathfrak{p}_F^t/\mathfrak{p}_F$ given by

$$\widehat{\Upsilon}_\xi^\alpha(a, b) = \sum_{\chi,\mu} \Upsilon_\xi^\alpha(\chi, \mu)\, \bar{\chi}(a)\, \bar{\mu}(b).$$

15.4. We appeal directly to the results of §11 to show that the map

$$\begin{aligned} \mathrm{Gal}(E/F)\backslash E(n)/U_E^{n+1} &\longrightarrow U_F/U_F^{[n/e]+1} \times \mathfrak{p}_F^t/\mathfrak{p}_F, \\ x &\longmapsto \big(\mathrm{N}_{E/F}(\alpha^{-1}x), \mathrm{Tr}_{E/F}(x)\big), \end{aligned}$$

is injective on minimal elements.

Comment. When E/F is totally ramified, the set $E(n)$ consists of minimal elements, since n is necessarily odd. When E/F is unramified and $n \geqslant 1$, a simple Gauss sum argument shows that, in the integral defining the local constant, we can replace $E(n)$ by its subset $E'(n)$ of minimal elements, introducing only a positive constant factor. (We have $E'(n) = \bigcup_\zeta \zeta\varpi_F^{-n/2}U_{\mathfrak{A}}^1$, where ζ ranges over all roots of unity in $E \smallsetminus F$ of order relatively prime to p.) When $n = 0$, the situation is more subtle: a direct argument, using the character table of §1, is required. We henceforward ignore this awkward case.

We define a function Φ on $U_F/U_F^{[n/e]+1} \times \mathfrak{p}_F^t/\mathfrak{p}_F$ as follows. If there exists a minimal element $x \in E(n)$ with $\mathrm{N}_{E/F}(\alpha^{-1}x) = a$ and $\mathrm{Tr}_{E/F}(x) = b$, we put $\Phi(a, b) = \xi(x){+}\xi^\sigma(x)$. Otherwise, we put $\Phi(a, b) = 0$. We then have

$$\Phi = k\,\widehat{\Upsilon}_\xi^\alpha,$$

for a positive constant k depending only on choices of measures. The function $\xi{+}\xi^\sigma$ is therefore determined (up to a positive constant) on the set of minimal elements in $E(n)$ while $\xi \mid F^\times$ is known. This data is sufficient to determine the Galois orbit of ξ. $\square$

15.5. We need to extend the scope of Theorem 15.1.

Corollary. *Let $\rho_1, \rho_2 \in \mathcal{G}_2^0(F)$. Suppose:*

(a) *the representations ρ_1, ρ_2 are imprimitive, and*
(b) *$\varepsilon(\chi \otimes \rho_1, s, \psi) = \varepsilon(\chi \otimes \rho_2, s, \psi)$, for every character χ of $F^\times$.*

We then have $\rho_1 \cong \rho_2$.

Proof. There is a quadratic extension E/F and a quasicharacter ξ of $\mathcal{W}_E$ such that $\rho_2 = \mathrm{Ind}_{E/F}\, \xi$. Let κ be the unique non-trivial character of $F^\times$ which is trivial on $\mathrm{N}_{E/F}(E^\times)$. Thus $\kappa \otimes \rho_2 \cong \rho_2$. In particular, we have $\varepsilon(\chi \otimes \rho_1, s, \psi) = \varepsilon(\chi \otimes \kappa \otimes \rho_1, s, \psi)$, for all characters χ of $F^\times$. The theorem, applied to ρ_1, shows that $\kappa \otimes \rho_1 \cong \rho_1$, with the result that ρ_1 is also induced from $\mathcal{W}_E$. The equality $\rho_1 \cong \rho_2$ now follows from the theorem. $\square$

15.6. We get some useful properties by combining these two "multiplicity one theorems". Let us write $\mathcal{G}_2^{\mathrm{im}}(F)$ for the set of imprimitive $\rho \in \mathcal{G}_2^0(F)$. Similarly, we say that $\pi \in \mathcal{A}_2^0(F)$ is imprimitive if there exists a character $\chi \neq 1$ of $F^\times$ such that $\chi\pi \cong \pi$. We write $\mathcal{A}_2^{\mathrm{im}}(F)$ for the set of imprimitive $\pi \in \mathcal{A}_2^0(F)$.

Theorem. *Let $\mathcal{K} : \mathcal{G}_2^{\mathrm{im}}(F) \to \mathcal{A}_2^0(F)$ be a map such that*

$$\varepsilon(\chi\mathcal{K}(\rho), s, \psi) = \varepsilon(\chi \otimes \rho, s, \psi), \tag{‡}$$

for all characters χ of $F^\times$. The map $\mathcal{K}$ is then injective, and its image is contained in $\mathcal{A}_2^{\mathrm{im}}(F)$. It is the unique map $\mathcal{G}_2^{\mathrm{im}}(F) \to \mathcal{A}_2^0(F)$ with property (‡).

Proof. Arguing as in (13.1), we have $\mathcal{K}(\chi \otimes \rho) = \chi\mathcal{K}(\rho)$, for all χ and ρ; Theorem 15.1 then implies that $\mathcal{K}$ is injective, and so its image consists of imprimitive representations.

If $\mathcal{K}'$ is another map $\mathcal{G}_2^{\mathrm{im}}(F) \to \mathcal{A}_2^0(F)$ with property (‡), we have

$$\varepsilon(\chi\mathcal{K}(\rho), s, \psi) = \varepsilon(\chi\mathcal{K}'(\rho), s, \psi) = \varepsilon(\chi \otimes \rho, s, \psi),$$

for all ρ and χ. Theorem 11.1 gives $\mathcal{K}'(\rho) = \mathcal{K}(\rho)$, as required. $\square$

15.7. This theorem has an important consequence. Suppose we have another field F_1 and an isomorphism $t : F_1 \to F$. We extend t, somehow, to an isomorphism $\bar{t} : F_1^{\mathrm{sep}} \to F^{\mathrm{sep}}$, which gives an isomorphism $\bar{t} : \mathcal{W}_{F_1} \to \mathcal{W}_F$. We can also use t to get an isomorphism $\mathrm{GL}_2(F_1) \to \mathrm{GL}_2(F)$. Composition with t or $\bar{t}$ then gives bijections $\mathcal{G}_2^0(F) \to \mathcal{G}_2^0(F_1)$ and $\mathcal{A}_2^0(F) \to \mathcal{A}_2^0(F_1)$. These bijections preserve imprimitivity of representations.

Suppose we have a map $\mathcal{K}$ satisfying (‡). This gives a map $\mathcal{K}_1 : \mathcal{G}_2^{\mathrm{im}}(F_1) \to \mathcal{A}_2^0(F_1)$ by

$$\mathcal{K}_1 : \rho \circ \bar{t} \longmapsto \mathcal{K}(\rho) \circ t,$$

satisfying the analogue of (‡). It is the only such map.

We apply this in the following context, with $F_1 = F$ and $t \in \mathrm{Aut}(F)$:

Corollary. *Let F/F_0 be a finite, Galois extension. Let $\mathcal{K} : \mathcal{G}_2^{\mathrm{im}}(F) \to \mathcal{A}_2^0(F)$ satisfy* (‡). *We then have*

$$\mathcal{K}(\rho \circ \sigma) = \mathcal{K}(\rho) \circ \sigma,$$

for all $\rho \in \mathcal{G}_2^{\mathrm{im}}(F)$ and all $\sigma \in \mathrm{Gal}(F/F_0)$.

16. The tamely ramified correspondence

We will now construct the Langlands correspondence $\mathcal{G}_2^0(F) \to \mathcal{A}_2^0(F)$ in the "tame" case $p \neq 2$. Part of the construction applies equally well when $p = 2$.

16.1. The overall scheme is as follows. We take $\rho = \mathrm{Ind}_{E/F}\,\xi \in \mathcal{G}_2^{\mathrm{im}}(F)$ and, in the case $p \neq 2$, we assume that ρ, indeed the extension E/F, is *unramified.* We assume that the representation ρ has minimal level, with respect to twisting by characters of $F^\times$. From the character ξ, we construct an extended simple type (J, Λ_0) in $G = \mathrm{GL}_2(F)$. We then modify this to produce another type (J, Λ), where $\Lambda = \Lambda_0 \otimes \Delta$, for a certain character Δ of J trivial on J^1. We thus get an irreducible cuspidal representation $\pi_\rho = c\text{-}\mathrm{Ind}_J^G\,\Lambda$ of G with the properties

$$\omega_{\pi_\rho} = \det \rho, \qquad \varepsilon(\pi_\rho, s, \psi) = \varepsilon(\rho, s, \psi),$$

for a fixed character ψ of F of level one. Because of the first property, the second holds for all non-trivial characters ψ of F. We then *define* $\pi_{\chi\otimes\rho} = \chi\pi_\rho$, for a character χ of $F^\times$. A straightforward manipulation of Gauss sums (which we omit) then gives the desired property

$$\varepsilon(\pi_{\chi\otimes\rho}, s, \psi) = \varepsilon(\chi \otimes \rho, s, \psi),$$

for all χ.

Thus π_ρ is the only possible candidate for the Langlands correspondent $\mathcal{L}(\rho)$, so we put $\pi_\rho = \mathcal{L}(\rho)$.

When $p \neq 2$, we thus have a map $\mathcal{L} : \mathcal{G}_2^{\mathrm{im}}(F) = \mathcal{G}_2^0(F) \to \mathcal{A}_2^0(F)$. By (15.6), it is injective. It will be obvious from the construction that it is surjective, and we have proved Theorem 13.1 in this case.

When $p = 2$, this definition gives an injective map from the set of unramified representations $\rho \in \mathcal{G}_2^0(F)$ to the unramified representations $\pi \in \mathcal{A}_2^0(F)$. It will likewise be clear that it is bijective.

From now on, we work relative to a fixed character ψ of F of level one.

16.2. In this paragraph, we allow the case $p = 2$. We take $\rho \in \mathcal{G}_2^0(F)$ such that $n(\rho) = n(\rho, \psi) = 0$. Thus $\rho \cong \mathrm{Ind}_{E/F}\,\xi$, where E/F is unramified and ξ has level zero. The restriction $\xi \mid U_E$ is the inflation of a character $\tilde{\xi}$ of $\boldsymbol{k}_E^\times$ which is *regular:* $\tilde{\xi}^\sigma \neq \tilde{\xi}$, for $\sigma \in \mathrm{Gal}(\boldsymbol{k}_E/\boldsymbol{k})$, $\sigma \neq 1$. As in §1, this gives rise to an irreducible cuspidal representation $\tilde{\lambda}$ of $\mathrm{GL}_2(\boldsymbol{k})$. We inflate $\tilde{\lambda}$ to a representation λ of $\mathrm{GL}_2(\mathfrak{o})$. We set $J = F^\times \mathrm{GL}_2(\mathfrak{o})$, and extend λ to a representation Λ_0 of J so that $\Lambda_0 \mid F^\times$ is a multiple of $\xi \mid F^\times$.

We let χ_2 be the unramified quadratic character of $F^\times$. We extend this to a character of J, trivial on $\mathrm{GL}_2(\mathfrak{o})$. We set $\Lambda = \chi_2 \otimes \Lambda_0$. We put $\pi_\rho = c\text{-}\mathrm{Ind}\,\Lambda$. The isomorphism class of π_ρ depends only on that of ρ.

Proposition. *The map $\rho \mapsto \pi_\rho$ gives a bijection between the set of $\rho \in \mathcal{G}_2^0(F)$ of level zero and the set of $\pi \in \mathcal{A}_2^0(F)$ of level zero. We have*

(1) $\omega_\rho = \det \rho$*;*
(2) $\varepsilon(\pi_\rho, s, \psi) = \varepsilon(\rho, s, \psi)$.
(3) *Moreover,* $\varepsilon(\chi \otimes \rho, s, \psi) = \varepsilon(\chi\pi_\rho, s, \psi)$*, for all characters χ of $F^\times$.*

Proof. The bijectivity of $\rho \mapsto \pi_\rho$ is clear from the classification theory, and (1) follows from the construction: $\delta_{E/F} = \chi_2$. The relation (2) has been proved in Proposition 10.1.

In (3), if χ is tamely ramified then $\chi \otimes \rho$ is of level zero, and the construction gives $\pi_{\chi\otimes\rho} = \chi\pi_\rho$. The local constant relation is then a special case of (2). If χ is not tamely ramified, a straightforward Gauss sum calculation is required. □

16.3. We now take $\rho \in \mathcal{G}_2^0(F)$ to unramified of minimal level $n = n(\rho) > 0$. Thus $\rho = \mathrm{Ind}_{E/F}\,\xi$, where E/F is unramified and ξ has level n. We set $\psi_E = \psi \circ \mathrm{Tr}_{E/F}$: thus ψ_E is a character of E of level one. We choose $\alpha \in E$ so that $\xi(1{+}x) = \psi_E(\alpha x)$, $2v_E(x) \geqslant n{+}1$. Thus $v_E(\alpha) = -n$. We think of E as embedded in $A = \mathrm{M}_2(F)$, and we let $\mathfrak{A}$ be the unique hereditary order stable under conjugation by $E^\times$. Since ξ has minimal level, $[\mathfrak{A}, n, \alpha]$ is a simple stratum in A. We form the group

$$J = E^\times U_{\mathfrak{A}}^{[(n+1)/2]}.$$

The construction now splits into two cases, depending on whether n is odd or even.

Suppose first that $n = 2m{+}1$ is *odd.* Therefore $J = E^\times U_{\mathfrak{A}}^{m+1}$. We define a character Λ_0 of J by

$$\begin{aligned} \Lambda_0 \mid E^\times &= \xi, \\ \Lambda_0(1{+}x) &= \psi(\mathrm{tr}(\alpha x)), \quad 1{+}x \in U_{\mathfrak{A}}^{m+1}. \end{aligned}$$

We next let χ_2 be the character of J which is trivial on $U_{\mathfrak{A}}^{m+1}$ and such that $\chi_2 \mid E^\times$ is unramified of order 2. We set $\Lambda = \chi_2 \otimes \Lambda_0$ and $\pi_\rho = c\text{-}\mathrm{Ind}\,\Lambda$.

Proposition 1. *With the preceding hypotheses, we have $\pi_\rho \in \mathcal{A}_2^0(F)$, and π_ρ is unramified.*

The map $\rho \mapsto \pi_\rho$ gives a canonical bijection between the set of unramified $\rho \in \mathcal{G}_2^0(F)$ of odd, minimal level and the set of unramified $\pi \in \mathcal{A}_2^0(F)$ of odd minimal level, with the property

$$\omega_{\pi_\rho} = \det \rho, \qquad \varepsilon(\pi_\rho, s, \psi) = \varepsilon(\rho, s, \psi).$$

Moreover,

$$\varepsilon(\chi\pi_\rho, s, \psi) = \varepsilon(\chi \otimes \rho, s, \psi),$$

for all characters χ of $F^\times$.

Proof. The surjectivity of the map $\rho \mapsto \pi_\rho$ is clear. The relation $\omega_{\pi_\rho} = \det \rho$ is given by the construction. The local constant relations follow from simple Gauss sum calculations. □

We now have the case where $n = 2m$ is *even.* Thus $J = E^\times U_{\mathfrak{A}}^m$. We shall also need the groups $H^1 = U_E^1 U_{\mathfrak{A}}^{m+1}$ and $J^1 = U_E^1 U_{\mathfrak{A}}^m$. We define a character θ of H^1 by

$$\begin{aligned} \theta \mid U_E^1 &= \xi \mid U_E^1, \\ \theta(1{+}x) &= \psi(\operatorname{tr}(\alpha x)), \quad 1{+}x \in U_{\mathfrak{A}}^{m+1}. \end{aligned}$$

Next, there is a unique irreducible representation η of J^1 such that $\eta \mid H^1$ contains (indeed, is a multiple of) the character θ. We have to extend η to a representation of J.

For this purpose, we let $\boldsymbol{\mu}_F$ denote the group of roots of unity in F of order relatively prime to p. The group $\boldsymbol{\mu}_E/\boldsymbol{\mu}_F$ is cyclic. There is then a unique irreducible representation Λ_0 of $U_E J^1$ such that $\Lambda_0 \mid J^1 = \eta$ and

$$\operatorname{tr} \Lambda_0(\zeta u) = \epsilon\, \xi(\zeta u),$$

for $u \in U_E^1$ and $\zeta \in \boldsymbol{\mu}_E$ generating $\boldsymbol{\mu}_E/\boldsymbol{\mu}_F$, where $\epsilon = \pm 1$ is a constant sign. We extend Λ_0 to a representation Λ_0 of $J = F^\times U_E J^1$ by demanding that $\Lambda_0 \mid F^\times$ be a multiple of $\xi \mid F^\times$.

We now define a character χ_2 of $E^\times/U_E^1 = J/J^1$ as follows. If ϖ_F is a prime of F, we set $\chi_2(\varpi_F) = -1$. On $\boldsymbol{\mu}_E$, the character χ_2 is to be trivial if $p = 2$, and of order 2 if $p \neq 2$. In all cases, $\chi_2 \mid F^\times$ is unramified of order 2.

We set $\Lambda = \chi_2 \otimes \Lambda_0$, $\pi_\rho = c\text{-Ind}\, \Lambda$.

Proposition 2. *With the preceding hypotheses, we have $\pi_\rho \in \mathcal{A}_2^0(F)$, and π_ρ is unramified.*

The map $\rho \mapsto \pi_\rho$ gives a canonical bijection between the set of unramified $\rho \in \mathcal{G}_2^0(F)$ of even, positive, minimal level and the set of unramified $\pi \in \mathcal{A}_2^0(F)$ of even, positive, minimal level, with the property

$$\omega_{\pi_\rho} = \det \rho, \qquad \varepsilon(\pi_\rho, s, \psi) = \varepsilon(\rho, s, \psi).$$

Moreover,

$$\varepsilon(\chi\pi_\rho, s, \psi) = \varepsilon(\chi \otimes \rho, s, \psi),$$

for all characters χ of $F^\times$.

The argument is exactly the same as before, although the Gauss sum calculations are a little more intricate and require some understanding of the sign ϵ. We remark that $\epsilon = +1$ when p is odd, and $\epsilon = -1$ when $p = 2$.

16.4. Let us summarize our results so far. Let $\mathcal{G}_2^{\mathrm{nr}}(F)$ denote the set of unramified $\rho \in \mathcal{G}_2^0(F)$, and similarly define $\mathcal{A}_2^{\mathrm{nr}}(F)$.

Theorem. *There is a unique bijection*

$$\mathcal{L}^{\mathrm{nr}} : \mathcal{G}_2^{\mathrm{nr}}(F) \longrightarrow \mathcal{A}_2^{\mathrm{nr}}(F)$$

such that, for $\rho \in \mathcal{G}_2^{\mathrm{nr}}(F)$ and $\pi = \mathcal{L}^{\mathrm{nr}}(\rho)$, we have

$$\varepsilon(\chi\pi, s, \psi) = \varepsilon(\chi \otimes \rho, s, \psi),$$

for all characters χ of $F^\times$.

Proof. Let $\tau \in \mathcal{G}_2^{\mathrm{nr}}(F)$; we may write $\tau = \chi \otimes \rho$, for a character χ of $F^\times$ and a representation $\rho \in \mathcal{G}_2^{\mathrm{nr}}(F)$ of minimal level. We set $\mathcal{L}^{\mathrm{nr}}(\rho) = \pi_\rho$, in the notation of the preceding paragraphs, and $\mathcal{L}^{\mathrm{nr}}(\tau) = \chi\pi_\rho$. We then have the property

$$\varepsilon(\zeta \otimes \tau, s, \psi) = \varepsilon(\zeta\mathcal{L}^{\mathrm{nr}}(\tau), s, \psi),$$

for all characters ζ of $F^\times$. The map $\mathcal{L}^{\mathrm{nr}}$ is therefore well-defined and injective. We know it is surjective on representations of minimal level. It is therefore surjective. $\square$

16.5. We now write $\mathcal{G}_2^{\mathrm{tr}}(F) = \mathcal{G}_2^0(F) \smallsetminus \mathcal{G}_2^{\mathrm{nr}}(F)$ and $\mathcal{A}_2^{\mathrm{tr}}(F) = \mathcal{A}_2^0(F) \smallsetminus \mathcal{A}_2^{\mathrm{nr}}(F)$. (We call the elements of these sets "totally ramified".) We construct the Langlands correspondence $\mathcal{G}_2^{\mathrm{tr}}(F) \to \mathcal{A}_2^{\mathrm{tr}}(F)$ in the case $p \neq 2$.

The sequence of events parallels that of the first part of (16.3). We take $\rho \in \mathcal{G}_2^{\mathrm{tr}}(F)$, of minimal level n. Thus $n \geqslant 1$, and $n = 2m+1$ is odd. We write

$\rho = \mathrm{Ind}_{E/F}\,\xi$, for a ramified quadratic extension E/F and a character ξ of $E^\times$, which is of minimal level n. We choose $\alpha \in E$ such that

$$\xi(1{+}x) = \psi(\mathrm{Tr}_{E/F}(\alpha x)),$$

for $x \in \mathfrak{p}_E^{m+1}$.

We view E as embedded in $A = \mathrm{M}_2(F)$, and we let $\mathfrak{A}$ be the unique hereditary order in A which is stable under conjugation by $E^\times$.

We define $J = E^\times U_{\mathfrak{A}}^{m+1}$, and we define a character Λ_0 of J by

$$\Lambda_0 \mid E^\times = \xi,$$
$$\Lambda_0(1{+}x) = \psi(\mathrm{tr}(\alpha x)), \quad 1{+}x \in U_{\mathfrak{A}}^{m+1}.$$

Lemma. *There is a unique character χ_2 of $E^\times$ with the following properties:*

(1) $\chi_2 \mid U_E^1 = 1$ *and* $\chi_2 \mid F^\times = \delta_{E/F}$*;*

(2) *if we extend χ_2 to a character of J trivial on $U_E^1 U_{\mathfrak{A}}^{m+1}$, and put $\Lambda = \chi_2 \otimes \Lambda_0$, $\pi_\rho = c$-$\mathrm{Ind}\,\Lambda$, then*

$$\varepsilon(\pi_\rho, s, \psi) = \varepsilon(\rho, s, \psi).$$

This again reduces to a simple Gauss sum calculation. One can specify χ_2 explicitly: it depends only on the coset $\alpha U_{\mathfrak{A}}^1$. We then get:

Proposition. *With the preceding hypotheses, we have $\pi_\rho \in \mathcal{A}_2^0(F)$, and π_ρ is totally ramified.*

The map $\rho \mapsto \pi_\rho$ gives a canonical bijection between the set of totally ramified $\rho \in \mathcal{G}_2^0(F)$ of minimal level and the set of totally ramified $\pi \in \mathcal{A}_2^0(F)$ of minimal level, with the property

$$\omega_{\pi_\rho} = \det \rho, \qquad \varepsilon(\pi_\rho, s, \psi) = \varepsilon(\rho, s, \psi).$$

Moreover,

$$\varepsilon(\chi\pi_\rho, s, \psi) = \varepsilon(\chi \otimes \rho, s, \psi),$$

for all characters χ of $F^\times$.

16.6. We can now extend the map $\rho \mapsto \pi_\rho$ on elements $\rho \in \mathcal{G}_2^{\mathrm{tr}}(F)$ of minimal level by setting

$$\mathcal{L}^{\mathrm{tr}}(\chi \otimes \rho) = \chi\pi_\rho.$$

As in 16.4, we see that $\mathcal{L}^{\mathrm{tr}}$ gives a bijection $\mathcal{G}_2^{\mathrm{tr}}(F) \to \mathcal{A}_2^{\mathrm{tr}}(F)$, satisfying the desired local constant relation. We have now proved Theorem 13.1, in the case $p \neq 2$.

17. The dyadic case

We assume, from now on, that $p = 2$. Our treatment will be incomplete.

17.1. We consider the class $\mathcal{G}_2^{\mathrm{tr}}(F)$ of totally ramified, irreducible representations of $\mathcal{W}_F$. If K/F is a finite tamely ramified field extension and $\rho \in \mathcal{G}_2^{\mathrm{tr}}(F)$, the representation ρ_K is irreducible, and lies in $\mathcal{G}_2^{\mathrm{tr}}(K)$. Thus restriction gives a map

$$\begin{aligned} \mathcal{R}_{K/F} : \mathcal{G}_2^{\mathrm{tr}}(F) &\longrightarrow \mathcal{G}_2^{\mathrm{tr}}(K), \\ \rho &\longmapsto \rho_K. \end{aligned}$$

If the Langlands conjecture holds, the correspondence $\mathcal{L}$ induces a bijection $\mathcal{G}_2^{\mathrm{tr}}(F) \to \mathcal{A}_2^{\mathrm{tr}}(F)$, and similarly over K. So, there should exist a map $\ell_{K/F} : \mathcal{A}_2^{\mathrm{tr}}(F) \to \mathcal{A}_2^{\mathrm{tr}}(K)$, corresponding to the restriction map $\mathcal{R}_{K/F}$, so that the diagram

$$\begin{array}{ccc} \mathcal{G}_2^{\mathrm{tr}}(F) & \xrightarrow{\mathcal{L}_F} & \mathcal{A}_2^{\mathrm{tr}}(F) \\ {\scriptstyle \mathcal{R}_{K/F}}\downarrow & & \downarrow{\scriptstyle \ell_{K/F}} \\ \mathcal{G}_2^{\mathrm{tr}}(K) & \xrightarrow[\mathcal{L}_K]{} & \mathcal{A}_2^{\mathrm{tr}}(K) \end{array}$$

commutes.

Using a global method, one can describe the map $\ell_{K/F}$, *without first establishing the existence of the Langlands correspondence.* We now outline the construction.

17.2. The map $\ell_{K/F}$ must surely have the property

$$\ell_{K/F}(\chi\pi) = \chi_K\, \ell_{K/F}(\pi),$$

for a character χ of $F^\times$ where, as usual, $\chi_K = \chi \circ \mathrm{N}_{K/F}$. It is therefore enough to describe $\ell_{K/F}(\pi)$, when π has minimal normalized level.

We therefore take a simple stratum $[\mathfrak{A}, n, \alpha]$ in $A = \mathrm{M}_2(F)$, with $n = 2m+1$ odd and $E = F[\alpha]/F$ totally ramified. We form $J = E^\times U_{\mathfrak{A}}^{m+1}$. We may assume that π contains a character Λ of J such that $\Lambda \mid U_{\mathfrak{A}}^{m+1} = \psi_\alpha$.

Let K/F be a finite, tamely ramified field extension. The algebra $KE = K \otimes E = K[\alpha]$ is a field, which we think of as a sub-algebra of $A_K = \mathrm{M}_2(K)$. There is a unique hereditary $\mathfrak{o}_K$-order $\mathfrak{A}_K$ in A_K which is stable under conjugation by $KE^\times$. We then have a simple stratum $[\mathfrak{A}_K, n_K, \alpha]$, where $n_K = ne(K|F)$ and we identify $\alpha = 1 \otimes \alpha$. We observe that $e(K|F)$ is odd, so $n_K = 2m_K+1$ is odd. We set $J_K = KE^\times U_{\mathfrak{A}_K}^{m_K+1}$, and we define a character Λ_K of J_K as follows. First

$$\Lambda^0 \mid U_{\mathfrak{A}_K}^{m_K+1} = \psi_{A_K,\alpha}.$$

Explicitly, $\psi_{A_K,\alpha}$ is the character of $U_{\mathfrak{A}_K}^{m_K+1}$ given by $1+x \mapsto \psi_K(\mathrm{tr}_{A_K/K}(\alpha x))$. We next define a character Λ' of $E^\times$, by

$$\Lambda'(x) = \Lambda(x)\,\delta_{K/F}(\mathrm{N}_{E/F}(x)), \quad x \in E^\times.$$

We complete our definition by setting

$$\Lambda_K \mid KE^\times = \Lambda' \circ \mathrm{N}_{KE/E}.$$

The pair (J_K, Λ_K) is then an extended simple type in $G_K = \mathrm{GL}_2(K)$. We accordingly define

$$\ell_{K/F}(\pi) = c\text{-}\mathrm{Ind}_{J_K}^{G_K} \Lambda_K.$$

Then certainly $\ell_{K/F}(\pi) \in \mathcal{A}_2^{\mathrm{tr}}(K)$. It is not clear, *a priori*, that this map is well-defined: a more careful definition is required to reveal this. We summarize the main properties:

Proposition.

(1) *Let K/F, L/K be finite, tamely ramified field extensions; then $\ell_{L/F} = \ell_{L/K} \circ \ell_{K/F}$.*

(2) *Let $\pi_1, \pi_2 \in \mathcal{A}_2^{\mathrm{tr}}(F)$, and let K/F be a finite, tamely ramified field extension. We have $\ell_{K/F}(\pi_1) \cong \ell_{K/F}(\pi_2)$ if and only if $\pi_1 \cong \chi\pi_2$, for a character χ of $F^\times$ trivial on $\mathrm{N}_{K/F}(K^\times)$.*

(3) *Let K/F be a finite, tamely ramified extension, and let L/F be the normal closure of K/F. For $\tau \in \mathcal{A}_2^{\mathrm{tr}}(K)$, the following conditions are equivalent:*

  (a) *there exists $\pi \in \mathcal{A}_2^{\mathrm{tr}}(F)$ such that $\tau = \ell_{K/F}(\pi)$;*

  (b) *the representation $\ell_{L/K}(\tau)$ is fixed by $\mathrm{Gal}(L/F)$ and ω_τ factors through $\mathrm{N}_{K/F}$;*

  (c) *there exists $\pi \in \mathcal{A}_2^{\mathrm{tr}}(F)$ such that $\ell_{L/F}(\pi) = \ell_{L/K}(\tau)$.*

17.3. We now assume:

Theorem. *There exists a bijection*

$$\mathcal{L}_F^{\mathrm{im}} : \mathcal{G}_2^{\mathrm{im}}(F) \xrightarrow{\approx} \mathcal{A}_2^{\mathrm{im}}(F)$$

such that

$$\varepsilon(\chi \otimes \rho, s, \psi) = \varepsilon(\mathcal{L}_F^{\mathrm{im}}(\chi \otimes \rho), s, \psi),$$

for all $\rho \in \mathcal{G}_2^{\mathrm{im}}(F)$ and all characters χ of $F^\times$.

We shall comment below on the methods available for constructing this bijection. We have already remarked (§15) that there is at most one such bijection; it has to commute with automorphisms of the base field and satisfy $\mathcal{L}_F^{\mathrm{im}}(\chi \otimes \rho) = \chi\mathcal{L}_F^{\mathrm{im}}(\rho)$.

17.4. We now take a *primitive* representation $\rho \in \mathcal{G}_2^0(F)$, and define its Langlands correspondent $\pi = \mathcal{L}(\rho)$. We know (14.6) that there is a cubic (therefore tamely ramified) extension K/F such that $\rho_K = \rho \mid \mathcal{W}_K$ is imprimitive. We may therefore define $\pi_K = \mathcal{L}_K^{\mathrm{im}}(\rho_K) \in \mathcal{A}_2^{\mathrm{tr}}(K) \cap \mathcal{A}_2^{\mathrm{im}}(K)$.

Lemma. *There is a unique representation $\pi_\rho = \pi \in \mathcal{A}_2^{\mathrm{tr}}(F)$ such that $\pi_K \cong \ell_{K/F}(\pi)$ and $\omega_\pi = \det \rho$.*

Proof. As ω_{π_K} is $\det \rho_K = \det \rho \circ \mathrm{N}_{K/F}$, it certainly factors through $\mathrm{N}_{K/F}$. If the image of $\rho(\mathcal{W}_F)$ in $\mathrm{PGL}_2(\mathbb{C})$ is A_4, the extension K/F is cyclic. Since ρ_K is fixed by $\mathrm{Gal}(K/F)$, the same is true of π_K, whence $\pi_K = \ell_{K/F}(\pi)$, for some $\pi \in \mathcal{A}_2^{\mathrm{tr}}(F)$. There are exactly three choices for π, but only one of these has central character equal to $\det \rho$.

We assume therefore that the image of $\rho(\mathcal{W}_F)$ in $\mathrm{PGL}_2(\mathbb{C})$ is S_4. Thus K/F is non-Galois let L/F be its normal closure. We have $\ell_{L/K}(\pi_K) = \mathcal{L}_L^{\mathrm{im}}(\rho_L) \in \mathcal{A}_2^{\mathrm{tr}}(L)^{\mathrm{Gal}(L/F)}$, so there exists $\pi \in \mathcal{A}_2^{\mathrm{tr}}(F)$ such that $\ell_{K/F}(\pi) = \pi_K$. Since K/F is not Galois, there is only one such π, and $\omega_\pi = \det \rho$. $\square$

Thus $\rho \mapsto \pi_\rho$ gives us a canonical map

$$(\star) \qquad \mathcal{L}_F : \mathcal{G}_2^{\mathrm{tr}}(F) \smallsetminus \mathcal{G}_2^{\mathrm{im}}(F) \longrightarrow \mathcal{A}_2^{\mathrm{tr}}(F) \smallsetminus \mathcal{A}_2^{\mathrm{im}}(F)$$

which is injective. We next show it is surjective.

Lemma. *Let $\pi \in \mathcal{A}_2^{\mathrm{tr}}(F)$ be primitive. There exists a ramified cubic extension, unique up to isomorphism, such that $\ell_{K/F}(\pi)$ is imprimitive.*

(1) *The extension K/F is cyclic if and only if $\ell_{K/F}(\pi)$ is triply imprimitive.*
(2) *Suppose $\ell_{K/F}(\pi)$ is not triply imprimitive, and let L/F be the normal closure of K/F; then $\ell_{L/F}(\pi)$ is triply imprimitive.*

The proof involves a delicate analysis of simple types, and we omit it. Given this result, we can reverse the argument above to prove surjectivity.

17.5. We have now produced a canonical bijection $\mathcal{L} : \mathcal{G}_2^{\mathrm{tr}}(F) \to \mathcal{A}_2^{\mathrm{tr}}(F)$, which commutes with twisting by characters of $F^\times$. We have to show that it has the correct properties with respect to the local constant, on primitive representations. This relies on certain general properties of local constants and Gauss sums. We let $\boldsymbol{\mu}_2(\mathbb{C})$ denote the group of roots of unity in $\mathbb{C}$ of order a power of 2.

Lemma 1. *Let $\rho \in \mathcal{G}_2^{\mathrm{tr}}(F)$.*

(1) *There exists $a_\rho \in F^\times$, uniquely determined modulo U_F^1, such that*

$$\varepsilon(\chi \otimes \rho, s, \psi) = \chi(a_\rho)^{-1}\, \varepsilon(\rho, s, \psi),$$

for all tamely ramified characters χ *of* $F^\times$.

(2) *We have*

$$\varepsilon(\rho, \tfrac{1}{2}, \psi) \equiv \det \rho(a_\rho)^{-1/2} \pmod{\boldsymbol{\mu}_2(\mathbb{C})}.$$

(3) *If* K/F *is a finite, tamely ramified field extension, then*

$$a_{\rho_K} \equiv a_\rho \pmod{U_K^1}.$$

There is an exactly parallel result on the other side.

Lemma 2. *Let* $\pi \in \mathcal{A}_2^{\mathrm{tr}}(F)$.

(1) *There exists* $c_\pi \in F^\times$, *uniquely determined modulo* U_F^1, *such that*

$$\varepsilon(\chi\pi, s, \psi) = \chi(c_\pi)^{-1}\, \varepsilon(\pi, s, \psi),$$

for all tamely ramified characters χ *of* $F^\times$.

(2) *We have*

$$\varepsilon(\pi, \tfrac{1}{2}, \psi) \equiv \omega_\pi(c_\pi)^{-1/2} \pmod{\boldsymbol{\mu}_2(\mathbb{C})}.$$

(3) *If* K/F *is a finite, tamely ramified field extension, then*

$$c_{\pi_K} \equiv c_\pi \pmod{U_K^1},$$

where $\pi_K = \ell_{K/F}(\pi)$.

These lemmas allow us to express cleanly the properties of local constants relative to restriction and tame lifting:

Proposition. *Let* $\rho \in \mathcal{G}_2^{\mathrm{tr}}(F)$ *and* $\pi \in \mathcal{A}_2^{\mathrm{tr}}(F)$. *Let* K/F *be a tamely ramified field extension of finite degree* d, *and write* $\rho_K = \rho \mid \mathcal{W}_K$, $\pi_K = \ell_{K/F}(\pi)$. *Then*

$$\frac{\varepsilon(\rho, s, \psi)^d}{\varepsilon(\rho_K, s, \psi_K)} = \delta_{K/F}(a_\rho)\lambda_{K/F}^2$$
$$\frac{\varepsilon(\pi, s, \psi)^d}{\varepsilon(\pi_K, s, \psi_K)} = \delta_{K/F}(c_\pi)\lambda_{K/F}^2.$$

17.6. Let $\rho \in \mathcal{G}_2^{\mathrm{tr}}(F)$ be primitive, and set $\pi = \mathcal{L}_F(\rho)$. Let K/F be cubic, so that ρ_K is imprimitive. We thus have $\ell_{K/F}(\pi) = \mathcal{L}_K(\rho_K)$: we abbreviate $\pi_K = \ell_{K/F}(\pi)$. We have

$$\varepsilon(\eta \otimes \rho_K, s, \psi_K) = \varepsilon(\eta\pi_K, s, \psi_K),$$

for all characters η of $K^\times$. It follows that $c_\pi \equiv a_\rho \pmod{U_F^1}$. We deduce that

$$\varepsilon(\rho, \tfrac{1}{2}, \psi) \equiv \varepsilon(\pi, \tfrac{1}{2}, \psi) \pmod{\boldsymbol{\mu}_2(\mathbb{C})}.$$

On the other hand,

$$\varepsilon(\rho, s, \psi)^3 = \varepsilon(\pi, s, \psi)^3.$$

The powers of $(\tfrac{1}{2}-s)$ occurring in the two local constants are therefore the same and so, finally, $\varepsilon(\rho, s, \psi) = \varepsilon(\pi, s, \psi)$.

17.7. It remains only to comment on the construction of the map $\mathcal{L}_F^{\mathrm{im}}$. There are basically two approaches.

Let E/F be a separable quadratic extension, $\mathrm{Gal}(E/F) = \{1, \sigma\}$, and let κ be the unique non-trivial character of $F^\times/\mathrm{N}_{E/F}(E^\times)$. Let $\mathcal{G}_1(E)$ be the group of characters of $E^\times$, and let $\mathcal{G}_2^{\mathrm{ss}}(F)$ be the set of equivalence classes of two-dimensional, smooth, *semisimple* representations of $\mathcal{W}_F$. Induction thus gives a map $\mathcal{I}_{E/F} : \mathcal{G}_1(E) \to \mathcal{G}_2^{\mathrm{ss}}(F)$. The key properties are the obvious ones: the representation $\mathcal{I}(\xi)$ is irreducible if and only if $\xi \neq \xi^\sigma$, and an irreducible representation $\rho \in \mathcal{G}_2^0(F)$ lies in the image of $\mathcal{I}$ if and only if $\kappa \otimes \rho \cong \rho$.

Using a global method, one can construct an analogous map $A_{E/F} : \mathcal{G}_1(F) \to \mathcal{A}_2(F)$, called *automorphic induction.* It has similar properties: the representation $A(\xi)$ is cuspidal if and only if $\xi \neq \xi^\sigma$, while a cuspidal representation $\pi \in \mathcal{A}_2^0(F)$ lies in the image of $A_{E/F}$ if and only if $\kappa\pi \cong \pi$.

Automorphic induction is expressed via a relation on characters which, in the case $p = 2$, is difficult to understand in local terms. However, it gives the Langlands correspondence straightaway on imprimitive representations: if $\rho = \mathrm{Ind}_{E/F}\,\xi \in \mathcal{A}_2^{\mathrm{im}}(F)$, we put $\mathcal{L}_F^{\mathrm{im}}(\rho) = A_{E/F}(\xi)$. The compatibility of automorphic induction with the tame lifting map also requires global arguments.

One can construct $\mathcal{L}_F^{\mathrm{im}}(\rho)$ in purely local terms, using a technique called the "Weil representation". This gives a bijection, with the right properties relative to local constants, but it is not easy to express in terms of the classification we have used. It is not so eay to show that it is compatible with tame lifting. The Weil representation technique also relies on a rather different approach to the representation theory of $\mathrm{GL}_2(F)$, which is not as effective as the one we have used.