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RATIONAL CURVES ON K3 SURFACES

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1. BASICS

1.1. A K3 surface X is a compact Kähler surface which is simply connected and has trivial canonical bundle. A quick computation gives the Hodge numbers:

$$(1.1) \quad \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Here $H^0(X, \mathbb{Z}) = \mathbb{Z}$, $H^1(X, \mathbb{Z}) = 0$ and $H^{2,0}(X) = H^0(K_X) = \mathbb{C}$ follow directly from the definition, while $h^{1,1}(X) = 20$ follows from Noether's formula:

$$(1.2) \quad \chi_{top}(X) = 12(K_X^2 + \chi(\mathcal{O}_X)) = 24$$

So $H^2(X) = \mathbb{C}^{22}$. A subtle point here is that $H^2(X, \mathbb{Z}) = \mathbb{Z}^{22}$ is torsion free. This follows from Lefschetz (1, 1) theorem and Riemann-Roch.

Proposition 1.1. $H^2(X, \mathbb{Z})$ is torsion free for a K3 surface X .

Proof. By Lefschetz (1, 1) theorem, every torsion element of $H^2(X, \mathbb{Z})$ lies in the image of $\text{Pic}(X) \rightarrow H^2(X, \mathbb{Z})$. If $H^2(X, \mathbb{Z})$ is not torsion free, there exists a line bundle L such that $L \neq \mathcal{O}_X$ and $L^{\otimes m} = \mathcal{O}_X$ for some $m > 1$. By Riemann-Roch,

$$(1.3) \quad h^0(L) - h^1(L) + h^0(L^{-1}) = 2$$

Therefore $h^0(L) + h^0(L^{-1}) \geq 2$. So at least one of L and L^{-1} is effective. WLOG, assume that $h^0(L) > 0$. Let $s \in H^0(L)$. Then $s^m \in H^0(\mathcal{O}_X)$. Consequent, s nowhere vanishes and hence $L = \mathcal{O}_X$. Contradiction. \square

There are (at least) two conclusions we can draw from $h^{1,1}(X) = 20$. First, since $H^1(X, \mathcal{O}_X) = 0$, we see that

Proposition 1.2. $\text{Pic}(X)$ is a lattice of rank at most 20 contained $H^2(X, \mathbb{Z}) = \mathbb{Z}^{22}$.

Second, by Serre duality

$$(1.4) \quad H^{1,1}(X) = H^1(\Omega_X) = H^1(T_X)^\vee = \mathbb{C}^{20}$$

This, along with $H^2(T_X) = H^0(\Omega_X)^\vee = 0$, implies that the versal deformation space of X is smooth of dimension 20. That is, the moduli space of K3

surfaces, if exists, has dimension 20. In case that X is projective, a general deformation of X is, however, no longer algebraic.

1.2. Examples. The simplest examples of K3 surfaces are complete intersections in \mathbb{P}^n . Let $X \subset \mathbb{P}^n$ be a complete intersection cut out by hypersurfaces of degrees d_1, d_2, \dots, d_{n-2} . By weak Lefschetz, $H^1(X, \mathbb{Z}) = 0$. By adjunction,

$$(1.5) \quad K_X = \mathcal{O}_X \Leftrightarrow d_1 + d_2 + \dots + d_{n-2} = n + 1$$

We also require that X be nondegenerate, i.e., $d_i \geq 2$. Therefore, here are all the possibilities:

- (1) $X \subset \mathbb{P}^3$ a quartic surface;
- (2) $X \subset \mathbb{P}^4$ a complete intersection of a quadric and a cubic;
- (3) $X \subset \mathbb{P}^5$ a complete intersection of three quadrics.

Theorem 1.3 (Noether-Lefschetz). *For a very general surface $X \subset \mathbb{P}^3$ of degree $d \geq 4$, $\text{Pic}(X) \cong \mathbb{Z}$ is generated by the hyperplane section $\mathcal{O}_X(1)$.*

A consequence of this theorem is

Corollary 1.4. *Let X_1 and X_2 be two very general quartic surfaces and let $f : X_1 \rightarrow X_2$ be an isomorphism. Then f is induced by an action of $\mathbb{P}GL(4)$.*

Proof. f induces an isomorphism $\text{Pic}(X_2) \rightarrow \text{Pic}(X_1)$. Obviously, $f^*\mathcal{O}_{X_2}(1) = \mathcal{O}_{X_1}(1)$ and it also induces a linear map $|\mathcal{O}_{X_2}(1)| \rightarrow |\mathcal{O}_{X_1}(1)|$. Both $|\mathcal{O}_{X_i}(1)| \cong |\mathcal{O}_{\mathbb{P}^3}(1)| = \mathbb{P}^3$. Therefore, f induces an automorphism of \mathbb{P}^3 , say $\sigma \in \mathbb{P}GL(4)$. In return, it is easy to see that σ induces f . \square

Now we can compute the dimension of the moduli space of quartic surfaces, if it exists

$$(1.6) \quad \dim \mathcal{M} = \dim(|\mathcal{O}_{\mathbb{P}^3}(4)| / \sim) = \binom{7}{3} - 1 - \dim \mathbb{P}GL(4) = 19$$

Similarly, we can compute the dimension of the space of the complete intersections $X = Q \cap C \subset \mathbb{P}^4$ of type $(2, 3)$:

$$(1.7) \quad \begin{aligned} & \dim |\mathcal{O}_{\mathbb{P}^4}(2)| + \dim |\mathcal{O}_{\mathbb{P}^4}(3)| - \dim |I_X(2)| - \dim |I_X(3)| \\ &= \binom{6}{4} + \binom{7}{4} - 1 - (5 + 1) = 43 \end{aligned}$$

where I_X is the ideal sheaf of X and $h^0(I_X(d))$ can be computed via Kozul complex:

$$(1.8) \quad 0 \rightarrow \mathcal{O}_{\mathbb{P}^4}(-5) \rightarrow \mathcal{O}_{\mathbb{P}^4}(-2) \oplus \mathcal{O}_{\mathbb{P}^4}(-3) \rightarrow I_X \rightarrow 0$$

Again we can show that the isomorphism between two very general complete intersections of such type is induced by $\mathbb{P}GL(5)$. Hence the moduli space of $X = Q \cap C$ has dimension $43 - 24 = 19$.

Exercise 1.5. Compute the dimension of the moduli space of the complete intersection $X \subset \mathbb{P}^5$ of type $(2, 2, 2)$.

Here is another example. Let $\pi : X \rightarrow \mathbb{P}^2$ be the double cover of \mathbb{P}^2 ramified over a smooth sextic curve $D \subset \mathbb{P}^2$. To see that X is a K3 surface, we first prove

Proposition 1.6. *Let X_0 be a smooth hypersurface of $\mathbb{P}^1 \times \mathbb{P}^2$ of type $(2, 3)$. Then the projection $X_0 \rightarrow \mathbb{P}^2$ is a double cover ramified along a sextic curve.*

By weak Lefschetz and adjunction, X_0 is a K3 surface. Obviously, every double cover X of \mathbb{P}^2 ramified along a smooth sextic curve can be deformed to an $X_0 \subset \mathbb{P}^1 \times \mathbb{P}^2$ of type $(2, 3)$. Hence X is a K3 surface.

The dimension of the moduli space of such X is the same as the dimension of the moduli space of sextic curves:

$$(1.9) \quad \dim |\mathcal{O}_{\mathbb{P}^2}(6)| - \dim \mathbb{P}GL(3) = 27 - 8 = 19.$$

Let $A = \mathbb{C}^2/\Lambda$ be a 2-dimensional complex torus. We have \mathbb{Z}_2 action on A by sending $(x, y) \rightarrow (-x, -y)$. Let $X = A/\mathbb{Z}_2$ and \tilde{X} be the minimal resolution of X . Then \tilde{X} is a special K3 surface called Kummer surface.

Proposition 1.7. *Let $G = \mathbb{Z}_2$ act on \mathbb{A}_{xy}^2 by sending $(x, y) \rightarrow (-x, -y)$. Then \mathbb{A}^2/G is a hypersurface in \mathbb{A}_{uvw}^3 given by $uv = w^2$.*

Proof. $\mathbb{A}^2/G = \text{Spec } k[x, y]^G = \text{Spec } k[x^2, y^2, xy] = \text{Spec } k[u, v, w]/(uv - w^2)$. \square

The action \mathbb{Z}_2 on A has sixteen fixed points. Therefore, X has sixteen singularities where X is locally given by $\text{Spec } \mathbb{C}[u, v, w]/(uv - w^2)$, i.e., X has sixteen rational double points.

Proposition 1.8. *Let $X = (xy = z^2) \subset Y = \mathbb{A}_{xyz}^3$ and let $\pi : \tilde{Y} \rightarrow Y$ be the blowup of Y at the origin p . Let \tilde{X} be the proper transform of X under π and E_X be the exceptional divisor of $\pi : \tilde{X} \rightarrow X$. Then \tilde{X} is smooth, E_X is a smooth rational curve with $E_X^2 = -2$ on \tilde{X} and $K_{\tilde{X}} = \pi^*K_X$.*

Proof. $\tilde{Y} \subset \mathbb{A}^3 \times \mathbb{P}^2$ is given by $x/X = y/Y = z/Z$ and \tilde{X} is by

$$(1.10) \quad \frac{x}{X} = \frac{y}{Y} = \frac{z}{Z} \text{ and } XY = Z^2$$

It is straightforward to check that \tilde{X} is smooth. The exceptional divisor is a smooth conic curve $XY = Z^2$ in \mathbb{P}^2 . Let E_Y be the exceptional divisor of $\tilde{Y} \rightarrow Y$. By

$$(1.11) \quad \pi^*X = \tilde{X} + 2E_Y \Rightarrow \pi^*X \cdot E_Y^2 = \tilde{X} \cdot E_Y^2 + 2E_Y^3 \Rightarrow \tilde{X} \cdot E_Y^2 = -2E_Y^3$$

we see that $E_X^2 = \tilde{X} \cdot E_Y^2 = -2$. Since

$$(1.12) \quad K_{\tilde{Y}} = \pi^*K_Y + 2E_Y,$$

$$(1.13) \quad K_{\tilde{X}} = (\pi^*K_Y + 2E_Y + \tilde{X})|_{\tilde{X}} = \pi^*(K_Y + X)|_{\tilde{X}} = \pi^*K_X$$

□

So we see that \tilde{X} contains sixteen (-2) -curves. And since $\nu^*K_X = K_{\tilde{A}}$ and $\pi^*K_X = K_{\tilde{X}}$, $K_{\tilde{X}}$ is trivial. By classification of complex surfaces, \tilde{X} can be either a K3 surface or abelian surface. Since an abelian surface does not contain any rational curves, \tilde{X} must be a K3 surface.

1.3. Deformation of K3 surfaces. As we pointed out in the examples of K3 surfaces as complete intersections in \mathbb{P}^n or double covers of \mathbb{P}^2 , the corresponding moduli space of K3 surfaces has dimension 19, while the versal deformation of a K3 surface has dimension 20. So where does the extra dimension go? The answer is that a general deformation of a K3 surface is not algebraic; 19 is the dimension of the deformation of a polarized K3. Let us illustrate this using the example of quartic surfaces.

Let $X \subset \mathbb{P}^3$ be a smooth quartic surface. We have the exact sequence

$$(1.14) \quad 0 \rightarrow T_X \rightarrow T_{\mathbb{P}^3}|_X \rightarrow N_X \rightarrow 0$$

where N_X is the normal bundle of X . The induced long exact sequence is

$$(1.15) \quad H^0(N_X) \rightarrow H^1(T_X) \xrightarrow{\beta} H^1(T_{\mathbb{P}^3}|_X)$$

By Euler sequence

$$(1.16) \quad 0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X(1)^{\oplus 4} \rightarrow T_{\mathbb{P}^3}|_X \rightarrow 0$$

we see that $H^1(T_{\mathbb{P}^3}|_X) \cong H^2(\mathcal{O}_X) = \mathbb{C}$. Consider the dual map β^\vee of β :

$$(1.17) \quad \begin{array}{ccc} H^1(T_X) & \xrightarrow{\beta} & H^1(T_{\mathbb{P}^3}|_X) \\ \times & & \times \\ H^1(\Omega_X) & \xleftarrow{\beta^\vee} & H^1(\Omega_{\mathbb{P}^3}|_X) \\ \downarrow & & \downarrow \\ \mathbb{C} & & \mathbb{C} \end{array}$$

Since $H^1(\Omega_{\mathbb{P}^3}|_X) = H^1(\Omega_{\mathbb{P}^3}) = \mathbb{C}$ is generated by $c_1(L)$,

$$(1.18) \quad \text{Im } \beta^\vee = \{\lambda c_1(L) : \lambda \in \mathbb{C}\}$$

where L is the hyperplane bundle. Consequently,

$$(1.19) \quad \begin{aligned} \text{Im}(H^0(N_X) \rightarrow H^1(T_X)) &= \ker \beta \\ &= \{\varepsilon \in H^1(T_X) : \langle \varepsilon, c_1(L) \rangle = 0\} = c_1(L)^\perp \end{aligned}$$

By deformation theory, $H^0(N_X)$ classifies embedded deformations of $X \subset \mathbb{P}^3$ and $H^1(T_X)$ classifies the deformations of X as a complex manifold. So the image $\text{Im}(H^0(N_X) \rightarrow H^1(T_X))$ classifies the deformations of the pair (X, L) , i.e., a polarized K3 surface. The above argument actually applies to any polarized K3 surface, not only quartic surfaces. Therefore, the versal deformation space of a polarized K3 surface (X, L) is a hyperplane in $H^1(T_X)$ that is perpendicular to $c_1(L)$.

The above technique can be generalized to prove the following:

Proposition 1.9. *Let X/Δ be a family of smooth projective surfaces over disk Δ with central fiber $S = X_0$ and let $D \subset S$ be an effective divisor on S . Suppose that D can be extended to X , i.e., there exists a flat family Y/Δ with the commutative diagram*

$$(1.20) \quad \begin{array}{ccc} Y & \xrightarrow{\pi} & X \\ \downarrow & & \downarrow \\ \Delta & \longrightarrow & \Delta \end{array}$$

such that Y_0 embeds into X_0 with image D . For each $w \in H^0(K_S)$, let μ_w be the map

$$(1.21) \quad \mu_w : H^1(\Omega_S) \xrightarrow{\otimes w} H^1(\Omega_S(K_S))$$

where K_S is the canonical class of S . Then the Kodaira-Spencer class $\text{ks}(\partial/\partial t) \in H^1(T_S)$ of X lies in the subspace

$$(1.22) \quad \{v \in H^1(T_S) : \langle v, \mu_w(c_1(D)) \rangle = 0 \text{ for all } w \in H^0(K_S)\}$$

where $\langle \cdot, \cdot \rangle$ is the pairing $H^1(T_S) \times H^1(\Omega_S(K_S)) \rightarrow \mathbb{C}$ given by Serre duality.

Using the above proposition, we can give a proof of Noether-Lefschetz for quartic surfaces.

Let $M = |\mathcal{O}_{\mathbb{P}^3}(4)|$ and $S = \{(p, X) : p \in X\} \subset \mathbb{P}^3 \times M$. Let $\mathcal{L} = \pi_1^* L$ be the pullback of the hyperplane bundle of \mathbb{P}^3 . If for a very general X , there is a line bundle $D \in \text{Pic}(X)$ such that D is not a multiple of L , then after a possible base change of M , there exists a line bundle \mathcal{D} on S such that \mathcal{D}_X is not a multiple of \mathcal{L}_X when restricted to a very general point $[X] \in M$. Then by the above proposition, the image of

$$(1.23) \quad T_{M, [X]} \xrightarrow{\text{ks}} H^0(N_X) \rightarrow H^1(T_X)$$

is contained in $c_1(D)^\perp$. Also $\text{Im}(H^0(N_X) \rightarrow H^1(T_X))$ is $c_1(L)^\perp$ and ks is obviously surjective. Therefore, we necessarily have

$$(1.24) \quad c_1(L)^\perp = c_1(D)^\perp$$

That is, $c_1(L)$ and $c_1(D)$ are linearly dependent over \mathbb{Q} . Therefore, $\text{Pic}(X) = \mathbb{Z}$. Let J be a generator of $\text{Pic}(X)$. Then $L = mJ$ for some $m \in \mathbb{Z}$. WLOG, assume that $m > 0$. Since $L^2 = 4$, $m = 1$ or $m = 2$. We are done if $m = 1$. If $m = 2$, $J^2 = 1$ and $(K + J)J = 1$. This is impossible by Riemann-Roch.

Exercise 1.10. Let M be the moduli space of the tuple $(X, L_1, L_2, \dots, L_m)$, where X is a K3 surface, $\{L_k\}$ are m linearly independent line bundles on X and L_1 is ample. Then $\dim M \leq 20 - m$.

So far we reach the conclusion that the moduli space of polarized K3 surfaces (X, L) has dimension at most 19. Then how many components does this moduli space have?

We call a polarized K3 surface (X, L) a primitive K3 surface if there does not exist $D \in \text{Pic}(X)$ and $m > 1$ such that $L = mD$ and in this case, we say L is a primitive line bundle over X . Let $C \in |L|$. Then $2p_a(C) - 2 = L^2$. This number $g = p_a(C)$ is called the genus of X . By a K3 surface of genus g , we mean a polarized K3 surface (X, L) with a primitive line bundle L and $L^2 = 2g - 2$.

Theorem 1.11. *For each $g \geq 2$, there exists a moduli space \mathcal{M}_g parameterizing genus g K3 surfaces; \mathcal{M}_g is quasi-projective, smooth and irreducible of dimension 19.*

Genus 2 K3 surfaces are double covers of \mathbb{P}^2 ramified along a smooth sextic curves. Genus 3 K3 surfaces are quartic surfaces in \mathbb{P}^3 . Genus 4 K3 surfaces are complete intersections in \mathbb{P}^4 of type $(2, 3)$. Genus 5 K3 surfaces are complete intersections in \mathbb{P}^5 of type $(2, 2, 2)$. Here I will give an elementary proof of existence of K3 of any genus g .

Proposition 1.12. *For every $g \geq 2$, there exists a K3 surface X of genus g and $\text{Pic}(X) = \mathbb{Z}$.*

Proof. Let X be a smooth surface in $\mathbb{P}^1 \times \mathbb{P}^2$ of type $(2, 3)$. We embed $X \hookrightarrow \mathbb{P}^{3k+2}$ by the very ample linear series $|\pi_1^* \mathcal{O}_{\mathbb{P}^1}(k) \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^2}(1)|$ with $k > 0$. In the exact sequence

$$(1.25) \quad H^1(T_X) \rightarrow H^1(T_{\mathbb{P}^g}|_X) \rightarrow H^1(N_X) \rightarrow H^2(T_X)$$

we have already seen that $H^1(T_X) \rightarrow H^1(T_{\mathbb{P}^g}|_X)$ is surjective, where $g = 3k+2$. And since $H^2(T_X) = 0$, $H^2(N_X) = 0$ and the embedded deformations of $X \subset \mathbb{P}^g$ are unobstructed. Therefore, there exists a flat family $Y \subset \mathbb{P}^g \times \Delta^m$ such that $Y_0 = X \subset \mathbb{P}^g$ and the Kodaira-Spencer map $T_{\Delta^m, 0} \rightarrow H^0(N_X)$ is an isomorphism. We have proved that $\text{Im}(H^0(N_X) \rightarrow H^1(T_X))$ is $c_1(L)^\perp$, where $L = \pi_1^* \mathcal{O}_{\mathbb{P}^1}(k) \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^2}(1)$. For a general fiber Y_t of $Y \rightarrow \Delta^m$, $\text{Pic}(Y_t) = \mathbb{Z}$ is generated by L .

So this proves the proposition when $g = 3k + 2$. For $g = 3k, 3k + 1$, see the following exercise. \square

Exercise 1.13. Let $E = \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(1)$ be a rank three vector bundle over \mathbb{P}^1 and $Y = \mathbb{P}E$. Let $X \in |-K_Y|$ be a smooth anti-canonical surface in $\mathbb{P}E$. Show that the complete linear series $|\mathcal{O}_{\mathbb{P}E}(1) \otimes \pi^* \mathcal{O}_{\mathbb{P}^1}(k)|$ ($k \geq 1$) embeds X into \mathbb{P}^{3k} .

Change E to $E = \mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(1)$ and do the same thing.

2. RATIONAL CURVES ON K3

2.1. Existence. First of all, we have

Proposition 2.1. *There are at most countably many rational curves on a K3 surface X .*

Proof. Otherwise, X is covered by rational curves, i.e., X is uniruled. There exists a dominant rational map $\mathbb{P}^1 \times \Gamma \rightarrow X$, where Γ is a smooth projective curve. This rational map can be resolved by a sequence of blowups. Let $f : Y \rightarrow \mathbb{P}^1 \times \Gamma \rightarrow X$ be such a resolution. So Y is a fibration over Γ whose general fibers are \mathbb{P}^1 . Since f is surjective, we have the injection

$$(2.1) \quad f^* : H^0(K_X) \hookrightarrow H^0(K_Y).$$

So K_Y is effective since K_X is. Let $Y_p = C$ be a general fiber of $Y \rightarrow \Gamma$. Then

$$(2.2) \quad K_Y|_C = K_C = -2$$

Yet $K_Y|_C = K_Y \cdot C \geq 0$ since K_Y is effective. Contradiction. \square

Exercise 2.2. Let X be a smooth projective variety satisfying that mK_X is effective for some $m > 0$. Show that X is not uniruled.

Yet the existence of rational curves are more subtle. The existence of rational curves on K3 surfaces was established by S. Mori and S. Mukai. I made it more precise:

Theorem 2.3 (Chen). *For any integers $n \geq 3$ and $d > 0$, the linear system $|\mathcal{O}_S(d)|$ on a general K3 surface S in \mathbb{P}^n contains an irreducible nodal rational curve.*

The idea for the proof is to degenerate a K3 surface to a union of rational surfaces. It is best illustrated by quartic surfaces.

Let us consider $X_0 = Q_1 \cup Q_2 \subset \mathbb{P}^3$ be a union of two quadrics. This is a “special” quartic surface and any smooth quartics can be degenerated to it. It is a common knowledge that $Q_i \cong \mathbb{P}^1 \times \mathbb{P}^1$ is embedded into \mathbb{P}^3 by $|H_1 + H_2|$, where H_i are two rulings of Q_i . Let $E = Q_1 \cap Q_2$. Then E is an elliptic curve in the linear series $|2H_1 + 2H_2|$. In addition we have

$$(2.3) \quad \mathcal{O}_{Q_1}(H_1 + H_2)|_E = \mathcal{O}_{Q_2}(H_1 + H_2)|_E$$

Let $X \subset \mathbb{P}^3 \times \Delta$ be a pencil of quartics whose central fiber is X_0 . So the defining equation of X looks like

$$(2.4) \quad F_{Q_1}F_{Q_2} + tF = 0$$

where F_{Q_i} are the defining equations of Q_i . We choose X to be general enough. The idea is to find a curve $Y_0 \in |\mathcal{O}_{X_0}(d)|$ and show that Y_0 can be deformed to a nodal rational curve $Y_t \in |\mathcal{O}_{X_t}(d)|$.

For example, let us work out the case $d = 1$. Obviously, there exists $r \in E$ such that

$$(2.5) \quad \mathcal{O}_E(H_1 + H_2) = \mathcal{O}_E(4r)$$

Actually there are exactly 16 such points. There exists a unique curve $C_i \in |\mathcal{O}_{Q_i}(H_1 + H_2)|$ such that $C_i \cdot E = 4r$. This is due to the fact

$$(2.6) \quad H^0(\mathcal{O}_{Q_i}(H_1 + H_2)) = H^0(\mathcal{O}_E(H_1 + H_2))$$

Also for E general, C_i is irreducible and smooth.

Let $U^{d,\delta}(S)$ be the subset of $|\mathcal{O}_S(d)|$ consisting of irreducible nodal curves with δ nodes on a quartic surface S . Let

$$(2.7) \quad W_{d,\delta} = \bigcup_{t \neq 0} U^{d,\delta}(X_t) \subset |\mathcal{O}_X(d)|$$

and let $\overline{W}_{d,\delta}$ be the closure of $W_{d,\delta}$ in $|\mathcal{O}_X(d)|$

A theorem of Caporaso-Harris-Ran shows that

Proposition 2.4. *The following are true:*

- (1) $[C_1 \cup C_2] \in \overline{W}_{1,3}$;
- (2) $\overline{W}_{1,3}$ has an ordinary singularity of multiplicity 4 at $[C_1 \cup C_2]$;
- (3) for any open neighborhood O_r of $r \in \mathbb{P}^3$, there exists an open neighborhood $V_{[C_1 \cup C_2]}$ of $[C_1 \cup C_2] \in \overline{W}_{1,3}$ such that for any $[C] \in V_{[C_1 \cup C_2]}$, the nodes of C lies in O_r .

From the above proposition, we see that $U^{1,3}(X_t)$ is nonempty for $t \neq 0$. So there exists an irreducible curve Y_t with 3 nodes in $|\mathcal{O}_{X_t}(1)|$. This curve Y_t is obviously a rational curve.

For $d \geq 2$, a slight different construction is needed but the basic idea is the same. For example, let us work out the case $d = 2$.

The threefold X has sixteen rational double points lying on E . Let p be one of them. We let $Y_0 = C_{11} \cup C_{12} \cup C_{21} \cup C_{22}$ with

- (1) $C_{i1} \in |\mathcal{O}_{Q_i}(H_1)|$ and $C_{i2} \in |\mathcal{O}_{Q_i}(H_1 + 2H_2)|$;
- (2) $C_{i1} \cdot E = p + q_i$ and $C_{i2} \cdot E = q_{3-i} + 5r$

where q_1, q_2, r are determined by p up to 25 different choices.

Again we can show there is a flat family $Y \subset X$ of nodal curves after a base change such that $Y_t \in |\mathcal{O}_{X_t}(2)|$ has 9 nodes, with 4 of them approaching r , 1 of them approaching p , 2 of them approaching $C_{11} \cap C_{12}$ and 2 of them approaching $C_{21} \cap C_{22}$ as $t \rightarrow 0$. Obviously, Y_t is a rational curve. For details, please see [C1].

2.2. Counting rational curves. The next natural question following the existence problem is how many irreducible rational curves there are in $|\mathcal{O}(d)|$ on a general K3 surface in \mathbb{P}^n . The number for $d = 1$ has been successfully calculated in [Y-Z]. They give the following remarkable formula

$$(2.8) \quad \sum_{g=1}^{\infty} n(g)q^g = \frac{q}{\Delta(q)}$$

where $\Delta(q) = q \prod_{n=1}^{\infty} (1 - q^n)^{24}$ is the well-known modular form of weight 12 and $n(g)$ is the nominated number of rational curves in $|\mathcal{O}(1)|$ on a general K3 surface in \mathbb{P}^g for $g \geq 3$. More precisely, $n(g)$ is the sum of the Euler characteristics of the compactified Jacobians of all rational curves in $|\mathcal{O}(1)|$. Since the compactified Jacobian of a rational curve with singularities other than nodes is not very well understood, we only know this sum equals the

number of rational curves in $|\mathcal{O}(1)|$ on a K3 surface in the case that all these rational curves are nodal.

Later J. Bryan and N.C. Leung redid and generalized Yau-Zaslow's counting via a different approach. Basically, they used a degeneration argument by degenerating a general K3 surface to a K3 surface S of Picard lattice

$$(2.9) \quad \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let C and F be the generators of $\text{Pic}(S)$ with $C^2 = -2$, $C \cdot F = 1$ and $F^2 = 0$. We also assume that F is effective.

The good thing about S is that $(S, C + gF)$ is a K3 surface of genus g and every member of the linear series $|C + gF|$.

Exercise 2.5. Show that $h^0(C) = 1$, $h^0(F) = 2$ and the map $\pi : S \rightarrow \mathbb{P}^1$ given by $|F|$ realizes S as an elliptic fibration. For S general, there are exactly 24 singular fibers of π .

Exercise 2.6. Show that every curve in $D \in |C + gF|$ is a union $C \cup F_1 \cup \dots \cup F_g$ with $F_i \in |F|$.

Using this degeneration, I proved the following theorem:

Theorem 2.7. *All rational curves in the primitive class of a general K3 surface of genus $g \geq 2$ are nodal.*

This justifies the number obtained by Yau-Zaslow is the number of rational curves.

Question 2.8. *Compute the number of rational curves in $|\mathcal{O}_S(d)|$.*

2.3. Hodge-D-Conjecture. As another application of rational curves on K3, J. Lewis and I proved the following theorem, originally a conjecture of Beilinson:

Theorem 2.9 (Chen, Lewis). *Hodge-D conjecture holds for a general K3 surface X (general under the real analytic topology). That is, the regulator map*

$$(2.10) \quad r_{2,1} : \text{CH}^{2,1}(X) \rightarrow H^{1,1}(X, \mathbb{R})$$

is surjective.

Here the rational curves are used to construct nontrivial classes in $\text{CH}^{2,1}(X)$. By definition,

$$(2.11) \quad \text{CH}^k(X, 1) = \frac{\left\{ \sum_j (f_j, Z_j) : \begin{array}{l} \text{cd}_X Z_j = k - 1, \ f_j \in \mathbb{C}(Z_j)^\times \\ \sum_j \text{div}(f_j) = 0 \end{array} \right\}}{\text{Image}(\text{Tame symbol})}.$$

Choose two rational curves C_1 and $C_2 \subset X$. Suppose that there are two points $p, q \in C_1 \cap C_2$. Then there exists two rational functions f_1 and f_2 on C_1 and C_2 , respectively, such that $(f_1) = p - q$ and $(f_2) = q - p$. Then $(f_1, C_1) + (f_2, C_2)$ is a class in $\text{CH}^{2,1}(X)$.

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