Complex geometry: Its brief history and its future Personal perspective

Shing-Tung Yau

Once complex number is introduced as a field, it is natural to consider functions depending only on its "pure" holomorphic variable z. As it is independent of $\bar{z}$,

$$\frac{\partial f}{\partial \bar{z}} = 0.$$

There are surprising rich properties of these holomorphic functions.

The possibility of holomorphic continuation of holomorphic functions forces us to consider multi-valued holomorphic functions.

The concept of Riemann Surfaces was introduced to understand such phenomena.

The ideas of branch cuts and branch points immediately relate topology of these surfaces to complex variables.

The possibility of two Riemann surfaces can be homeomorphic to each other without being equal was realized in nineteenth century where remarkable uniformization theorems were proved by Riemann for simply connected surfaces. Although it took Hilbert many years later to make Riemann's work on variational principle to be rigorous, the Dirichlet principle of constructing harmonic functions and hence holomorphic functions has tremendous influence up to modern days.

Koebe finally proved that every abstractly defined simply connected Riemann surface is either the disk, the complex line or the Riemann sphere.

There are proofs based on complex function theory, variational principle and geometric deformation equations.

The uniformization theorem allows one to identify space of complex structures to space of discrete groups of $SL(2, \mathbb{R})$ which acts on the disk by linear fractional transformations.

The problem of how to parametrize all possible Riemann surfaces with a fixed topology has been one of the most interesting problems in mathematics.

A very important distinction between two dimensional geometry and higher dimensional geometry is that every two dimensional orientable Riemannian manifold admits a complex structure so that the metric has the form $h(dx^2+dy^2)$. For genus greater than one, it was found by Poincare that each of these metric can be conformally deformed to a unique metric with curvature equal to -1.

Hence the space of conformal structures on a surface of genus g is the same as the space of metrics with constant negative curvature $= -1$ on the surface. It is of course important to realize that the group of diffeomorphism acts on this space. The quotient space is the moduli space of conformal structure. It is denoted by M_g. If we restrict the diffeomorphisms to those that are isotopic to identity, then the quotient space is called the Teichmuller space and is denoted by T_g.

Naturally, T_g covers M_g and the covering transformation is the mapping class group which is the quotient of the above two diffeomorphism groups.

It is not hard to prove that T_g is contractible. The topology and the geometry of M_g is far more complicated.

Teichmuller has studied T_g extensively by introducing the concept of extremal conformal map between Riemann surfaces. Bers demonstrated that it is possible to embed T_g into $\mathbb{C}^{3g-3}$ as a domain of holomorphy. It would be interesting to find a meaningful extension of extremal conformal map to higher dimensional complex manifolds.

However, there is no precise description of how bad the boundary of the Bers' embedding is. It is also not clear what is the "optimal" embedding of T_g into $\mathbb{C}^{3g-3}$.

The geometry of M_g is more algebraic in nature. It is quasiprojective in the sense that there is algebraic variety $\overline{M}_g$ so that $\overline{M}_g \smallsetminus M_g$ are given by subvarieties. The most basic construction of $\overline{M}_g$ was due to Deligne–Mumford who introduced the concept of stable curves (concept of stable manifolds that are derived from geometric invariant theory).

It is known that for large genus, M_g is difficult to describe in the sense that it is of "general type" and there is no nontrivial holomorphic maps from complex projective space onto $\overline{M}_g$.

Study of $\overline{M}_g$ has been a fundamental subject in complex geometry and mathematics in general.

There are many natural complex bundles over M_g. In fact there is a universal curve over M_g, i.e., a complex manifold fibered M_g so that each fiber is the given Riemann surface. On the universal curve, we can take tangent bundles along the fiber and we can form the Hodge bundle by taking holomorphic one forms along the fiber. The Chern classes of these natural bundles give important cohomology classes of $\overline{M}_g$. The Mumford conjecture says that low dimensional (related to g) cohomology of $\overline{M}_g$ is generated by Chern classes. Madsen has settled this problem recently. But it is still an interesting problem to understand such cohomology in the unstable range.

The Chern numbers of these bundles can be organized nicely and has been a very active area of study. In the past fifteen years, string theory contributed a great deal of understanding into these numbers. There are Witten conjecture (proved by Kontsevich), Mariño–Vafa formula (proved by Liu–Liu–Zhou) and many other exciting works.

The concept of holomorphic functions of one variable can be readily generalized to functions of several variables. The naive generalization of uniformization fails completely as the equations $\frac{\partial f}{\partial \bar{z}^i} = 0$ for all i form an overdetermined system.

We call a manifold M to be complex if there are coordinates charts $(z_1, \dots, z_n)$ so that their coordinate transformations are holomorphic.

A complex manifold M has the property that the complexified tangent bundle admits a linear operator J so that $J^2 = -$ identity such that $\{v \,|\, Jv = \sqrt{-1}\, v\}$ form holomorphic tangent space $\{\frac{\partial}{\partial z^i}\}$ and $\{v \,|\, Jv = -\sqrt{-1}\, v\}$ form antiholomorphic tangent space.

A manifold admits such an operator J is called an almost complex manifold.

It is said to satisfy the complex Frobenius condition if for any complex vector field v_j so that $Jv_j = \sqrt{-1}\, v_j$, we know that

$$J[v_j, v_k] = \sqrt{-1}\, [v_j, v_k].$$

The celebrated Newlender–Nirenberg theorem says that an almost complex manifold which satisfies the complex Frobenius condition is a complex manifold.

While there is an effective method to determine which smooth manifold admits an almost complex structure, it is a great mystery and fundamental question to find a topological condition to determine which even dimensional orientable manifold admits complex structure.

Most tools in studying complex manifolds come from Kähler geometry.

Kähler observed the importance of existence of Hermitian metric $\sum g_{ij}\, dz^i\, d\bar{z}^j$ so that $d\left(\sqrt{-1}\sum g_{ij}\, dz^i \wedge d\bar{z}^j\right) = 0$. Kähler metric has the important property that there is a holomorphic coordinate system so that it can be approximated by the flat metric up to first order.

Since the introduction of the concept of complex manifolds, the first important contribution was the introduction of Chern classes. Coupling with the classical theory of Riemann-Roch theorem and sheaf theory, Chern classes was used in a prominent way by Hirzebruch to prove the Riemann-Roch formula for higher dimensional algebraic manifold. The formula of Hirzebruch was interpreted and generalized by Grothendieck in functorial setting and K-theory was developed as a fundamental tool.

Based on this formula and the idea of Bochner's vanishing formula, Kodaira proved the embedding theorem for Kakler manifolds of special type. Note that once a Kähler manifold is holomorphically embedded into complex projective space, a fundamental theorem of Chow says that it must be defined by an ideal of homogeneous algebraic polynomials. Hence they are algebraic manifolds.

Chow also introduced fundamental tools to study algebraic cycles. The Chow coordinates were introduced. The concept of Chow variety is one of most important concept in modern algebraic and arithmetic geometry.

The work of Hodge on the Hodge structures of Kähler manifolds was also used extensively by Kodaira. At the same time it puts the old theory of Picard and Lefschetz on a new setting. The conjecture of Hodge on algebraic cycles is perhaps the most elegant and important question in algebraic geometry. Due to its relation to arithmetic question, a lot of number theorists made contribution to it.

The development of Hodge structure was due to many people: Hodge, Atiyah, Grothendieck, Deligne, Shafarevich, Borel, Dwork, Katz, Schmid, Griffiths, Clemens, and others. A very important question is its relation to monodromy and the Torelli theorem. The establishment of suitable form of Torelli theorem has been an important direction. It has been a fundamental tool in the study of Calabi-Yau manifolds.

Kodaira proved that every Kähler surface can be deformed to an algebraic surface.

According to Kodaira's classification (with later work by Siu on $K3$ surfaces), the only unknown non-

Kähler complex surfaces would be so called class VII_0 surfaces.

Such surfaces are not Kähler and it would be good to classify them. There are two subclasses of such surfaces:

(1) Those with no holomorphic curves. This was classified by Bogomolov and Jun Li–Yau–Zheng.

(2) Those with finite number of curves.

Hopefully the method of Li-Yau-Zheng can be used to clarify this remaining class of non-Kähler surfaces.

How to describe topology of algebraic surfaces?

Riemann–Roch formula and Atiyah–Singer Index formula have played fundamental roles.

When $b_1 \neq 0$, the formula provides information on holomorphic one forms and hence one can integrate the one form to obtain nontrivial information.

Van de Ven was the first one to observe that Riemann–Roch implies

$$8C_2(M) \geq C_1^2(M).$$

Bogomolov used his idea of stable bundles and symmetric tensors to improve Van de Ven inequality to

$$4C_2(M) \geq C_1^2(M).$$

Immediately afterwards, I used the newly developed existence of Kähler–Einstein metric to prove

$$3C_2(M) \geq C_1^2(M)$$

which was optimal as the inequality is achieved by quotient of the complex ball.

Miyaoka then also sharpened Bogomolov's method to achieve similar inequality.

However up to now, analytic method is the only way to prove that $3C_2(M) = C_1^2(M)$ implies that either M is $\mathbb{C}P^2$ or quotient of the ball.

The generalization of this kind of inequality to orbifolds is rather straightforward and was achieved by Cheng–Yau, Kobayashi and Tian–Yau.

My observation that Kähler–Einstein metrics become metrics with constant holomorphic sectional curvature when $3C_2(M) = C_1^2(M)$ makes me realize the relevance of Mostow rigidity theorem. It immediately implies that the only complex structure over such a manifold is the standard one.

Therefore I conjectured that compact Kähler manifold with negative curvature has unique complex structure. I proposed to use harmonic map to settle this problem. The idea was that curvature of the target should force the rigidity of harmonic map. It is inspired by the way to prove uniformization theorem by Dirichlet principle. I proposed to Siu this program who observed that the special form of the curvature of Kähler metric helps to solve an important case of my conjecture.

Application of harmonic map to prove existence of incompressible minimal surfaces was initiated by Schoen and myself a few years earlier. In that theory, the collar theorem of Linda Keen was used and Schoen and I realized that the energy of harmonic map can be turned around to provide an important exhaustion function of the Teichmuller space. After my talk in Utah in 1976, this idea was picked up by other people. The beautiful work of Michael Wolf demonstrated how harmonic map can be used to give Thurston compactification of Teichmuller space.

Jost and I then found that harmonic map can be used to demonstrate that a topological map from a compact Kähler manifold to a curve of higher genus can be homotopic to a holomorphic map if we change the complex structure of the curve.

While harmonic map is effective for manifolds with large fundamental group, its existence for simply connected manifold is not known.

Let $f : M \longrightarrow N$ be a map from a compact Kähler manifold M to another one such that its induced map on $\Pi_2(M)$ is nontrivial. I conjectured that there is always one harmonic map from M to N whose induced map on $\Pi_2(M)$ is nontrivial.

The reason that this may be true come from understanding of the celebrated theorem of Sacks–Uhlenbeck on harmonic maps of two dimensional spheres.

Siu and I studied the structure of bubbling of Sacks–Uhlenbeck sphere in the proof of Frenkel conjecture. Similar study was also used later by Parker-Wolfson and Ruan-Tian to understand the compactification of stable maps and Gromov-Witten invariant. The final formulation was due to Kontsevich on the concept of moduli space of stable maps.

Even when existence theorem for harmonic map can be proved, it still remains to find properties of such harmonic maps. Under what conditions that these maps are unique up to holomorphic endomorphisms of M and N?

In general, methods from linear and nonlinear partial differential equations can be used to produce holomorphic objects. However, the analogue construction for algebraic varieties over characteristic p will be difficult to be carried out. This can be an interesting direction as Mori was able to construct rational curves through methods of characteristic p. This spectacular method is still needed to be understood through analytic means.

Let us now discuss ideas from nonlinear analysis.

Kähler–Einstein metrics are Kähler metrics so that

$$R_{i\bar{\jmath}} = c\, g_{i\bar{\jmath}}.$$

For $c \leq 0$, it is unique if we fix the Kähler class. If $c > 0$, it is also unique up to automorphisms of the manifold, due to the work of Bando–Mabuchi.

Hence when the metric exists, it provides important invariants for the complex structure of the manifold.

It is not hard to show that the Kähler–Einstein metric in fact determines the complex structure of the underlying manifold unless it is hyperkähler. This follows by studying the pull back of the Kähler form under the isometry.

The existence of Kähler–Einstein metrics therefore provides a way to understand complex structure by metrics.

A very important question is therefore the full spectrum of Laplacian acting on the space of (p, q) forms should determine the structure (polarized complex structure if $c = 0$). Some contribution of these spectrum would give rise to important invariants of the manifold, e.g., holomorphic torsion. While we can embed the moduli space of complex structures into the space of spectrum, there is no obvious way to give complex structure to the later space which makes the embedding to be holomorphic.

Kähler–Einstein metric with $c \leq 0$ has been very powerful in understanding the complex structure of the manifold. There were the following major ways:

(1) Using curvature representation of Chern classes, one can represent $c_2\omega^{n-2}$ by L^2 integrals of curvature which is clearly non-negative and trivial only if the manifold is flat. If $\omega = \pm c_1$, there is then an inequality between $c_2c_1^{n-2}$ and c_1^n with equality only when the manifold is complex projective space or the quotient of the complex ball.

(2) By using curvature decreasing property, one can prove that the tangent bundle is slope stable in the sense of Mumford. (This kind of work was motivated by Bogomolov's work.) From tangent bundle and cotangle bundle, we can take tensor product and wedge product and build natural bundles that come from natural representation of $GL(n, \mathbb{C})$. They all have natural Kähler–Einstein metric induced from the tangent bundle.

If a natural bundle V comes from an irreducible representation of $GL(n, \mathbb{C})$ and if $c_1(V) = 0$, then any nontrivial holomorphic section of V is parallel and the holonomy group of the original connection must be reduced to a smaller group.

In this way, one can characterize those Kähler manifolds that are locally symmetric.

The fact that we can give a complete algebraic geometric characterization of Shimura varieties. It gives a way to prove Galois conjugation of Shimura variety is still Shimura. This is a theorem due to Kazdhan using representation theory.

It should be possible to characterize sub manifolds whose metrics are Kähler–Einstein.

It should also be interesting to characterize by algebraic geometric means of those submanifolds which are locally symmetric.

(3) Deformation of complex structure using parallel forms.

For $K3$ surfaces, one can mix up the $(2,0)$ form, $(0,2)$ form and $(1,1)$ form to find P^1 family of complex structures.

Bogomolov observed that for hyperkähler manifolds, complex structures are unobstructed. This was followed by Tian–Todorov closely with basically the same argument.

(4)Since we know the Ricci curvature of such manifolds, one can apply Schwarz lemma to study holomorphic maps between Kähler manifolds.

One should be able to compute Weil-Petersson metric associated to the canonical KE metric. The moduli space should have rich properties to be studied. This include the volume of the Weil-Peterson geometry and its L^2-cohomology. For Calabi-Yau manifolds, the cohomology classes are called BPS states and should have interest in string theory.

(4) It is clear that the tangent bundle is stable when the manifold has Kähler–Einstein metric. However it has not exhausted the stength of Kähler–Einsten metric yet. At the time when I applied KE metric to algebraic geometry, I realized that existence of KE metric should be equivalent to stability of manifolds in the sense of geometric invariant theory. (Besides the obvious obstruction that come from the sign of first Chern class.)

Only until recently, Donaldson has made definite progress on this problem.

While there are some activities on extremal metrics or metrics with constant scaler curvature recently, the fundamental focus of the research should not be shifted away from KE metrics with nonpositive scalar curvature. The case of KE metrics with positive scalar curvature is more relevant to the above mentioned question of stability and also to understand existence of Ricci flat manifolds.

In 1978 Helsinki Congress, I outline the existence of complete noncompact Ricci-flat manifold. The detail was written up with Tian later. KE metrics with positive scalar curvature played a role in the later construction.

So far, no significant contribution of such metrics to algebraic geometry has been found. When my question of stability can be settled, the situation may be different.

In order to understand geometric stability of Kähler–Einstein manifolds, one would like to relate the metric with respect to induced metrics from projective embeddings, I initiated this program more than twenty years ago to find projective embeddings by high powers of ample line bundles to approximate KE metrics.

Several of my students follow this programm. As was guided by me in his thesis, Tian applied my idea with Siu on characterization of non-compact Kähler manifolds which are $\mathbb{C}^n$. He proved that such embedding is possible. The perturbation analysis was followed by Lu, Zelditch, Phong–Sturm. Tian made some partial contribution to my question of stability, based on works of Donaldson.

In both thesis of Luo and X. Wang continued such studies on the balanced condition.

Basically, Donaldson settled the important necessary part of my conjecture. There are some works related to existence of KE metric with positive scalar curvature for toric manifolds. (Recently Zhu and Wang made contributions by proving existence of the real Monge–Ampere equation that comes from the reduction of Donaldson.)

What Donaldson has done should be applied towards understanding of manifolds with nonpositive first Chern class. This is especially true for manifolds that come from arithmetic geometry, moduli problem and questions related to algebraic cycles and algebraic bundles.

Moduli space of polarized algebraic manifolds should support Kähler–Einstein metrics with negative scalar curvature. It may admit orbifold type singularities. When the deformation space is obstructed, it can be very challenging to describe the metric structure of the singularity.

When the moduli space is compactified, the KE metric should behave in a suitable form asymptotically. It will be important to understand such behaviour in terms of periods of integrals.

The simplest problem of this sort appears already in one dimension. Only recently, Liu–Sun–Yau was able to identify the behaviour of KE metric on the Teichmuller space.

While the boundary of Teichmuller space may be complicated complex analytically, it is interesting to know that, based on the work of Shi, we proved that the curvature and all its covariant derivatives are bounded. This is in contrast to my previous work with S.-Y. Cheng on KE metrics on strictly pseudoconvex domains.

Besides KE metric, the Bergman metric is a natural metric to be studied on the moduli space. Its relation with KE metric and the covering space should be interesting.

There are many interesting subvarieties of moduli space, even in the case of a curve. Kefeng Liu, Sun and others exploit the Schwarz lemma, Kang Zuo studies variation of Hodge structures.

It is a fascinating problem to characterize those moduli problems where the moduli space is a Shimura variety or Calabi–Yau space.

Moduli space of algebraic cycles coupled with stable bundles should be an interesting topic to study.

Based on idea from string theory, it should be interesting to understand this moduli space under the following duality

$$\begin{array}{ccccc} & & T^k \times (T^k)^* & & \\ & & \downarrow & & \\ T^k & & M \underset{N}{\times} \widehat{M} & & (T^k)^* \\ \downarrow & \swarrow & & \searrow & \downarrow \\ M & & & & \widehat{M} \\ & \searrow & & \swarrow & \\ & & N & & \end{array}$$

The maps from M to N, from $\widehat{M}$ to N are holomorphic fibration that may have singularity. There should be a rank one holomorphic sheaf over $M \underset{N}{\times} \widehat{M}$ that serves as fiberwise Poincare line bundle. By applying Fourier–Mukai transform via such a sheaf, one should map the above moduli space from M to $\widehat{M}$.

In the above picture, we can allow the torus to be realspccial Lagrangian. In that case ,we shall obtain the mirror map from M to $\widehat{M}$. This is called the SYZ construction.

String theory has provided a very rich background to study geometry of Ricci flat metrics. Duality concepts have provided very powerful tools. The construction of SYZ needs to be explored much further, both in terms of construction of special Lagrangian cycles and the perturbation of semi-flat Ricci flat metrics to Ricci flat metrics in terms of holomorphic disks.

Fundamental question in complex geometry is

(1) To find a topological condition so that an almost complex manifold admits an integrable complex structure.

(2) To find a way to determine which integrable complex structure admits Kähler metrics, or weaker form of Kähler metrics, e.g., balanced metrics. There are Hermitian metrics ω so that

$$d(\omega^{n-1}) = 0.$$

(3) To find a way to deform a Kähler manifold to a projective manifold.

(4) To characterize those projective manifolds in terms of algebraic geometric data that can be defined over $\bar{Q}$.

(5) Study algebraic cycles and algebraic vector bundles(or more generally, derived category of algebraic manifolds).

(6) To understand moduli space of algebraic structures and the above algebraic objects.

For $\dim_{\mathbb{C}} \geq 3$, all these problems would be quite different from $\dim_{\mathbb{C}} = 2$.

(1) Is it possible that every almost complex manifold admits an integrable complex structure for $\dim_{\mathbb{C}} \geq 3$? Prof. Chern has made significant progress on this problem.

(2) For balanced manifolds, one should study the system of equations introduced by A. Strominger where the coupled holomorphic bundle is coupled with the Hermitian metric.

A. Strominger.

There is a holomorphic bundle V over complex three dimensional manifold with Hermitian metric whose curvature F_h satisfies

$$\partial\bar{\partial}\omega = \sqrt{-1}\,\mathrm{tr}\,F_h \wedge F_h - \sqrt{-1}\,\mathrm{tr}\,F_g \wedge F_g$$
$$F_h^{2,0} = F_h^{0,2} = 0$$
$$\mathrm{tr}\,F_h = 0$$

and ω is conformally balanced.

We expect "mirror symmetry" on such class of manifolds also.

Jun Li and I were able to solve the Strominger system in a small neighborhood of Calabi–Yau manifolds. It should be possible to solve it in a global setting.

There are several important operations in complex geometry

(1) Blowing up

(2) Blowing down

(3) Deformation (local or global)

Neither projective nor Kähler geometries are preserved under all these operations. It will be certainly desirable to find some kind of geometry that admits such operations.

This is particularly significant if we start from a projective manifold and perform these operations successfully. Can we reach the class of all Kähler manifolds? (Note that Voison did construct Kähler manifolds that cannot be deformed to projective manifolds.) What is the largest category that can be reached in this way?

Based on twistor's construction, many non-Kähler complex manifolds were constructed from the work of Taubes on the existence of anti-self-dual structure on all four dimensional manifolds after taking connected sum with enough copies of $S^2 \times S^2$. The construction of Clemens' by blowing down curves with negative normal bundle and smoothing the blowed down manifolds allows us to construct many interesting non-Kähler complex manifolds. One cannot ignore the theory of non-Kähler complex manifolds anymore.

In studying Kähler structures, Hodge theory did play the most fundamental role. The important point is that the Laplacian acting on the k-forms split covariantly on (p, q) forms with $k = p + q$. It allows us to link the topology of the Kähler manifold to complex structure of the manifold. It would be important to seek similar statement for more general class of complex manifolds which may include those that support the Strominger's structure.

It is conjectured by M. Reid that the moduli space of Calabi-Yau manifolds is connected if we allow to deform through non-Kähler structure. Is it possible that such structure supports strominger's structure?

The most outstanding question in algebraic geometry has been the Hodge conjecture. The desire to find a characterization of algebraic cycles by (p, q) type Hodge classes is fundamental.

If we enlarge the scope of geometry, we may have to enlarge the scope of Hodge conjecture. The most notably example in this regard is that in the case of Calabi–Yau manifolds we have covariant constant n-forms. We can look for those Lagrangian cycles so that the restriction of these n-forms become a constant multiple of the volume form. These are called special Lagrangian cycles.

On the construction of Strominger–Yau–Zaslow of mirror manifolds, special Lagrangian cycles play fundamental role. A fundamental question is that for an n-dimensional homology class in an n-dimensional Calabi–Yau manifold, is some integer multiple of it representable by special Lagrangian cycles.

It is believed that special Lagrangian cycles are "mirror" to stable holomorphic bundles over the mirror manifold. Hence construction of such cycles may be helpful to understand the Hodge conjecture. It is proposed by Thomas–Yau that starting from the Lagrangian cycles, stable in a well-defined sense, we can deform it to special Lagrangian cycle by the mean curvature flow. Mu-tao Wang had made significant progress on this problem.

It is also a fundamental question to construct holomorphic structures over a complex vector bundle. After stabilizing with trivial bundles, such question may be easier to handle. Only in the case of complex two dimensional surfaces, the works of Taubes and Donaldson give effective answers. The work of Jun Li and Gieseker–Li gave many important contributions for understanding the geometry of moduli space of algebraic bundles. It would be useful to construct a flow on almost complex structures on the bundle to an integrable structure.

Special Lagrangian torus is supposed to be abundent for Calabi–Yau manifolds where they can give a fibration. In case of complex three dimension, the base of this fibration may look like $S^3 \setminus G$ where G is a trivalent graph. The SYZ geometry call for existence of flat affine structure over $S^3 \setminus G$ where certain real Monge–Ampere equation needs to be solved and the monodromy belong to $SL(3, \mathbb{Z})$. Recently, Loftin–Yau–Zaslow was able to solve these equations in a neighborhood of G with nontrivial monodromy.

When the manifold is Kähler–Einstein with scalar curvature not equal to zero, special Lagrangian cycle should be replaced by those Lagrangian cycle whose mean curvature form is harmonic. It should be interesting to develop the corresponding SYZ geometry for such cycles. The moduli space of them would give new invariants for the Kähler manifold. The understanding of holomorphic curves whose boundaries form homology classes on these Lagrangian cycles would be important to be studied also.

The Donaldson–Uhlenbeck–Yau theorem on the existence of Hermitian–Yang–Mills connections on stable holomorphic bundles have been generalized when there are special structures. The most important one is the Higgs bundle structure by C. Simpson. It is related to the variation of Hodge structure. The theory is not completely satisfactory when the base manifold is noncompact but quasiprojective.

It is a challenging question to construct Kähler–Einstein metrics with zero or negative scalar curvature or Hermitian–Yang–Mills connections over quasiprojective manifold where the complementary divisors is not smooth but normal crossing.

Hermitian–Yang–Mills connection can be used to reduce the holonomy group of a holomorphic bundle when suitable algebraic geometric condition is verified. They should be used extensively in studying moduli space of bundles and non-Kähler complex manifolds.

Smith, Thomas and Yau studied the possible mirror manifold of a non-Kähler complex manifold. Some concrete example of symplectic manifolds were constructed. Perhaps one can explore such duality in more detail.

Recently Jun Li made fundamental contribution towards the understanding of moduli space of stable maps of an algebraic variety. Quantities over such moduli are very important for future study.