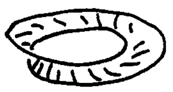
June 7. -July 9. 2004

List of Reference Zheda

Lecture 1

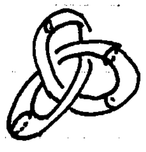
- surfaces, 3-manifold: Hausdorff, locally Euclidean, countable basis
- orientable surface: no Möbius band



Thm (classification thm)

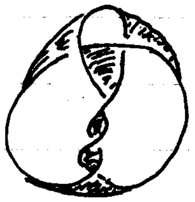
idea.

Example.

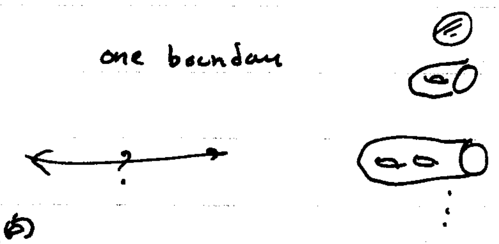


reality

X=



是否定向



Hw: Find the genus of X. Show X orientable.

• Geometry & Geometric Structures	pts	lines	transformations
3-basic Geometries			
1) Spherical			
2) Euclidean			
3) Hyperbolic			

An interesting formulation:

Σ^2 closed orientable $\Rightarrow$ $\Sigma \cong S^2, S^1 \times S^1$ (admitting S^1 action)

∂

or Σ can be decomposed into 3-holed spheres

$\Sigma_{3.0} =$

whose interior has a complete hyperbolic metric of finite area.

$$\Sigma_{3.0} \cong \mathbb{C} - \{0,1\} \underset{\longleftarrow}{\cong} \mathbb{H}/\Gamma(2) \qquad \Gamma(2) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \quad \right\}$$

modular function (Elliptic Function)

A picture of gluing

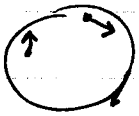
Lecture 1

3-Manifold.

Examples: S^3, $T^3 = S^1 \times S^1 \times S^1$, $\Sigma_g \times S^1$, or $UT\Sigma_g$ (unit tangent bundle)

$$\overset{\shortparallel}{} \{ v \in T\Sigma_g \mid |v| = 1 \}$$

$$UTS^2 = \{ (x, u) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid |x| = 1, |u| = 1 \quad x \cdot u = 0 \} = SO(3) = \mathbb{R}P^3 = S$$



All admitting S^1-action

Construction From Gluing: or conn. sum.

Take a knot $K \subset S^3$, $N(K)$ small regular nbhd.

$$P(K) = S^3 - \mathring{N}(K) \qquad \text{(3-manifold w/ bound}$$

$$\partial P(K) = S^1 \times S^1$$

$$M(K) = P(K) \sqcup P(K) \Big/ \underset{x \in \partial P(K)}{x \sim x} = P(K) \underset{\partial_{id}/_{\partial}}{\cup} P(K)$$

$$= \mathbb{R}^3 \cup \{\infty\}$$

Proposition $M(K) \cong M(K') \Leftrightarrow K \sim K' \Leftrightarrow \exists$ homeo, $\varphi: S^3 \to S^3$

$$\varphi(K) = K'$$

(Knot equivalence)

Classification of 3-manifold is at least as complicated as classification of Kn

Thurston (1978) Conjecture.

$$T^3 \underset{S^2}{\#} T^3$$

Connected Sum Construction

Def M^3 irreducible if it contains no essential 2-sphere. $\Leftrightarrow$

S^2- decomposition.

(921)

Definition $\Sigma_g \hookrightarrow M^3$ surface, incompressible means. $i_*: \pi_1(\Sigma) \to \pi_1(M)$ injective

Each loop as, not null homotopic in Σ $\Rightarrow$ not null homotopic in M.

Lecture 1

$M^3 = N_1 \# \cdots \# N_k$ N_i irreducible, unique

Thurston's G.C.

M^3 closed, irreducible, orientable 3-manifold.

There exists a finite collection of disjoint, incompressible tori $T_1, \cdots, T_k$ in

so that (1) each component of $M^3 - (T_1 \cup \cdots \cup T_k)$ ~~is~~ has

(1) a non-trivial S^1-action . or ([illegible], SFS)

(2) a complete hyperbolic metric of finite ~~area~~ volum. ($\mathbb{H}^3 = \{x \in \mathbb{R}^3 \mid x_3 > 0\} /$

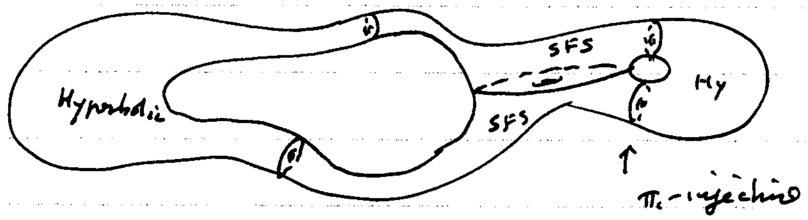
$\underline{\Gamma < PSL(2, \mathbb{C})}$

Thm (Thurston) 1982 G.C. holds if $k \geq 1$. (McMullen) 1998

Perelman. Hamilton Ricci flow may resolved it by now.

G.C $\Rightarrow$ Poincaré Conjecture : M^3 closed , $\pi_1(M) = 1 \Rightarrow M \cong S^3$

Picture



$M = N_1 \# \cdots \# N_k \Rightarrow \pi_1(M) = \pi_1(N_1) * \cdots * \pi_1(N_k)$

$\Rightarrow \pi_1(N_i) = 1$

May assume N_i irreducible, M. $\pi_1(M) = 1 \Rightarrow$ NO incompressible tori

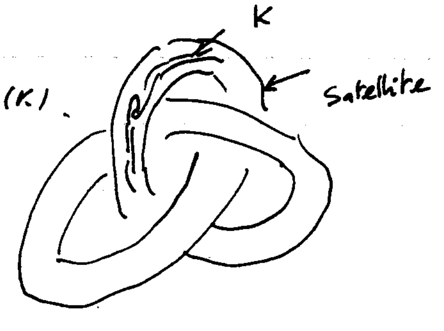
$\pi_1(S^1 \times S^1) = \mathbb{Z} \oplus \mathbb{Z}$

$\overset{G.C.}{\Rightarrow}$ M either SFS. or

hyperbolic $= \mathbb{H}^3 / \Gamma$ $\Gamma \cong \pi_1(M) \Rightarrow$ Not hy

SFS must be S^3 ([illegible])

Back to Knot Complement : Thurston G.C. applied to M(K) .



G.C. knots : — torus knot.

(S^1-action

Thurston Thm

prime knot — satellite knots

hyperbolic,

Knot compositions

6-10-04

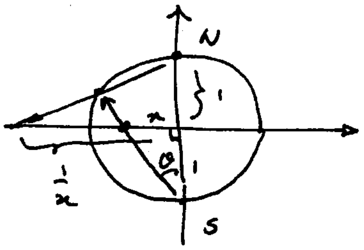
F. Luo

L2. Manifolds & Examples

Def. M^n manifold: Hausdorff, countable basis, locally Euclidean: M^n covered by charts $\{(U_\alpha, \phi_\alpha) \mid \phi_\alpha : U_\alpha \to V_\alpha \subset \text{open } \mathbb{R}^n$ homeo. $\}$
call $\phi_\alpha \circ \phi_\beta^{-1}$ the transition functions

M^n – smooth if $\forall$ $\phi_\alpha \circ \phi_\beta^{-1} \in C^\infty$
– analytic if $\forall$ $\phi_\alpha \circ \phi_\beta^{-1}$ analytic
– P.L. (piecewise linear) if $\forall$ $\phi_\alpha \circ \phi_\beta^{-1}$ P.L.

Example 1 of the S^n $S^n = \{ x \in \mathbb{R}^{n+1} \mid |x| = 1 \}$



$N = (0, \cdots, 0, 1)$
$S = (0, \cdots, 0, -1)$
$U = S^n - \{N\}$ $\phi: U \to \mathbb{R}^n$
$V = S^n - \{S\}$ $\psi: V \to \mathbb{R}^n$
stereographic projection

Check $\phi \circ \psi^{-1} : \mathbb{R}^n - 0 \to \mathbb{R}^n - 0$ $x \mapsto x/\|x\|^2$
↑
inversion about $|x| = 1$

S^n is smooth.

Exercise for S^1: Several different ways of putting charts.

Example 2 The quotient space $\mathbb{R}^1/\mathbb{Z}^1$ w/ the quotient topology

Definition (Quotient Top.) X topological space $f: X \to Y$ ANY onto map ($Y = f(X)$). Then the quotient topology on Y: $U \subset Y$ open $\Leftrightarrow$ $f^{-1}(U) \subset X$ ope

We give $Y = \mathbb{R}^1/\mathbb{Z}^1$ the quotient topology from the standard quotient map
$\pi : \mathbb{R}^1 \to \mathbb{R}^1/\mathbb{Z}^1 = \{ [x] \mid x \in \mathbb{R} \}$
$[x] = \{ x + n \mid n \in \mathbb{Z} \}$.

L2

This example is a very special case of a group Γ acting on a topological space Ω by homeomorphisms:

$n \in \mathbb{Z}$, $x \in \mathbb{R}$ $\qquad n * x \hat{=} x + n$.

This action has the following two properties

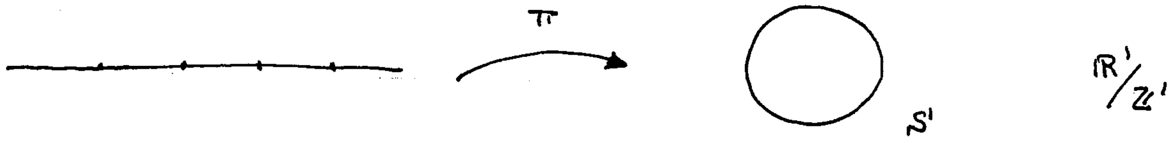
(1) (free action) $\forall \ \gamma \in \Gamma - \{id\}$ $\quad \gamma(x) \neq x \quad \forall \ x \in \Omega$

(2) (properly discontinuous) $\forall$ cpt $K \subset \Omega$,

$\{\gamma \in \Gamma \mid \gamma(K) \cap K \neq \phi\}$ is finite

Example 2 (Cont.) The quotient space $\mathbb{R}^1/\mathbb{Z}^1$ is a 1-dim. manifold. (S^1)

Proof.



Easy way: if (a, b) interval w/ $b - a < 1 \Rightarrow$

(3) $\qquad \forall \ \gamma \in \mathbb{Z} - \{0\} \qquad \gamma(a, b) \cap (a, b) = \phi$

Thus

$\pi(a, b) = U_{a,b}$ has:

$$\pi^{-1}(U_{a,b}) = \bigcup_{n \in \mathbb{Z}} \left((a, b) + n \right) \text{ is open}$$

$+ \qquad \pi|_{(a,b)}: (a, b) \longrightarrow U_{a,b}$ $\qquad$ 1-1, onto, continuous inverse continuous

Thus $\phi_{a,b} : U_{a,b} \longrightarrow (a, b)$ $\quad \pi|_{(a b)}^{-1}$ are the charts.

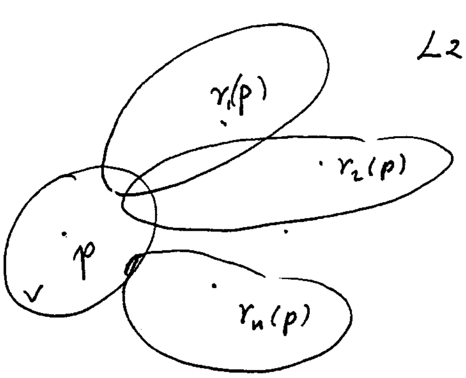
It is easy to show Hausdorff property (HW).

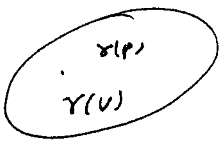
Difficult way: Uses the properties (1), (2) only. Key to find open nbhds satisfying (3):

$\forall \ p \in \mathbb{R}$. take nbhd V of p w/ $\bar{V}$ cpt.

$\Rightarrow \exists$ only finitely $\gamma_1, \cdots, \gamma_n \in \mathbb{Z}_1 - 0$ w/ $\quad \gamma_i V \cap V \neq \phi$

L2





Now $\gamma_i(p) \neq p$ by (1)

We may choose $W \subset V$ nbhd of p st (1) $\gamma_i(W) \cap W = \phi$

[illegible]

$\Rightarrow \forall \ \gamma \in \mathbb{Z} - \{0\}$ $\gamma W \cap W = \phi$

Define $\pi(W)$ to be the basic open sets in $\mathbb{R}^1/\mathbb{Z}^1$

$\phi: \pi(W) \to W$ to be $(\pi/_W)^{-1}$ the chart map

$\Rightarrow$ manifold.

Thus we have

<u>Thm</u> Γ a group acting topologically (or smoothly) on an open set $\Omega \subset \mathbb{R}^n$ freely and properly discontinuously, then

Ω/Γ in the quotient topology is a manifold

<u>Remark</u> 1. The <u>transition function</u> for Ω/Γ are in fact elements in Γ!

Thus: $\mathbb{R}^1/\mathbb{Z}^1$ is a smooth manifold

2. The map: $\pi: \Omega \to \Omega/\Gamma$ is a covering map.
3. We may even assume that Ω is a manifold, the thm still holds.

<u>Example</u> 3: $\Omega = S^n$ $\Gamma = \mathbb{Z}_2$ generated $A(x) = -x$

$$S^n/\Gamma \cong S^n/_{x \sim -x} = \mathbb{RP}^n$$

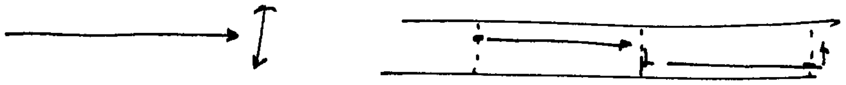
is the projective space.

<u>Example 4</u> $\mathbb{Z}$ action on $\mathbb{R} \times (-1, 1)$ $n \cdot (x, y) = (x+n, y)$

$\mathbb{R} \times (-1,1)/\mathbb{Z}$ =  is the open cylinder annulus

Example 5. $\mathbb{Z}$ acting on $\mathbb{R}\times(-1,1)$ generated by γ

$$\gamma(x,y) = (x+1,-y)$$



Check freely and discontinuously. $\mathbb{R}\times(-1,1)/\mathbb{Z} \cong$  $\shortparallel$

Möbius band. M_{ob}

Example 6. $M_{ob}\times \mathring{I}$ is called the solid Möbius band

A 3-manifold is orientable iff it contains no solid Möbius band

Example 7. The torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2 = \mathbb{C}/(\mathbb{Z}+i\mathbb{Z})$

Example 8. $SL(2,\mathbb{R})/SL(2,\mathbb{Z})$ $= \mathbb{R}$

More generally, a Lie group G and a discrete subgroup $\overset{\mathbb{Z}}{\overset{\shortparallel}{\Gamma}}$
Γ acts on G by left multiplication.
Then G/Γ is a manifold

$\mathbb{R}/\mathbb{Z}$, $\mathbb{R}^2/\mathbb{Z}^2$, $SL(2,\mathbb{R})/SL(2,\mathbb{Z})$ (check discreteness!)

Example 9. $Nil = \left\{ \begin{bmatrix}1&a&b\\0&1&c\\0&0&1\end{bmatrix} \;\middle|\; a,b,c\in\mathbb{R} \right\}$ Nilpotent group (Heisenberg

$\Gamma = \left\{ \begin{bmatrix}1&a&b\\0&1&c\\0&0&1\end{bmatrix} \;\middle|\; a,b,c\in\mathbb{Z} \right\}$

Γ is discrete! freely is obviously by definition

Check discreteness

$$Nil \cong \mathbb{R}^3$$

$$(a,b,c)*(x,y,z) = (a+x,\ b+y+az,\ c+z)$$

Γ acting on $\mathbb{R}^3$ properly discont.

$\forall\ K\subset\mathbb{R}^3$ cpt. $\|k\|\le N$ $\forall\ k\in K$

Check. $(a,b,c)\in\Gamma$

$\mathbb{Z}\subset\mathbb{R}$ discrete: if $\gamma_i\in\mathbb{Z}$ $\gamma_i\to a \Rightarrow \gamma_i\equiv a$ for $i>>1$ □

-L2.5-

<u>Example 10.</u> $\Gamma(2) = \{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2, \mathbb{Z}) \mid a, d \text{ odd}, b, c \text{ even} \} / \pm 1.$

acts $\mathbb{H} = \{ z \in \mathbb{C} \mid \operatorname{Im} z > 0 \}$ properly discon. freely.

$\mathbb{H}/\Gamma(2)$ is a surface.

<u>Example 11.</u> $SL(2, \mathbb{R}) / SL(2, \mathbb{Z})$.

<u>Thm</u> $\forall$ connected oriel surface Σ. $\exists$ discrete group $\Gamma \subset SO(3)$, $\mathbb{E}^2 \rtimes SO(2)$ or $PSL(2, \mathbb{R})$ so that $\mathbb{H}/\Gamma$ or $\mathbb{C}/\Gamma$ or S^2/Γ is homeo to Σ.

<u>Example</u> Γ discrete subgroup of a Lie group G (G closed subgroup of $GL(n, \mathbb{R}$

Then the left multiplication by Γ on G is

(1) free

(2) properly discontinuous

(1) $\gamma \in \Gamma - \{id\}$ $\gamma(x) = \gamma x = x \Rightarrow \gamma = id$

(2) Otherwise $\exists$ cpt $K \subset G$ + infinitely distinct $\gamma_i \in \Gamma$

$$\gamma_i K \cap K \neq \phi$$

$$\Leftrightarrow \exists \ k_i \in K \quad \gamma_i k_i \in K$$

K cpt. by choosing subseq. (subsequence of a subseq

we may assume

$$\begin{aligned} k_i &\to a \\ \gamma_i k_i &\to b \end{aligned} \quad \text{in } K$$

$$\Rightarrow \quad \gamma_i = (\gamma_i k_i) k_i^{-1} \longrightarrow b a^{-1}$$

$$\Rightarrow \quad \gamma_i \equiv b a^{-1} \quad \text{for } i >> 1$$

L 3. How to Recognize the Topology

Goal: We want to "visualize" the space like $\mathbb{R}^1/\mathbb{Z}^1$, $\mathbb{R}^2/\mathbb{Z}^2$, $SL(2,\mathbb{R})/SL(2,\mathbb{Z})$ Nil/Γ etc.

3.1. Lemma. X, Z top. spaces $f: X \to Y$ onto map so that Y has the quotient topology. Then $g: Y \to Z$ is cont. $\Leftrightarrow$ $g \circ f: X \to Z$ is cont.

Proof. "$\Rightarrow$" clear, compositions
"$\Leftarrow$" Clear as well. (this is the special property of quotient top.) □

3.1 Example 1. $\mathbb{R}^1/\mathbb{Z}^1 \cong S^1$

Pf Let $\pi: \mathbb{R}^1 \to \mathbb{R}^1/\mathbb{Z}^1$ quotient map $\pi(x) = [x]$
Define $g: \mathbb{R}^1/\mathbb{Z}^1 \to S^1$ $g([x]) = e^{2\pi i x}$

By definition g is 1-1, onto (point set)

Claim 1 g continuous.
$g \circ \pi: \mathbb{R}^1 \to S^1$ $x \mapsto e^{2\pi i x}$ continue

Claim 2 g^{-1} continuous or g homeomorphism (need)

3.2 Lemma Suppose X cpt, Y Hausdorff $g: X \to Y$ cont, 1-1, onto
Then g is a homeomorphism.

Evidently $\mathbb{R}^1/\mathbb{Z}^1$ is compact ($\mathbb{R}^1/\mathbb{Z}^1 = \pi[0,1]$)

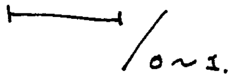
Proof. It suffices to show g sends closed sets to closed sets
$C \subset X$ closed
$\Rightarrow$ C cpt ($\because$ X cpt)
$\Rightarrow$ $g(C)$ cpt
$\Rightarrow$ $g(C)$ closed in Y (Y Hausdorff). □

Corollary X cpt Y Hausdorff $f: X \to Y$ onto cont. $\Rightarrow$ $\tilde{f}: X/R \to Y$ is a homeomorphism R: $x \sim x'$ iff $f(x) = f(x')$ in the quotient top.
(i.e., $Y \cong$ quotient topology)

Lecture 3 Quotient Topology

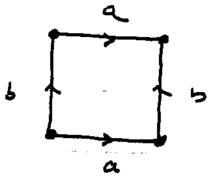
Example 2. $\pi : [0,1] \to S^1$ $\pi(x) = e^{2\pi i x}$

$\Rightarrow$ Quotient topology on $[0,1]$ ~



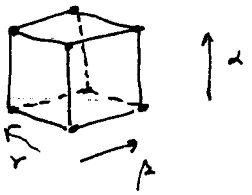
S^1 Hausdorff

3. $\pi : [0,1] \times [0,1] \to S^1 \times S^1$ $\pi(x,y) = (e^{2\pi i x}, e^{2\pi i y})$ onto.



≈

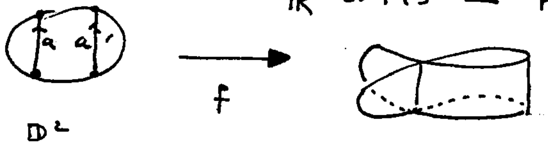
3. $S^1 \times S^1 \times S^1$ homeomorphic to



3 translations.

Note: $V = 1$, $E = 3$ $F = 3$ Tetr = 1 $\chi(M) = 0$!

4. Singular map:



$\mathbb{R}^3$ or M3 — Hausdorff!

subspace top

We can recognize $f(\mathbb{D}^2)$ as the quotient space $\mathbb{D}^2/a \sim a'$ $f(a) = f(a')$

Important: We can recover $f(\mathbb{D}^2)$ from the singularities of f on $\mathbb{D}$

3.3. **Lemma** X. Y cpt Hausdorff spaces $A \subset Y$ closed $h : A \to X$ continuous, Then the quotient space

$$Z = X \sqcup Y / a \sim f(a),\ a \in A \ \overset{\Delta}{=} \ \underline{X \cup_f Y}$$

is Hausdorff,

Gluing Construction.

Pf. Let $\pi : X \sqcup Y \to X \sqcup Y / a \sim f(a)$ be the quotient map $\pi(p) = [p]$

Take $[p_1] \neq [p_2]$ in Z.

Lecture 3

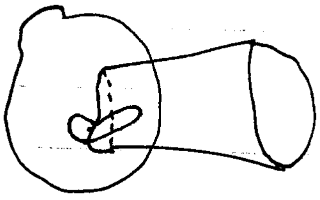
want two disjoint nbhds U_1, U_2 of $[p_1], [p_2]$ in Z.

Case 1. $[p_1], [p_2] \notin \pi(A)$

clear

Case 2 $[p_1] \notin \pi(A)$ $[p_2] \in \pi(A)$

clear



Case 3 $[p_1], [p_2] \in \pi(A)$

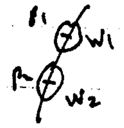
May assume that $p_1\ p_2 \in f(A)$ for simplicity.

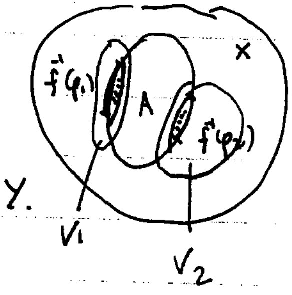
Easy lemma

4. Lemma If V is a nbhd of $f^{-1}(p) \subset A$, then $\exists$ nbhd W' of p in X st

$$f^{-1}(W') \subset V$$

(Indeed $W' = X - f(V^c).$)





Now take two disjoint nbhds V_1, V_2 of $f^{-1}(p_1), f^{-1}(p_2)$ in Y.
For this V_i, find open nbhd W_i of p_i in X, st.

$$f^{-1}(W_i) \subset V_i$$

Let $f^{-1}(W_i) = A \cap V_i'$ where V_i' open in Y

Then $\mathcal{U}_i = \pi(\ W_i \cup (V_i' \cap V_i))$ satisfies the property

□

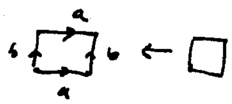
Homework (Poinset Topology)

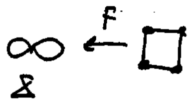
More examples of Gluing

Example 5 (Homework) Let $h: \mathbb{R}^1 \to \mathbb{R}^1$ homeomorphism onto, show that

$$\mathbb{H} \cup_h (-\mathbb{H}) \cong \mathbb{C}$$

(Example 4 (Mapping cylinder)

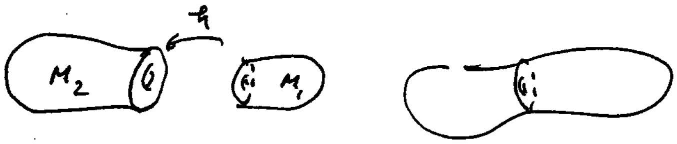




$A = \partial \mathbb{D}^2$. Handcuff.)

Example 6. Two 3-manifolds M_1, M_2 (compact) $h: \partial M_1 \to \partial M_2$ homeo then

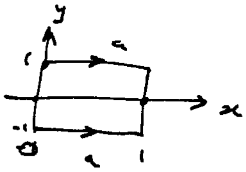
$$M_2 \cup_h M_1$$ is a 3-manifold.



Combining them 3.

Lecture 3.

Example 6. I- bundles: $I \times [-1, 1]$ $\quad I = [0,1]$



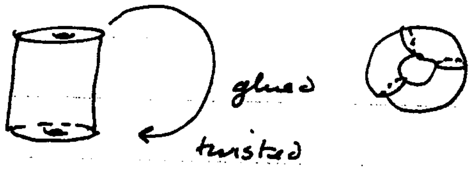
(drop it) {
trivial: product $\quad I \times [-1,1] \big/ (x,-1) \sim (x,1)$

Möbius: $\quad I \times [-1,1] \big/ (x,-1 \sim (1-x,1)$

(Twisted I bundle ZOZ
z acts on Σ freely
... Then $\Sigma \times I \big/ (z,-1)$)

Example 7. Mapping Cylinder: $h: X \to X$ homeomorphism. Then

$$X \times I \big/ (x,0) \sim (h(x),1)$$



Take $X = I$ $\quad h = id \Rightarrow$ Product

$X = I$ $\quad h(x) = 1-x \Rightarrow$ Möbius band

RM. These are all bundles over S^1. The same as $X \times \mathbb{R} \big/ \mathbb{Z}$ $\quad \underline{\alpha: (x,t) = (h(x), t+1)}$

Example 8. Understanding Nil/Γ

$$\Gamma = \left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \Bigg| \; a, b, c \in \mathbb{Z} \right\} \overset{\text{as set}}{\underset{\downarrow}{=}} \{(a,b,c) \mid a,b,c \in \mathbb{Z}\}$$

normal subgroup $(x,y,z)\cdot(a,b,c) = (x+a,\; y+b+xc,\; z+c)$

$\Gamma_0 = \{(a,b,0) \mid a,b \in \mathbb{Z}\} \cong \mathbb{Z} \oplus \mathbb{Z}$ standard.

normal subgroup w/ $\Gamma/\Gamma_0 \cong \mathbb{Z}$ generated by $(0,0,1)$

(Easy lemma)

3.5. Lemma Γ acting on X properly discontinuously and $\Gamma_0 < \Gamma$ normal $\Rightarrow$

$$X/\Gamma \cong (X/\Gamma_0) \big/ (\Gamma/\Gamma_0)$$

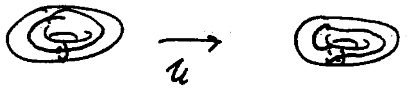
Now: $\quad Nil/\Gamma_0 \cong \mathbb{R}^3 \big/ \mathbb{Z} \oplus \mathbb{Z} \oplus 0 \cong T^2 \times \mathbb{R}^1$

The quotient group $\Gamma/\Gamma_0 \cong \mathbb{Z}$ generated by $\gamma \sim (0,0,1)$

$$\gamma(x,y,z) = (x,y,z) * (0,0,1) = (x,\; y+x,\; z+1)$$

Thus, $\quad Nil/\Gamma \cong T^2 \times \mathbb{R}^1 \big/ \mathbb{Z} \cong T^2 \times I \big/ h(x) \sim v$

Where $h: T^2 \to T^2$ induced by $(x,y) \mapsto (x, y+x)$ (Dehn twist)



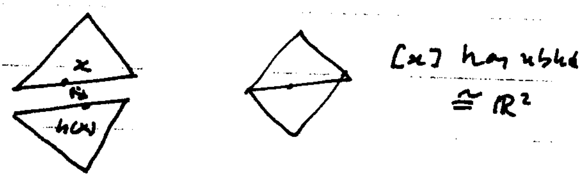
L3-4 How to understand the topology

__Conclusion__ Nil/Γ is the torus bundle over S^1 w/ monodromy matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$
(or the mapping cylinder of $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}: T^2 \to T^2$)

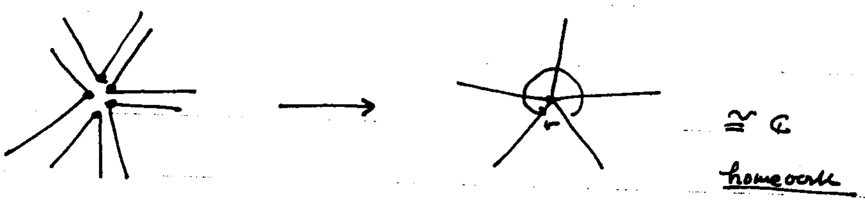
__Introduction to triangulations of low dimensional manifolds__

A finite collection of triangles $\Delta_1, \cdots, \Delta_n$ w/ their edges identified in pairs by homeomorphisms is called a __triangulation of a surface.__

__Problem of manifold nbhd at $x \in e^0$:__



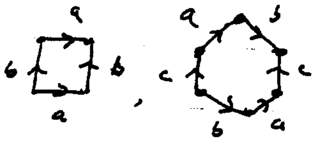
__Problem at vertex__



Inductively ⇒ the quotient space is Always Hausdorff using lemma 3.3.

__Example__: 1. The quotient space of a polygon with edges identified in pairs is a triangulable surface.

The best identification: identify opposite sides in pairs by orientation reversing homeomorphisms



The Euler characteristic of a triangulated (or CW-decomposed) surface X

$$\chi(X) = V - E + F$$

__Note__: Möbius band

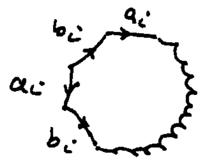
__Thm__ (Rado) Each compact topological surface can be triangulated

__Thm__ (Moise) Each compact topological 3-manifold can be triangulated.

Lecture 4 Triangulations

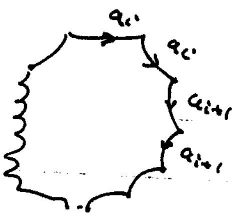
Thm (Möbius) All closed surfaces are homeomorphic to one of

Σ_g $g \geq 0$ or $P_g = \mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2$



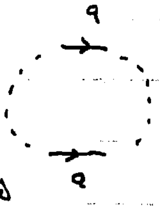
4g-gon

$\chi(\Sigma_g) = 2-2g$



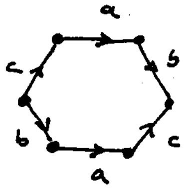
2g

$\chi(P_g) = 1-g$



Several variation: opposite side identification of 4g gon gives Σ_g (orientation reversing w.r.t. the induced orientation)

Example 1.



What is the quotient (2-sided as you can see)?

orientability (see it), Can you draw the ∂D^2 in the surface!

Euler Characteristic $\chi(X)$ of a triangulated (or cw) complex.

2-dim X cut into 2-cells (simply connected pieces) by a graph. Then

$$\chi(X) = V - E + F \qquad F \doteq \# \text{ 2-cells}. \quad E = \# \text{ 1-cells}, etc$$

Above case: Understand the $V=1$, $E = 2g$, $F = 1$

Example 2

$$\chi(X) = 0$$

Basic properties:

Thus

(1). $\chi(X \cup Y) = \chi(X) + \chi(Y) - \chi(X \cap Y)$, $\chi(S^1) = 0$

(2). $X \simeq Y \Rightarrow \chi(X) = \chi(Y)$

$$\chi(\Sigma_g - (2\text{-disk})) = 2-2g-1 \qquad \text{etc}$$

Example 2 What is

$X =$

What is the surface? orientable

∂X one circle, $X \simeq$  $\simeq$ 

$\chi(X) = 2-3 = -1$

$\Rightarrow X \cong$ .

Lecture 4. Triangulations

Corollary X a compact surface (with possibly $\partial X \neq \phi$), then $\chi(X) \leq 2$ so that $\chi(X) = 2$ iff $X = S^2$

Homework. (1) Find out the surface obtained by identifying opposite sides of a $(4n+2)$-gon by orientation reversing homeomorphisms

(2) Show that $\mathbb{Z}_{4g}$ $+ \mathbb{Z}_{4g+2}$ acts non-trivially on Σ_g

(3). The Möbius band B can be embedded in a solid torus $D^2 \times S^1$ as shown.

What is the space $D^2 \times S^1 - \overset{\circ}{N}(B)$? Is it connected ? What is the boundary of $D^2 \times S^1 - \overset{\circ}{N}(B)$?

What is $N(B)$? (the regular nbhd)?

RM: All topological surfaces can be triangulated (Rado)

Triangulation of 3-manifold

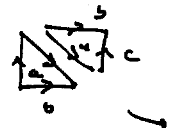
A *triangulation* of a 3-manifold M is given by a homeomorphism between M and the quotient space of a finite set of 3-simplices (tetrahedrons) $\sigma_1, \cdots, \sigma_n$ by identifying pairs of faces of σ_i by affine homeomorphisms. (preserving vertices)

$$K = \sigma_1 \sqcup \cdots \sqcup \sigma_n / \sim \quad \underset{\text{homeo}}{\xrightarrow{\cong}} \quad M$$

$\sim$ the identification

Note, unlike 2-dim, Not every such identifications of faces by homeomorphisms gives 3-manifold.

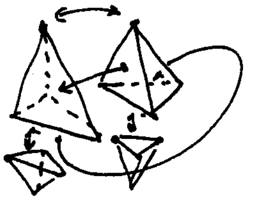
Example 3 2-dim



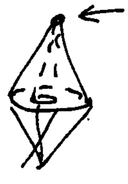
$\Rightarrow$ quotient torus

$= T^2$: Cone construction

now: cone $\rightarrow$



quotient space cone over T^2



$\leftarrow$ non-mfd pt.

Lecture 4 Triangulations

Let K be the quotient space of $\sigma_1^3, \cdots, \sigma_n^3$ by identifying pairs of faces by homeomorphisms preserving vertices. Suppose every faces of $\sigma_1^3, \cdots \sigma$ is identified and K is connected.

Theorem (Seifert) K is a closed 3-manifold $\Leftrightarrow$ $\chi(K) = 0$.

Proof "$\Rightarrow$" Poincare duality $\mathbb{Z}_2$ coefficient:

$$b_i(M, \mathbb{Z}_2) = b_{3-i}(M, \mathbb{Z}_2)$$

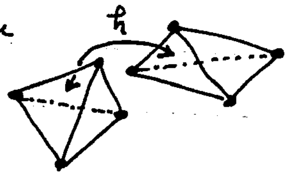
so $\chi(K) = b_0 - b_1 + b_2 - b_3 = 1 - b_1 + b_1 - 1 = 0$

"$\Leftarrow$" Let us understand the local structure.

$p \in K$,

(1) $p \in \overset{\circ}{\sigma}{}_i^3$ $\Rightarrow$ manifold pt

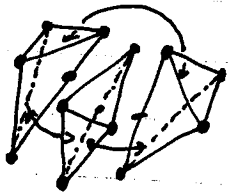
(2) $p \in \overset{\circ}{\sigma}{}_i^2$ · $\sigma_i^2 < \sigma_j^3$ interior of a hyplee $\Rightarrow$ manifold pt.



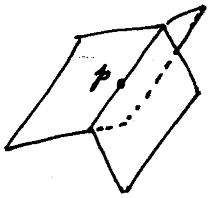
(3) $p \in \overset{\circ}{\sigma}{}_i^1$ interior of an edge:

$\Rightarrow$ manifold pt (by 2-dim argument!)

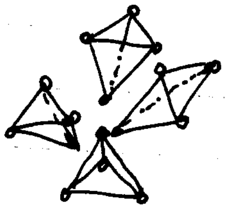
etc.



locally



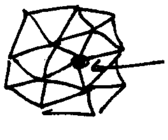
(4) p a vertex! trouble:



For each vertex v in K, the link of v, denoted by $\underset{K}{lk(v)} = lk(v)$ is the triangulated closed surface whose triangles $\Delta_1, \cdots, \Delta_n$ are in 1-1 correspondence w/ those faces of 3-simplexes in σ_3^i having v as a vertex; i.e., opposite to v.

$$v * \Delta_i = \sigma_3^j \quad \text{for some } j$$

Example 4 (2-dim)

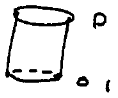


$lk(v)$: triangulation s.

The star of v, $st(v)$

$= v * st(v)$

Lecture 4 Triangulations



Cone construction X space. $C(X) = X\times I / _{X\times 0 \sim v}$

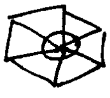
Also denoted by $v * X$ where $v = [X\times 0]$

The star $st(v)$ contains a nbhd of v.

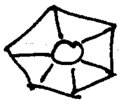
Let $N = K - \bigcup_{v\in K^0} \overset{\circ}{st}(v)$ (K^0 set of all vertices)

(Removing small star open)

Examples, 2-dim



removed



N is a 3-manifold w/ boundary $\partial N = \bigsqcup_{v\in K^0} lk(v) = F_1 \cup\cdots\cup F_m$

Claim $\chi(K)=0 \Rightarrow \forall v\in K^0 \ \ lk(v) \cong S^2 \Rightarrow$ Done.

$M = N \cup_{id} N$ closed 3-manifold $\Rightarrow$

$$0 = \chi(M) = \chi(N) + \chi(N) - \chi(\partial N)$$

so

$$2\chi(N) = \chi(\partial N)$$

Now suppose K has m vertices, $\chi(st(v)) = \chi(pt) = 1$

$K = N \cup (\bigcup_v st(v))$

So

$$\begin{aligned} 0 = \chi(K) &= \chi(N) + \chi(\bigcup_v st(v)) - \chi(\partial N) \\ &= \chi(N) + k - \chi(\partial N) \\ &= \frac{1}{2}\chi(\partial N) + k - \chi(\partial N) \\ &= k - \frac{1}{2}\chi(\partial N) \end{aligned}$$

so

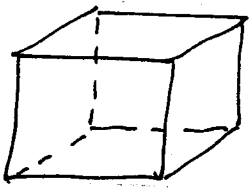
$$2k = \sum \chi(\partial N) = \sum_{i=1}^{k} \chi(F_i) \leq 2k$$

w/ equality iff $F_i \cong S^2$ #

Corollary. M is a quotient space obtained by identifying pairs of faces of a convex polygon by homeomorphisms preserving vertices. Then if $\chi(M)=0 \Rightarrow M$ is a 3-manifold

Lecture 5. Poincaré Homology Spheres

Example 6. The cube w/ opposite faces identified by translations



$V = 1$

$E = 3$

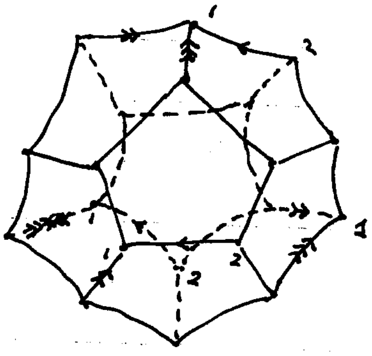
$F = 3$ $\qquad$ $F = \#$ triangles

$T = 1$ $\qquad$ $T = \#$ tetrahedron

$$\chi(M) = 1-3+3-1 = 0 \qquad \Rightarrow M \text{ is a 3-mfd.}$$

Example 7. The Poincaré homology sphere.

Take a Dodecahedron, identify opposite faces by $R_{\frac{4\pi}{5}} \circ A$ where $A(x) = -x$ $\quad R_\Theta$ = left-hand rotation by angle Θ (outward normal vector)



Dodecahedron	quotient space
$V = 20$	$V = 5 = 20/4.$
$E = 30$	$E = 30/3 = 10$
$F = 12$	$F = 6$
$\chi(\partial S) = 2$	$T = 1$

I am going to check only E edges

Thus the quotient space is a 3-mfd. called M.

This is a very famous one:

(It is the quotient of S^3/Γ. $|\Gamma| = 120$) actually

$$H_1(S^3/\Gamma, \mathbb{Z}) = H_2(S^3/\Gamma, \mathbb{Z}) = 0.$$

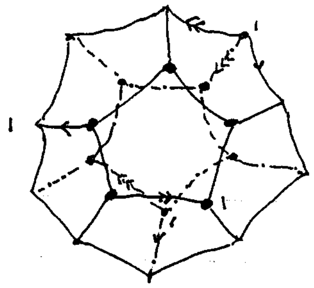
Example 8 (Seifert-Weber) Do the same w/ $R_{2\pi/5} \circ A$ identification

Homework Show that the quotient space is a 3-manifold

This is the first known closed 3-mfd w/ hyperbolic structure (to my best knowledge).

Lecture 5. The Poincaré Homology Sphere

5.1 Example Take a dodecahedron: identify a face F w/ its opposite face F′ as follows:



$$R_{\frac{2\pi}{5}} \circ A \qquad A(x) = -x$$

R_φ right-hand rotation by angle φ (left-hand.)

order 3 (each edge)

Total face : $12/2 = 6$ F

Total edge : $30/3 = 10$ E

total vertices : $20/4 = 5$

$\chi(\ ) = 5 - 10 + 6 - 1 = 0$

Thus, it is a 3-manifold M^3 (In fact $H_i(M) = 0$ $i = 1, 2$)

5.2 Example. Do the identification $R_{2\pi/5} \circ A$. Check that it is still a manifold.

5.3. Example. The trefoil knot complement.  $\cong SL(2, \mathbb{R})/SL(2, \mathbb{Z})$

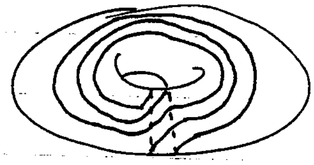
$S^3 = \{ (z, w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1 \} \cong \mathbb{R}^4$

Consider the torus $|z| = |w| = \frac{1}{\sqrt{2}}$ This is standard.

The trefoil knot:

$$\sqrt{2}\, z^3 = w^2 \quad \text{in} \quad |z| = |w| = \frac{1}{\sqrt{2}}$$

$$\left(\begin{matrix} S^1 \times S^1 & \text{trefoil} \\ (\underset{\underset{s}{\|}}{e^{i\alpha}}, \underset{\underset{t}{\|}}{e^{i\varphi}}) & (e^{3i\theta}, e^{2i\theta}) \\ & t^3 = s^2 \ \text{Trefoil} \end{matrix} \right)$$



$0 \le \theta \le 2\pi$

The same argument: $\forall\, a > 0$ $az^3 = w^2$ in $|z|^2 + |w|^2 = 1$ is trefoil.

Now Let us understand $SL(2, \mathbb{R})/SL(2, \mathbb{Z})$

$$\begin{matrix} SL(2, \mathbb{R})/SL(2, \mathbb{Z}) & & \\ \cap & & \\ GL(2, \mathbb{R})/GL(2, \mathbb{R}) & \cong & SL(2, \mathbb{R})/SL(2, \mathbb{Z}) \times \mathbb{R}_{\ge 0} \\ & & \uparrow \\ & & |\det| \end{matrix}$$

$GL(2, \mathbb{Z}) = \{ \det A = \pm 1 \}$.

Lecture 5 Poincaré Homology Sphere

A lattice $L = \mathbb{Z}u \oplus \mathbb{Z}v \subset \mathbb{C}$ is a discrete subgroup u, v indep / $\mathbb{R}$

$[u, v] \in GL(2 \cdot \mathbb{R})$

$GL(2 \cdot \mathbb{R}) / GL(2\, \mathbb{Z}) = \{$ space of all lattices L in $\mathbb{C} \}$

For each L, form the Weierstrass P-function (Elliptic function theory)

$$P_L(z) = \frac{1}{z^2} + \sum_{\omega \in L \setminus 0} \left[\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2} \right]$$

, convergent, meromorphic, per

P_L satisfies the eq.

$$(P')^2 = 4P^3 - g_2 P - g_3$$

where $g_2, g_3 \in \mathbb{C}$ satisfies

$$27 (g_3)^2 \neq g_2^3$$

(i.e, $4\omega^3 - g_2\omega - g_3 = 0$ has no repeated roots).

$$\{L\} \longrightarrow \{ (g_2, g_3) \in \mathbb{C}^2 / \ 27 g_3^2 \neq g_2^3 \}$$
$$L \longmapsto \{(g_2, g_3)\}$$

is a diffeomorphism (Weierstrass theory)

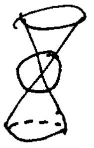
Thus $$SL(2.\mathbb{R}) / SL(2\,\mathbb{Z}) \times \mathbb{R}_{>0} \cong GL(2\,\mathbb{R}) / GL(2.\mathbb{Z}) \cong \mathbb{C}^4 - \{ (z, \omega) \mid 27 z^2 \neq \omega^3 \}$$

$$\cong (S^3 - \mathcal{B}) \times \mathbb{R}_{>0}$$

One checks easily that the homeomorphism preserves the $\mathbb{R}_{>0}$ factors

$$\Rightarrow \quad SL(2.\mathbb{R}) / SL(2.\mathbb{Z}) \cong S^3 - \mathcal{B}.$$

Example $$\mathbb{R}^3 - \{ z^2 = x^2 + y^2 \} \cong \left(S^2 - \{ z^2 = x^2 + y^2 \mid z^2 = 1 \} \right) \times \mathbb{R}_{>0}$$



L6
Hyperbolic Geometry in Dim 2 + 3

$\mathbb{H}^2 = \{ z \mid \operatorname{Im} z > 0 \}$ upper half plane $\qquad z = x + iy$

Hyperbolic metric (Riemannian) $\qquad$ + area

$$ds^2 = \frac{dx^2 + dy^2}{y^2} \qquad\qquad dA = \frac{dx\,dy}{y^2}.$$

meaning a smooth curve $\alpha : [0,1] \longrightarrow \mathbb{H}$ has lengths $L(\alpha) = \int_0^1 \frac{|\alpha'(t)|}{\operatorname{Im} \alpha(t)} dt$

__Transformations of R-metrics__. suppose $z = -\frac{1}{\omega} =$ $\qquad \omega = u + iv$. then $\underline{\overset{*}{F}(ds) = ds}$!

__Isometries__:

self map $F : \mathbb{H} \to \mathbb{H}$ preserving ds^2 i.e., $F^*(ds) = ds$

__Example__: (1). $F(z) = 2z$ or more generally $F(z) = \lambda z$, $\lambda \in \mathbb{R}_{>0}$ (4) $z \mapsto -\bar{z}$

(2). $F(z) = z + a$ $\quad a \in \mathbb{R}$

(3) $F(z) = -\frac{1}{z}$, __check__ directly.

__Def__ __Inversion__ of a sphere $\qquad S = \{ x \in \mathbb{R}^n \mid |x - c| = R \}$

$c = 0 \quad R = 1$: $\qquad x \mapsto \frac{x}{\|x\|^2} \qquad 0 \mapsto \infty$

$c = 0, \quad R > 0 \qquad x \mapsto \frac{R^2 x}{\|x\|^2} \qquad \infty \mapsto 0$

__General__ $\qquad x \mapsto R \frac{(x - c)}{\|x - c\|^2} + c \qquad \begin{matrix} c \mapsto \infty \\ \infty \mapsto \infty \end{matrix}$

__Def__ $\qquad \mathbb{H}^n = \{ (x_1, \cdots x_n) \in \mathbb{R}^n \mid x_n > 0 \}$ upper-half-space

__Hyperbolic metric__ $\qquad ds^2 = \dfrac{dx_1^2 + \cdots + dx_n^2}{x_n^2}$

6.1 __Lemma__. $\quad x \mapsto \frac{x}{\|x\|^2}$ preserves the hyperbolic metric.

__Pf__. $\quad$ say $\quad y_i = \frac{x_i}{\|x\|^2}$

$$dy_i = \frac{dx_i}{\|x\|^2} + x_i\, d\left(\frac{1}{\|x\|^2}\right) = \frac{dx_i}{\|x\|^2} - 2x_i \frac{\sum_j x_j dx_j}{\|x\|^4} = \frac{dx_i}{\|x\|^2} - \frac{2 \sum x_j dx_j}{\|x\|^4} x_i$$

$$(dy_i)^2 = \frac{(dx_i)^2}{\|x\|^4} - 4 \frac{1}{\|x\|^6} \sum_j x_i x_j dx_j dx_i + \left[4 \sum_j x_i^2 x_j^2 dx_j^2 + 4 \sum_{j \neq k} x_i x_j x_k dx_j dx_k \right] / \|x$$

so $\sum_i dy_i^2 = \frac{1}{|x|^4} \sum_i dx_i^2 - \frac{4}{|x|^6} \sum_{i,j} x_i x_j dx_i dx_j + \frac{4}{|x|^6} \sum_{i=j} x_i x_j dx_i dx_j$

$+ \frac{4}{|x|^6} \sum_{j \neq k} x_j x_k dx_j dx_k = \frac{1}{|x|^4} \sum_i (dx_i)^2$

$$\Rightarrow \quad \frac{\sum_i dy_i^2}{y_n^2} = \frac{\sum (dx_i)^2}{x_n^2}. \qquad \square$$

Evidently $x \mapsto \lambda x$ $\lambda > 0$ leaves ds^2 invariant $x \mapsto x + a$, $a \in \mathbb{R}^{n-1} \times 0$

Corollary. Inversions about spheres centered at $x_n = 0$ leave ds^2 invariant.

Homework. (1) Show that ds^2 in $\mathbb{H}^2$ is invariant under $z \mapsto \underset{=: \gamma(z)}{\frac{az+b}{cz+d}}$, $a, b, c, d \in \mathbb{R}$ $ad - bc = 1$.

(2) Show that all $\gamma(z)$ above are compositions of: $z \mapsto \lambda z$, $\lambda > 0$ $z \mapsto z + a$, $a \in \mathbb{R}$ and $z \mapsto -\frac{1}{z}$.

Thus we have proved.

Lemma $SL(2, \mathbb{R})/_{\pm 1} \subset Iso^+(\mathbb{H}^2)$

Lines in $\mathbb{H}^2$

Lemma 1. The positive y-axis is a geodesic in $\mathbb{H}^2$

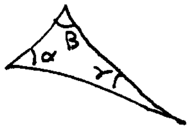
Lemma 2. All circles and lines $\perp$ x-axis are geodesics in $\mathbb{H}^2$

Lemma 3. All geodesics in $\mathbb{H}^2$ are as listed in lemma 2.

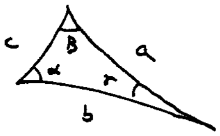
Lemma 4. $d(z, w) = \log (z, w, \tilde{w}, \tilde{z})$ the cross ratio

Lemma 5. $PSL(2\ \mathbb{R}) = Iso^+(\mathbb{H}^2)$.

Lemma 6. (The Gauss-Bonnet) The area of a triangle is $\pi - (\alpha + \beta + \gamma)$



Lemma 7 (The cosine Law)



$$\cosh a = \frac{\cos\alpha + \cos\beta \cos\gamma}{\sin\beta \sin\gamma}$$

$$\cos\alpha = \frac{-\cosh a + \cosh b \cosh c}{\sinh b \sinh c}, \qquad \frac{\sin\alpha}{\sinh a} = \frac{\sin\beta}{\sinh b}$$

L6. Hyperbolic Geometry in Dim 2 + 3.

The disk model $\mathbb{D}^2 = \{ z \in \mathbb{C} \mid |z|<1 \} \xrightarrow[i\frac{z-1}{z+1}]{} \mathbb{H}$

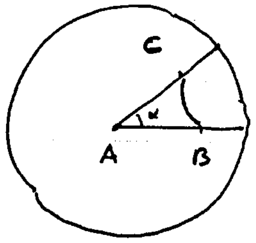
The pull back metric is $\dfrac{2|dz|}{1-|z|^2}$

Isometries: $z \mapsto e^{i\theta}\left(\dfrac{z-a}{\bar{a}z-1}\right)$ $|a|<1$. $\theta\in\mathbb{R}$

Geodesics:

Distances. $d(0,z) = d(0,|z|) = \ln\left(\dfrac{1+|z|}{1-|z|}\right) \Leftrightarrow \cosh d = \dfrac{1+|z|^2}{1-|z|^2}$

Example Hyperbolic cosine Law: $\triangle ABC$ as shown $A=0$; $B=x$ $c = ye^{i\alpha}$



$c = \ln\dfrac{1+x}{1-x}$, $\cosh c = \dfrac{1+x^2}{1-x^2}$

$b = \ln\dfrac{1+y}{1-y}$, $\cosh b = \dfrac{1+y^2}{1-y^2}$

$$a = d(x, ye^{i\alpha}) = d\left(0, \frac{ye^{i\alpha}-x}{xye^{i\alpha}-1}\right)$$

So
$$\cosh a = \frac{|xye^{i\alpha}-1|^2+|ye^{i\alpha}-x|^2}{|xye^{i\alpha}-1|^2-|ye^{i\alpha}-x|^2}$$

$$= \frac{[x^2y^2+1+x^2+y^2 - xy(e^{i\alpha}+e^{-i\alpha}) - xy(e^{i\alpha}+e^{-i\alpha})]}{x^2y^2+1-x^2-y^2 - xy(e^{i\alpha}+e^{-i\alpha}) + xy(e^{i\alpha}+e^{-i\alpha})}$$

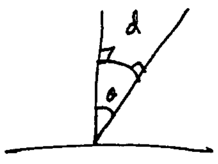
$$= \frac{(1+x^2)(1+y^2) - 4xy\cos\alpha}{(1\mp x^2)(1\mp y^2)}$$

$$= \left(\frac{1+x^2}{1-x^2}\right)\left(\frac{1+y^2}{1-y^2}\right) - \frac{2x}{1-x^2}\cdot\frac{2y}{1-y^2}\cos\alpha$$

$$= \cosh b\cosh c - \sinh b\sinh c \cdot \cos\alpha$$

□

Homework:



- Find the distance d in terms of θ
- (Gromov) show that the area of a hyperbolic triangle is less than the length of a side.

__Corollary__ $\forall n \geq 5$. $\exists$ regular hyperbolic n-gon with inner angle $\frac{2\pi -}{n}$

As a consequence, each orientable surface of genus ≥ 2 has a hyperbolic metric.

__Example__ Hyperbolic cylinder $h(z) = \lambda z$ $\mathbb{H}/_{z \sim \lambda z}$

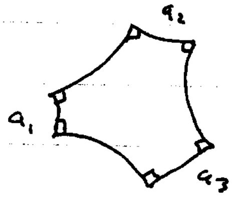
Geodesics in the cylinder

__Example__ Hyperbolic cusps $\mathbb{H}/_{z \sim z+1}$.

No geodesics (closed and simple)

__Example__ Hyperbolic right angled hexagon.

__Thm__ (F-N) $\forall a_1, a_2, a_3 > 0$ $\exists!$ hyperbolic right-angled hexagon with non-adjacent edge lengths a_1, a_2, a_3



__Corollary__ The hyperbolic metrics with geodesic boundary on the 3-holed sphere $\Sigma_{0,3}$ are completely determined up to isometry by its boundary lengths.

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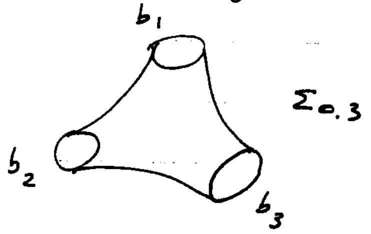
L 7. Hyperbolic Geometry

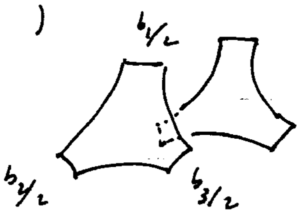
Last time.

Thm (Fenchel-Nielsen) $\forall\ a_1, a_2, a_3 > 0$, $\exists$: right-angled hyperbolic hexagon w/ non-adjacent edge lengths a_1, a_2, a_3.

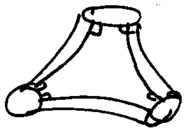
Corollary. $\forall\ b_1, b_2, b_3 > 0$, $\exists!$ hyperbolic structure on the cpt 3-holed sphere $\Sigma_{0,3}$ so that its boundary are geodesics of lengths b_1, b_2, b_3



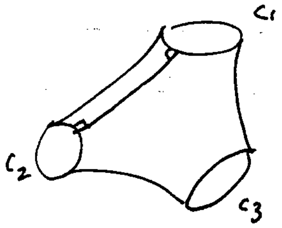
pf: (1) Existence. Take two right-angled hexagons of edge lengths ($b_1/2$, $b_2/2$, $b_3/2$)



glued by isometries

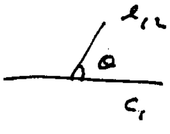


(2) Uniqueness, Suppose $(\Sigma_{0,3}, d)$ is such a metric w/ geodesic boundaries c_1, c_2, c_3



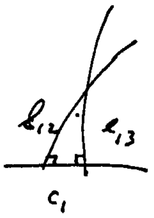
Joint c_i, c_j by the shortest path l_{ij}

Then (1) $l_{ij} \perp c_i$ (since it is the shortest)



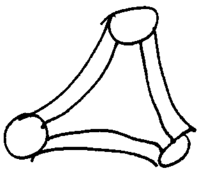
$\theta < \pi/2$
$\Rightarrow$ can be shorten

(2) $l_{ij} \cap l_{ik} = \phi$:

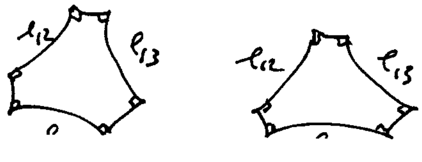


otherwise, $\Rightarrow \exists$ hyperbolic triangle of inner angle $(\pi/2, \pi/2, 0)$.

Thus, it must be:



$\Rightarrow$ cut open to obtain two right-angled hexgons. P_1 P_2

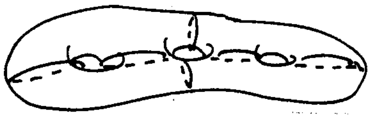


uniqs of F N thm
$\Rightarrow$ P_1 isometri to P

Hyperbolic Geometry

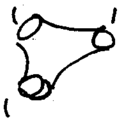
<u>Thm</u>: For all $g \geq 2$, there exists a hyperbolic structure on Σ_g.

<u>Pf 1</u>: Each surface Σ_g, ≥ 2 can be decomposed into a union of $\Sigma_{0,3}$'s by disjoint s.c.c.



(<u>Hw</u>: For Σ_g, you need $3g-3$ s.c.c's to cut them into $2g-2$ 3-holed spheres)

Now assign any positive number, say 1 to each curve and take



hyperbolic 3-holed sphere and glue them along the boundary by the identity maps.

<u>RM</u>: 1. By assigning arbitrary positive numbers $l_i \overset{\in \mathbb{R}_{>0}}{}$ to the i-th circle & gluing by a twist (rotation) $t_i \in \mathbb{R}$.

We obtain a hyperbolic metric $(l_1\ t_1, \cdots, l_{3g-3}\ t_{3g-3}) \in (\mathbb{R} \times \mathbb{R}_{>0})^{3g-3} =$

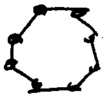
All hyperbolic metrics are of these forms

The space of all hyperbolic metrics modulo isometries is of dim $6g-6$.

(The above is called the Fenchel-Nielsen coordinate of the Teichmüller space

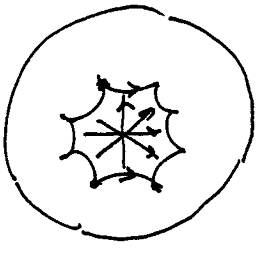
2.

<u>Pf 2</u>. The surface Σ_g is the quotient of the $4g$-gon by identifying opposite sides by translations. ($g=2$:



one vertex in Σ_2)

Now take a regular hyperbolic $4g$-gon P of inner angles $2\pi/4g$. identify the opposite sides by hyperbolic isometries $\gamma_i \in PSL(2, \mathbb{R})$.

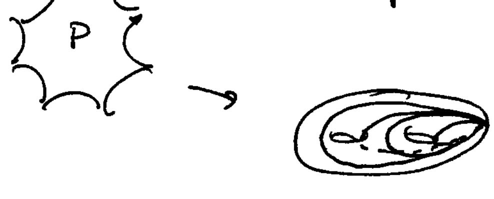


Then the quotient space Σ_g has a hyperbolic structure.

<u>The charts</u>: $[p] \in \Sigma_g$

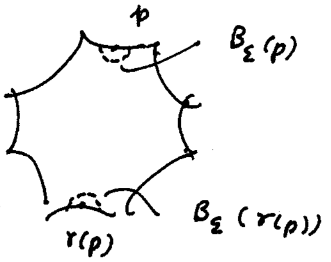
(i) $p \in \text{int}(P)$. $\mathcal{U} = [\text{int}(P)]$

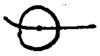
$\phi: \mathcal{U} \longrightarrow \text{int}(P)$



Hyperbolic Geometry

(ii) $p \in \text{int}(e)$ e an edge in P





~~$U_p = (P \cap \mathring{B}_\varepsilon(p)) \cup \gamma($~~

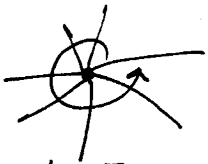
$$U_p = [P \cap \mathring{B}_\varepsilon(p) \cup \mathring{B}_\varepsilon(\gamma(p))]$$

$$\phi_p : U_p \longrightarrow \mathring{B}_\varepsilon(p).$$

$$[x] \longrightarrow x \quad \text{or} \quad \gamma^{-1}(x)$$

(iii). Since the sum of inner angles at ~~$[v]$~~ all vertices v vertex of $P = 2\pi$

the quotient space



We can define the hyperbolic charts ϕ at $[v] \in \Sigma_g$ in the same way.

using all corners of P and the gluing map. *

Hw. Check that the transition functions are in $\text{Iso}^+(\mathbb{H})$. (In fact, transitions are in $\Gamma = \langle \gamma_1, \cdots, \gamma_4 \rangle$ generated by the identifications of the edges).

□

We have in fact proved a theorem of Poincaré

Theorem (Poincaré Polyhedron thm)

Suppose P is a compact convex polygon in $\mathbb{H}^2$ whose sides are identified in pairs by elements $\gamma_1, \ldots, \gamma_n$ in $\text{Iso}^+(\mathbb{H}^2)$ so that if $\{v_{i_1}, \ldots, v_{i_m}\}$ are vertices in P identified to be one vertex in $P/\sim$ then the sum of the inner angles at $v_{i_1}, \ldots, v_{i_m}$ is 2π.

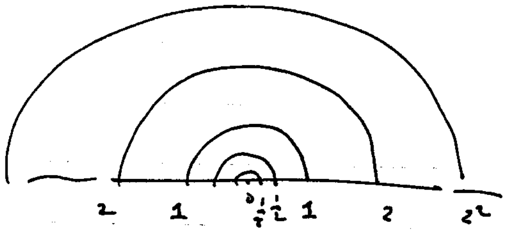
Then the quotient space $P/\sim$ has a hyperbolic structure whose transition functions are in $\langle \gamma_1 \cdots \gamma_n \rangle = \Gamma$

(RM $\Gamma \subset PSL(2.\mathbb{R})$ discrete and $\mathbb{H}/\Gamma \underset{\text{isometric}}{\cong} P/\sim$)

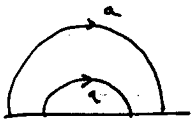
6-22-04

Lecture 7   Hyperbolic Surfaces

Example 1. The surface $\gamma(z) = 2z$ $\mathbb{H}/z\sim 2z$

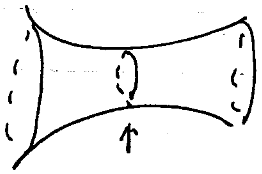


So



all area = π

It is a cylinder



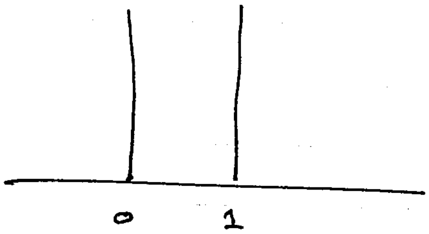
Area = infinite. why?

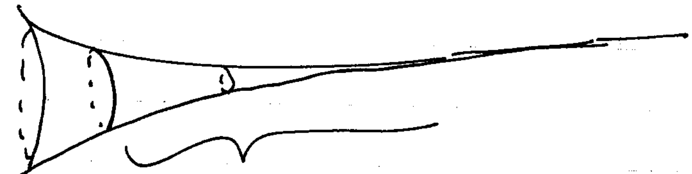
$\cong$ $S^1\times(-1,1)$

Hw. Can you find All closed geodesics in $\mathbb{H}/z\sim 2z$? (close = cpt)

Example 2 The cusp:

$\gamma(z) = z+1$ $\mathbb{H}/z\sim z+1$



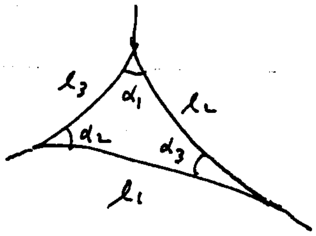


Finite area length → 0 very fast.

The Cosine Law for Hyperbolic Triangles

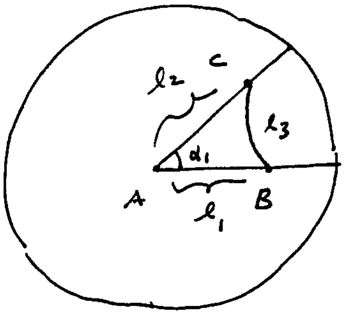
of edge lengths l_1, l_2, l_3 and inner angles $\alpha_1, \alpha_2, \alpha_3$. Then

$$\cosh l_i = \frac{\cos\alpha_i + \cos\alpha_j \cos\alpha_k}{\sin\alpha_j \sin\alpha_k}$$

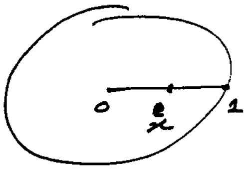


$i\neq j\neq k\neq i$

Proof. Put it as: ΔABC $A = 0$ center B positive x-axis.



Recall



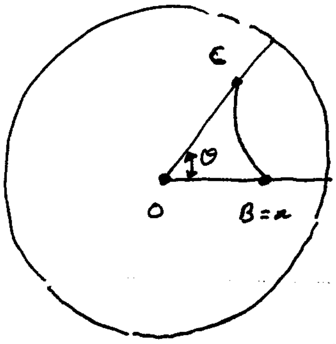
$d_{\mathbb{D}}(0,x) = d$

$$\cosh d = \frac{1+x^2}{1-x^2}$$

6-22-04

Hyperbolic Surfaces

-7.5-



$B = x \quad \in (0,1) \qquad C = ye^{i\theta}, \quad y \in (0,1) \quad \theta \in (0,\pi)$

Let $b = d_{\mathbb{D}^2}(0, B)$, $c = d_{\mathbb{D}^2}(0, C)$

$a = d_{\mathbb{D}^2}(B, C)$

distance formula: $\cosh b = \dfrac{1+x^2}{1-x^2}$

Now:
$$a = d_{\mathbb{D}^2}(x, ye^{i\theta}) \underset{\substack{z \mapsto \frac{z-x}{\bar{x}z-1} \\ x \mapsto 0 \\ y \mapsto}}{=\!=} d_{\mathbb{D}^2}\left(0, \frac{ye^{i\theta}-x}{xye^{i\theta}-1}\right)$$

So
$$\cosh a = \frac{1+|z|^2}{1-|z|^2} = \frac{|ye^{i\theta}-x|^2 + |xye^{i\theta}-1|^2}{|xye^{i\theta}-1|^2 - |ye^{i\theta}-x|^2}$$

$$\underset{\text{+ simplify}}{\overset{\text{expand}}{=\!=}} \frac{(1+x^2)(1+y^2) - 4xy\cos\theta}{(1-x^2)(1-y^2)}$$

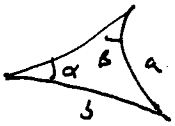
$$= \cosh b \cosh c - \sinh b \sinh c \cos\theta$$

So
$$\cos\theta = \frac{-\cosh a + \cosh b \cosh c}{\sinh b \sinh c.}$$

#

HW: Use the cosine Law ⇒

(1) Sine Law, $\dfrac{\sinh a}{\sin\alpha} = \dfrac{\sinh b}{\sin\beta}$



(2) cosine Law II. $\cosh l_i = \dfrac{\cos\alpha_i + \cos\alpha_j \cos\alpha_k}{\sin\alpha_j \sin\alpha_k}$.

Lecture 8. $\mathbb{H}^3$

Define $\mathbb{H}^3 = \{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 > 0 \}$ upper-half-space

$= \mathbb{C} \times \mathbb{R}_{>0}$ (z, t) $t > 0$

The metric $ds^2 = (dx_1^2 + dx_2^2 + dx_3^2) / x_3^2$

The isometries:

Example. $x \mapsto x / \|x\|^2$ or even $x \mapsto \frac{R^2 (x-c)}{\|x-c\|^2} + c$, $c \in \mathbb{R}^2 \times 0$

are isometries.

$x \mapsto x + c$ $c \in \mathbb{R}^2 \times 0$ $x \mapsto SO(3).x$ · K $O(3).x$

These are all isometries

* Key Property. Inversion preses ags, the set of .....

Back to $\mathbb{C}$, (more precisely the Riemann sphere $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$)

Each fractional linear transformation $z \mapsto (az+b)/(cz+d)$ $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2,\mathbb{C})$

induces $\bar{\mathbb{C}} \to \bar{\mathbb{C}}$ bijection. $\quad$ " $\gamma(z)$

is a composition of $z \mapsto z + a$, $z \mapsto \lambda z$, and $z \mapsto \frac{1}{z}$ $z \mapsto \bar{z}$

As a consequence, $\lambda \in \mathbb{C}$

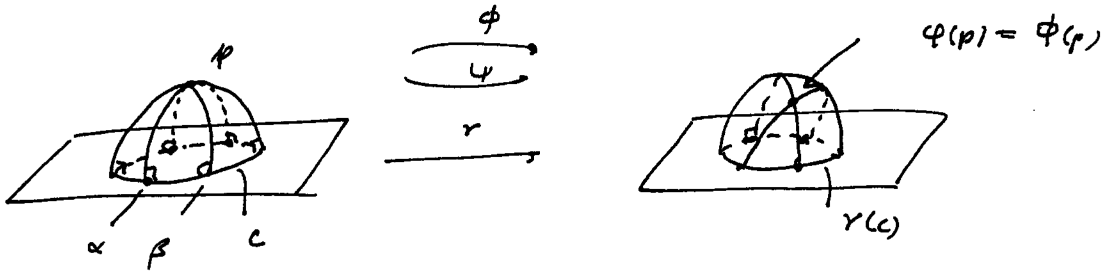
Each $\gamma : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ can be extended to a map $\mathbb{H} \to \mathbb{H}$ as a diffeomorphism, which are (Möbius transformation):

- preserving angles
- preserving the set of all circles and lines
- preserving the set of 2-spheres & 2-planes.

8.1. Lemma The extension of $\gamma(z) = \frac{az+b}{cz+d} : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ to $\tilde{\varphi} : \mathbb{H}^3 \to \mathbb{H}^3$ with above property is unique.

Proof. Suppose φ, ψ are two such extensions w/ $\varphi|_{\bar{\mathbb{C}}} = \psi|_{\bar{\mathbb{C}}}$.

Take a point $p \in \mathbb{H}^3$



\#

α, β two circles on the sphere $\perp$ c. $\Rightarrow$ Done

6-22-04

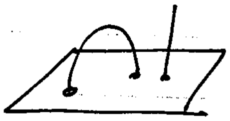
Lecture 8. $\mathbb{H}^3$

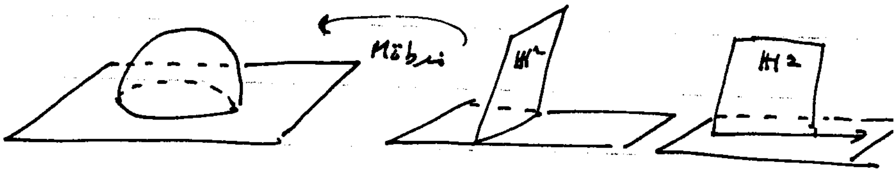
Thus, we have

<u>Prop</u>1 $PSL(2, \mathbb{C}) \subseteq Iso^+(\mathbb{H}^3)$

The same proof shows they are equal.

<u>Prop</u>2. All geodesics in $\mathbb{H}^3$ are circles and lines $\perp$ $\mathbb{C}$ $(= \mathbb{R}^2 \times 0)$





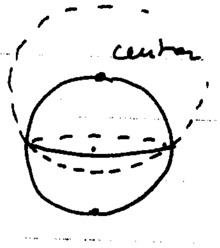
<u>The same proof</u>

<u>Proposition</u>. All <u>totally geodesic</u> surfaces in $\mathbb{H}^3$ are 2-spheres or planes $\perp$ $\mathbb{C}$

<u>Recall</u>: $\Sigma \subset M^n$. Riemann is called <u>totally geodesic</u> if $\forall$ geodesic γ so that $\gamma(0) \in \Sigma$, $\gamma'(0) \in T_{\gamma(0)}\Sigma \Rightarrow \gamma \subset \Sigma$.

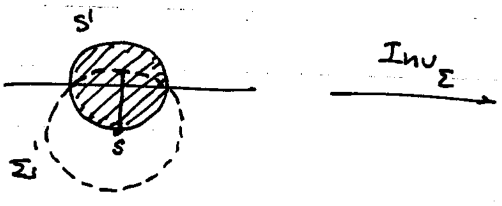
<u>Example</u>. $\mathbb{R}^3$ planes are totally geodesics.

<u>The ball model</u>: Let $\mathring{B}^3 = \{ x \in \mathbb{R}^3 \mid |x| < 1 \}$ open ball.



center $N = (0,0,1)$ radius $\sqrt{2}$. Σ New sphere

$Inv_\Sigma(S^2)$ = the plane



Then

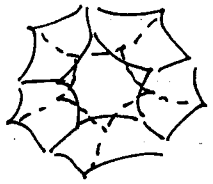
<u>We obtain the ball model</u>, $Iso^+(\mathring{B})$ hard to

But is all generated by inversion on S^2's $\perp \partial B^2$

- Geodesics = circles + lines $\perp \partial B^2$
- angles are the same as $\mathbb{E}^3$
- metric is $4(dx_1^2 + dx_2^2 + dx_3^2) \big/ (1-|x|^2)^2$: $\mathbb{E}^3$ rotations about a diameter is an isometry $SO(3) \subset Iso^+(\mathbb{H}$

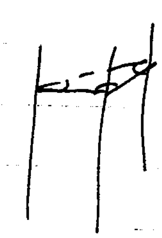
Lecture 8. $\mathbb{H}^3$

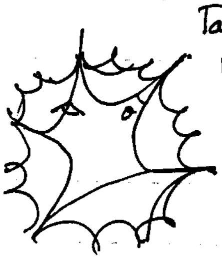
Proposition There exists a regular hyperbolic dodecahedron of dihedral angle $2\pi/5$. Dod

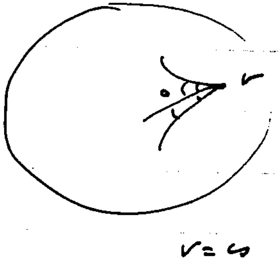


Proof Consider a time dependent family. where $t=$ edge length.

As $t \to \infty$



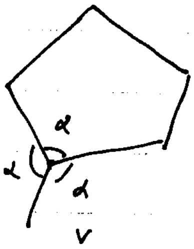




$\Rightarrow \boxed{\theta = \pi/3}$ ideal one $\pi/3 < 2\pi/5$

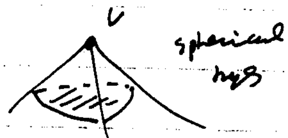
As $t \to 0$. it becomes Euclidean.

Question What is the dihedral angle of a Euclidean regular dodechedron?

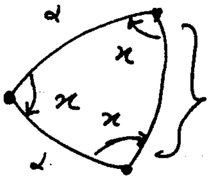


$\alpha = 3\pi/5$

vertex link



$\Rightarrow$ regular sphrical tri



$\}$ $3\pi/5 = \alpha$

$$\cos x \underset{\substack{\uparrow \\ \text{cosine Law.}}}{=} \frac{+\cos\alpha \mp \cos^2\alpha}{\sin^2\alpha} = \frac{\cos\alpha(1-\cos\alpha)}{(1+\cos\alpha)(1-\cos\alpha)} = \frac{\cos\alpha}{1+\cos\alpha}$$

Use calculator $\cos^{-1}\left(\dfrac{\cos 3\pi/5}{1+\cos 3\pi/5}\right) > \dfrac{2\pi}{5}$.

※

We use the 3-Dim:

Poincaré Polyhedron Theorem

There exists a closed 3-manifold w/ a hyperbolic structure obtained by identifying opposite faces of $P_{2\pi/5}$ by isometry of $\mathrm{Iso}^+(\mathbb{H}^3)$

(1) $A: x \mapsto -x \in \mathrm{Iso}^+($

(2) $R_{2\pi/5} \in \mathrm{Iso}^+(\mathbb{B}^3)$

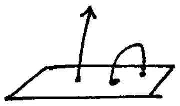
$\mathbb{I}$

Lecture 9. $\mathbb{H}^3$

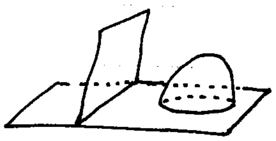
Recall $\mathbb{H}^3 = \mathbb{C} \times \mathbb{R}_{>0}$

$Isot(\mathbb{H}^3) \cong PSL(2,\mathbb{C})$

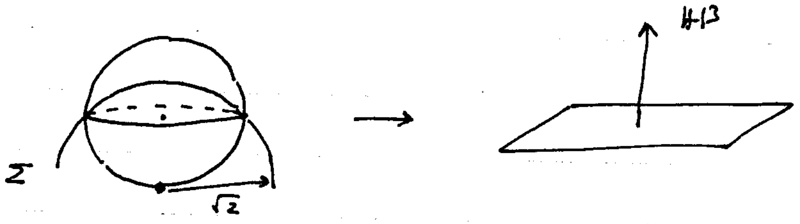
geodesics = circles + Lines $\perp \mathbb{C}$.



totally geodesic surfaces = 2-spheres + 2-planes $\perp \mathbb{C}$,



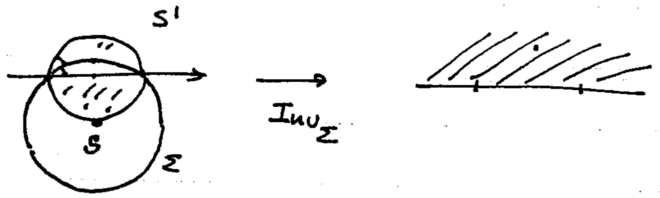
The ball model: $B^3 = \{ |x|<1 \text{ in } \mathbb{R}^3 \}$.



Let Σ be the 2-sphere centered at $S=(0,0,-1)$ of radius $\sqrt{2}$. $\Sigma \cap \mathbb{C} = \partial B^3 \cap \Sigma$

Then $Inv_\Sigma(B^3) = \mathbb{H}^3$

(Better see it in dim = 2



)

- $Iso^+(\mathring{B}^3)$ möbius transformations preserving ∂B^2 (same angle)
- geodesics
- totally geodesic surfaces
- Riemannian metric $ds^2 = 4\left(dx_1^2+dx_2^2+dx_3^2\right) \Big/ \left(1-|x|^2\right)^2$

Example. Rotations about any line through 0 : preserves ∂B^3, —.

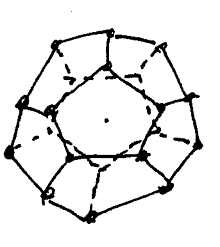
$\Rightarrow$ $SO(3) \subset$ Möb( ) or $Iso^+(\mathring{B}^3)$ $\quad x \mapsto -x \in SO(3)$

$O(3) \subset Iso(B^3)$

Proposition. There exists a regular hyperbolic dodecahedron of dihedral angle $2\pi/$

P

Pf. Take the regular Euclidean ($\mathbb{E}^3$) dodecahedron inscribed in ∂B^3.



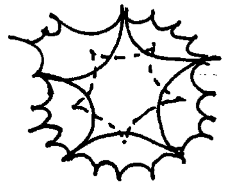
Join each vertex of P to 0

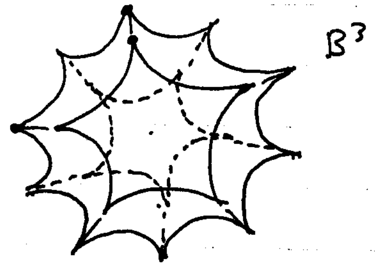
Now shrinking P toward 0

parametrized by t.

where $t = d_{\mathbb{H}^3}(\text{vertex}, o)$

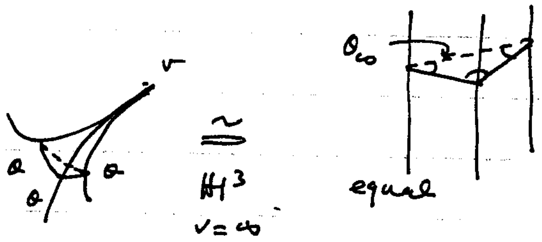
θ_t = dihedral angle





As $t \to +\infty$ $\Rightarrow$ ideal regular

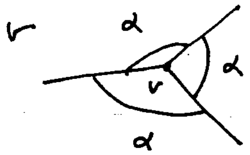
$$\theta_t \longrightarrow \pi/3 < 2\pi/5$$



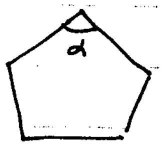
Euclidean triangle!

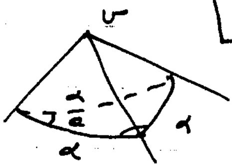
As $t \to 0$ $\Rightarrow$ $\mathbb{E}^3$ regular dodecahedral $\theta_0 \to ? > 2\pi/5$.

How to see it?

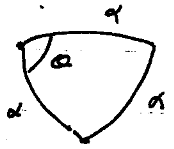


$\alpha = 3\pi/5$





$\Rightarrow$ Spherical triangle



Cosine Law:

$$\cos\theta = \frac{\cos\alpha - \cos^2\alpha}{\sin^2\alpha} = \frac{\cos\alpha}{1+\cos\alpha}$$

$$\theta = (\arccos)\left(\frac{\cos 3\pi/5}{1+\cos 3\pi/5}\right) > \frac{2\pi}{5}$$

#

Corollary The Seifert-Weber closed 3-manifold has a hyperbolic structure.

identifying opposite faces by rotating: $R_{2\pi/5} \circ A$ $\qquad R_{2\pi/5}, A \subset SO(3) \subset \mathrm{Isom}(B^3)$

$\Rightarrow$ dihedral angles match.

(Each edge is identified w/ exactly 5 others)

3-Dim. Poincaré Polyhedron thm

$\Rightarrow$ the quotient space.

L9. Hyperbolic 3-Space

Classification of Isometries

Each $SL(2,\mathbb{C})$ has two eigenvalues $\lambda, \frac{1}{\lambda}$ $\quad \lambda \in \mathbb{C} \setminus \{0\}$.

If $\lambda \neq \frac{1}{\lambda} \Leftrightarrow \lambda^2 \neq 1$, then Linear algebra $\Rightarrow$

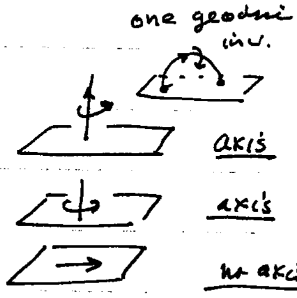
A is conjugate to $\begin{bmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{bmatrix}$ by an element in $SL(2,\mathbb{C})$

If $\lambda = \frac{1}{\lambda}$ i.e. $\lambda = \pm 1$, then either $A = \pm I$ or $A \sim \pm\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Classification Thm

(a) Each $A \in Iso^+(\mathbb{H}^3)$ is conjugate to

- (a1) $z \mapsto \lambda z$ $\quad |\lambda| > 1, |\lambda| < 1$ loxodromic
- (a2) $z \mapsto e^{i\theta} z$ $\quad$ elliptic
- (a3) $z \mapsto z + 1$ $\quad$ parabolic
- (a4) $z = z$ identity (could be think of elliptic)



In the ball model:


loxodromic


elliptic rotation

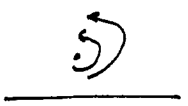

parabolic

Classification $A \in Iso^+(\mathbb{H}^2)$

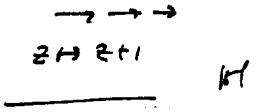
Each $A \in Iso^+(\mathbb{H}^2)$ is conjugate in $Iso^+(\mathbb{H}^2)$ to

- (b1) $z \mapsto \lambda z$ $\quad \lambda \in \mathbb{R}$ $\quad 0 < \lambda < 1$ or, $1 < \lambda < \infty$ (hyperbolic)
- (b2) $z \mapsto e^{i\theta} z$ in the disk model $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ elliptic
- (b3) $z \mapsto z + 1$ parabolic
- (b4) $z \mapsto z$ identity

hyperbolic



$$z \mapsto \frac{\cos\theta\, z + \sin\theta}{-\sin\theta\, z + \cos\theta} \qquad i \mapsto i$$



parabolic

The hyperbolic isometry in $\mathbb{H}$ has an <u>axis</u> which projects to a closed geodesics.

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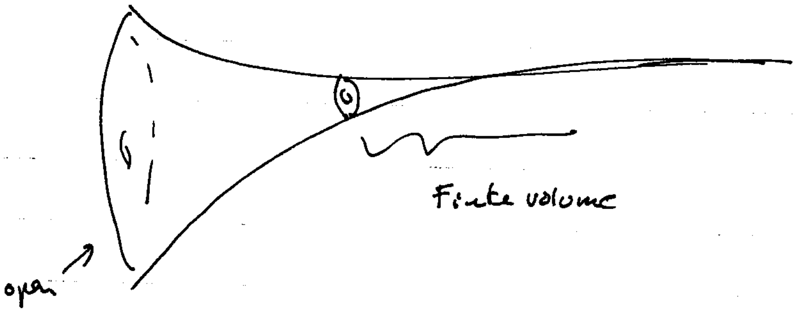
Lecture 9. $\mathbb{H}^3$

Example. Hyperbolic 3-dim cusp: $\Gamma = \mathbb{Z} + i\mathbb{Z}$ translations $\begin{bmatrix}1&1\\0&1\end{bmatrix}, \begin{bmatrix}1&i\\0&1\end{bmatrix}$

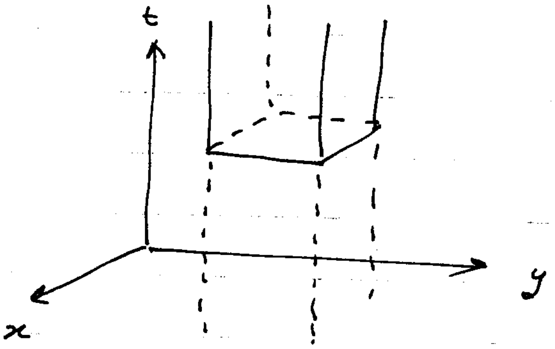
$\mathbb{H}^3/\Gamma$ = topologically

$T^2 \times \mathbb{R}_{>0}$

Geometrically:



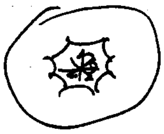
$dv_{\mathbb{H}^3} = \frac{dx\,dy\,dt}{t^3}$ volume of a cusp $\int_{\substack{0\le x\le 1\\0\le y\le 1}}\int_1^\infty \frac{dx\,dy\,dt}{t^3} < +\infty$



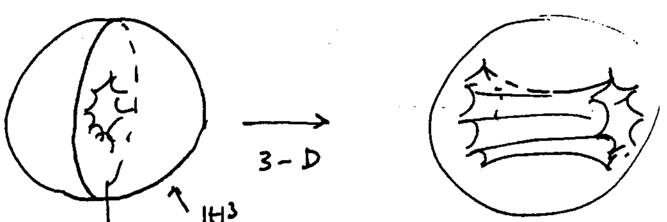
Example Γ discrete group of $SL(2,\mathbb{R})$ st $\mathbb{H}^2/\Gamma$ =

Σ_g. hyperbolic surface of genus 2.

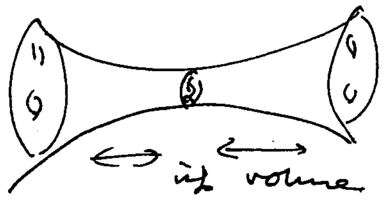
What is $\mathbb{H}^3/\Gamma$?



3-dim Ball:



$\mathbb{H}^3/\Gamma$:



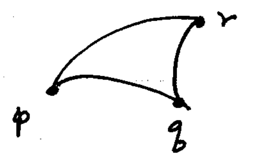
Lecture 10 Hyperbolic $\mathbb{H}^3$, Ideal Tetrahedron

Convex hull construction

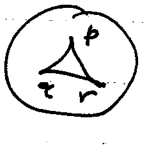
<u>Convex set</u>: $X \subset \mathbb{H}^3$ iff $\forall\, p, q \in X$, the geodesic segment $\overline{pq} \subset X$

Convex hull (X) $\quad ch(X) = \cap$ (all closed convex set contain X)

<u>Example</u> $p, q, r \in \mathbb{H}^3$, not in a geodesic then $ch(p, q, r) =$ hyperbolic Δpqr

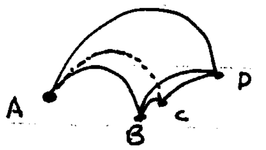


How to find:

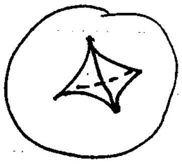


<u>$H^2 \subset H^3$</u>

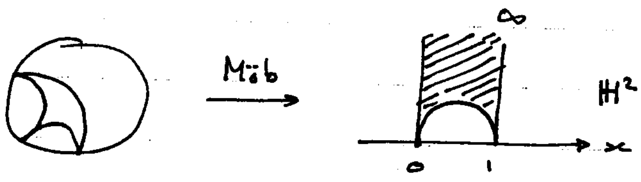
<u>Example</u> <u>Four pts</u> A, B, C, D in $\mathbb{H}^3$



$\mathbb{H}^3$: $\quad B^3$



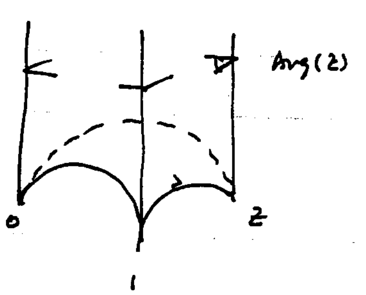
<u>Ideal Triangle</u>: $p, q, r \in \partial\mathbb{H}^2$



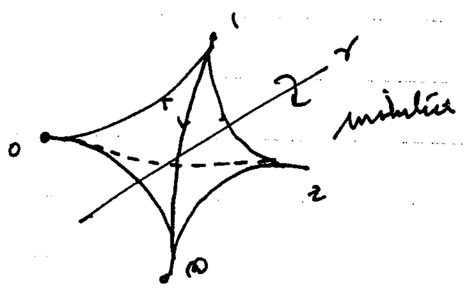
$\exists !$ hyperbolic ideal triangle of area $= \pi$

<u>Example</u> Ideal tetrahedron: $P, Q, R, S \in \overline{\mathbb{C}}$ distinct. $\exists\, \gamma \in PSL(2, \mathbb{C})$ st

$\gamma(P) = 1 \quad \gamma(Q) = 0, \quad \gamma(S) = \infty \quad \gamma(R) = z \in \mathbb{C} \setminus \{0, 1\}$

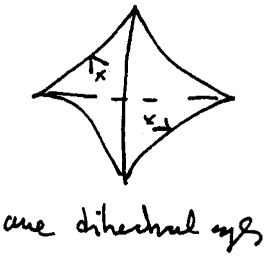


$\mathbb{H}^3$



Let its dihedral angles be $x, y, z \ldots$, then $x + y + z = \pi$ at each vertex

Linear algebra $\Rightarrow$



opposite sides have the same dihedral angles.

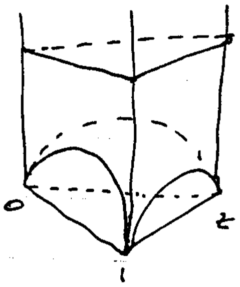
<u>Another way to see</u>

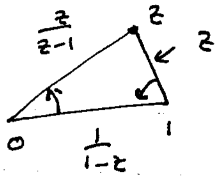
$(0, \infty, 1; z) \mapsto (\infty, 0, z, 1)$

$f: w \mapsto \frac{z}{w}$

Lecture 10. Hyperbolic $\mathbb{H}^3$, Ideal Tetrahedron

The ideal σ^3 can be parametrized by $z \in \mathbb{C} \setminus \{0, 1\}$, s.t. $z, \frac{1}{1-z}, \frac{z}{z-1}$ Represent the same (isometric) ideal σ^3.





$z \in \mathbb{H}^2 \Rightarrow \frac{1}{1-z}, \frac{z}{z-1} \in \mathbb{H}^3$

$\gamma(z) = \frac{1}{1-z} \quad \gamma \circ \gamma(z) = \frac{z}{z-1}$

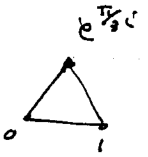
$\gamma \circ \gamma \circ \gamma(z) = z$

It is the rotation $\left[\begin{matrix} \cos\frac{\pi}{3} & \sin\frac{\pi}{3} \\ \sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{matrix}\right.$

Its ∂-boundary (components))

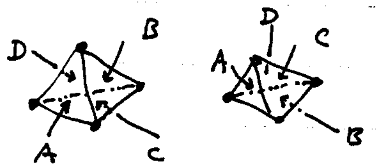
Its 4 boundary simplices are ideal triangles. All are pairwise isometric.

The most symmetric ideal σ^3: $z = e^{\frac{\pi}{3}i}$ regular



Lecture 11. Thurston's Example

Take σ_1^3, σ_2^3 + identify their faces by affine homeomorphisms so that the arrows are match ⇸ ⇻

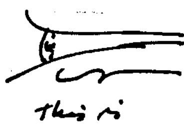


Hw: Show that the quotient space $P - [v]$ is orientable.

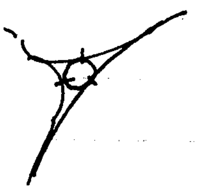
Known $P - [v]$ 3-manifold w/ end $T^2 \times \mathbb{R}$

Thm (Riley) $P - [v]$ has a complete hyperbolic structure of finite volume By taking two regular ideal σ_1^3. σ_2^3. and glued by isometries

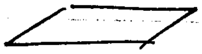
Why complete: $\Leftrightarrow$ A nbhd of ∞ ($\Leftrightarrow$ $(P-[v]) - (\text{cpt})$) $\overset{\text{isometric}}{\cong}$ cusp

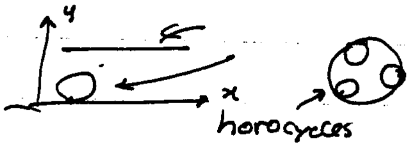


Quick reason: mid-pts of edges of an ideal triangle
Regular ideal tetrahed = all midpts match.

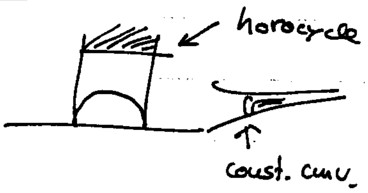


Def. A **horocycle** (**horosphere**) in $\mathbb{H}^3$ is a 2-sphere tangent to $\partial\mathbb{H}^3$:

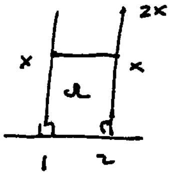




We usually take the boundary of a cusp = a horocycle:



Example:



incompleteness of $\{z \mid 1 \le \mathrm{Re}(z) \le 2\}/z \sim 2z$: horocycles do not match

Hw: Check that this does not occur for Thurston's example.
(Take $d > 0$, and a horocycle sphere of distance $= d$ from the mid-pts)

Thurston's Example

We construct the cusp end of $P - [v]$ directly.

Thurston Thm. $P - [v] \cong S^3 -$ 

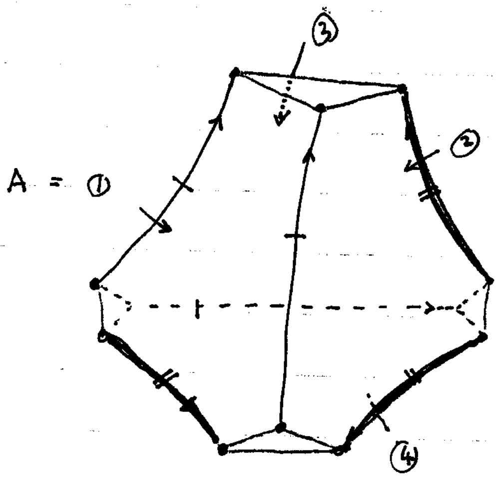
Lecture 11. Thurston's Example

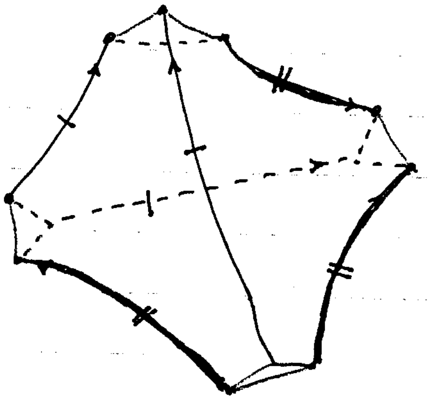
We obtain a complete hyperbolic 3-manifold (of finite volume) w/ a cusp end.

(This manifold was first obtained by R. Rèlley)

<u>Theorem</u>. 1. (Thurston) $M \cong S^3 -$  (the next simplest knot. 8-knot)

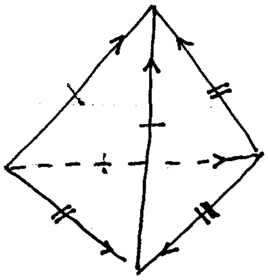
Removed

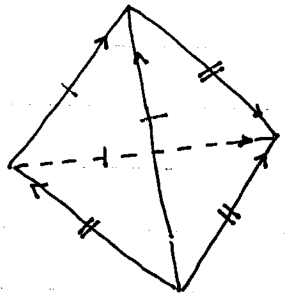


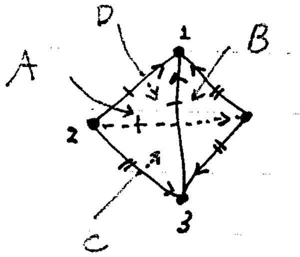


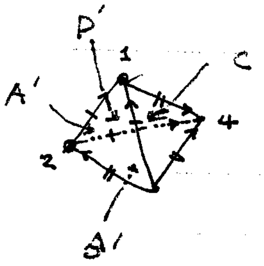
Lecture 11. Thurston's example.

Take two 3-simplexes σ_1^3, σ_2^3 and identify their faces by affine homeomorphisms so that the arrows are matched, ⇸ , ⇻.



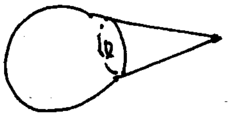






<u>Vertices.</u> $V = 1.$ $E = 2.$ $F = 4.$ $T = 2$

$$\chi = 1 - 2 + 4 - 2 = 1.$$



N the quotient

$M = N - \overset{\circ}{N}(v)$ nbhd of vertex v

$\chi(M) = \chi(N) -$ <u>$N(v) \simeq pt$</u>

$$\chi(M) + \chi(N(v)) - \chi(\partial M) = \chi(N) = 1$$

<u>∂M connected</u>:

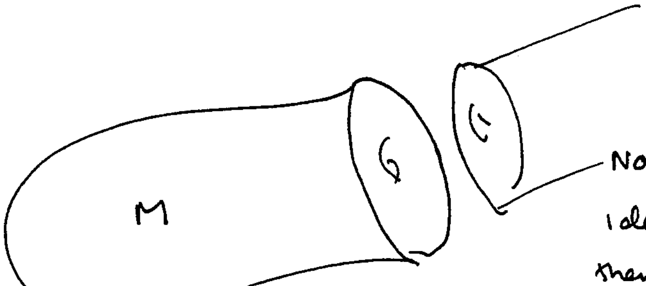
$$\chi(M) = \chi(\partial M)$$

$$0 = \chi(M \cup_{\partial M} M) = \chi(M) + \chi(M) - \chi(\partial M) = 0$$

$\Rightarrow \chi(M) = 0$

$\Rightarrow \chi(\partial M) = 0$

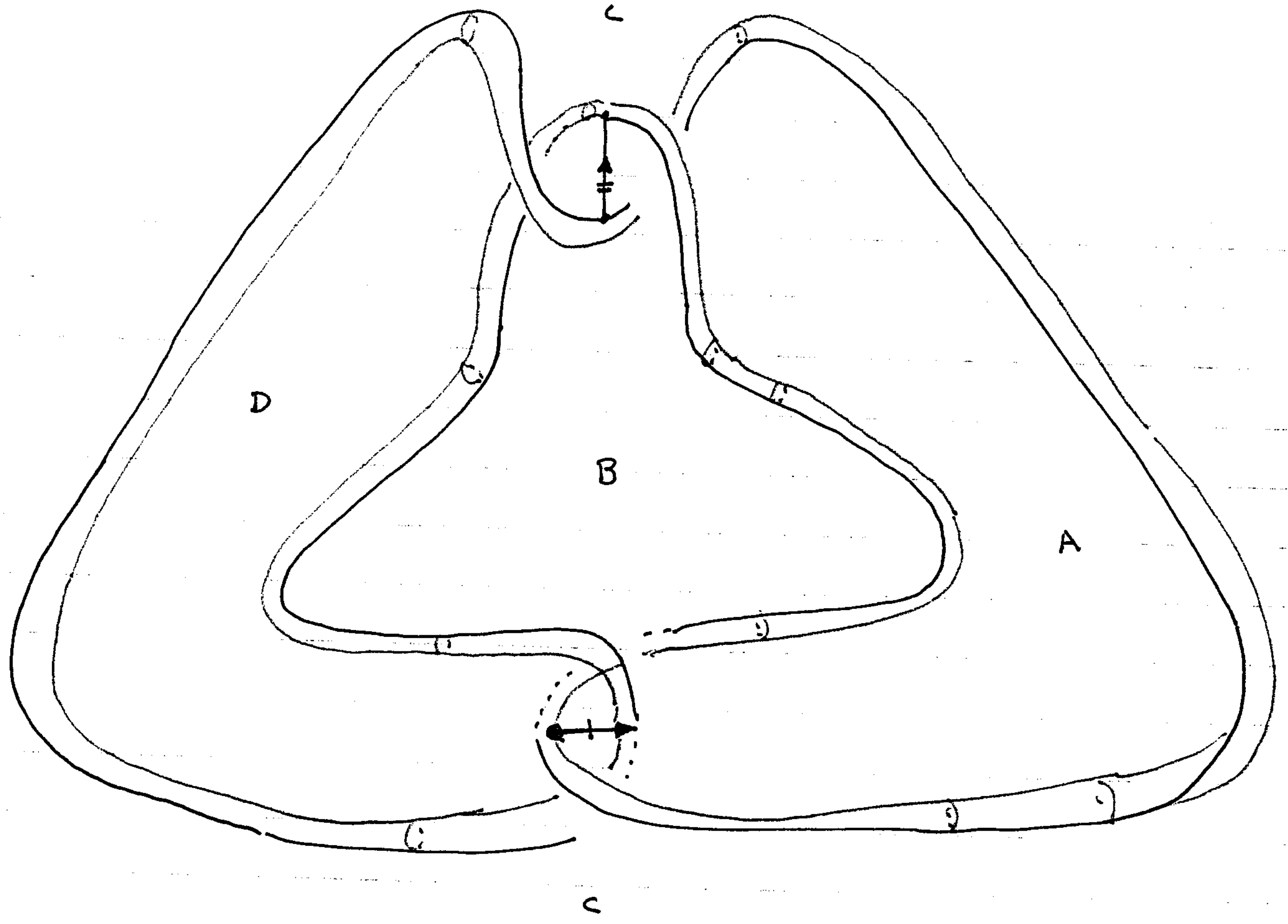
$\Rightarrow \partial M$ <u>torus</u>



Now take two regular ideal tetrahedra and glue their faces by the unique hyperbolic isometry as indicated

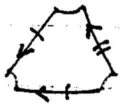
Angle sum at edges $= 6 \cdot \frac{2\pi}{\,} = 2\pi$

Lecture 11. Thurston's Example

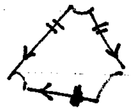


σ^3 = 3-simplex
= Tetrahedron

A :

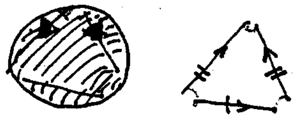


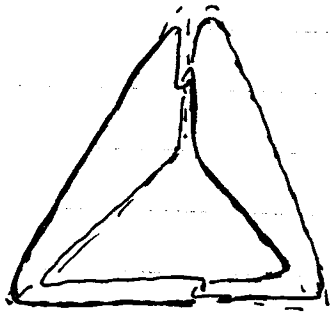
B :



C :

D .

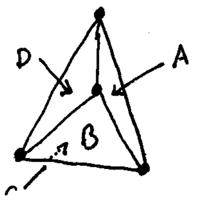




right-hand

⇒ produces a triangulation of T^2: w/ 8 triangles : 12 edges. 4 verts

$S^3 = \sigma^3 \underset{id|_{\partial\sigma^3}}{\cup} \sigma^3$,



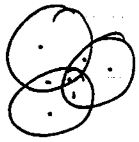
Lecture ~~13~~ 12. Some Basic Properties of Hyperbolic 3-manifolds

M^n 3-manifold, orientable hyperbolic. closed. ⟺ $M^n = \mathbb{H}^3/\Gamma$ Γ discrete PSL(2, ℂ)
properly discontinuously, freely

Since elliptic isometries $\gamma \in \mathrm{Iso}^+(\mathbb{H}^3)$ have fixed pts ⇒ $\forall \gamma \in \Gamma - \{id\}$,
γ loxodromic or parabolic.

Proposition M^3 cpt. ⇒ γ cannot be parabolic.

pf: ⇒ ∃ loop $\alpha \subset M^3$ $\alpha \not\simeq pt$ but length$(\alpha) \to 0$.
On the other hand, M^3 closed. ⇒ hyperbolic. can be covered by finitely many ε-balls $(\varepsilon > 0)$. $\exists\, \varepsilon' > 0$ so that $\forall\, p \in M^3$ $B_{\varepsilon'}(p) \subset B_\varepsilon(x_i)$ contractible some i.



□

Example Cusps (2-dim or 3-dim) contains essential loops of arbitrary small lengths.

Mostow Rigidity. If M^n, N^n are two homotopic closed hyperbolic n-manifolds, $(n \geq 3)$ then they are isometric.

$\mathrm{Iso}^+(\mathbb{H}^n) \cong SO_0^+(n+1, 1)$

(Margulis ⇒ generalized to all semi-simple Lie groups of rank 1)

As a consequence: All geometric invariants of the hyperbolic geometry are topological invariants: (volume, the shortest geodesic, the Chern-Simons inv. of hyperbolic 3-manifolds)

Theorem ~~Proposition~~ If $\overset{=\mathbb{H}^3/\Gamma}{M^3}$ closed hyperbolic ⇒ $\overset{\cong\Gamma}{\pi_1(M)}$ contains no subgroup $\cong \mathbb{Z} \oplus \mathbb{Z}$.

Proof Suppose otherwise, say $\alpha, \beta \in \pi_1(M) - \{id\}$ satisfy

(1) $\alpha^n \neq \beta^m$ $n, m \in \mathbb{Z} \setminus \{0\}$

(2) $\alpha\beta = \beta\alpha$

After a conjugation, say $\alpha(z) = \lambda z$. $|\lambda| > 1$ Two fixed pts $0, \infty$

Now
$$\alpha\beta(0) = \beta(0)$$
$$\alpha\beta(\infty) = \beta(\infty)$$
$$\Rightarrow \quad \beta(\{0,\infty\}) = \{0,\infty\} \quad \Rightarrow \quad \beta^2(0)=0,\ \beta^2(\infty)=\infty$$

By replacing β by β^2, we may assume
$$\beta(0)=0 \quad \beta(\infty)=\infty$$
$$\Rightarrow \quad \beta(z)=\mu z \qquad |\mu|\neq 1$$

<u>Homework</u> Suppose $\lambda,\mu\in\mathbb{C}$ $\quad |\lambda|>1 \quad |\mu|\neq 1 \quad \lambda^n\neq\mu^m \quad n,m\in\mathbb{Z}\setminus 0$

Then the group
$$\alpha^n\beta^m = \begin{bmatrix}\lambda^n\mu^m & 0\\ 0 & \lambda^{-n}\mu^{-m}\end{bmatrix}$$
is Not discrete.

(it conveys to some matrix)

(hint: show that $\mathbb{Z}1 \oplus \mathbb{Z}(\pi)$ is not discrete in $\mathbb{R}$).

Lecture ~~10~~ 13

Part II Basic Topological Technology in 3-Manifolds

We will work in smooth C^∞ or PL categories.

(C^∞ – transition function smooth. P.L. piecewise linear.)

Def <u>$S \subset M$ a submanifold</u> if $\forall\, x \in S$. $\exists$ nbhd $\mathcal{U}$ of x in

and chart $(\mathcal{U}, \phi)$ so that $(\mathcal{U}, \mathcal{U} \cap S) \xrightarrow{\phi} (\mathbb{R}^n, \mathbb{R}^k \times 0)$. ($C^\infty$ or

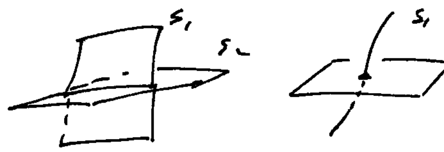
Def <u>Embedding</u> $f: S \to M$ a homeomorphism from S to the submanifold $f(S)$

<u>Example</u>.

<u>Transverse</u> If $S_1, S_2 \subset M^3$ two submanifolds $\dim S_1 + \dim S_2 \geq 3$.

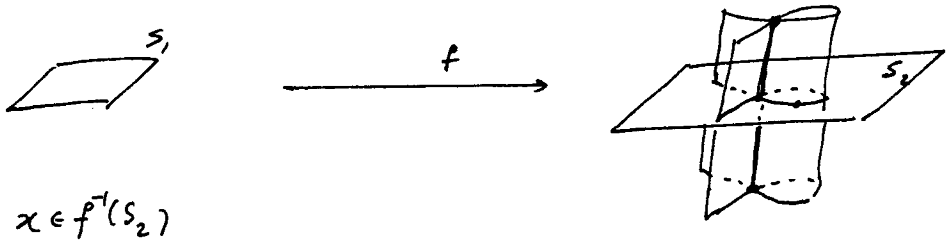
We say they are transverse if $\forall\, x \in S_1 \cap S_2$

$$T_x M = T_x S_1 + T_x S_2$$



Non-transverse.

Def A map $f: S_1 \to M$ is transverse to a submanifold $S_2 \subset M$ if



$\forall\, x \in f^{-1}(S_2)$

$$T_{f(x)} M = T_{f(x)} S_2 + D(f)(T_x S_1)$$

<u>Basic Theorem</u> (Hirsch)

(a) If S_2 is a submfd of M, then $\forall\, f: S_2 \to M$ can be approximated by $g: S_2 \to M$ which is transverse to S_1 ($g \simeq f$ homotopic) dim

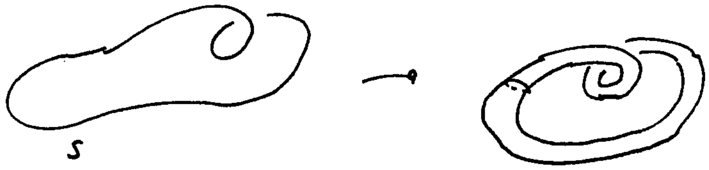
(b). if $f: S_1 \to M$ is transverse to S_2, then $f^{-1}(S_2)$ is a submfd of S_1 of dim =

$\dim S_2 - \dim f^{-1}(S_2) = \dim M - \dim S_2.$

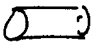
For instance S_1, S_2 surfaces in M $\Rightarrow$ $f^{-1}(S_2)$ 1-sobmanifold in S_1.

Tubular nbhd. $S \subset M^3$ sobmanifold.

(1) S 1-dimensional compact, then S has arbitrary small nbhd diffeomorphic to $S^1 \times \mathbb{R}^2$ (if M orientable.)



$\cong$ $I \times \mathbb{R}^2 / 0 \times \mathbb{R}$

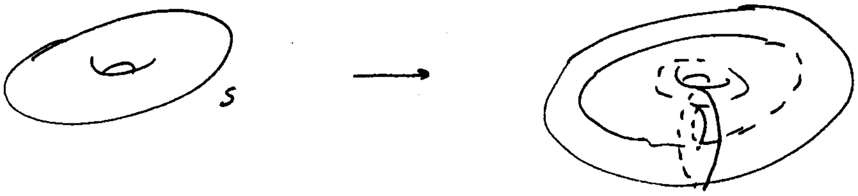


(0, z

(Solid Kleinbottle,

$I \times \mathbb{R}^2 / 0 \times \mathbb{R}^2 \sim 1 \times \mathbb{R}^2$
$(0, z) \sim (1, z)$)

(2) orientable surface $S \subset M^3$, M^3 orientable has arbitrary small nbhd diffeomorphic to $S \times \mathbb{R}^1$



N.

Schönflies Theorem

Notations

B^n = (n-ball) = D^n (n-disk) = $\{ x \in \mathbb{R}^n \mid |x| \leq 1 \}$

$S^n = \partial B^{n+1}$, $S^n \cong \mathbb{R}^n \cup \{\infty\}$ stereographic

(topological)

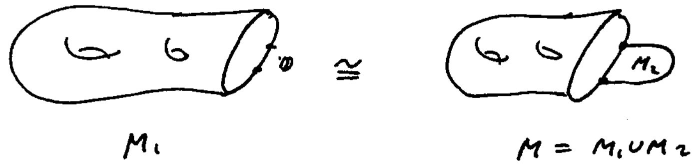
Thm Any (C^∞ or PL) 2-sphere in $\mathbb{R}^3$ bounds a 3-ball. (Not true
top)

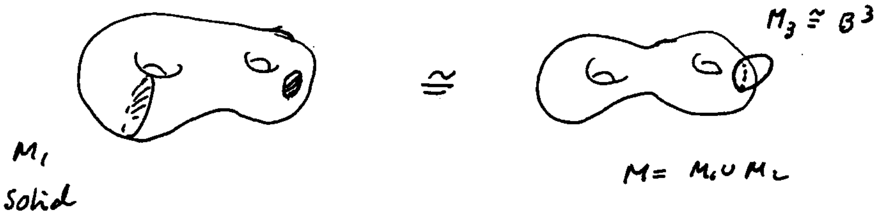
Jordan Curve Any C^∞ 1-sphere in $\mathbb{R}^2$ bounds a 2-ball.

proof. We need the following proposition

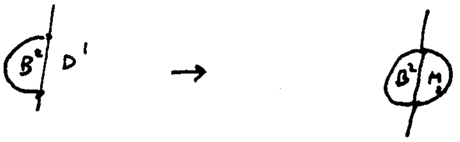
Proposition Suppose $M = M_1 \cup M_2$ M_1 3-manifold $\partial M_1 \neq \phi$. $M_2 \cong B^3$ s
that $M_1 \cap M_2 = \partial M_1 \cap \partial M_2 \cong D^2$. Then $M \cong M_1$. ($\cong$ homeom

n=2





Basic n=2



$\exists$ a homeomorphism

$h: \{ x \in \mathbb{R}^2 \mid |x| \leq 1, \ Re(x) \leq 0 \} \to \{ x \mid |x| \leq 1 \}$

so that $h(e^{i\theta}) = e^{i\alpha}$, $\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}$

α closed arc in ∂D^2

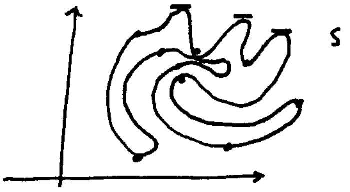
□

Main reason: $(D^2, \alpha) \cong (D^2, \{|z|=1 \mid Re(z) \geq 0\}) \cong (D^2 \cap \{Im z \leq 0\}, D^2 \cap \{Im z$

Schönflis Theorem

Proof of Jordan curve. $S \subset \mathbb{C}$ C^∞ simple closed curve

S can be perturbed to so that $h(x,y)=y$ restricted to S has only finitely many local max + min pts. Also the local max + min values are distinct (transvers

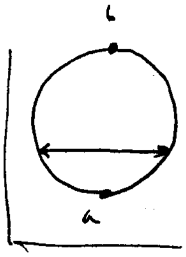


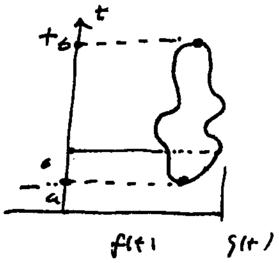
Fix

Let n be the number of local max.
(= # of local minimal pts) (why?)

Induction on n

$n=1$:



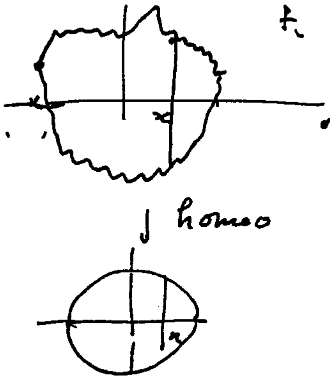


$a \le t \le b$

linear

$(-\sqrt{1-t^2}, \sqrt{1-t^2})$

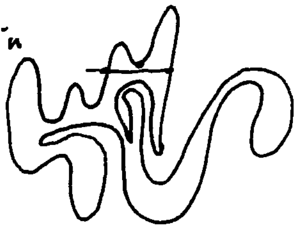
done Write down the homeomorphism directly

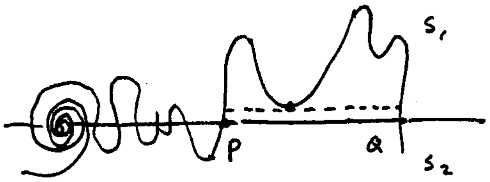


f, g two smooth functi
on $[-1,1]$

should deal w/ $n=2$.

$n \ge 2$. Let y_0 be the largest local minimal value
For $\varepsilon > 0$ small, st $\{y_0 - \varepsilon < y < y_0\}$ contains no local min
consider the Line $L = \{y = y_0 - \varepsilon\}$ cutting S



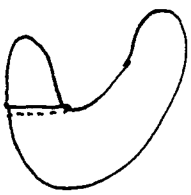


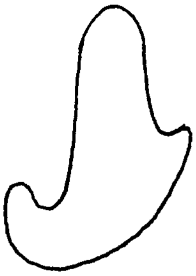
Let $P. Q$ be the segment in L (inner most) so that $[P. Q] \cap S$ only at the end pt (inner
and the local min pt is above $[P. Q]$

$P. Q$ cut S into two disks S_1, S_2 as shown

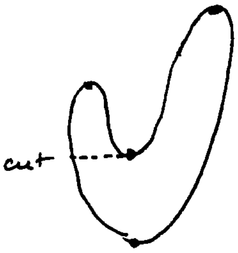
Consider $C_1 = S_1 \cup [P. Q]$
$C_2 = S_2 \cup [P. Q]$

Then by induction (C_1 contains only a local min). C_2. $(n-1) \Rightarrow$ do n=2:

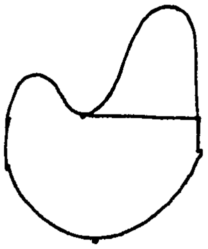




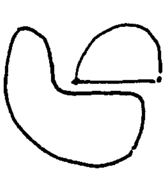
?



reduce



should win



n = 2

Schönflies Thm.

<u>$n=3$ Schönflies (Alexander)</u>

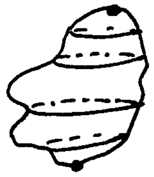
Let $S \subset \mathbb{R}^3$ C^∞ 2-sphere. S may be perturbed so that "height function" $h:(x,y,z)=z : \mathbb{R}^3 \to \mathbb{R}$ has a finite number of maxima, minima & saddle pts and no other critical pts, and the values are distinct



Let n be the number of saddle pts.

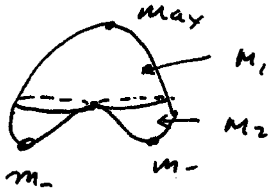
Hw. <u>Morse theorem (Easy)</u> $m_+ + m_- - n = 2 \overset{\chi(S)}{=}$ or $\underline{n = m_+ + m_- - 2}$

<u>(1) $n=0$</u> so $m_+ = m_- = 1$

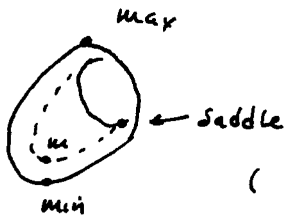


$\simeq$ ball B^3 (Caratheodory)

(2) $n=1$. Say $m_+ = 1$, $m_- = 2$



<u>or</u>



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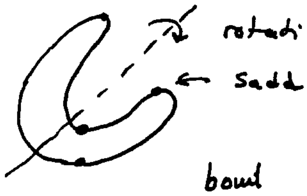
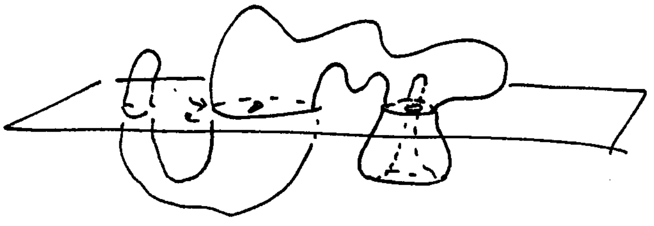


(3) <u>$n \geq 2$</u> Suppose every 2-sphere w/ $<n$ saddle pts bound a 3-ball. Take the horizontal plane H disjoint from critical pts with ≥ 1 saddle pts on each side of H.

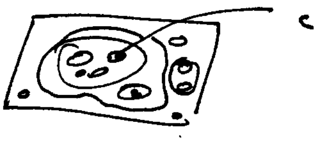
$H \cap S$ is a compact 1-manifold without boundary, so $H \cap S$ is a finite union of disjoint circles ($= S^1$) in H

Let C be the innermost component of $H \cap S$, $\exists$ disk D in

$\partial D = C$ $\quad D \cap S = \partial D \cap S = e$

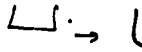


H



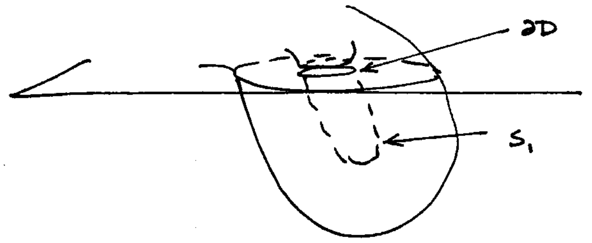
C cuts S into two disks D_1 D_2 (Jordan curve thm)

so $S_1 = D_1 \cup D$, $S_2 = D_2 \cup D_2$ are two embedded 2-spheres in $\mathbb{R}^3$.

<u>Case 1</u>. Both S_1, S_2 contain saddle pt ($S_1 \to$  $\to$ make it smooth) 

$\Rightarrow$ Both bound balls B_1, B_2 $\Rightarrow$ done prop

<u>Case 2</u>. If S_1 contains no saddle pt. $\Rightarrow$ S_1 bounds a ball B_1



Prop. $\Rightarrow$ S bounds ball if S_2 bounds.

Now use B_1 to modify S

<u>Focus on S_2.</u> modify S_2 so that

$S_2 \cap H = H \cap S - \{c\}$

Next. repeat the same step 1 or step 2 to S_2

Since step 2 does not eliminate any saddle pts, we must have step 1 in some steps. <u>Done</u>.

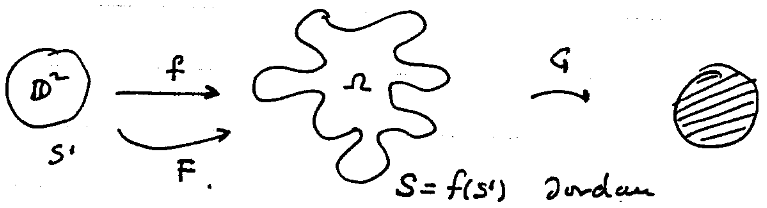
Lecture 14. Schönflis Theorem)

Consequence of Jordan curve theorems

Lemmas (Alexander Trick) If $h: S^{n-1} \to S^{n-1}$ is a homeo, then it can be extended to a homeomorphism $H: D^n \to D^n$ where $r \in [0,1]$, $x \in S^{n-1}$.

$H(rx) = rh(x)$
$H = c(h)$ cone.

Corollary: (a) Any homeomorphism $f: S^1 \to S \subset \mathbb{C}$ extends to an embedding $F: D^2 \to \mathbb{C}$

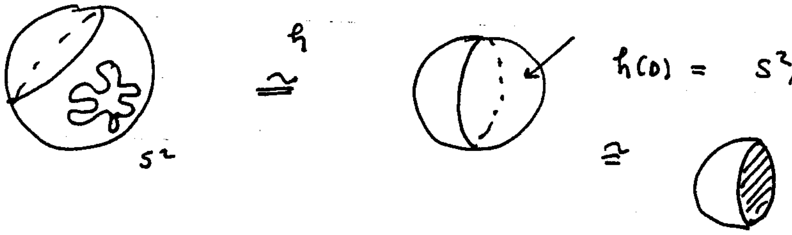


Proof. Let Ω be the component of $f(S^1)$ so that $\bar{\Omega} \xrightarrow[\cong]{G} D^2$. The extension is then:

$$G^{-1} \circ c(G \circ f)$$ gives the result.

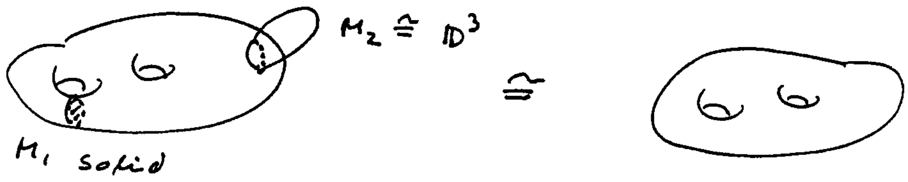
∗

(b) if $D \subset \hat{\mathbb{C}}$ is homeomorphic to D^2, then $\exists\, h: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ homeomorphism st $h(D) = \{|z| \le 1\}$ (standard)



$h(D) = S^2 \cap \{x_1 \ge 0\}$.

Proposition M_1, M_2 3-mfd s.t. $M_2 \cong D^3$ and $M_1 \cap M_2 = \partial M_1 \cap \partial M_2 \cong D^2$. Then $M_1 \cup M_2 \cong M_1$



Lecture 13.

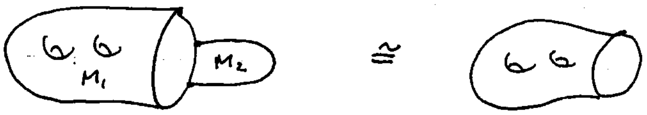
Jordan curve theorem + Schönflis theorem

<u>Notation</u>. $\mathbb{D}^n = \{ x \in \mathbb{R}^n \mid |x| \leq 1 \}$,
$\partial \mathbb{D}^n = S^{n-1} = \{ |x| = 1 \}$.

We work in C^∞ or P.L. category, the conclusion is homeomorphism $\cong$.

<u>Prop 1</u>. Suppose $M = M_1 \cup M_2$ surface where $M_1, M_2 \cong \mathbb{D}^2$ so that $M_1 \cap M_2 = \partial M_1 \cap \partial M_2 =$ arc. Then $M \cong M_1$. (Topological surfaces)

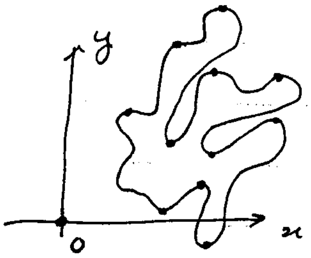
<u>Proof</u>.



<u>Jordan Curve Thm</u>. Suppose $S \subset \mathbb{C}$ is a smooth (C^2) simple closed curve. Then S bounds a disk.

<u>Pf</u>. Let $h(x,y) = y : \mathbb{C} \to \mathbb{R}$ be the height function.

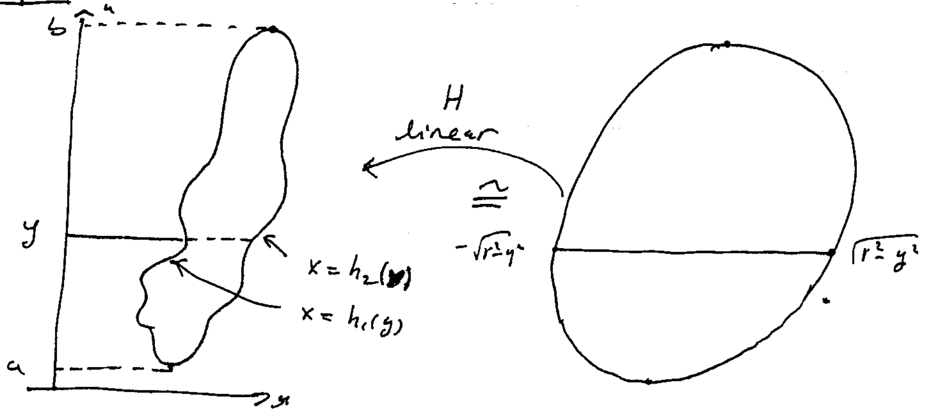
Perturb S so that $h|_S : S \to \mathbb{R}$ has only f. many local max, local min + their vales are distinct.



<u>Fact</u>: the theorem holds for perbations. (Hirsch).
($\exists$ isotopy of S in $\mathbb{C}$ making it true)

Let $n = \#$ local max. $\underline{n \geq 1}$. (= # of local min).
Induction on n.

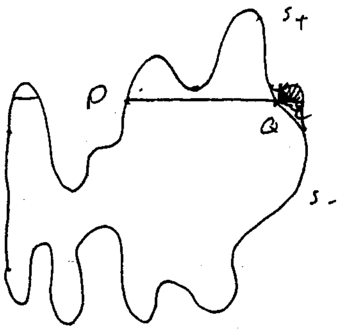
<u>Step 1</u> $n = 1$.



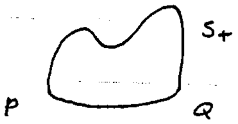
circle of radius b-a
height preserving $r = \frac{b-a}{2}$
homeomorphism

Step 3

For $n \geq 2$. Let y_0 be the largest local min value.



cut S^1 by the line $y = y_0 - \varepsilon$ for $\varepsilon > 0$ small to obtain a segment P. Q as shown. Say P. Q cuts S^1 into S_+ + S_- where S_+ contains only one local min whose val is y_0

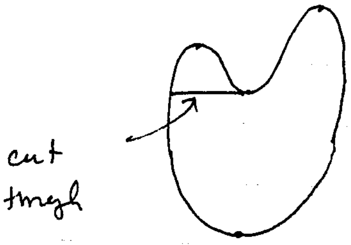


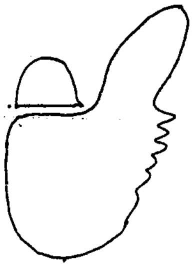
+ $\overline{PQ} \cup S_-$

has fewer loca max.

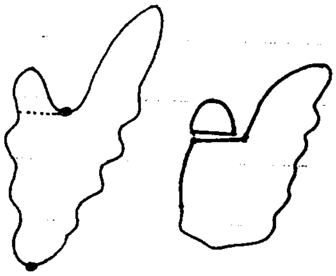
(S_+ must contain two local max!) ⇒ by the property done

Step 2. If it $n=2$





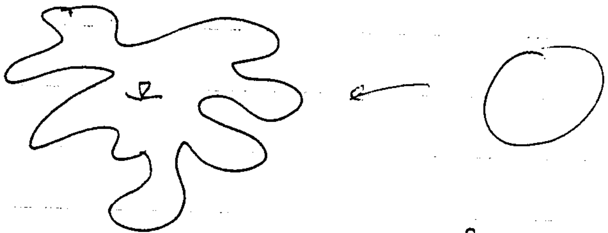
Picture.



(perturbation).

Done.

RM. The best proof. Riemann mapping theorem.
$f: S^1 \to \mathbb{C}$ embedding.



Then the Riemann mapping theorem $\varphi : \{|z|<1\} \to \Omega$ extends to a homeomorphism from $\{|z| \leq 1\}$ to $\bar{\Omega}$. (Caratheodory)

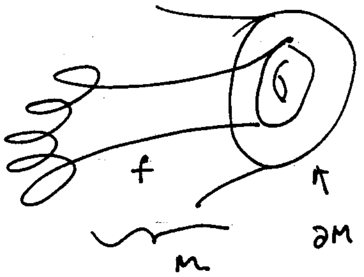
- Jordan curve thm for surfaces
- Analytic Proof

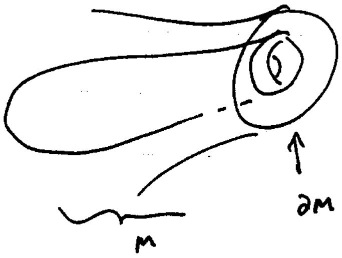
15-16-17

Lecture ~~12/13~~ Loop & Sphere Theorems in Dimension 3. (Papakyriakopoulos)

M 3-mfd. P.L or C^∞ category.

Dehn's lemma If $f: (\mathbb{D}^2, \partial\mathbb{D}^2) \longrightarrow (M, \partial M)$ s.t $f|_{\partial\mathbb{D}^2}$ is an embedding, then $\exists$ an embedding $g: (\mathbb{D}^2, \partial\mathbb{D}^2) \longrightarrow (M, \partial M)$ st $g|_{\partial\mathbb{D}^2} = f|_{\partial\mathbb{D}^2}$ $g(\mathring{\mathbb{D}}^2) \cap$





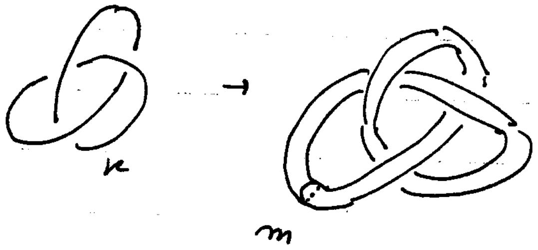
Dehn's original work (We work in C^∞ or P.L category)

Corollary (Dehn) A knot $K \subset S^3$ is trivial $\Leftrightarrow$ $\pi_1(S^3 - K) \cong \mathbb{Z}$.

Pf "$\Rightarrow$" trivial since $S^3 - K \cong K \times \mathring{\mathbb{D}}^2 \cong S^1 \times \mathbb{D}^2$

"$\Leftarrow$" Let $N(K)$ be a regular nbhd of K $N(K) \cong S^1 \times \mathbb{D}^2$

Let $m, l \subset \partial N(K)$ be simple loops : m null homotopic in $N(K)$
l intersects m in one point



$|l \pitchfork m| = 1.$

Note $H_1(\partial N(K), \mathbb{Z}) \cong \mathbb{Z}[m] + \mathbb{Z}[l]$

$M = S^3 - \mathring{N}(K)$ cpt $\partial M \cong S^1 \times S^1$

Consider inclusion $i: \partial M \to M$

$$i_*: \underset{\mathbb{Z}\oplus\mathbb{Z}}{\underset{\parallel}{\pi_1(\partial M)}} \to \underset{\mathbb{Z}}{\underset{\parallel}{\pi_1(M)}}$$

it has kernel; say $p[m] + q[l]$ is a <u>generator</u> of $\ker(i_*)$

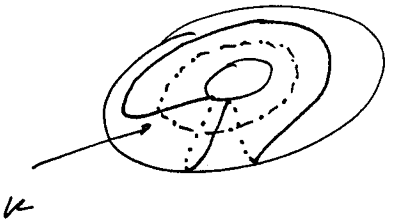
Homework: show that $q = \pm 1$. (use M-V seq. $S^3 = M \cup N$, $M \cap N = \partial N$)
or use Alexander duality)

Thus, $\exists$ $f: (\mathbb{D}^2, \partial\mathbb{D}^2) \to (M, \partial M)$ $f|_{\partial\mathbb{D}^2}$ embedding representing <u>$pm \pm l$</u>

$\Rightarrow$ Dehn's lemma $\exists$ we may assume $f: \mathbb{D}^2 \to M$ is an embedding
st $f(\mathring{\mathbb{D}}^2) \cap \partial M = \phi$

L14 loop Thm.

Now there is an annulus A in $N(K)$ $\partial A = f(\partial D^2) \cup k$ (since $q = \pm 1$



Using $f(D^2) \cup A$ $\Rightarrow$ $\exists$ an embedded disk D in S^3 w/ $\partial D = k$
$\Rightarrow$ K is trivial. ✻

Loop Thm. M 3-mfd if $f : (D^2, \partial D^2) \to (M, \partial M)$ s.t. $f|_{\partial D^2} \not\simeq pt$, then $\exists$ $g : (D^2, \partial D^2) \to (M, \partial M)$ an embedding s.t $g|_{\partial D^2} \not\simeq pt$.

Sphere Thm. M orientable 3-mfd. if $\exists$ $f : S^2 \to M^3$ s.t. $f \not\simeq pt$ $\Rightarrow \exists$ embedding $g : S^2 \to M$ s.t $g \not\simeq pt$.

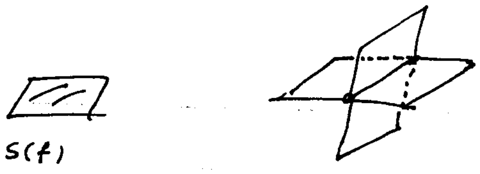
A Proof of Dehn's lemma (Papakyriakopoulos 1958)

- Singularity of P.L maps Σ surface, M 3-mfd
-

Definition. $f : \Sigma \to M^3$ P.L map, its singularity
$S(f) = \overline{\{ x \in \Sigma \mid \exists y \in \Sigma \ y \neq x \ f(x) = f(y) \}}$ closure

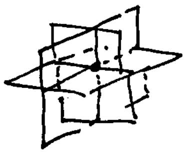
Example. Basic example

(1) double pts :



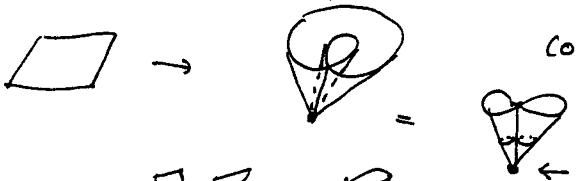
$S_2(f) = \{ x \in S(f) \mid \# f^{-1}(f(x)) =$

(2) triple pts :



$S_3(f)$

(3) branch pts :



Cone over $\infty \subset \mathbb{C}$

$S_1(f)$.

(4) non-gen

These singularities cannot be eliminated by a small perturbation of f.

Lecture 14-15. Dehn's Lemma

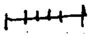
Thm Suppose Σ is a compact surface, M a P.L manifold w/ a metric d. and $f : \Sigma \to M$ a P.L map. Then $\forall \varepsilon > 0$. $\exists$ P.L map $g.\ \Sigma \to M$ (with possible subdivision of Σ) so that

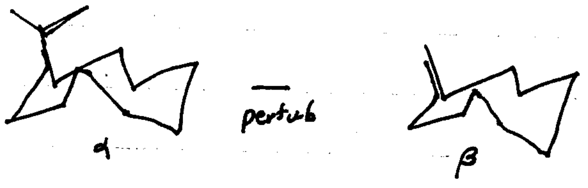
(1) $dist(f-g) < \varepsilon$

(2) g has only I, II, III singularities

Pf (Sketch)

step 1. A generic P.L map $\alpha : S^1 \to \mathbb{C}$ has only one type of singularities +

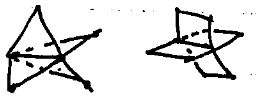




step 2 By perturbation, and subdivision, we may assume that $\forall \sigma^2 \in T$ (triple) $f| \quad f(\sigma^2)$ still a triple in M^3 st

(1). $f(\mathring{\sigma}^2) \cap f(\mathring{\tau}^2) = \phi \qquad \forall \sigma^2 \neq \tau^2$

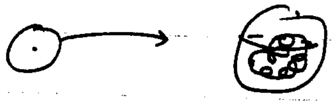
(2). $\Sigma(f) \cap \sigma^1$ is at most the double pts.



step 3 At each vertex $v \in T$. the map:

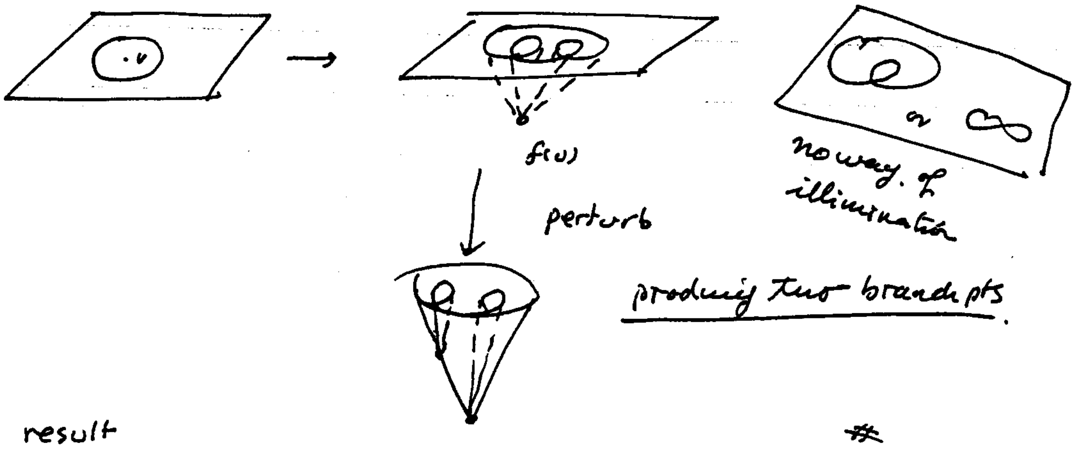
$$\begin{tikzcd} lk(v) \arrow[r, "f"] \arrow[d, phantom, "\cong" rotate=90] & lk(f(v)) = S^2 \\ S^1 & \end{tikzcd}$$

is generic having only double pts.



$$f : St(v) = N(v) \longrightarrow St(f(v)) = N(f(v))$$

$St(v) = C(lk(v))$, $N(v) \cong \mathbb{D}^2$, $N(f(v)) \cong \mathbb{D}^3$ is the cone construction.



⇒ result. #

By the construction, we can always producing a subdivision of Σ_1 st $S(g)$ is a subcomplex. We will assume from now on the triplation is fixed

L14. A Proof of Dehn's lemma

Recall of covering space theory (X, Y say path connected) Y manifold.

(1) $f: X \to Y$ is a covering if —

(2) lifting of simply connected space

(3) the existence of covering of Y.

} Alg. Top.

Proof of Dehn's lemma

May assume $f: (\mathbb{D}^2, \partial\mathbb{D}^2) \to (M, \partial M)$ generic w/ 3-singularities type I D
Also, we may assume that $f(\mathring{\mathbb{D}}^2) \cap \partial M = \phi$ by a small perturbation.

Lemma 1 N^3 compact 3-mfd, $\partial N \neq \phi$ then if $H_1(N^3, \mathbb{Z}) = 0 \Rightarrow \partial N^3$ union of S^2's.

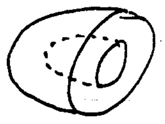
Proof. Take $\mathbb{Z}_2$ coefficients. a surface connected Σ is $S^2 \Leftrightarrow H_1(\Sigma, \mathbb{Z}_2) = 0$
(Möbius classification)

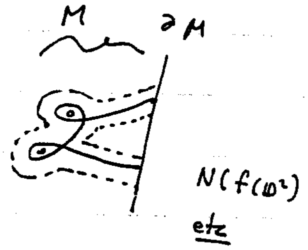
Now
$$H_1(N) \underset{\text{lefschetz}}{\cong} H_2(N, \partial N) = 0$$

But M-V
$$\to \underset{\underset{0}{\shortparallel}}{H_2(N, \partial N)} \to \underset{\underset{0}{\wr\shortparallel}}{H_1(\partial N)} \to \underset{\underset{0}{\shortparallel}}{H_1(N)} \to \cdots$$

#

Note if f embedding, $N(f(\mathbb{D}^2)) \cong \mathbb{D}^3$, (1-handle)





Let $V_0 = N(f(\mathbb{D}^2))$ be a regular nbhd of $f(\mathbb{D}^2)$. If $H_1(V_0) = 0$. shp

Let $p_1: N_1 \to V_0$ be a 2-fold cover ($z \mapsto z^2$ ⭕ → o)

$$\begin{tikzcd}
 & & M_1 \arrow[dl] \\
f_0: \mathbb{D}^2 \arrow[urr] \arrow[r] & V_0 \arrow[r, hook] & M_0 = M
\end{tikzcd}$$

Dehn's lemma

$$\begin{tikzcd}
 & & & M_2 \arrow[dll] \\
 & V_1 \arrow[r, phantom, "\subset"] & \; & M_1 \arrow[dll, "p_1"'] \\
f_0\colon \mathbb{D}^2 \arrow[ur, "f_1"] \arrow[r] & V_0 \arrow[r, phantom, "\subset"] & \; & M_0 = M
\end{tikzcd}$$

Let f_1 be a lifting of $f_0: \mathbb{D}^2 \to V_0$, and $V_1 = N(f_1(\mathbb{D}^2))$ $(\simeq f_1(\mathbb{D}^2))$

Inductively: suppose a boundary component of $\underset{\underset{V_i}{\shortparallel}}{N(f_i(\mathbb{D}^2))}$ is not S^2, then $\exists$ two-fold cover $M_{i+1} \xrightarrow{p_{i+1}} V_i$.

$$\begin{tikzcd}
 & V_{i+1} \arrow[r, phantom, "\subset"] & M_{i+1} \arrow[dl, "p_i"] \\
\mathbb{D}^2 \arrow[ur, "f_{i+1}"] \arrow[r, "f_i"] & V_i \arrow[r, phantom, "\subset"] & M_i =
\end{tikzcd}$$

$\forall$ a lift $f_{i+1}: \mathbb{D}^2 \to M_{i+1}$.

Here: $p_i \circ f_{i+1} = f_i$

$\Rightarrow$ $S(f_{i+1}) \subset S(f_i)$ (all are subcomplexes of $\mathbb{D}^2$ w/ a fixed triangulation)

Claim $S(f_{i+1}) \neq S(f_i)$

If not, $S(f_{i+1}) = S(f_i)$ $\Rightarrow$ $f(\mathbb{D}_i) \overset{\sim}{=} f(\mathbb{D}_{i+1})$ homeomorphic by p_{i+1}

$p_{i+1}: f(\mathbb{D}_{i+1}) \downarrow f(\mathbb{D}_i)$ homeo. onto

Since the topology of $f(\mathbb{D}_i)$ are determined by $S(f_i)$ as the quotient space.

This contradicts $(p_i)_*: \pi_1(M_{i+1}) \to \pi_1(V_i)$ has proper image. (image not equal to $\pi_1(V_i)$. in fact index 2 subgroup)

$\Rightarrow$ after finitely many construction, there is V_n so that ∂V_n consists of S^2's. by lemma 1.

Lecture 14-15.

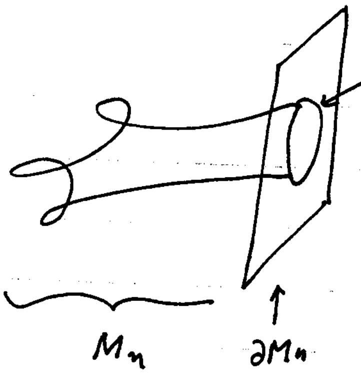
Dehn's lemma

-13.6-
-14.6-

Lemma 2. Dehn's lemma holds for $f_n: (D^2, \partial D^2) \longrightarrow (V_n, \partial V_n)$.

Pf. By definition, $V_{n+1} = N(f_n(D^2))$. its boundary consisting of 2-spheres.

Take the boundary comp. $S^2 \subset \partial V_{n+1}$ so that



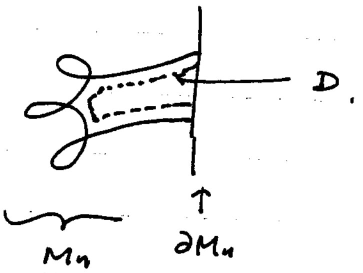
$f_n(\partial D^2) \subset S^2$

By Jordan curve thm, $f_n(\partial D^2)$ cuts S into two disks D_+ + D_-

Now perturbe D_+ near $f_n(\partial D^2)$ so that $\mathring{D}_+ \cap \partial M_n = \phi$

⇒ done

Picture

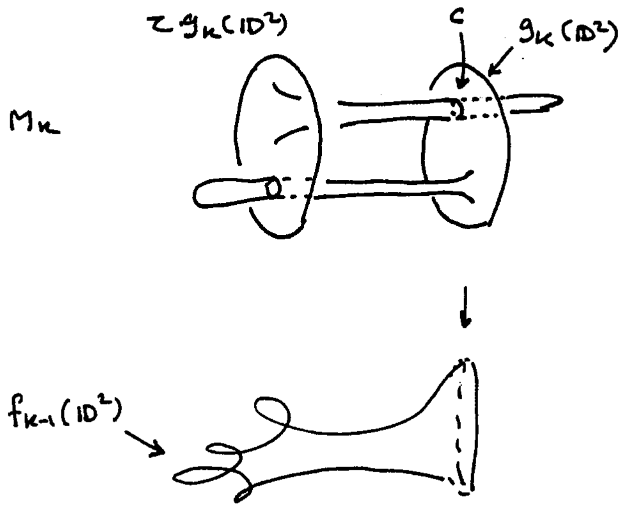


□

Lemma 3. If Dehn's lemma holds for $f_k: (D^2, \partial D^2) \longrightarrow (V_k, \partial V_k)$, it holds for $f_{k-1}: (D^2, \partial D^2) \longrightarrow (V_{k-1}, \partial V_{k-1})$.

Pf. Suppose $g_k: (D^2, \partial D^2) \longrightarrow (V_k, \partial V_k)$ be an embedding w/ $g_k|_{\partial D^2} = f_k|_{\partial D^2}$.

$V_k \subset M_k \xrightarrow{p} V_{k-1}$ is a 2-fold covering w/ Deck transformation τ



$\tau \circ \tau = id$ $\quad \tau(x) \neq x$. $\quad V_{k-1} = M_k/_{\tau(x)\sim x}$

(covering space theory)

By perturbing g_k we may assume that

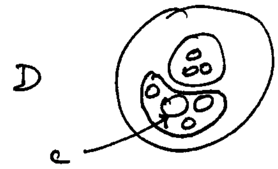
$g_k \pitchfork \tau \circ g_k$ $\quad D \pitchfork \tau(D)$

no singularity at ∂D^2. $\quad D = g(D^2)$

Thus $g_k(D) \cap \tau g_k(D^2)$

14.15-7.

Now $D\cap\tau(D)$ consists of 1-submanifolds without boundary



Note. if $D\cap\tau(D)=\phi \Rightarrow \varphi(D)$ is the required embedded disk for F_{k-1}

To eliminate $D\cap\tau(D)\neq\phi$. Take $c\subset D\cap\tau(D)$ an inner most component of $D\cap\tau(D)$ in D. Let Δ be the disk in D bounding c

$$\mathring{\Delta}\cap\tau(D)=\phi. \Leftrightarrow \tau(\mathring{\Delta})\cap D=\phi.$$

Suppose $\tau(c)$ bound disk D' in D. Now modify D. to

$$\tilde{D}=(D-D')\cup\tau(D)$$ and a little perturbation of it

produces a new disk, still denoted by $\tilde{D}$ s.t.

(1). $\partial\tilde{D}=\partial D$

(2). $\tilde{D}\cap\tau(\tilde{D})$ has one components fewer than that of $D\cap\tau$

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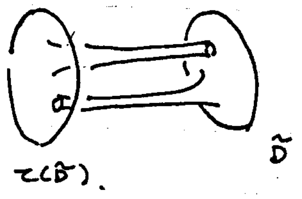


#

Lectures 18.19 Finding Geometric Structures on Surfaces + 3-Manifolds

Ricci Flow. 3-dim. 2-Dim:

Low-Dimension:

Basic geometric objects: cpt convex polytopes $P \subset \mathbb{R}^3$  . it carries the combinatorics : a graph on ∂P ($\cong S^2$)

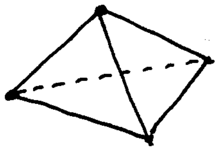
Def A CW-decomposition of a surface Σ is a graph G in Σ st each component of $\Sigma - G$ is s.c.

Example 1. (G.C.) Given a CW-decomposition (S^2, G) of S^2, when does there exist a cpt convex poytope $P \subset \mathbb{R}^3$ s.t its combinatorics is $\cong (S^2, G)$?

Steinitz Thm P exists iff $\forall$ vertex $u, v \in G$, $G - \dot{st}(u) \cup \mathring{st}(v)$ is connected.

Question Is the space of all such P's modulo isometry a Euclidean space $\mathbb{R}^N$?
(Still open)

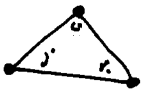
Example 2. How to understand a $\mathbb{E}^3$ - 3-simplex σ^3 metrically ?
(total space 6-dim)



Method 1. Assign (measure) edge lengths $x_1, \cdots, x_6$

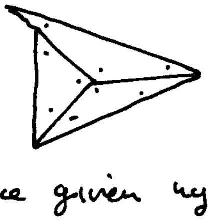
- $x_i + x_j > x_k$ i, j, k form Δ σ^2
- sum of vertex angles $< 2\pi$ (convexity)

Method 2 Measure face angles $x_1, \cdots x_{12} \in (0, \pi)$ s.t.

linear Alg. { (1) $x_i + x_j + x_k = \pi$ i, j, k forms a σ^2 

(2) $x_i + x_j + x_k < 2\pi$ at vertex v 

Analysis. (3) Matching at the edges (six equation, + equations)

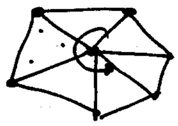


Rivin: Consider the space given by (1), (2) (8-dim). (Locally Euclidean spaces)
make (3) to be the critical pt equation of a natural "volume":
(Ann. of Math. 1994, 553-580) $\mathbb{E}^2$, $\mathbb{H}^2$, S^2 ques

Lecture 18, 19 Rivin's Work

Take the torus $T^2 = S^1\times S^1$ and a triangulation T of T^2 w/ the V, E, F the set of vertices, edges and triangles. (topological)

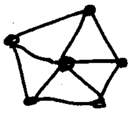
Def. A geometric triangulation of T^2: a flat metric d $(T^2 \underset{\text{iso}}{\cong} \mathbb{E}^2/\mathbb{Z}+\mathbb{Z})$ so that each $f\in F$ becomes a Euclidean triangle.



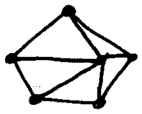
v: Σ angles $= 2\pi$

f: Σ angle $= \pi$

For fixed T, (topological), if it is isotopic to a geometric triangulation, it is not unique:



(move vertices)



(moduli space)

Define the dihedral angle Δ_d of it to be

$$\Delta_d: E \longrightarrow (0, \infty) \qquad \Delta_d(e) = \alpha+\beta$$

the sum of two opposite inner angles facing e.

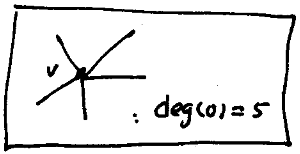
Fix topological (T^2, T).

Thm 1 (Rivin). If $\Delta_{d_1} = \Delta_{d_2}$, $\Rightarrow$ $d_1 = \lambda d_2$ $\lambda\in\mathbb{R}_{>0}$ two metrics are isometric upto a dilatation. (Rigidity)

RM. (open for hyperbolic $\mathbb{H}^2$. holds for S^2.)

Thm 2 (Rivin). Given any $\Delta: E \to (0,\pi]$ so that

(1) $\forall\ v\in V$ $\quad \sum_{v<e_i} \Delta(e_i) = \pi(\mathrm{Deg}(v)-2)$

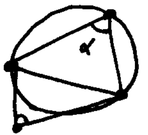


(2) $\forall$ subset $X \subsetneq F$, let $E(X) = \{e\in E \mid e<f,\ f\in X\}$ then

$$\sum_{e\in E(X)} \Delta(e) > \pi|X|. \qquad (3)\ \sum_{e\in E}\Delta(e) = \pi|F|$$

there exists a unique flat metric d on T^2 so that $\Delta_d = \Delta$.

RM: 1. $\Delta_d(e) \le \pi$ $\Leftrightarrow$ Delaunay triangulation



2. The realization problem is still open for $\Delta: E \to (0, 2\pi)$.

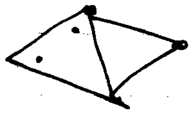
3. Construction of d: reduces to find $\Delta: E \to (0.\pi]$ w/ (1), (2) (linear algebra)

4. True $\forall$ surfaces, without (1), metric d has singularity at $v\in V$.

L18.-19

Local Euclidean Structure

Rivin's Sett Up. T a triangulation of a surface Σ. V. E. F. we say $v \in V$ $f \in F$ satisfies $v < f$ if v is a vertex of f. ($v<e$. $e<f$ etc)



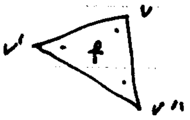
Def A corner of T is a pair (v,f) $v<f$ $v \in V$, $f \in F$

A local Euclidean structure of (Σ, T):

$$x: \{(v,f) \mid v \in V,\ f \in F\ \ v<f\} \to \mathbb{R}_{>0}$$

s.t.,

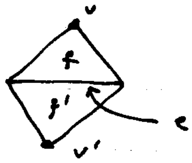
$$x(v,f) + x(v',f) + x(v'',f) = \pi$$



The dihedral angle of x $\Delta_x: E \to \mathbb{R}$

$$\Delta_x(e) = x(v,f) + x(v',f')$$

where the two corners (x,f) + (v',f) are facing e.



Obviously. A flat metric d w/ a geometric triangulation ⟹ a local Euclidean structure by measuring inner angles.

Fix a map $\Delta: E \to \mathbb{R}_{>0}$, let $\mathcal{A}(\Delta)$ be the space of all local Euclidean structures x so that $\Delta_x = \Delta$.

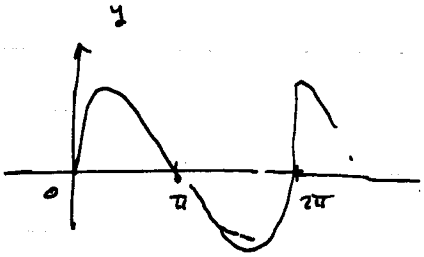
- Some Calculus of Euclidean triangles

Lobachevsky Function $L(x) = -\int_0^x \ln|2\sin t|\,dt$

L(x) continuous, period π and odd

$$L(x+\pi) = L(x), \qquad L(-x) = -L(x)$$

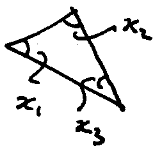
(Note $|\ln|t||$ is $L^1(\mathbb{R})$.)



A similarity triangle is a triple $(x_1, x_2, x_3) \in (0,\pi)^3$ $x_1+x_2+x_3 = \pi$

Its volume

$$V(x_1, x_2, x_3) = L(x_1) + L(x_2) + L(x_3)$$



RM: $V(x_1, x_2, x_3)$ = volume of hyperbolic ideal 3-simplex σ^3 w/ dihedral angles $x_1\, x_2\, x_3\, x_1\, x_2\, x_3$

HW.

(See Milnor's Lecture in Thurston's note, chapter 7).

L18-19 Rivin

Lemma 1 $V : \{(x_1, x_2, x_3) \in [0,\pi]^3 \mid \Sigma x_i = \pi\} \longrightarrow \mathbb{R}$ is strictly concave function.

(i.e. Hessian $H(V) < 0$)

Proof. Let $(x,y) \in \{(x,y) \mid 0 < y, x < \pi, 0 < x+y < \pi\}$

$$f(x,y) = L(x) + L(y) - L(x+y)$$

$$\Rightarrow \frac{\partial f}{\partial x} = \ln[\sin(x+y)/\sin(x)] \qquad \frac{\partial f}{\partial y} = \ln[\sin(x+y)/\sin(y)]$$

$$\frac{\partial^2 f}{\partial x^2} = \cot(x+y) - \cot(x) = -\frac{\sin(y)}{\sin(x+y)\sin x}, \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\cos(x+y)}{\sin(x+y)}$$

$$H(f) = -\frac{1}{\sin(x+y)} \begin{bmatrix} \sin(y)/\sin(x) & \cos(x+y) \\ \cos(x+y) & \sin(x)/\sin(y) \end{bmatrix} = (-k) \begin{bmatrix} \lambda & \mu \\ \mu & \frac{1}{\lambda} \end{bmatrix}$$

$\lambda > 0$. $|\mu| < 1$ done *

RM. Critical pts of V on $\Omega \Rightarrow x = y = \pi/3$.

For a local Euclidean structure x on (T^2, T), its volume

$$V(x) = \sum_f V(x(v,f), x(v',f), x(v'',f)).$$

sum of the volumes of its similarity triangles

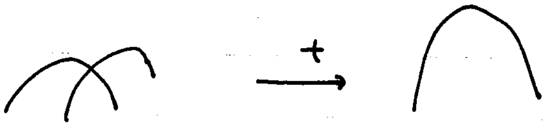
Thm (Rivin). (1) $V(x) : \mathcal{A}(\Delta) \rightarrow \mathbb{R}$ is strictly concave down

② The critical pts of $V : \mathcal{A}(\Delta) \rightarrow \mathbb{R}$ is equal to those local Euclidean structures induced from a flat metric.

RM 1. $\mathcal{A}(\Delta)$ is an open convex polytope. $\Rightarrow$ V has at most one critical pt (= max. pt)

$\Rightarrow$ theorem 1 (local rigidity)

Part (1) is trivial since sum of concave functions is concave



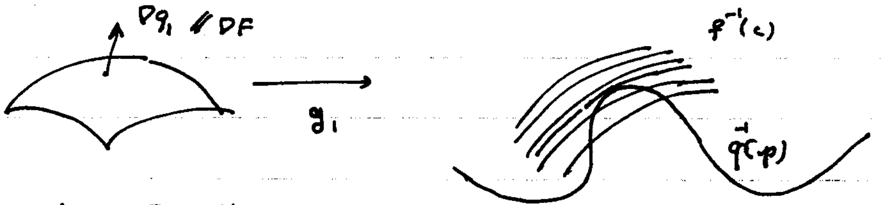
- 18,19,5=

L18-L19 Rivin

Pf: Recall the Lagrangian multiple: $F : \Omega \overset{\text{open}}{\subset} \mathbb{R}^n \longrightarrow \mathbb{R}$ smooth. $G : \Omega \to \mathbb{R}^k$ $= (g_1, \cdots, g_k)$ smooth, then the critical pts of $F| : G^{-1}(p) \to \mathbb{R}$ (p regular value) are

$$\nabla F = \sum_{i=1}^{k} \lambda_i \nabla g_i \qquad \lambda_i \in \mathbb{R} \text{ is the Lagrangian mult}$$

($k=1$:



)

RM if g_i linear it is EXACTLY the c.p

NOW: Let us label the set of all corners of T: $1, \cdots, n$.
There are constraints for each $f \in F$ + each $e \in E$.

The volume function $V(x_1, \cdots, x_n) = \sum L(x_i)$ is defined on $\mathbb{R}^n$.

w/

$$\frac{\partial V}{\partial x_i} = -\ln|2\sin(x_i)|$$

By the Lagrangian multipliers:

$$\exists \quad c : F \to \mathbb{R} \quad + \quad c' : E \to \mathbb{R} \quad \text{s.t}$$

(1) $$\nabla V = \sum_{f \in F} c_f \nabla_{\alpha_f} + \sum_{e \in E} c'_e \nabla_{\beta_e}$$

Note. α_f: $x_i + x_j + x_k = \pi$ $\qquad \nabla_{\alpha_f} = (0 \cdots \underset{i}{1}, \cdots \underset{j}{1}, \cdots \underset{k}{1} \quad)$

β_e : $x_i + x_j = \Delta(e)$ $\qquad \nabla_{\beta_e} = (\cdots 0, 1, \cdots, 1 \; 0 \cdots 0)$

Thus if $x \in \mathcal{A}(D)$ is a critical pt of $V : \mathcal{A}(D) \to \mathbb{R}$

(1) $\Longleftrightarrow$ $$\frac{\partial V}{\partial x_i} = c_f + c_e$$

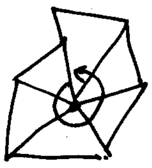
$i = (v, f)$

e faces v

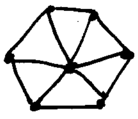
(2) i.e., $$-\ln|2\sin(x_i)| = c_f + c_e$$

To be derived from a Euclidean metric, it must have vanishing holonomies:

Example At each vertex v.



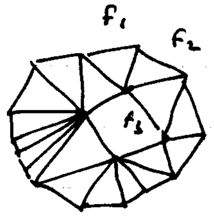
must match



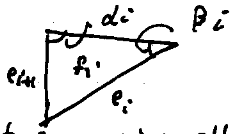
More generally, if we have a cycle of triangles

$f_1, \cdots, f_k \in F$

L18-L19 Rivin -18.19. 6-



Then, the edges must match.
hence
f_i, f_{i+1} share an edge e_i (i - mod n)



Lemma 2. Note the ratio of the lengths

$\ell_{f_i}(e_i)/\ell_{f_i}(e_{i+1})$ in a similarity type is well defined

For any cycle $\{f_1, e_1, f_2, e_2, \dots, f_k, e_k, f_1\}$, define its holonomy (in the local Euclidean structure) to be

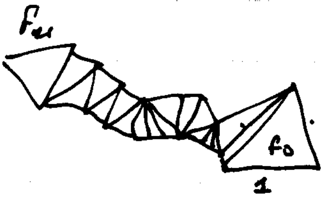
$$H(c) = \sum_{i=1}^{n} \ln\left(\ell_{f_i}(e_i)/\ell_{f_{i+1}}(e_{i+1})\right) = \sum \ln\left(\frac{\sin(\alpha_i)}{\sin(\beta_i)}\right)$$

Lemma 2 A local Euclidean structure x is derived from a Euclidean metric iff its holonomy $\equiv 0$ for all cycles

Pf "⇒" there exists a uniform metric d inducing it.

(3) Thus $H(c) = \sum \ln \ell_d(e_i) - \sum \ln \ell_d(e_{i+1}) = 0$

"⇐" Take a triangle $f_0 \in F$ and declare an edge of it to be 1



Now propagate the the Euclidean metric from f_0 to any other $f_m \in F$ through paths β of triangles $f_0, f_1, \dots, f_m$

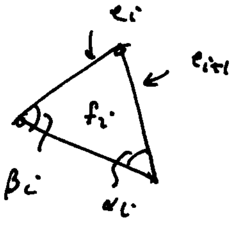
Where f_i, f_{i+1} share an edge e_i. Define a Euclidean metric d on f_m (depending on the path). But the vanishing of the holonomy ⇒ well defined.

#

if without ∂S, then

Lemma 3 Equations (2) + Equations (3) are equivalent.

Pf (2) ⇒ (3):

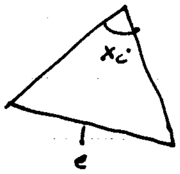


$$\ln \sin(\alpha_i) = -c_{f_i} - c'_{e_i} - \ln 2$$
$$\ln \sin(\beta_i) = -c_{f_i} - c'_{e_{i+1}} - \ln 2$$
$$\ln \sin(\alpha_i) - \ln \sin(\beta_i) = c_{e_{i+1}} - c_{e_i} \Rightarrow H(\text{cycle}) = 0$$

Rivin's Work

Pf "⇐" (3) holds ⇒ ∃ flat metric d inducing it.

Now for each triangle f:



$$\frac{l_d(e)}{\sin x_i} = \eta_f$$ depends only on f

(sine law)

$$\Rightarrow \quad -\ln \sin x_i = \ln \eta_f - \ln l_d(e)$$

so

$$-\ln 2\sin x_i = \left(-\ln 2 + \ln \eta_f\right) + (-\ln l_d(e))$$
$$= c_f \quad + c'_e$$

□

Rivin also proved

Thm 2 If $A(\Delta)$ contains more than one pt and $\Delta(e) \leq \pi$ ∀ e. Then V has its maximal pt in $A(\Delta)$.

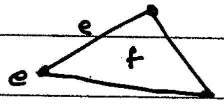
$\boxed{\text{3-connected}}$

(79)

Lecture 20-21. Local Euclidean Structure + Volume (Rivin)

Σ closed surface, T a triangulation w/ V, E, F vertices, edges + faces

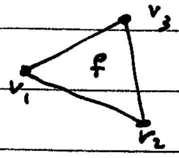
$v < e < f$ means v a vertex of e, e an edge of f



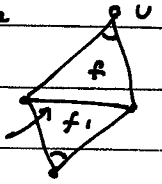
Corner (v, f) $v \in V$, $f \in F$ $v < f$.

Def A local E^2-structure on (Σ, T): $\overset{\text{angle}}{\downarrow} x : \{(v,f) \mid v < f\} \to (0, \pi)$ s.t

$$x(v_1, f) + x(v_2, f) + x(v_3, f) = \pi$$



Its dihedral angles $D_x(e) = x(v, f) + x(v', f)$ $e < f$, $e < f_1$

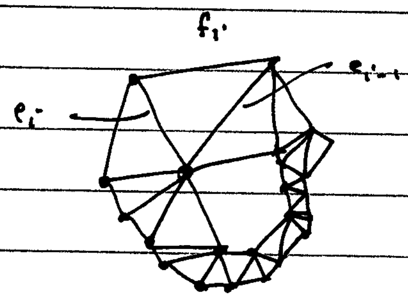


cycle of triangles $f_1, \dots, f_n \in F$ f_i, f_{i+1} share an edge e_i.

Its holonomy. $H_x(\{f_1 e_1 \dots f_n, e_n\})$

$$= \sum_i \ln \frac{\sin \alpha_i}{\sin \beta_i}$$

where α_i, β_i are the angles in x facing e_i + e_{i-1} in f_i



Def A local E^2 structure is called derived from a flat metric d on Σ if $\exists$ metric d on Σ d/f Euclidean st. x is the induced angle measure.

Lemma 1. A local E^2 structure x is derived from a flat metric $\overset{d}{\Leftrightarrow}$ $H_x = 0$ at all cycles

Pf 1. if d exists $\Rightarrow$ $H_x = 0$

Indeed

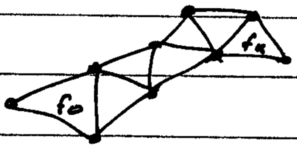
$$H(\text{cycle}) \quad \sum \ln \frac{\sin \alpha_i}{\sin \beta_i} = \sum_i \ln \frac{l_d(e_i)}{l_d(e_{i-1})} = \sum \ln l_d(e_i) - \ln l_d(e_{i-1}) = 0$$

Local Euclidean Structure

Conversely Suppose holonomy $\equiv 0$.

Take $f_0 \in F$ + fix a flat metric d_0 on f_0 s.t. its inner angs are given by x.

For any path of triangles ρ: $f_0, e_0, f_1, e_1, \cdots, e_{n-1}, f_n$ from f_0 to f_n where f_i, f_{i+1} share an e_i

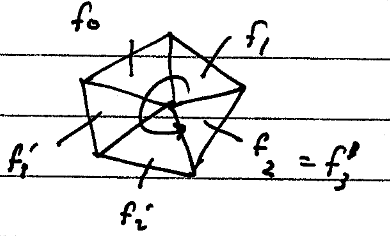


Extend d_0 to a flat metric d_ρ on f_n along ρ

The vanishing of $H(\text{cycle}) \Rightarrow$ $d_\rho = d_{\rho'}$ when

~~ρ, ρ' are two paths from f_0 to f_n (Check it~~

Say:



Vanishing of holonomy.

□

Given (Σ, T),

Thm. Let $\Delta : E \to (0, \infty)$ be a fixed map. Suppose $\mathcal{A}(D) = \{ x,$ local $\mathbb{E}^2$ structures w/ dihedral angle $D \}$ contains two pts.

Then the critical pts of the vol. $\mathcal{A}(D) \to \mathbb{R}$ are exactly those structures derived from a Euclidean metric.

Proof. . multipliers.

Recall Lagrange multiplier: $\Omega \subset \mathbb{R}^n$ open $F : \Omega \to \mathbb{R}$ smooth.

$G = (g_1, \cdots, g_m) : \Omega \to \mathbb{R}^m$ smooth, where p a regular value. Then

if c is a critical pt of $F|_{G^{-1}(p)}$ on $G^{-1}(p)$, we have

$$(*) \qquad \nabla F(c) = \sum_{i=1}^{m} \lambda_i \nabla g_i \Big|_c$$

for some $\lambda_1, \cdots, \lambda_m \in \mathbb{R}$ (Lagrange multiplier)

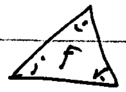
Furthermore, if $g_1, \cdots, g_m$ are linear, then all solutions of $(*)$ are critical pts of $F|_{G^{-1}(p)}$

Local Euclidean Structure

pf Say $x_1 \dots x_n$ are the set of all inner angles $(n = 3|F|)$

"(⇒)" If x is a critical pt. $V(x): \mathbb{R}^n \to \mathbb{R}$ is the volume

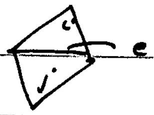
(1) $$V(x) = \sum_{\substack{i,j,k \\ \text{forms } f \in F}} \left(L(x_i) + L(x_j) + L(x_k)\right)$$



$$= \sum_i L(x_i)$$

Condition: At each $f \in F$: $x_i + x_j + x_k = \pi$ — α_f

At each edge $x_i + x_j = \Delta(e)$ — β_e

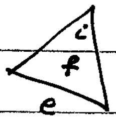


$\Rightarrow\ \exists\ c: F \to \mathbb{R}$ + $c': \underline{E} \to \mathbb{R}$ (Lag. Multiplier) st

(2) $$\nabla V = \sum_{f \in F} c_f \nabla \alpha_f + \sum_{e \in E} \widetilde{c}_e \nabla \beta_e$$

Compare:

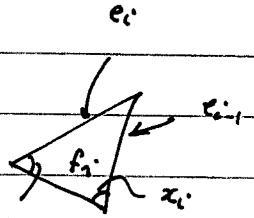
(3) $$\frac{\partial V}{\partial x_i} = c_f + \widetilde{c}_e$$



$$\shortparallel \uparrow \text{def}$$

$$-\ln|2\sin(x_i)|$$

Thus: along a cycle



the holonomy:

$$\ln\left(\frac{\sin x_i}{\sin y_i}\right) = \ln(2\sin x_i) - \ln(2\sin y_i)$$

$$= \left(c_{f_i} + \widetilde{c}_{e_i}\right) - \left(c_{f_i} + \widetilde{c}_{e_{i-1}}\right)$$

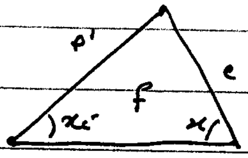
$$= \widetilde{c}_{e_i} - \widetilde{c}_{e_{i-1}}$$

$\Rightarrow$ Total sum $= 0$.

"⇐". Suppose on the other hand that x is derived from a flat metric d. Want to show (3) holds for some $c: F \to \mathbb{R}$ + $\widetilde{c}: E \to \mathbb{R}$.

Local Euclidean Structure

d the metric inducing x



Define $\tilde{c}: E \to \mathbb{R}$ to be $\tilde{c}(e) = -\ln(2\ell_d(e))$

Want: $-\ln(2\sin(x_i)) + \ln(2\ell_d(e))$

$= \ln\left(\frac{\ell_d(e)}{\sin x}\right) = c_f$ independent of the choice of which corner x_i in f that you chose.

$\|$

Since the sine law

$\ln \frac{\ell_d(e')}{\sin x'}$ □

3-Dim

Eight Geometries in Dim 3

constant sectional curvature

$\mathbb{H}^3$. S^3 (Iso = $O(4)$) $\mathbb{E}^3$ (Iso = $O(3) \ltimes \mathbb{E}^3$)

ref. $G \to \mathrm{Aut}(H)$ homomorphism, $G \ltimes H$ $(g, h) * (g', h') = (gg', h\,g(h'))$

Other 5: $\mathbb{H}^2 \times \mathbb{R}$, $S^2 \times \mathbb{R}$, $\widetilde{SL(2.\mathbb{R})}$, Nil. Sol $= \mathbb{R} \ltimes \mathbb{R}^2$

$PSL(2\mathbb{R}) \times \mathbb{R}$, $O(3)$

$t \cdot (x.y) = (e^t x, e^{-t} y)$

All discrete subgroup of finite volume in these geometries are classified (listed) except $\mathbb{H}^3$. $PSL(2.\mathbb{C})$.

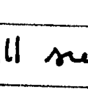
See P. Scott, Bull Lond. Math Soc. ~1983. (Geometries of 3-mfd. in Dim 3) vol 15. 401-

How to find hyperbolic structures? (Variational.)

For simplicity, assume M^3 cpt orientable and ∂M^3 consists of tori

Def An ideal 3-simplex := 3-simplex σ^3 - vertices removed

Def An ideal triangulation of M^3: $M^3 - \partial M^3$ is homeomorphic to the quotient space of finitely many ideal 3-simplices w/ faces identified in pairs by homeomorphisms.

Example S^3-  admits such

Thm (Moise) All such manifolds admits an ideal triangulation

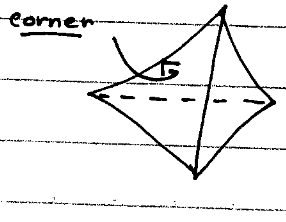
$(M, \mathcal{T})$ $(M, \mathcal{T})$ ideal triangulation

let V, E, F, T sets of vertices, edges, faces + 3-simplices

$x < y$ means x is a sub-simplex of y

A corner of $\mathcal{T}$: (e, t) $e \in E$, $t \in T$ $e < t$

it has 6 corners



3-Dimensional

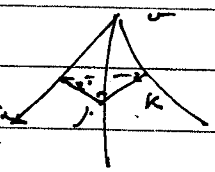
Def. A linear hyperbolic structure x on $(M, \mathcal{T})$.

$$x : \{ (e, t) \mid e \in E \quad t \in T \quad e < t \} \longrightarrow \mathbb{R}_{>0}$$

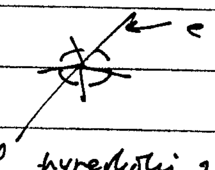
s.t.

(1) $x_i + x_j + x_k = \pi$

all three corners inside each $t \in T$ from a vertex



(2) $\Sigma x_i = 2\pi$ around each $e \in E$



$x_i > 0$

i.e. each $\sigma^3 \in T$ becomes a hyperbolic ideal σ^3.

Define its volume: $V(x)$ = sum of volumes of hyperbolic ideal σ^3 produced from x

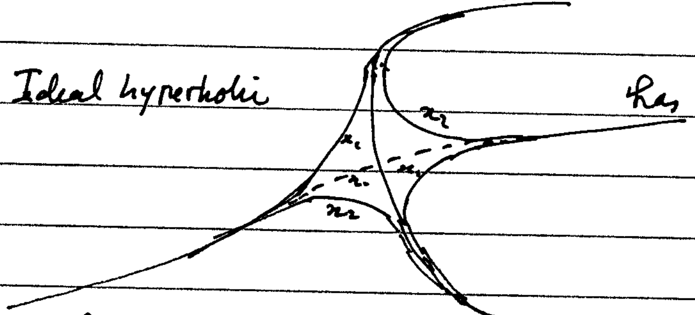
Let $A(T)$ be the space of all linear hyperbolic structures on $(M, \mathcal{T})$

(linear open convex polytope)

(If you don't care

$$V(x) = \frac{1}{2} \sum_i L(x_i) \qquad \text{over all corners}$$

RM Ideal hyperbolic



has volume

$L(x_1) + L(x_2) + L(x_3)$

Thm. Suppose $A(\mathcal{T})$ contains more than one point. ~~Then the [illegible]~~

(1) $V(x)$ is convex in $A(\mathcal{T})$

(2) The $\underbrace{\text{critical pt}}_{\text{max pt}}$ of V on $A(\mathcal{T})$ is exactly the complete hyperbolic structure on M.

Label the set of all corners by $1, \cdots, n$. Let $x_i = x$ (i-th corner)

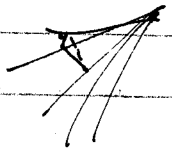
Pf: Given $x \in A(\mathcal{T})$

$$x = (x_1 \dots x_n) \in \mathbb{R}^n_{>0}$$

We can produce a hyperbolic structure on $M - \partial M$ by metric gluing of its (ideal) hyperbolic faces.

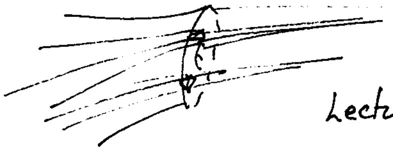
The main issue: the completeness of the structure.

Near each boundary component $S \subset \partial M$ ($\Leftrightarrow$ each vertex $v \in \mathcal{T}$).



$lk(v)$ = torus

with a hyperbolic ...

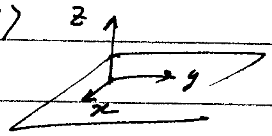


Lemma (Thurston). Thus, we may think ∂M is triangulated by $\mathcal{T}$.

x induces a local $\mathbb{E}^2$ structure on ∂M

Definition The structure x is called complete iff the local $\mathbb{E}^2$ structure on ∂M comes from a flat metric.

(Example Near ∞ of $\mathbb{H}^3/\Gamma$ $\Gamma = \langle z \mapsto z+1, \overset{z\mapsto}{z+\alpha} \rangle$



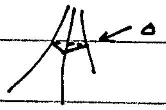
the cusp region is a flat torus

$$\frac{dx^2+dy^2+dz^2}{z^2} \quad \text{when } \underline{z = \text{const}} \quad \text{becomes} \quad \frac{dx^2+dy^2}{\lambda^2} \quad \underline{\text{flat}}$$

$\lambda > 0$

Pf. There are two sets of constraints:

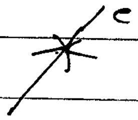
Δ: vertex triangle



$\Rightarrow$ Equation α_Δ: $x_i + x_j + x_k = \pi$

$E \ni e$

Equation β_e: $\Sigma x_i = 2\pi$

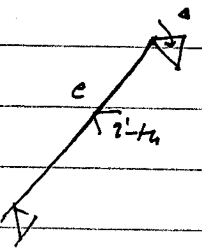


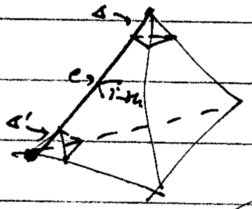
$(V: \mathbb{R}^4 \to \mathbb{R})$ $\exists$ c_2: $\{$ vertex trigs $\} \to \mathbb{R}$ $c_1: E \to \mathbb{R}$

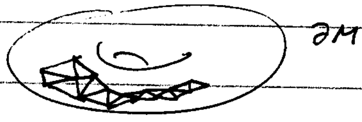
Thus, the Lagrangian multipliers: $\nabla V = \Sigma\, c_2(\Delta) \nabla \alpha_\Delta + \Sigma\, c_2(e) \nabla \beta_e$.

(1) $$\boxed{\frac{\partial V}{\partial x_i} = c_2(e) + c_1(\Delta) + c_1(\Delta')}$$

Where the i-th corner has edge e & adjoint to two vertex trigs Δ, Δ'



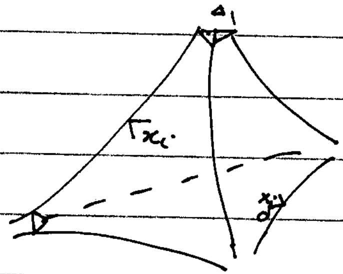




It is the critical pts

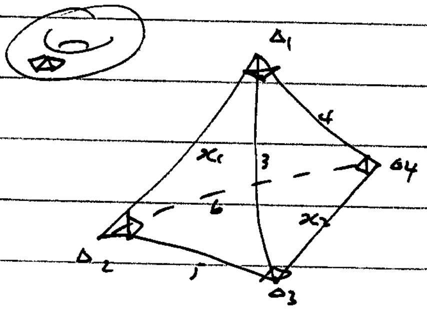
At the critical pt x:

$$\boxed{-\ln|2\sin x_j| = \text{[redacted]}\quad c_1(e) + c_2(\Delta_1) + c_2(\Delta_2')} \qquad (*)$$



Say at a vertex $v \ominus s$ of ∂M

Lemma 1: $x_i = x_j$ if i, j are two opposite corners in $t \in T$.



$$-\ln|2\sin x_1| = c_1(e_1) + c_2(\Delta_1) + c_2(\Delta_2)$$
$$-\ln|2\sin x_2| = c_1(e_2) + c_2(\Delta_3) + c_2(\Delta_4)$$

so $\ln \sin x_1 = \ln \sin x_2$

~~$-\ln|2\sin x_1| = \frac{1}{2}\left(c_1(e_1) + c_1(e_2)\right) + \sum_{i=1}^{4} c_2(\Delta_i)$~~

$\Rightarrow$ ~~$-\ln|\sin x_1| + \ln|\sin x_3|$~~

$$= \frac{1}{2}\left[c_1(e_1) + c_1(e_2) - c_1(e_3) - c_1(e_6)\right]$$

$$=$$

or $\ln\left(|\sin x_3| / |\sin x_1|\right) = $ [redacted] $= \xi_0 \pi S$

$$= \frac{1}{2}\left[c_1(e_1) + c_1(e_2) - c_1(e_3) - c_1(e_6)\right]$$
$$= \frac{1}{2}\left[c_1(e_2) - c_1(e_6)\right] + \frac{1}{2}\left[c_1(e_1) - c_1(e_3)\right]$$
$$= \frac{1}{2}\left[c_1(e_2) - c_1(e_6)\right] + \frac{1}{2}\left[\left(c_1(e_1) + c_1(e_4)\right) - \left(c_1(e_3) + c_1(e_4)\right)\right]$$

done.

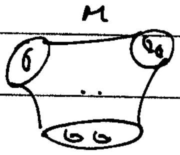
Lecture 22-23.

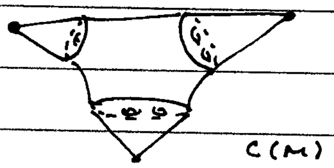
Steinitz

3-Manifolds & Hyperbolic Structures

M cpt 3-manifold $\partial M \neq \emptyset$ so that each component has $\chi(\cdot) \leq 0$

Let $C(M)$ be M w/ each boundary component cone off by a point.





Def An ideal triangulation of M = a triangulation $\mathcal{T}$ of $C(M)$ so that the vertices are cone pts.

$\Rightarrow$ ∂M is also triangulated by $\mathcal{T} \cap \partial M$.

Let V, E, F, T be the sets of all vertices, edges, triangles & tetrahedrons in $\mathcal{T}$ respectively.

A corner of $\mathcal{T}$: $\{(e,t) : e \in E,\ t \in T \quad e < t$

A vertex triangle of $\mathcal{T}$: (v,t): $v \in V,\ t \in T \quad v < t$.

Case 1 Assume all ∂M = tori

We label the set of $\{(v,t) \mid v \in V \ t \in T\}$ by $1, \dots, n$.

Def A linear hyperbolic structure on $(M, \mathcal{T})$:

$x: \{(e,t) \mid e \in E,\ t \in T \ e < t\} \longrightarrow \mathbb{R}_{>0}$ (dihedral angles)

s.t. (1) $x_i + x_j + x_k = \pi$ when i, j, k form a (v,t) $v < t$.

~~(2) $\forall e \in E$, $\sum x_i = 2\pi$ where x_i are corners landing at e~~

RM.

The volume of x: $V(x) = \sum_{v=1}^{n} L(x_i)$ (= 2· volume of all.)

The set of all linear hyperbolic structures on $(M, \mathcal{T})$ is denoted by $\mathcal{A}(T)$.

Vol: $\mathcal{A}(T) \longrightarrow \mathbb{R}_{\geq 0}$ $\mathrm{Vol}(x) = V(x)$

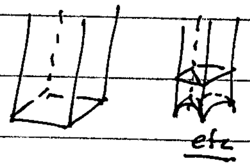
Thm 1. If $\mathcal{A}(T)$ contains more than one pt, then the critical pt of vol ~~one~~ exactly the complete hyperbolic metric on M.

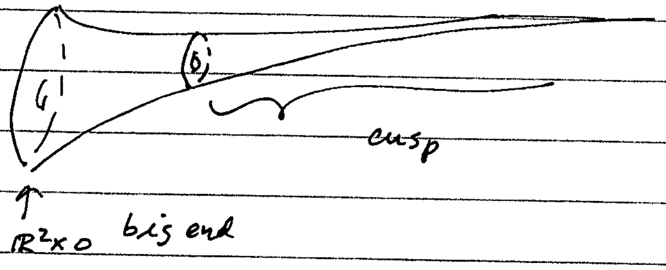
Proof. For each $x \in \mathcal{A}(T)$, we realized each $t \in T$ by an ideal hyperbolic σ^3 w/ dihedral angles given by x.

Now these geometric σ^3's can be glued w/ Isometries to obtain a hyperbolic (usually not complete) structure on $c(M) - (\text{cone pts}) \cong M - \partial M$.

Def. The hyperbolic structure is complete iff the local E^2 structure on each boundary component of ∂M is derived from a Euclidean metric.

Example. The cusp, $\mathbb{H}^3 / \begin{smallmatrix} z \sim z+1 \\ z \sim z+\alpha \end{smallmatrix}$ $\alpha \notin \mathbb{R}$ near ∞





has a flat torus boundary cut by $x_3 = \text{const.}$ Metric

$$\left(dx_1^2 + dx_2^2 + dx_3^2\right) / x_3^3 = \lambda\left(dx_1^2 + dx_2^2\right) \qquad \lambda > 0$$

We prove the theorem again using Lagrangian multipliers.

$\mathcal{A}(\mathcal{T}) \subset \mathbb{R}^n_{\geq 0}$ subject to two sets of linear constraints

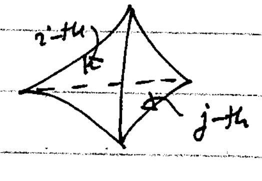
$\alpha_{(v,t)}$ $\qquad x_i + x_j + x_k = \pi$

β_e $\qquad \Sigma x_v = 2\pi$

$$V: \mathbb{R}^n_{>0} \longrightarrow \mathbb{R} \qquad V(x) = \sum_{i=1}^{n} L(x_i)$$

$$\frac{\partial V}{\partial x_i} = -\ln|2\sin x_i|.$$

Lemma: $x_i = x_j$ if i-th. and j-th corners are opposite inside $t \in T$



(This is a consequence of d.o.f.)

By the Lagrange Multipliers

$\exists$ multiplers. $c : E \longrightarrow \mathbb{R}$ and $\tilde{c} : \{(v,t) \mid v<t\} \longrightarrow \mathbb{R}$

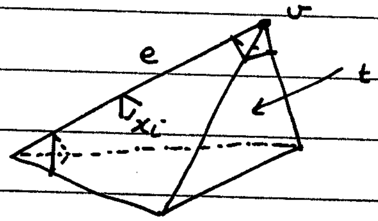
s.t

$$\nabla Vol = \sum_{e\in E} c(e)\nabla\beta_e + \sum_{\substack{(v,t)\\ v<t}} \tilde{c}(v,t)\nabla d_{v,t}$$

∇f = gradient = $\left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right)$

(i.e.).

(1) $\quad -\ln|2\sin x_i| = c(e) + \tilde{c}(v,t) + \tilde{c}(v',t) \qquad \partial e = \{v,v'\}$



where $x_i = x(e,t)$

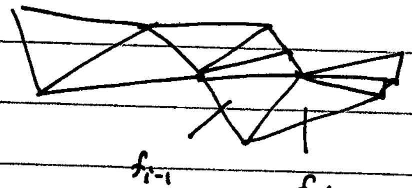
We claim that (1) is EXACTLY the vanishing of holonomy condition.

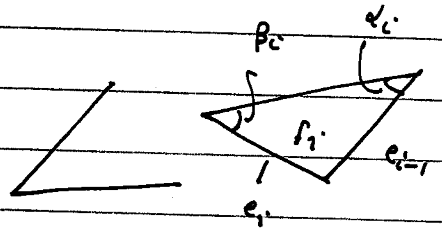
"⇒" Assume that x is a critical pt solving (1).

Take a boundary component of $\partial\Sigma$ corresponding to the vertex σ

Take a cycle (c) of triangles $\{f_1, e_1, f_2, e_2, \dots; f_n, e_n\}$ $\quad f_i \in$ triangles $\quad e_i$ edges

where f_i, f_{i+1} share e_i

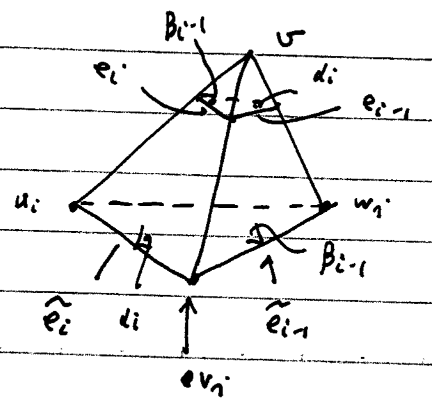




Let $t_i := v * f_i \in T$, $\quad v * e_i \in F$ be the corresponding simplices in T

and $\tilde{e}_{i-1}$ $\tilde{e}_{i-1}$ be the corresponding E (projected from σ of e_i, e_{i-1})

V_i = the common vertex shared by $\tilde{e}_i$, $\tilde{e}_{i-1}$



Hyperbolic Structure

By (1):

$$-\ln 2\sin(\alpha_i) \overset{\overset{e_{i-1}}{\downarrow}}{=} c(\overset{vw_i}{\cancel{\ \ \ }}) + \tilde{c}(v, t_i) + \tilde{c}(w_i, t_i)$$

$$-\ln 2\sin(\alpha_i) \overset{\tilde{e}_0}{=} c(\tilde{e}_i) + \tilde{c}(u_i, t_i) + \tilde{c}(v_i, t_i)$$

Sum + divide by 2

$$-\ln 2\sin(\alpha_i) = \frac{1}{2}\left(c(vw_i) + c(\tilde{e}_i)\right) + \frac{1}{2}\sum_{\substack{y<t_i\\ y\in v}} \tilde{c}(y, t_i)$$

By the same

$$-\ln 2\sin(\beta_i) = \frac{1}{2}\left(c(vu_i) + c(\tilde{e}_{i-1})\right) + \frac{1}{2}\sum_{\substack{y<t_i\\ y\in v}} \tilde{c}(y, t_i)$$

So

$$\ln\left(\sin(\alpha_i)/\sin(\beta_i)\right) = \frac{1}{2}\left(c(\tilde{e}_{i-1}) - c(\tilde{e}_i)\right) + \frac{1}{2}\left(c(vu_i) - c(vw_i)\right)$$

$$= \frac{1}{2}\left(c(\tilde{e}_{i-1}) - c(\tilde{e}_i)\right) + \frac{1}{2}\left\{\left[c(vu_i) + c(vv_i)\right] - \left[c(vw_i) + c(vv_i)\right]\right\}$$

$$= \frac{1}{2}\left(c(\tilde{e}_{i-1}) - c(\tilde{e}_i)\right) + \frac{1}{2}\left(D(\tilde{e}_{ia}) - D(\tilde{e}_{i-1})\right)$$

$$\Rightarrow \quad \sum \ln \sin(\alpha_i)/\sin(\beta_i) = 0.$$

#

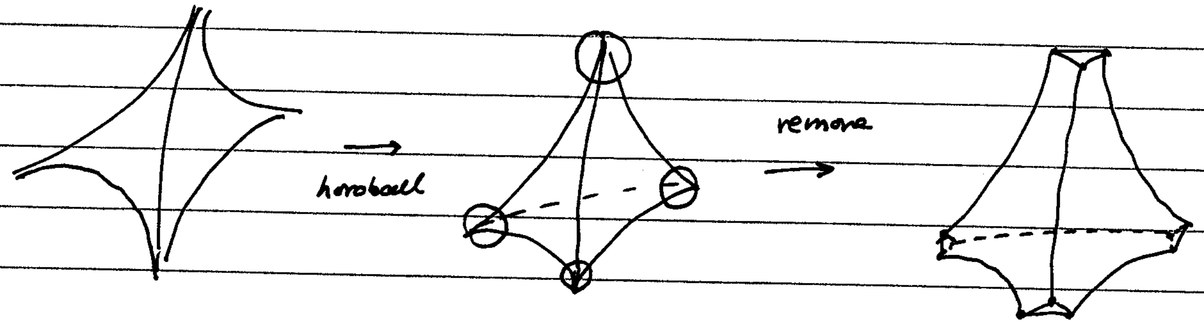
Conversely suppose $x \in \mathcal{A}(T)$ is a complete structure.

There exist a hyperbolic metric d on $M - \partial M$ so that each $t \in \mathcal{T}$ becomes ideal hyperbolic w/ inner (dihedral) angle given by x.

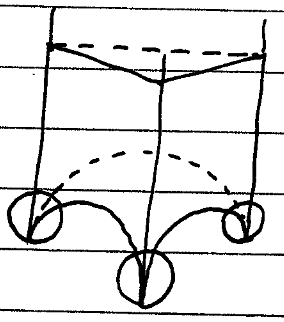
We claim $\exists$ c, $\tilde{c}$ so that (1) holds.

To this end, at each cusp, remove a horoball. (This is possible due to the completeness of d)

Thus, each hyperbolic ideal σ^3 in $\mathcal{C}(M)$



in $\mathbb{H}^3$:



For each $e \in E$, the distance $l_d(e)$ = distance between the horospheres measured along e. is well defined

<u>Goal</u>: to produce $c: E \to \mathbb{R}$ and $\tilde{c}: \{(v,t) \mid v < t\} \to \mathbb{R}$

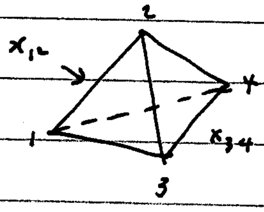
<u>s.t</u>

$$\tilde{c}(v,t) + \tilde{c}(v',t) = -\ln|2\sin x_{v}| + c(e) \qquad (2)$$

$x_i = x(e,t)$ $\partial e = v \cup v'$

Suppose $c(e)$ has been defined

(2) is individualized at each t (solve for each $t \in T$ individually !)



When can you write $\underline{x_{ij} = \tilde{c}_i + \tilde{c}_j}$?

<u>Lemma</u> Suppose x_{ij}, $i \neq j$ in $\{1,2,3,4\}$ are six given numbers $x_{ij} = x_{ji}$.

Then $x_{ij} = c_i + c_j$ for some c_1, c_2, c_3, c_4 iff

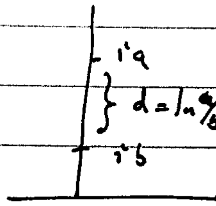
$$x_{ij} + x_{kl} = x_{ik} + x_{kl} = x_{il} + x_{jk}$$

where $\{i,j,k,l\} = \{1,2,3,4\}$ (对边和都为常数)

22.7

3-Manifolds & Hyperbolic Structures

$\Rightarrow \quad d_1 + d_2 - d_3 - d_4 \qquad d_2 = d_4 = 0$

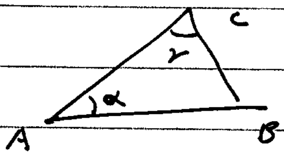


$$= d_1 - d_3 = \ln\left(\frac{h}{2r_A}\right) - \ln\left(\frac{h}{2r_C}\right)$$

$\left(d_{Hy}(ia, ib) = \ln b/a \right)$

$$= \ln\left(\frac{r_C}{r_A}\right)$$

$$= 2\ln \frac{|B-C|}{|A-B|}$$



$$\overset{\text{sine law}}{\underset{\triangle ABC}{=}} 2 \ln \frac{\sin\alpha}{\sin\gamma}$$

This establishes the existence

#

<u>Corollary</u>. 1. $S^3 -$  supports a complete hyperbolic structure:

Since it has an ideal triangulation s.t each edge has degree 6.

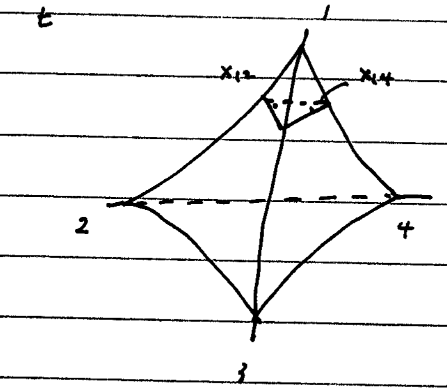
$$V(x) \leq \underset{\substack{\uparrow \\ \text{is the absolute maximal.}}}{2 \cdot \mathrm{vol}(\pi/3 . \pi/3 . \pi/3)}$$

2. If M^3 has an ideal triangulation so that each edge has degree 6. then $\mathring{M}$ has a complete hyperbolic structure of finite volume

(**Prasad** : Extend Mostow rigidity to finite volume)

Thm (Casson) If (M, T) supports a linear hyperbolic structure, then
M is irreducible and all incompressible tori are parallel to ∂M.

Applied it to (2): thus, we need $c: E \longrightarrow \mathbb{R}$ s.t $\forall t \in T$



x_{ij} = dihedral angle at edge e_{ij}:

$$-\ln 2\sin x_{ij} + c(e_{ij}) - \ln 2\sin x_{kl} + c(e_{kl})$$

$=$ depending only on t

$$= -\ln 2\sin x_{ik} + c(e_{ik}) - \ln 2\sin x_{jl} + c(e_{jl})$$

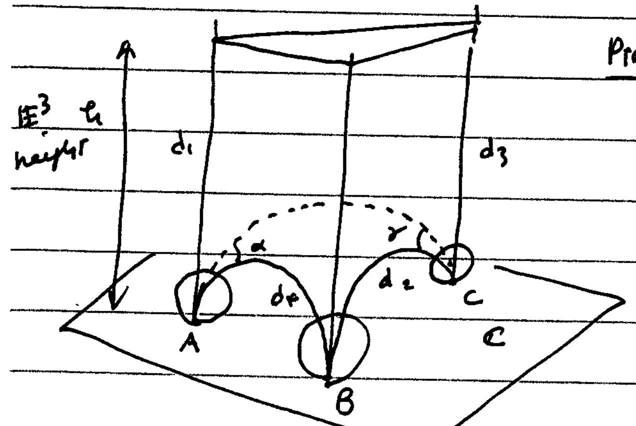
But opposite sides ~~are~~ have the same dihedral angles. $\Rightarrow$

~~(3)~~ $$\boxed{\ln \frac{\sin x_{ij}}{\sin x_{ik}} = c_{ik} + c_{jl} - c_{ij} - c_{kl}}$$

<u>Lemma</u>: Define $c_{ij} = -\frac{1}{2}\, l_d(e_{ij})$ (the hyperbolic distance between horospheres along e_{ij})

Then (3) holds.

<u>Pf</u>: if you change the size of horo-ball, the RHS of (3) remains unchanged.



<u>Proof</u> Take a Euclidean triangle $\Delta ABC \subset \mathbb{C}$

and consider an ideal hyperbolic σ^3 based on ABC

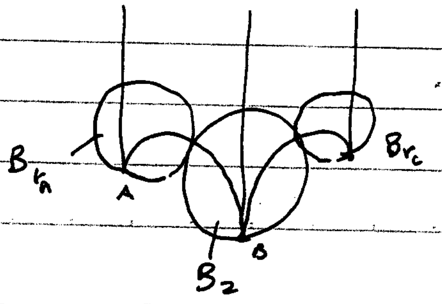
$\underline{d_1 + d_2 - d_3 - d_4}$

~~Lemma~~ B_{r_B}

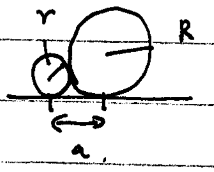
Fix a horoball of radius r_B at B

Take two horoballs of radii r_A & r_c at A, C

So that they are tangent to B_{r_B}



<u>Lemma</u>:



$\underline{r \cdot R = \left(\frac{a}{2}\right)^2}$

$$\Rightarrow \quad r_A r_B = \frac{|A-B|^2}{4} \qquad r_C r_B = \frac{(B-C)^2}{4}$$