Poisson Jumps in Credit Risk Modeling: a Partial Integro-differential Equation Formulation

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Outline

- Probabilistic approach to the heat equation
- Fokker-Planck equation and the boundary value problem
- Matching exit probability - free boundary formulation
- Application: modeling credit risk in finance
- Poisson jump-diffusion
- Partial integro-differential equation formulation
- Analysis issues

Probabilistic Approach to the Heat Equation

- Random walk: step sizes from normal distribution $N(0, T)$
- X: new position of the particle starting from 0

$$P[X \in (x, x + dx)] = \frac{1}{\sqrt{2\pi T}} e^{-\frac{x^2}{2T}} dx$$

- One step of random walk replaced by N steps

$$X_{n+1} = X_n + \epsilon_n, \;\; \epsilon_n \sim N(0, T/N), \;\; n = 0, 1, ..., N - 1,$$

$$X_0 = 0, \; \epsilon_n \text{independent}$$

- $P[X_N \in (x, x + dx)]$ same as above

Probabilistic Approach to the Heat Equation (Continued)

- Let $N \to \infty$, X_t follows the process

$$dX_t = dW_t, \quad X_0 = 0$$

  W_t : standard Brownian motion

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$$u(x,t) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} = P[X_t \in (x, x+dx)]/dx$$

  satisfies

$$u_t = \frac{1}{2} u_{xx}, \quad u(x,0) = \delta(x)$$

- Heat kernel

More general cases

- Itô process:

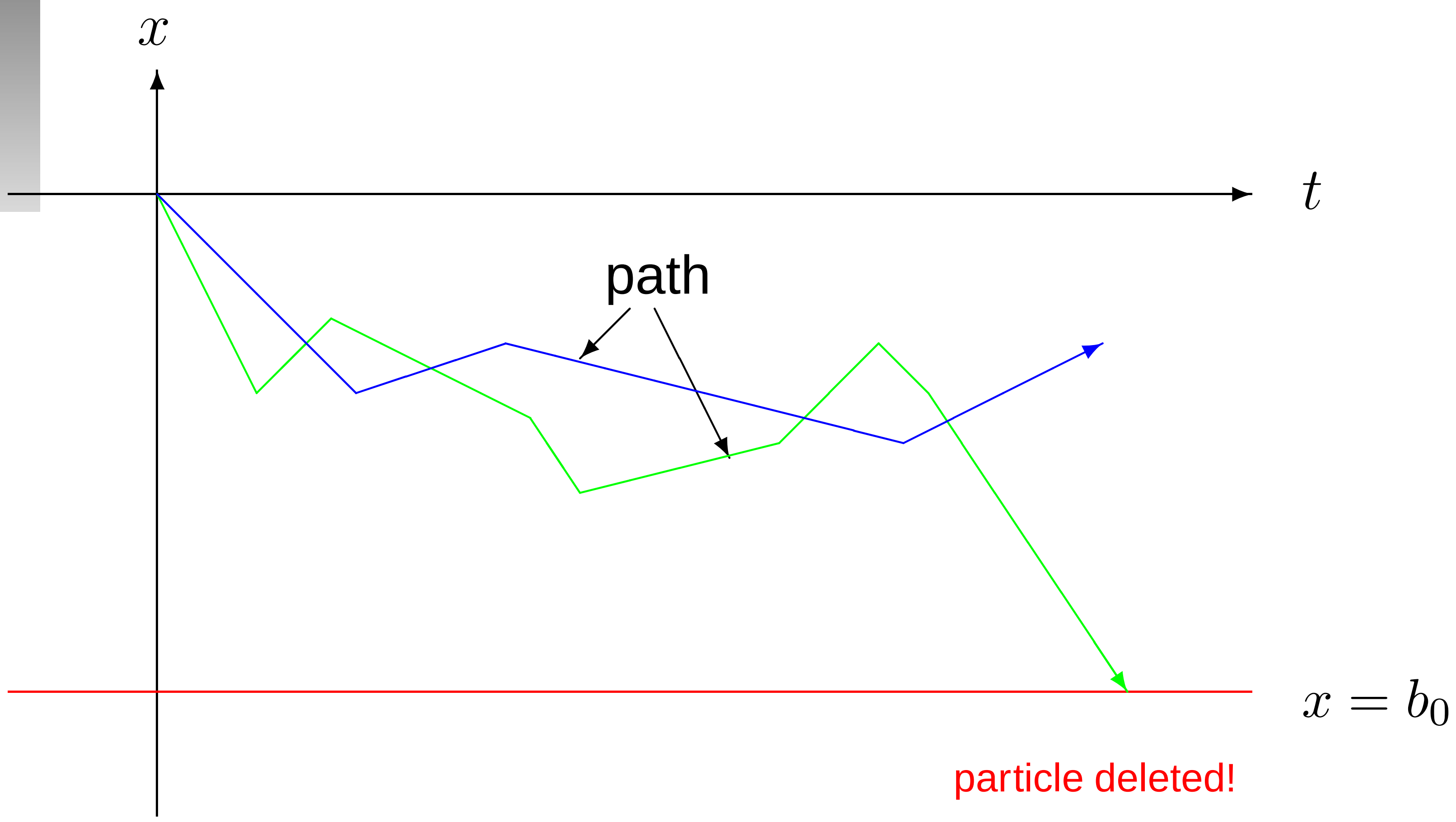
$$dX_t = a(X_t, t)dt + \sigma(X_t, t)dW_t, \quad X_0 = 0.$$

- Distribution of particles satisfies the Fokker-Planck equation

$$u_t = \frac{1}{2}(\sigma^2 u)_{xx} - (au)_x$$

$$u(x, 0) = \delta(x)$$

Killing of Particles



- Boundary condition for u at $x = b_0$: $u(b_0, t) = 0, \quad t > 0$

Solution to the Model Problem

- Consider the special case $a = 0$, $\sigma = const$:

$$u_t = \frac{1}{2}\sigma^2 u_{xx}, \quad x > b_0$$

$$u(x,0) = \delta(x), \quad u(b_0,t) = 0.$$

- Solution

$$u(x,t) = \frac{1}{\sqrt{2\pi\sigma^2 t}} \left[e^{-\frac{x^2}{2\sigma^2 t}} - e^{-\frac{(x-2b_0)^2}{2\sigma^2 t}} \right], \quad x \geq b_0$$

- Explicit solutions also available for linear b
- Standard existence theory for general barrier $b(t)$

Survival Probability and Free Boundary

- Probability of survival up to t:

$$Q(t) = \int_{b(t)}^{\infty} u(x,t)dx, \quad Q'(t) < 0$$

- Probability that X_t has exited the barrier by t:

$$P(t) = 1 - Q(t) = 1 - \int_{b(t)}^{\infty} udx$$

- Exit probability density:

$$P'(t) = -Q'(t) = \int_{b(t)}^{\infty} u_t dx = \frac{1}{2}\frac{\partial}{\partial x}(\sigma^2 u)|_{x=b(t)}$$

- Can we find $b(t)$ to generate a given exit probability density?

Application in Finance: Distance-to-default Model

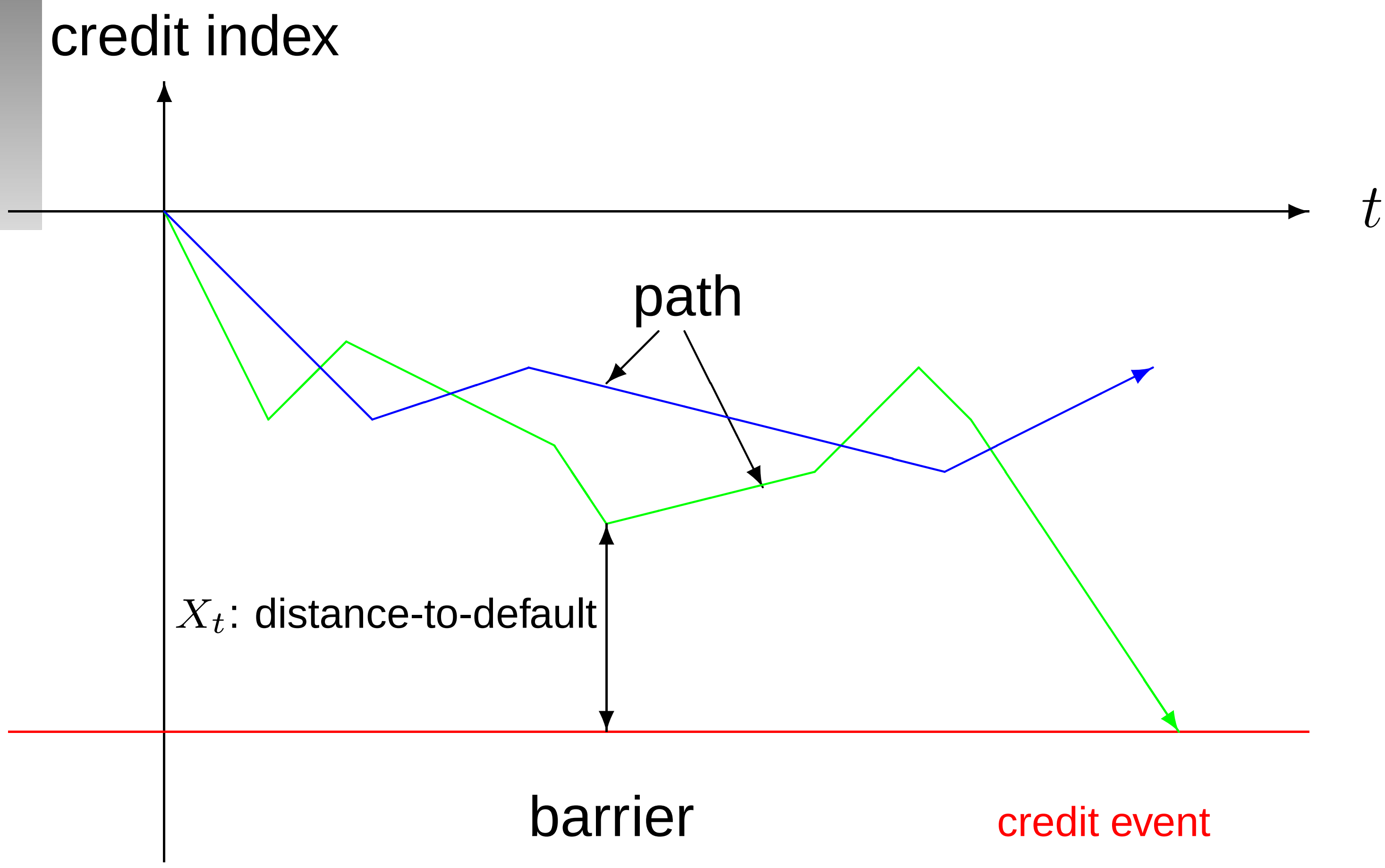
- (Ref. Avellaneda and Zhu, *Risk* , 2001)
- Distance-to-default: $X_t = V_t - b(t) \geq 0$,
  $V_t, b(t)$: generalized value of the firm and liability

$$dX_t = -b'(t)dt + \sigma(X_t, t)dW_t$$

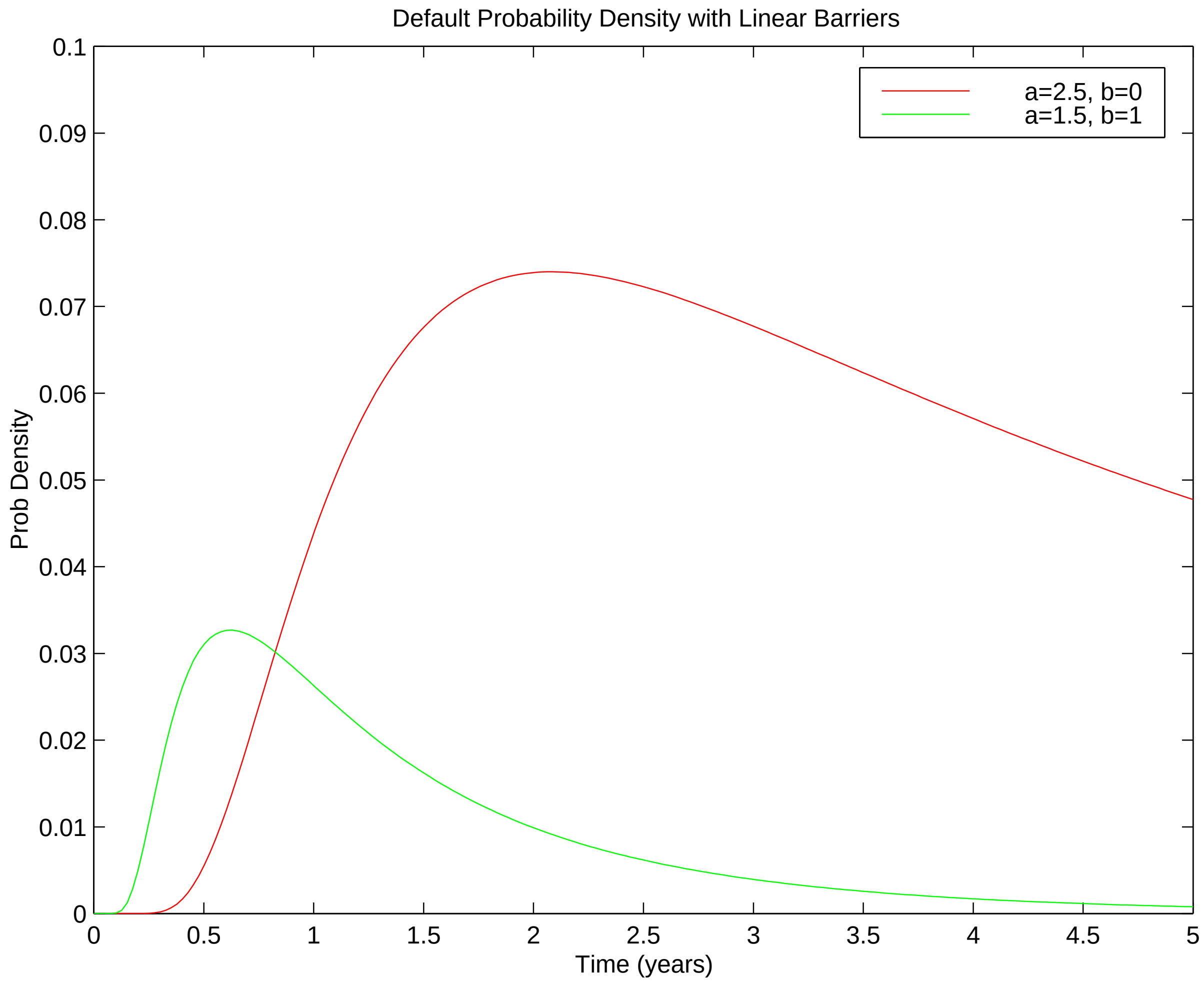
- Barrier interpretation: $b(t),\ t \geq 0$ to be determined
- First exit time: $\tau = \inf \{t \geq 0 : X_t \leq 0\}$
- $u(x,t)$: survival probability density at t:

$$u(x,t)dx = P\left[X_t \in (x, x+dx),\ t < \tau\right], \quad x \geq 0$$

Schematic illustration:



$P'(t)$ with Linear Barrier



$$b(t) = -\alpha - \beta t$$

Control Problem for $u(x,t)$

- Determine b so that the solution to

$$u_t = \frac{1}{2}(\sigma^2 u)_{xx} + b' u_x, \quad x > 0,\ t > 0$$

with initial and boundary conditions:

$$u_{|t=0} = \delta(x + b(0))$$

$$u_{|x=0} = 0, \quad t > 0$$

- satisfies the additional BC

data fitting $\longrightarrow \frac{1}{2}\left[\frac{\partial}{\partial x}\left(\sigma^2 u\right)\right]_{|x=0} = P'(t), \quad t > 0$

Calibration: Free Boundary Problem

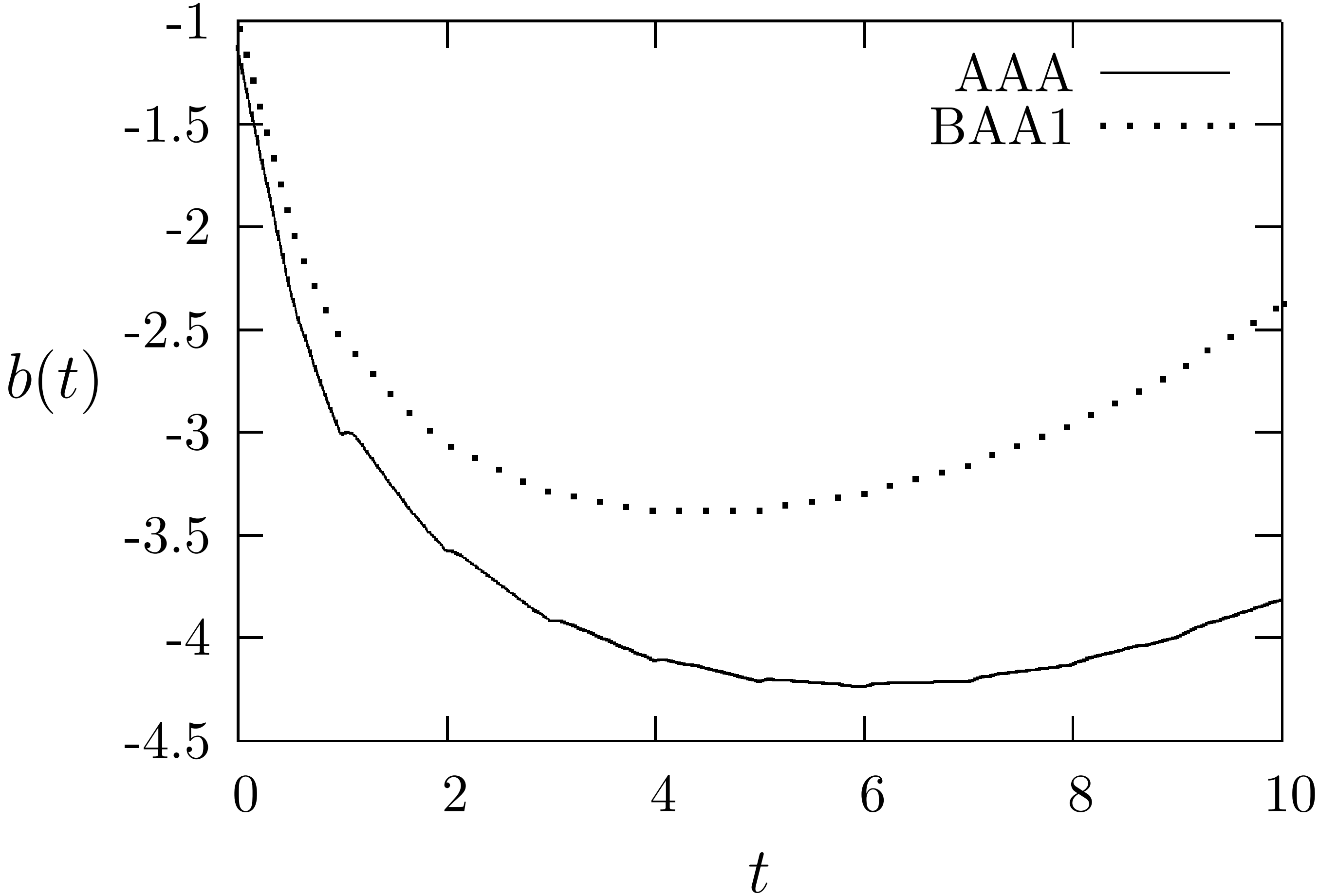
- Difficulties:
  - Starting from a δ-function initial data
  - Need to determine the correct $b(t)$
- Approaches:
  - Initial layer: $b(t), 0 \leq t \leq t_0$ for small t_0 approximated by a linear barrier
  - Second-order finite difference method to solve for $t > t_0$, using $u(x, t_0)$ from the initial layer as initial condition
- Matching two solutions at $t = t_0$:
  - Choose $\alpha,\ \beta$ so that $\bar{P}(t_0) = P(t_0),\ \bar{P}'(t_0) = P'(t_0)$

Examples

- Default probabilities for the bank industry with ratings in AAA and BAA1.

year	AAA	BAA1
1	0.0073	0.0222
2	0.0136	0.0285
3	0.0166	0.0315
4	0.0190	0.0339
5	0.0210	0.0360
6	0.0229	0.0380
7	0.0246	0.0396
8	0.0264	0.0415
9	0.0284	0.0437
10	0.0307	0.0466

Default Barriers for AAA and BAA1 Companies



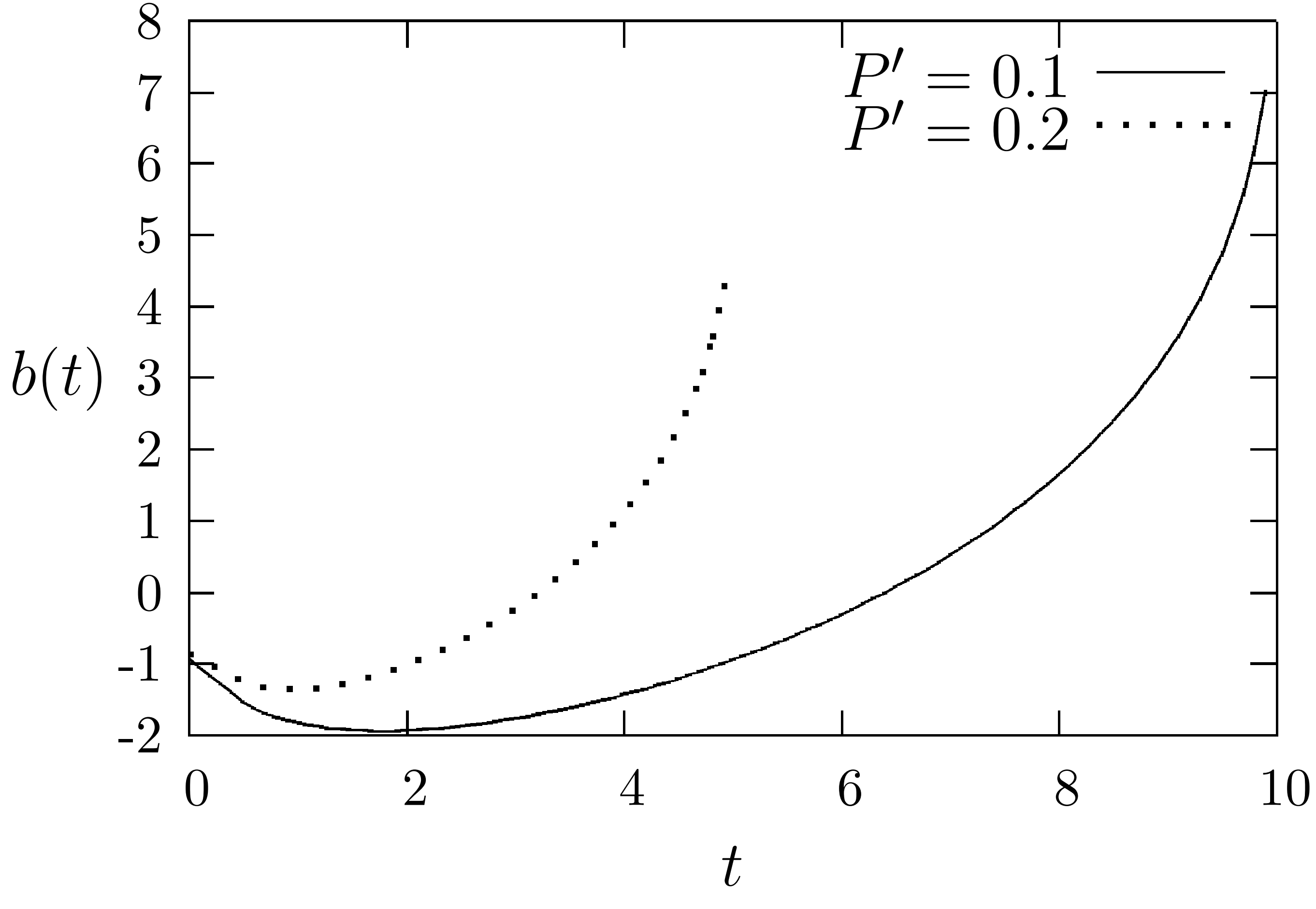
Questions

- Existence of solutions?

$$P'(t) > 0, \quad P(t) \leq 1, \text{for } t < T$$

- $u \geq 0$ for all $x \geq 0$?
- Stability of the barrier?

Blowup of Default Barrier



Linear Stability Analysis

- Small perturbations to P lead to small changes in $b(t)$?
- Perturbation analysis: for small ϵ

$$
\begin{aligned}
P(t) &= P_0(t) + \epsilon P_1(t) + O(\epsilon^2) \\
u(x,t) &= u_0(x,t) + \epsilon u_1(x,t) + O(\epsilon^2) \\
b(t) &= b_0(t) + \epsilon b_1(t) + O(\epsilon^2)
\end{aligned}
$$

- u_0 satisfies the equations with extra condition P_0
- Goal: bound $||b_1(t)||$ in terms of $||P_1(t)||$

Perturbation Equation

- Assume $\sigma = 1$
- u_1, if exists, should satisfy

$$v_t = \frac{1}{2} v_{xx} + b_0' v_x + b_1' u_{0,x}$$

$$v\big|_{x=0} = 0$$

$$v\big|_{t=0} = 0$$

$$v_x\big|_{x=0} = 2P_1'(t)$$

- b_1' chosen based on $P_1'(t)$

Heat Kernel

- Consider problem for $w(x,t,s)$, for arbitrary b_1:

$$w_t = \frac{1}{2} w_{xx} + b_0' w_x, \quad t > s$$

$$w|_{x=0} = 0$$

$$w|_{t=s} = b_1'(s) u_{0,x}(x,s)$$

- Express the solution

$$w(x,t,s) = K_{b_0} * \left(b_1'(s) u_{0,x}(x,s)\right)$$

K_{b_0} is the general time-dependent heat kernel

Duhamel's Principle

- Represent solution

$$u_1(x,t) = \int_0^t w(x,t,s)ds$$

- Compute $u_{1x}(0,t)$:

$$2P_1'(t) = \int_0^t w_x(0,t,s)ds = \int_0^t \tilde{K}_{b_0,u_0}(t,s)b_1'(s)ds$$

- b_1' and $P_1''(t)$ related through an integral equation

Discussions

- Challenges:

  Extremely low default probability for short time horizon

- Propositions:
  - Introduce time dependent volatility $\sigma(t)$
  - Allow the barrier to be stochastic (Pan, 2001)
  - Start from a probability distribution (Imperfect information)
  - Add Poisson jumps

Compound-Poisson Process

- Features:
  - Shocks (jumps) coming at uncertain times
  - Markov process
  - $Z(t)$: total number of occurrences before t,

$$P_n(t) = \mathbf{P}\{Z(t) = n\} = \frac{(\lambda t)^n}{n!} e^{-\lambda t}$$

- Intensity λ:
  - for small h, $P_1(h) = \lambda h + o(h), \; P_0(h) = 1 - \lambda h + o(h)$
- Applications in finance: stock option pricing (Merton, 1976), reduced-form models

Combined Wiener-Poisson Process

- Discontinuous process:

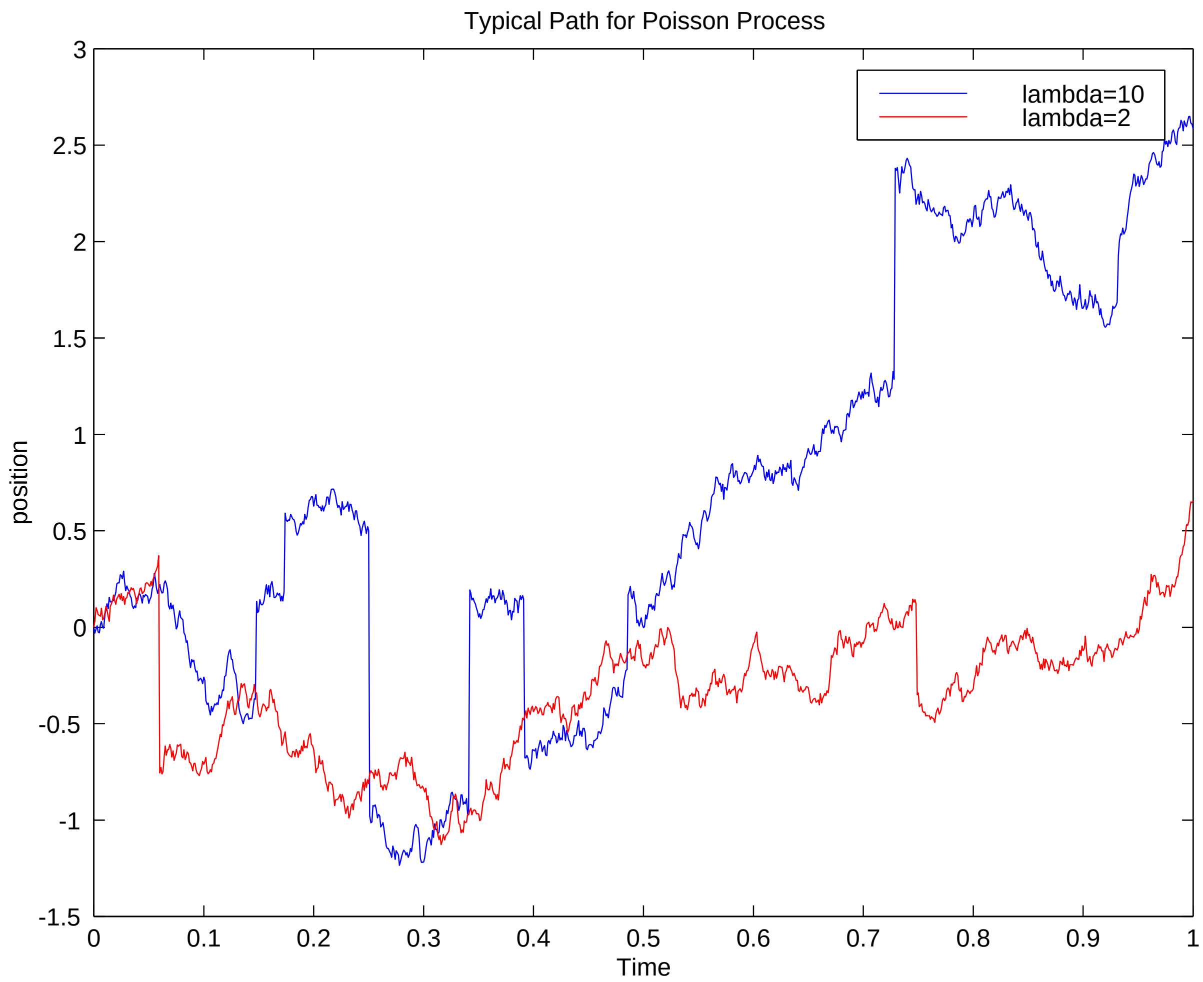
$$dX_t = -b' dt + \sigma dW_t + dq_t$$

- q_t experiences jumps with intensity λ,
- Once a jump occurs, probability measure of the jump amplitude

$$G(x, dy) = P[x \to (y, y + dy)]$$

is given

Sample Paths



Infinitesimal Generator of the Process

- For the Poisson process, with small $t > 0$

$$E^x[f(X_t)] \approx \lambda t \int f(y) G(x, dy) + (1 - \lambda t) f(x)$$

- For the combined process

$$\begin{aligned} \mathcal{A} f(x) &= \lim_{t \to 0+} \frac{E^x\left[f(X_t)\right] - f(x)}{t} \\ &= \frac{1}{2}\sigma^2 f_{xx} - b' f_x + \lambda \int \left(f(y) - f(x)\right) G(x, dy). \end{aligned}$$

- Kolmogorov backward equation:

$$\frac{\partial f}{\partial t} = \mathcal{A} f$$

Forward Equation with Boundary Condition

- Adjoint operator (assuming $G(x, dy) = g(x, y)dy$):

$$\mathcal{A}^* u = \frac{1}{2}\left(\sigma^2 u\right)_{xx} + b'(t)u_x + \lambda\left[\int_0^\infty u(y,t)g(y,x)dy - u\right]$$

- Boundary condition
- Forward equation (Fokker-Planck)

$$u_t = \mathcal{A}^* u, \;\; x > 0, \quad u(x,t)_{|x=0} = 0, \; t \geq 0.$$

- Killing of particles:

$$Q'(t) = -\frac{\sigma^2}{2}u_x(0,t) + \lambda\int_0^\infty u(y,t)\left[\int_0^\infty g(y,x)dx - 1\right]dy < 0$$

Partial Integro-Differential Equation

- Assume $g(y,x) = g(x-y)$,

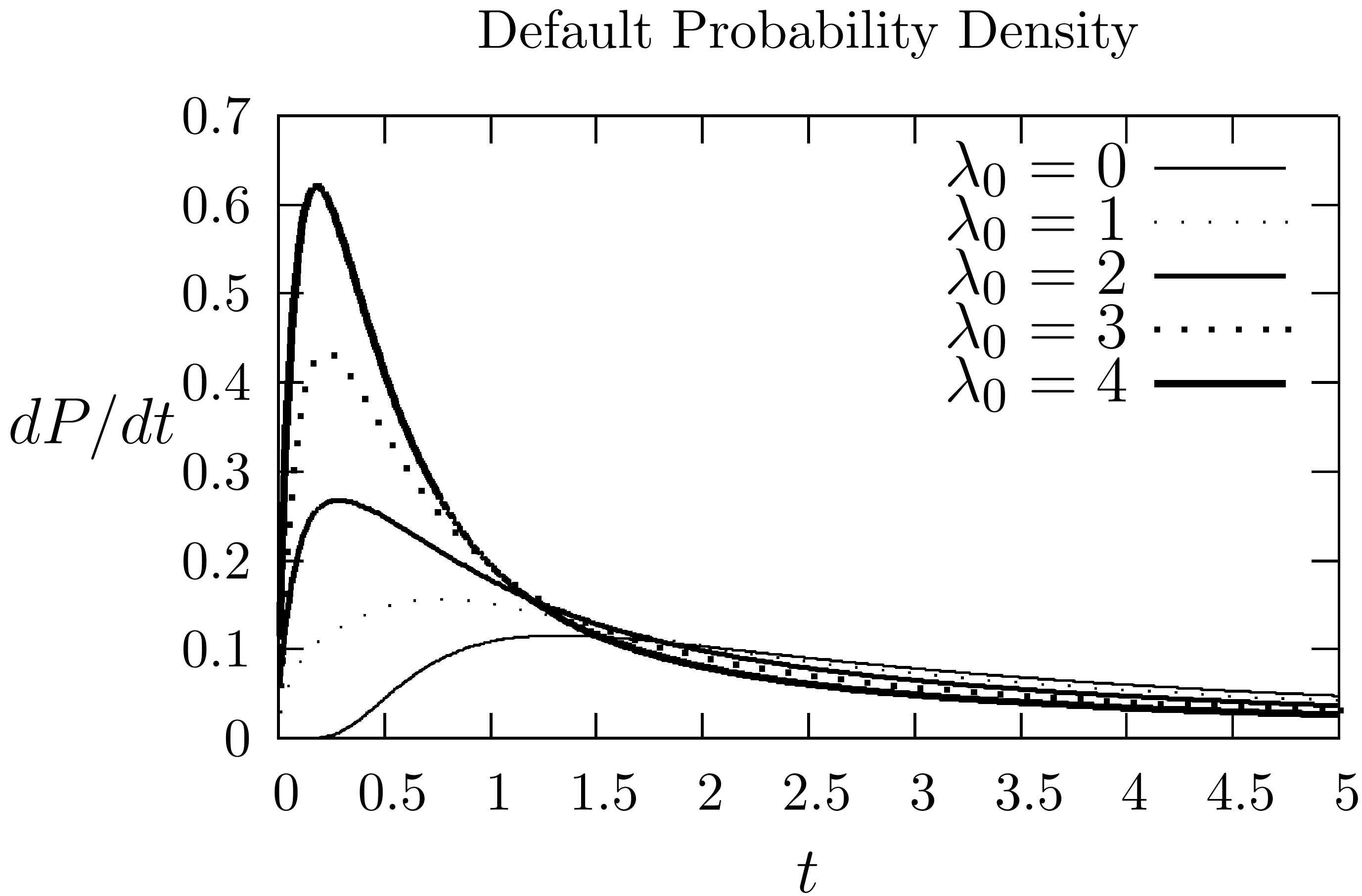
$$u_t = \frac{1}{2}(\sigma^2 u)_{xx} + b'(t)u_x - \lambda u + \lambda \int_0^\infty u(y)g(x-y)dy, \quad x > 0$$

- Boundary condition: $u(0,t) = 0$
- Probability density function $g(x)$ for jump size

$$g(x) = \frac{1}{\sqrt{2\pi\beta^2}} e^{-\frac{(x-\mu)^2}{2\beta^2}}$$

- Example: $\lambda = \lambda_0 e^{-t},\ \mu = -e^{-2t},\ \beta = \frac{1}{2}e^{-0.2t}$

Default Probability Density with Poisson Jumps



Matching Condition at the Boundary

- Matching condition is nonlocal

$$\frac{1}{2}\left(\sigma^2 u\right)_{x|_{x=0}} - \lambda \int_0^\infty \int_0^\infty u(y) g(x-y) dy dx = \lambda(P(t)-1) + P'(t), \quad t > 0$$

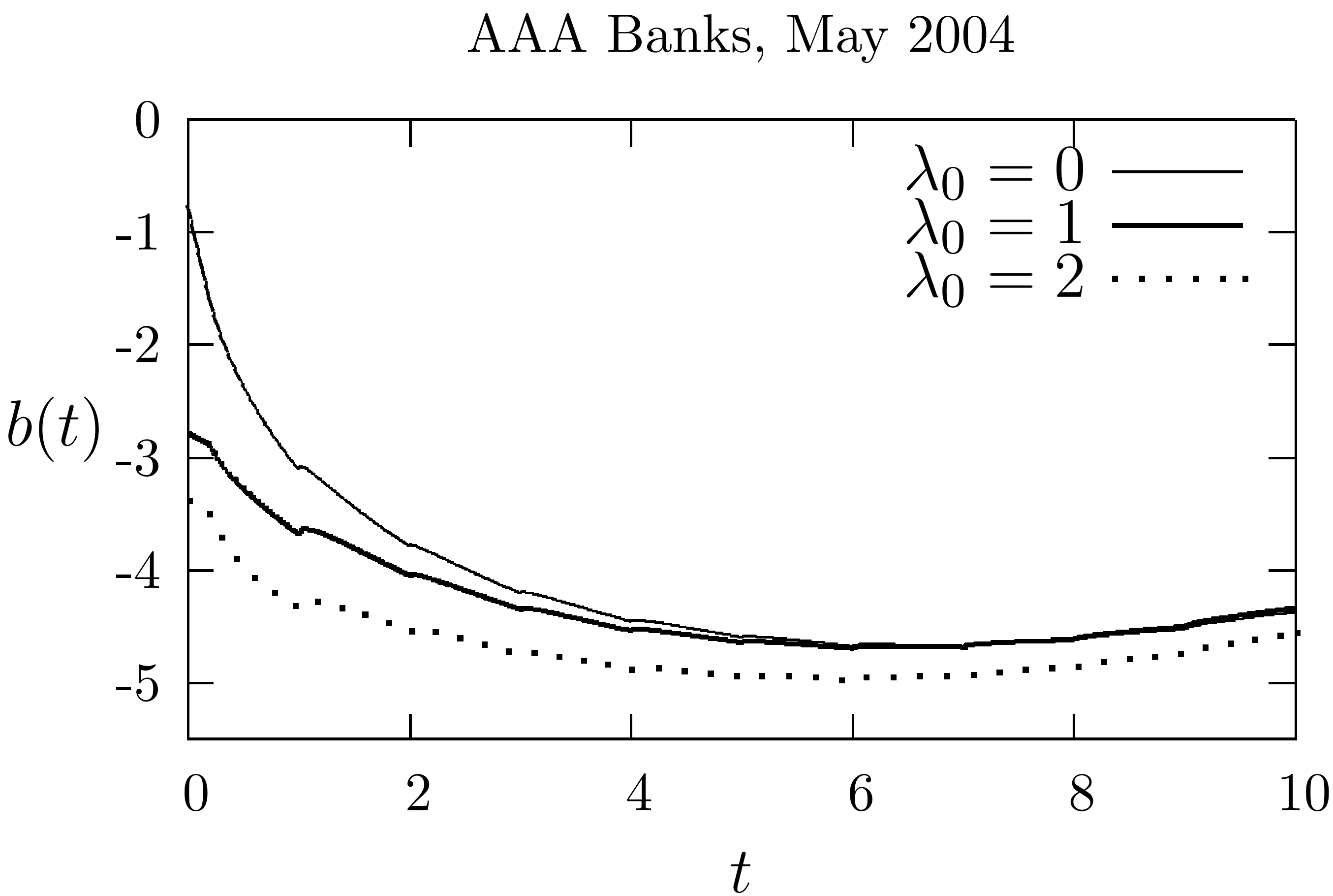
- Finite difference-Nyström approximation of the partial integro-differential equation
- Integral term treated as a source term
- Initial layer: a linear barrier for $0 < t < t_0$ is sought to match data ($P(t_0)$ and $P'(t_0)$), numerical solutions used
- Similar shooting technique to determine the free boundary for $t > t_0$

Example: Jump-diffusion

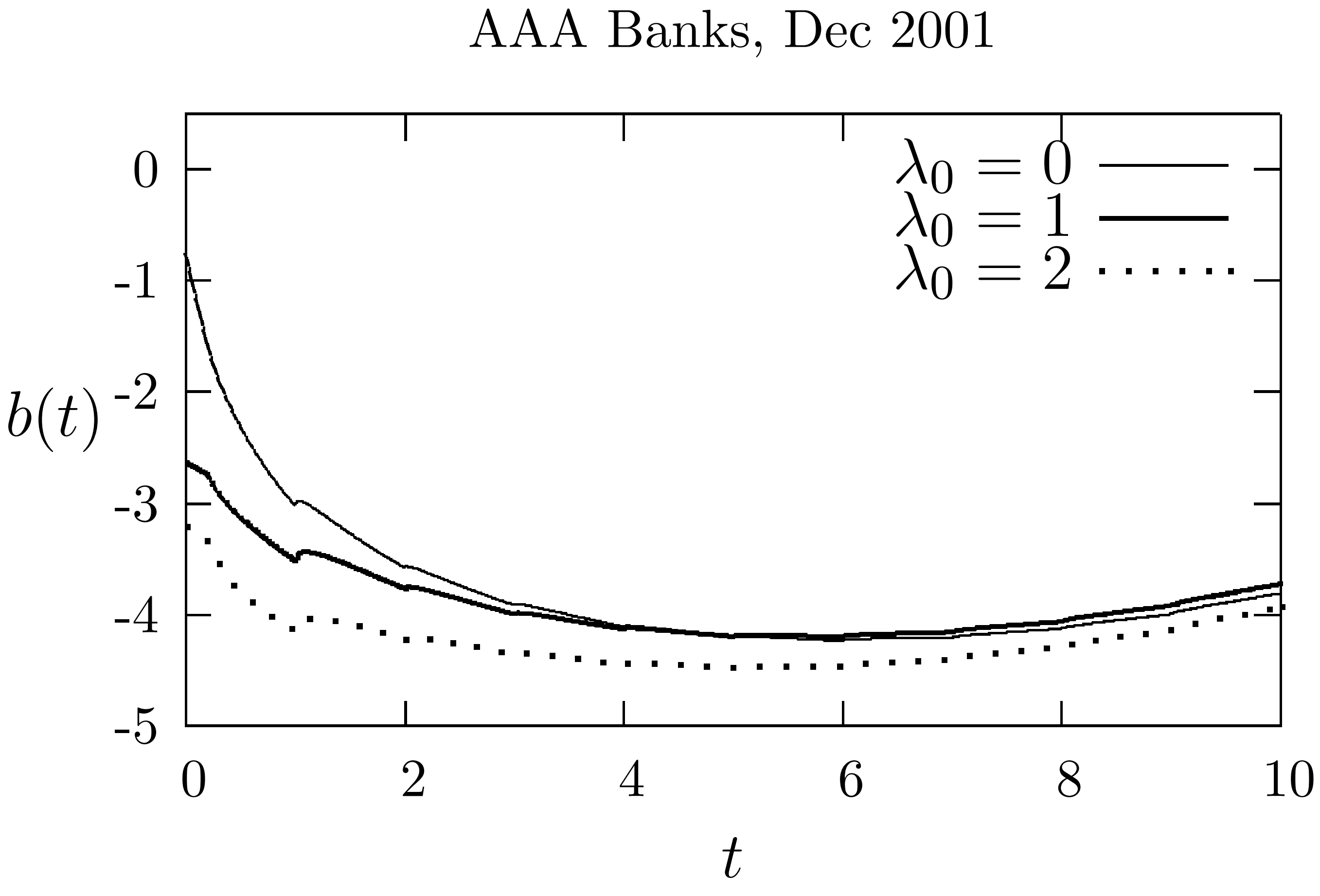
- Bank industry with AAA ratings

year	May 2004	Dec 2001
1	0.0058	0.0073
2	0.0094	0.0136
3	0.0115	0.0166
4	0.0135	0.0190
5	0.0155	0.0210
6	0.0171	0.0229
7	0.0190	0.0246
8	0.0210	0.0264
9	0.0232	0.0284
10	0.0256	0.0307

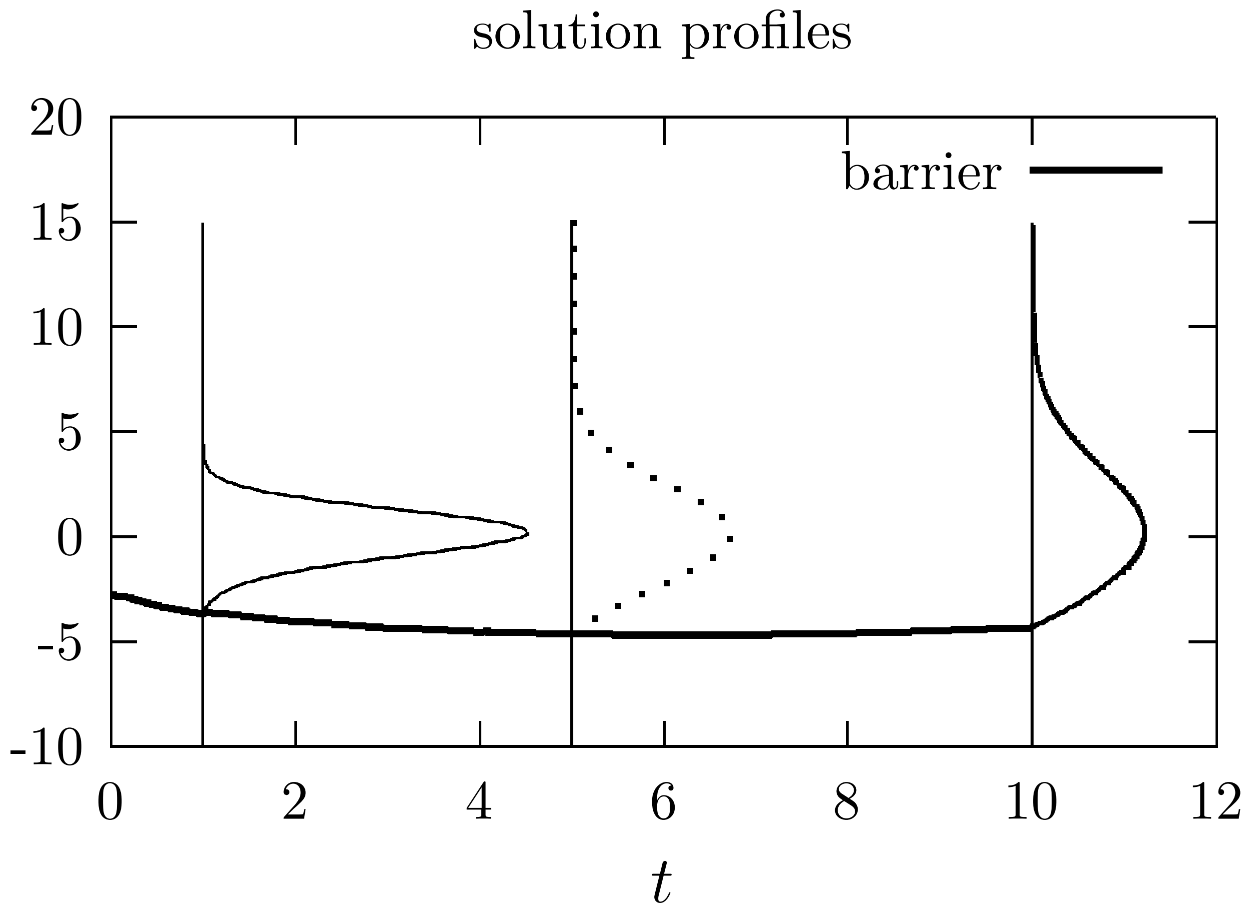
Barriers with Jump-diffusion (1)



Barriers with Jump-diffusion (2)



Survival density



Summary

- Allow jumps in distance-to-default
- Additional parameters to fit the data
- Partial integro-differential equation formulation
- Efficient and stable numerical solutions
- Stability analysis needed
- Study the equation with random killing term
- Build in correlation structures