Variation of Hodge Structures and Calabi-Yau Manifolds

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Prologue

In the note,we give a quick introduction to variation of Hodge structures and its applications to algebraic geometry.We has talked most of the note in working seminars at Harvard University and CMS. The revised version has been finished after lectures of Algebraic Geometry Workshop of 2004 Hangzhou-Beijing International Summer School.

Contents

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1. Introduction to Variation of Hodge Structure

1.1. Family of varieties and higher direct image of constant sheaf. Let $f : \mathfrak{X} \to M$ be smooth family with the projective manifolds as fiber. In general cases , $\mathfrak{X}$ and M are quasi-projective manifolds,such a family always comes from :

$$\begin{array}{ccc} \mathfrak{X} & \xrightarrow{\subset} & \overline{\mathfrak{X}} \\ \pi\downarrow & & \overline{\pi}\downarrow \\ M & \xrightarrow{\subset} & \overline{M} \end{array}$$

where $\overline{\mathfrak{X}}$ and $\overline{M}$ are smooth projective manifolds contain $\mathfrak{X}$ and M as Zariski open sets , $\overline{\pi}$ is a surjective proper morphism such that $\mathfrak{X}$ is the inverse image of M under $\overline{\pi}$ and the restricted map π is a smooth map over M.Moreover,by the well known Hironaka's desingularity theorem,one can assume that $S = \overline{M} \setminus M$ is a normal crossing divisor. In the case M is a noncompact algebraic curve,with the method of semi-stable reduction, one can obtain the smooth compactification such that over every point $s \in S$,$\overline{\pi}^{-1}(s)$ is at most a normal crossing divisor.

We know each close fiber $\pi^{-1}(s)$ for $s \in M(\mathbb{C})$ in some given moduli space,i.e. $[\pi^{-1}(s)] \in \mathfrak{M}$. By Matasusaki Big theorem, we can assume in this notes that *all smooth families are projective*.i.e., for the family $f : \mathfrak{X} \to M$,there exists a projective space $\mathbb{P}^N$ such that the diagram is commutative:

$$\begin{array}{ccc} \mathfrak{X} & \xrightarrow{i} & \mathbb{P}^N \times M \\ & {}_f\searrow \quad \swarrow_\pi & \\ & M & \end{array}$$

where i is an embedding and π is the natural projection.This actually means means that one have a family of manifolds embedding into $\mathbb{P}^N$.The restriction to $\mathfrak{X}_t$ of the canonical sheaf $\mathcal{O}_{\mathbb{P}^N}(1)$ defines a positive line bundle L_t.

Let $\omega_t = c_1(L_s)$,all ω_t is nonzero and torsion free in $H^2(\mathfrak{X}_t, \mathbb{Z})$. Actually,fitting all ω_t together one obtain a global flat section ω of the direct image sheaf $R^2\pi_*(\mathbb{Z})$.

Therefore, one obtains the induced C^∞ flat vector bundle from the family f as following:

Fixing a base point $t_0 \in M$,for each k with $0 \leq k \leq 2n$,the local system $R^k f_*(\mathbb{C})$ is just a representation of fundamental group

$$\rho : \pi_1(M, t_0) \to \mathrm{GL}(H^k(\mathfrak{X}_{t_0}, \mathbb{C})),$$

it means

$$R^n f_*(\mathbb{C}) = \widetilde{M} \times_\rho H^k(\mathfrak{X}_{t_0}, \mathbb{C})$$

where $\widetilde{M}$ is the universal covering of M.

Denote $\mathbf{H}^k = R^k f_*(\mathbb{C}) \otimes \mathcal{A}$ where $\mathcal{A}$ is the sheaf of C^∞ complex function over M, one actually obtains a C^∞ flat complex vector bundle $(\mathbf{H}^k, \nabla)$ over M whose sheaf of germs of flat sections is just the direct image sheaf $R^k f_*(\mathbb{C})$ and the fibre $(\mathbf{H}^k)_t$ over $t \in M$ can be identified with $H^k(\mathfrak{X}_t, \mathbb{C})$.

The flat bundle $\mathbf{H}^k$ contains a real flat bundle $\mathbf{H}^k_{\mathbb{R}}$ whose sheaf of germs of flat sections is $R^k f_*(\mathbb{R})$ and a flat lattice $\mathbf{H}^k_{\mathbb{Z}}$ which corresponds to $R^k f_*(\mathbb{Z})$,thus the representation really has $\mathbb{Z}$ structure,i.e.

$$\rho_{\mathbb{Z}} : \pi_1(M, t_0) \to \mathrm{GL}(H^k(\mathfrak{X}_{t_0}, \mathbb{Z})).$$

Important properties:

1. **Relative Hodge filtration**.
   For each $t \in M$,there is Hodge decomposition
   $$H^k(\mathfrak{X}_t, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(\mathfrak{X}_t)$$
   with respect to ω_t the induced Kähler structure.From the Grauert's coherent theorem,the integers $h_t^{p,q} = \dim_{\mathbb{C}} H^{p,q}(\mathfrak{X}_t, \mathbb{C})$ depend upper semicontinuously on t.On the other hand,$\dim_{\mathbb{C}} H^k(\mathfrak{X}_t, \mathbb{C})$ remains constant,thus the integers $h_t^{p,q}$ are constants.Therefore,one has a C^∞ decomposition of the bundle
   $$\mathbf{H}^k = \bigoplus_{p+q=k} H^{p,q}$$
   with $\overline{H^{p,q}} = H^{q,p}$. Setting $\mathbf{F}^p = \bigoplus_{i \geq p} H^{i,k-i}$ for $0 \leq p \leq k$,then all $\mathbf{F}^p$ are C^∞ subbundles of $\mathbf{H}^k$.
2. **Hodge-Riemann bilinear relation.**
   There is a natural nondegenerate bilinear form Q_t on $H^k(\mathfrak{X}_t, \mathbb{C})$ for every $t \in M$
   $$Q_t(\xi, \eta) = \int_{\mathfrak{X}_t} \xi \wedge \eta \wedge \omega_t^{n-k},$$
   which is defined over $\mathbb{Z}$ and satisfies the Hodge-Riemann bilinear relation **(HR)**:
   (1)$Q_t(H_t^{p,q}, H_t^{r,s}) = 0$,unless $r = n - p$ and $s = n - q$,
   (2)$(\sqrt{-1})^{p-q} Q_t(\xi, \overline{\xi}) > 0$, for $\forall\ \xi \in P_t^{p,q}$.
   Fitting all $\{Q_t\}$,$\mathbf{Q} = \{Q_t \mid t \in M\}$ is flat section of $(-1)^k$-$Sym^2(\mathbb{V}_{\mathbb{Z}})^*$.

3. **Griffiths transversality.**
From the Hodge-Riemann bilinear relation and the flatness of $\mathbf{Q}$, there is Griffiths infinitesimal relation :

$$\nabla : H^{p,q} \to \mathcal{R}^{0,1}(H^{p+1,q-1}) \oplus \mathcal{R}^{1,0}(H^{p,q}) \oplus \mathcal{R}^{0,1}(H^{p,q}) \oplus \mathcal{R}^{1,0}(H^{p-1,q+1}). \tag{1.0.1}$$

Decompose the operator $\nabla = \overline{\theta} + \partial + \overline{\partial} + \theta$ according to above decomposition **(HR)**, then $\nabla^{1,0}$ and $\overline{\partial}$ both obey the $\overline{\partial}$-Leibnitz rule and are integrable.So one obtains the holomorphic vector bundle $\mathcal{V}$ under the holomorphic structure $\nabla^{0,1}$ with integrable Gauss Manin connection $\nabla_{GM} = \nabla^{1,0}$,

$$\nabla_{GM} : \mathcal{O}^{an}(\mathbf{H}^k) \to \mathcal{O}^{an}(\mathbf{H}^k) \otimes \Omega^1_{M^{an}}.$$

The Griffiths infinitesimal relation 1.0.1 is equivalent to

Theorem 1.1 (Griffiths transversality)**.** *The subbundles $\mathbf{F}^p \subset \mathbf{H}^k$ are all holomorphic subbundles under the holomorphic structure $\nabla^{0,1}$,thus one has a corresponding holomorphic filtration*

$$0 \subset \mathcal{F}^k \subset \cdots \subset \mathcal{F}^0 = \mathcal{V} := R^k f_*(\mathbb{C}) \otimes \mathcal{O}^{an}_M.$$

Furthermore, for each $0 \leq p \leq k$,

$$\nabla_{GM} : \mathcal{F}^p \to \mathcal{F}^{p-1} \otimes \Omega^1_{M^{an}}.$$

Shown by Deligne[1],Katz[4] and Schmid[7], this holomorphic bundle has regular singularities around every component of the infinite normal crossing divisor $D_\infty = \overline{M} \setminus M$,so they are all *algebraic bundles.* It should be pointed out,the induced algebraic structure are isomorphic to the intrinsic algebraic structure shown by Grothendieck.

4. **Grothendieck's cohomology of direct image**.
It had been shown by Grothendieck that the above filtration of $\mathcal{V}$ can be defined in a purely algebraic manner as follows:
Let $\Omega^p_{\mathfrak{X}/M} := \Omega^p_{\mathfrak{X}} / (f^*\Omega^1_M \wedge \Omega^{p-1}_{\mathfrak{X}})$,then

$$\Omega^\bullet_{\mathfrak{X}/M} = \bigoplus_{i=0}^{\dim \mathfrak{X}_t} \Omega^q_{\mathfrak{X}/M}$$

is an algebraic coherent sheaf in the Zaraski topology on $\mathfrak{X}$. It is known that $\Omega^\bullet_{\mathfrak{X}/M}$ is a graded differential sheaf with $\mathcal{O}_M$-linear morphism

$$d^q : \Omega^q_{\mathfrak{X}/M} \to \Omega^{q+1}_{\mathfrak{X}/M}$$

induced from the exterior differentiation on $\Omega^\bullet_{\mathfrak{X}}$.
One will have De Rham complex of relative differential forms of the smooth family $f : \mathfrak{X} \to M$

$$\Omega^\bullet_{\mathfrak{X}/M} : 0 \longrightarrow \mathcal{O}_{\mathfrak{X}} \xrightarrow{d^0} \Omega^1_{\mathfrak{X}/M} \xrightarrow{d^1} \Omega^2_{\mathfrak{X}/M} \longrightarrow \cdots \tag{1.1.1}$$

then for all $k \geq 0$ one can define the Leray hypercohomology sheaves $\mathbb{R}^k f_*(\Omega^\bullet_{\mathfrak{X}/M})$ which are algebraic coherent sheaves on M.It is not difficult to show that the analytic coherent sheaves in M associated to $\mathbb{R}^* f_*(\Omega^\bullet_{\mathfrak{X}/M})$ is just

$$R^* f_*(\mathbb{C}) \otimes \mathcal{O}_{M^{an}}.$$

It is well known that the Leray spectral sequence $\{E_r^{p,q}\}$ with $E_1^{p,q} = R^q f_*(\Omega^p_{\mathfrak{X}/M})$ converges to the limit $\oplus_{p+q=k} E_\infty^{p,q}$ which is graded sheaf associated to a filtration of $\mathcal{V} = \mathbb{R}^k f_* \Omega^\bullet_{\mathfrak{X}/M}$. The algebraic definition of the filtration

$$0 \subset \mathcal{F}^k \subset \cdots \subset \mathcal{F}^0 = \mathcal{V}$$

is obtained by the Hodge filtration(often called stupid filtration) on the complex 1.1.1,i.e. $F^p \mathbb{R}^k f_* \Omega^\bullet_{\mathfrak{X}/M} \simeq \mathbb{R}^k f_*(F^p \Omega^\bullet_{\mathfrak{X}/M})$ where $F^p \Omega^\bullet_{\mathfrak{X}/M} := \Omega^{\bullet \geq p}_{\mathfrak{X}/M}$.

Furthermore,the spectral sequence degenerate at the first term

$$E_1^{p,k-q} \Rightarrow \mathbb{R}^k f_*(\Omega^\bullet_{\mathfrak{X}/M}) = R^k f_*(\mathbb{C}) \otimes \mathcal{O}_M,$$

thus

$$\mathcal{F}^p = \bigoplus_{p \leq r \leq k} R^{k-r} f_*(\Omega^r_{\mathfrak{X}/M}),$$

so they are compatible with the definition in item 3 and are algebraic subbundles of $\mathcal{V}$. Especially,

$$\mathcal{F}^k = R^0 f_*(\Omega^k_{\mathfrak{X}/M})$$

is an algebraic subbundle.

5. **Higgs Field and Yukawa coupling.**

Taking grading of Griffiths transversality ,because $\mathcal{O}_M(F^p/F^{p+1})$ is just the Leray sheaf $R^p f_*(\Omega^{k-p}_{\mathfrak{X}/M})$,the Gauss Manin connection ∇_{GM} induces an $\mathcal{O}_M$-linear mapping

$$\theta^{k-p,p} = \mathrm{Gr}(\nabla) : E^{k-p,p} = R^p f_*(\Omega^{k-p}_{\mathfrak{X}/M}) \to E^{k-p-1,p+1} = R^{p+1} f_*(\Omega^{k-p-1}_{\mathfrak{X}/M}) \otimes \Omega^1_M.$$

Lemma 1.2. *Let $\theta := \oplus \theta^{p,q}$, θ is a $\mathcal{O}_M$-linear map, we sometimes call it* **Higgs field**. *θ is algebraic and it has such important properties:*

$$\overline{\partial}(\theta) = 0,\ \theta \wedge \theta = 0.$$

Lemma 1.3 (Griffiths)**.** *θ is just the cup product with the Kodaira-Spencer infinitesimal deformation class*

$$\kappa \in H^0(M, R^1 f_*(T_{\mathfrak{X}/M}) \otimes \Omega^1_M).$$

Let complex dimension of fiber be n. **Yukawa coupling** of $\mathbb{R}^n f_*(\mathbb{C})$ is just the n-iterated Higgs Field

$$\theta^n : E \to E \otimes S^n \Omega^1_M,$$

where $E = R^n f_*(\mathbb{C}) \otimes \mathcal{O}_M$.

As (E,θ) can be splitting into (in C^∞ category)

$$(E,\theta) = (\bigoplus_{p+q=n} E^{p,q}, \bigoplus \theta^{p,q})$$

With

$$\theta^{p,q} : E^{p,q} \to E^{p-1,q+1} \otimes \Omega^1_M$$

and $\theta \wedge \theta = 0$. We always write the *Yukama coupling* as

$$\theta^n : S^n T_M \to \mathcal{H}om(E^{n,0}, E^{0,n}) = ((R^0 f_* \Omega^n_{\mathfrak{X}/M})^*)^{\otimes 2}.$$

6. **Polarization and Hodge metric.**

For some technical reasons, one hopes there is a positive Hermitian metric on the vector bundle $\mathbf{H^k}$.Fortunately,it will be given by the natural polarization on the primitive part of the cohomology group of the fibers $\mathfrak{X}_t$ with the Lefschetz decomposition and the Hodge-Riemann bilinear relation.

Fixed a base point t_0, let $L = L_{t_0}$, then $\omega_{t_0} = c_1(L) \in H^2(\mathfrak{X}_{t_0}, \mathbb{Z})$ is invariant under action of the globally monodromy $\pi_1(M)$. Therefore one can define the primitive cohomology $P^k(\mathfrak{X}_{t_0}, \mathbb{C})$ for $k \leq n = \dim_{\mathbb{C}} \mathfrak{X}_t$ to be the kernel of

$$L^{n-k+1} : H^k(\mathfrak{X}_{t_0}, \mathbb{C}) \to H^{2n-k+2}(\mathfrak{X}_{t_0}, \mathbb{C}).$$

For convenience, we will abuse the notations so that we do not tell apart L and $c_1(L)$ clearly.

Then,the **Lefschetz decompostion:**

$$H^k(\mathfrak{X}_{t_0}, \mathbb{C}) = \bigoplus_{i=0}^{[k/2]} L^i P^{k-2i}(\mathfrak{X}_{t_0}, \mathbb{C})$$

is a $\pi_1(M)$-invariant $\mathbb{Q}$-decomposition of $H^k(\mathfrak{X}_{t_0}, \mathbb{C})$ because L is defined over $\mathbb{Q}$.Therefore,globally there is a sheaf map between the local constant sheaves

$$\omega^{n-k+1} : R^k f_*(\mathbb{C}) \to R^{2n-k+2} f_*(\mathbb{C})$$

The the kernel sheaf $P^k f_*(\mathbb{C}) := R^k_{prim} f_*(\mathbb{C})$ is also local constant which is called Leray primitive cohomology sheaf and $P^k f_*(\mathbb{C})_t = P^k(\mathfrak{X}_t, \mathbb{C})$.One has the global Lefschetz decompostion defined over $\mathbb{Q}$:

$$R^k f_*(\mathbb{C}) = \bigoplus_{i=0}^{[k/2]} \omega^i \wedge P^{k-2i} f_*(\mathbb{C}).$$

Certanly, the fundamental group representation can be reduced to

$$\rho : \pi(M, t_0) \to \mathrm{GL}(P^k(\mathfrak{X}_{t_0}, \mathbb{C}))$$

which corresponds to $P^k f_*(\mathbb{C})$. So one will get a C^∞ flat vector bundle $(\mathbf{H^k}, \nabla)$ as same as the case of cohomology group.

Because of Hodge-Riemann relation **(HR)**, $(R^k_{prim} f_*(\mathbb{C}) \otimes \mathcal{O}_M, \nabla_{GM})$ has a positive Hermitian metric H (**Hodge metric**)defined by

$$H(\cdot, \cdot) := \mathbf{Q}(C\cdot, \bar{\cdot})$$

where the operator C is the Weil operator defined naturally by the filtration $\{\mathcal{F}^p\}_{p=0}^k$. Furthermore,by Lefschetz decomposition, we can define a positive Hermitian metric on $R^k f_*(\mathbb{C}) \otimes \mathcal{O}_M$. Therefor

At all, we have

Theorem 1.4. $(R^k_{prim} f_*(\mathbb{Q}), \nabla_{GM}), (R^k f_*(\mathbb{Q}), \nabla_{GM})$ *are both polarized variation of rational Hodge structures.*

Example 1.5 (Family of Calabi-Yau threefolds)**.** If X is a compact Kähler manifold with dimension 3,one have the Lefschetz decomposition:

$$H^3(X, \mathbb{C}) = P^3(X, \mathbb{C}) \oplus LP^1(X, \mathbb{C}).$$

If X a projective Calabi-Yau threefold with holonomy group SU(3),then

$$H^1(X, \mathbb{C}) = 0 \text{ and } H^3(X, \mathbb{C}) = P^3(X, \mathbb{C}).$$

Therefore,for a family of Calabi-Yau threefolds $f : \mathcal{X} \to M$

$$R^3 f_*(\mathbb{C}) = P^3 f_*(\mathbb{C}).$$

Thus $R^3 f_* \mathbb{Q}$ is q polarized $\mathbb{Q}$-variation of Hodge Structure.

Example 1.6 (Family of Abelian varieties)**.** Let $f : \mathcal{X} \longrightarrow M$ be smooth family of Abelian variety, then it is easy to check that the VHS $R^1 f_*(\mathbb{C})$ is a polarized VHS because of $R^1 f_*(\mathbb{C}) = P^1 f_*(\mathbb{C})$.

1.2. **Variation of Hodge structures and period map.** Now one can formulate the definition of VHS from the examples coming form geometry.

Definition 1.7. Let $\mathcal{R}$ be a subring of $\mathbb{R}$ that is stable under complex conjugate. A (polarized) variation of $\mathcal{R}$-Hodge Structure of weight k on the base manifold M consists of the datum

$$\{M, \mathbb{V}_{\mathcal{R}} \subset \mathbb{V}_{\mathbb{R}} \subset \mathbb{V}_{\mathbb{C}} \subset \mathbf{H}, \{\mathbf{F}^p\}_{p=0}^k, \nabla = \nabla^{1,0} + \nabla^{0,1}, \mathbf{Q}\}$$

where

(1) $\mathbb{V}_{\mathcal{R}}$ is local system of $\mathcal{R}$-module of finite rank and $(\mathbf{H} = \mathbb{V}_{\mathcal{R}} \otimes_{\mathcal{R}} \mathcal{A}, \nabla)$ is a flat complex vector bundle where $\mathcal{A}$ is sheaf of C^∞ complex functions over M.
(2) $\{\mathbf{F}^p\}_{p=0}^k$ is a filtration of $\mathbf{H}$ and the fiber $\{\mathbf{F}^p_t\}_{p=0}^k$ form a Hodge filtration of $\mathbf{H}_t$ of weight k with respect to the real structure $(\mathbb{V}_{\mathbb{R}})_t$ for all $t \in M$.

(3) According to holomorphic structure $\nabla^{0,1}$,all $\mathbf{F}^p$ become holomorphic subbundles (denote $\mathcal{F}^p$) of $\mathcal{V} = \mathbb{V}_{\mathcal{R}} \otimes \mathcal{O}_{M^{an}}$ and there is a decreasing filtration

$$0 \subset \mathcal{F}^k \subset \cdots \subset \mathcal{F}^0 = \mathcal{V};$$

Furthermore, Let $\nabla_{GM} = \nabla^{1,0}$ be the integrable Gauss-Manin connection, one have Griffiths tranversality

$$\nabla_{GM} : \mathcal{F}^p \to \mathcal{F}^{p-1} \otimes \Omega^1_{M^{an}}.$$

(4) (Polarization) $\mathbf{Q}$ is flat non-degenerate section of $(-1)^k$-$Sym^2(\mathbb{V}_{\mathcal{R}})^*$ and for all $t \in M$,Q_t polarizes the Hodge filtration $\{\mathbf{F}_t\}_{p=0}^k$. $\mathbf{Q}$ is called the polarization of the VHS.

If $\mathcal{R} = \mathbb{Z}$ one always call the VHS has rational structure (because V is of finite dimension, it is equivalent to have $\mathbb{Q}$-structure),as $\mathcal{R} = \mathbb{R}$ real VHS.

If one tick off the condition (4)(i.e.no polarization),one also call it a variation of Hodge Structure.

Remarks 1.8. (a) From the polarization, one can define the Hodge metric on $\mathbf{H}$ as follows:

$$H(\cdot, \cdot) := \mathbf{Q}(C\cdot, \bar{\cdot})$$

Where the operator C is the Weil operator defined naturally by the filtration $\{\mathbf{F}^p\}_{p=0}^k$.
(b) In general, Let $g : \mathcal{X} \to B$ be proper smooth morphism between connected complex manifolds,Assume both $\mathcal{X}$ and B have Kähler structure,then one will get a variation of Hodge Structure $R^k g_*(\mathbb{Z})$.As we have shown we can obtain polarized rational VHSs $R^k_{prim} g_*(\mathbb{Q})$, $R^k g_*(\mathbb{Q})$ with the Hodge metric.

The most important fact is that if (V, ∇, H) is a polarized VHS,then (V, ∇, H) has the property of polynomial growth directly near D_∞ from the result of nilpotent orbit theorem of Schmid(cf.[7]),so it has the canonical extension and becomes an algebraic vector bundle.Exactly,

Theorem 1.9 (**Deligne,Katz,Griffiths,Schmid**). *Let $\mathbb{V}$ be a polarizable variation of Hodge structures on a smooth quasi-projective variety $M/\mathbb{C}$.Then the holomorphic vector bundle $\mathcal{V} = \mathbb{V} \otimes \mathcal{O}^{an}_M$ carries a unique algebraic structure such that the connection ∇^{an} becomes algebraic and such that ∇ has regular singularities at infinity relative to any Hironaka completion $\overline{M}$. with respect to this structure, the holomorphic subbundles $\mathcal{F}^p \subset \mathcal{V}$ becomes algebraic.*

Let D be the classified space of weight k all real polarized (given by $\mathbf{Q}$) Hodge structures with Hodge number given $\{h^{p.k-p}\}_{p=0}^k$.

Example 1.10. Let M be a simply-connected, $\{M, \mathbb{V} \subset \mathbf{H}, \{\mathbf{F}^p\}_{p=0}^k, \nabla, (\mathbf{Q})\}$ be a polarization VHS on M, In this case $\mathbb{V} = M \times V$,$\nabla = d$, $\mathbf{Q}$ constant biliner form. So the holomorphic filtration $\{\mathcal{F}^p\}_{p=0}^k$ induce a holomorphic map

$$\Phi : M \to D$$

Moreover, as same as above,

$$\begin{array}{ccccc} M & \xrightarrow{\Phi} & D & \xrightarrow{\iota} & \prod_{p=0}^k \mathrm{Gr}(h^p, V) \\ & \searrow^{\phi^p} & & \swarrow_{\lambda^p} & \\ & & \mathrm{Gr}(h^p, V) & & \end{array}$$

For any $t \in M$ one also obtain

$$\iota_* \Phi_*(T_t) \subset \bigoplus_{p=0}^{k} \mathrm{Hom}(\mathcal{F}_t^p/\mathcal{F}_t^{p+1}, \mathcal{F}_t^{p-1}/\mathcal{F}_t^p)$$

More precisely, Φ has the horizontal property i.e.

$$\Phi_*(T_t) = (\phi_*^0(T_t), \cdots, \phi_*^p(T_t), \cdots, \phi_*^k(T_t)) \subset T_{*,\Phi(t)}^h$$

where $T_{*,\Phi(t)}^h$ is the horizonal tangent space,i.e.,

$$T_{*,\Phi(t)}^h := \iota_{*,\Phi(t)}^{-1}(\bigoplus_{p=0}^{k} \mathrm{Hom}(\mathcal{F}_t^p/\mathcal{F}_t^{p+1}, \mathcal{F}_t^{p-1}/\mathcal{F}_t^p)),$$

and $\phi_*^p(T_t) \subset \iota_{*,\Phi(t)}^{-1} \mathrm{Hom}(\mathcal{F}_t^p/\mathcal{F}_t^{p+1}, \mathcal{F}_t^{p-1}/\mathcal{F}_t^p)$.

In general, M is always not simply-connected.Let

$$\{M, \mathbb{V}_{\mathbb{Z}} \subset \mathbf{H}, \{\mathbf{F}^p\}_{p=0}^k, \nabla = \nabla_{GM} + \nabla^{0,1}, (\mathbf{Q})\}$$

be a polarization VHS.Fixing a point $t_0 \in M$,because $\mathbf{Q}$ is flat ,the a global monodromy representation is :

$$\rho : \pi_1(M, t_0) \to \mathrm{Aut}(\mathbb{V}_{\mathbb{C},t_0}, Q_{t_0})$$

The flatness of $\mathbf{Q}$ and the existence of the lattice guarantee that

$$\Gamma := \rho(\pi_1(M, t_0)) \subset \mathbb{G}_{\mathbb{Z}}(t_0) = \mathrm{Aut}(\mathbb{V}_{\mathbb{Z},t_0}, \mathbf{Q}_{t_0})$$

We call Γ the monodromy group of the VHS.

Let $\widetilde{M}$ be the universal coving of M,so as a local system $\mathbb{V}_{\mathbb{C}} = \widetilde{M} \times_\rho \mathbb{V}_{\mathbb{C},t_0}$ and the pullback VHS on $\widetilde{M}$ is canonical trivial as a vector bundle. Fixing $\widetilde{t_0} \in \widetilde{M}$ which is the lifting of $t_0 \in M$, one identify the fibers over t_0 and $\widetilde{t_0}$ are V. The pullback polarization is a constant section over $\widetilde{M}$,one identify $\mathbf{Q}_t$ with it which be denoted Q . Following the above example, one get the holomorphic map

$$\widetilde{\Phi} : \widetilde{M} \longrightarrow D$$

Moreover $\widetilde{\Phi}$ is ρ-equivalent.i.e.

$$\widetilde{\Phi}(\sigma\widetilde{t}) = \rho(\sigma)(\widetilde{\Phi}(\widetilde{t})) \quad for \quad \forall\sigma \in \pi_1(M, t_0)$$

Thus one have the induced mapping

$$\Phi : M \longrightarrow D/\Gamma$$

from M to the analytic space D/Γ, We call it Griffiths's period mapping corresponding to that VHS.

Remark: D/Γ will be a manifold if Γ is torsion free.In general, D/Γ is not necessarily a manifold,but it has a structure of stack.

Theorem 1.11 (Griffiths)**.** *The definition of polarized $\mathbb{Q}$-VHS is equivalent to the following results which are functional version of VHS.*

(I) Φ *is holomorphic;*

(II) Φ *is locally liftable,that is for each $t_0 \in M$ there exists a neighborhood $U \subset M$ and a holomorphic map $\widetilde{\Phi}_U$, such the diagram*

$$\begin{array}{ccc} U & \xrightarrow{\widetilde{\Phi}_U} & D \\ & {\scriptstyle\Phi}\searrow \quad \swarrow{\scriptstyle\pi} & \\ & D/\Gamma & \end{array}$$

commutes,here π is the canonical projective map.

(III) *By the property of tranversality of the VHS ,the mapping $\widetilde{\Phi}_U$ is horizontal.*

Here is key lemma which opens the door for applying hyperbolic analysis to Hodge Theory,

Lemma 1.12 (Giffiths-Schmid)**.** *On $\check{D}$,there is a G-invariant Hermitian metric whose curvature relative to the horizonal section is negative and bounded away from zero.*

Corollary 1.13 (Ahlfors)**.** *The lifting period map*

$$\widetilde{\Phi} : \widetilde{M} \to D$$

is uniformly continuous if $\widetilde{M}$ is given the standard Poincaré metric.

1.3. **Limit of the Hodge structure and Nilpotent orbit theorem.** What's the asymptotic behavior of the period map?This question is related to the degeneration of the family. Because the base M is a quasi-projective manifold with Hironaka completion $\overline{M}$, M always has this type $(\Delta^*)^m \times (\Delta)^{n-m}$ near the D_∞.

WLOG,let $M = (\Delta^*)^n$,then $\pi_1(M) = \prod^n \mathbb{Z}$ is generated by $\{\gamma_1, \cdots \gamma_n\}$ each γ_i corresponding to counterclockwise path around 0 of i-th component of $(\Delta^*)^n$.Let $(\mathbb{V}_{\mathbb{Z}})$ a polarized VHS over M, fix a $t_0 \in M$ the monodromy representation is

$$\rho : \pi_1(M, t_0) \to \operatorname{Aut}(\mathbb{V}_{\mathbb{C},t_0}), Q_{t_0})$$

Denote $T_i = \rho(\gamma_i)$ the Picard-Lefschetz transformation.

The universal covering $\widetilde{M}$ of M is U^n the product of Póincare upper planes by $U = \{z \mid \operatorname{Im} z > 0\}$.The covering mapping is

$$\tau : U^n \xrightarrow{(t_1,\cdots,t_n)} (\Delta^*)^n \ , \ \tau(z_1, \cdots, z_n) = (e^{2\pi\sqrt{-1}z_1}, \cdots, e^{2\pi\sqrt{-1}z_n}),$$

and the lifting period map $\widetilde{\Phi} : U^n \longrightarrow D$ is ρ-equivariant,i.e.

$$\widetilde{\Phi}(z_1, \cdots, z_i + 1, \cdots, z_n) = T_i(\widetilde{\Phi}(z_1, \cdots, z_i, \cdots, z_n)).$$

Lemma 1.14 (Landman,Katz,Borel [7])**.** *All eigenvalues of T_i are roots of unitary.Let T be one monodromy,T can written as $T = T_u T_s = T_s T_u$ where T_s is semisimple and T_u nilpotent, there are positive integers β and $l \le k$(k is the weight of the VHS) such that*

$$(T^\beta - \mathrm{id})^{l+1} = (T_u^\beta - \mathrm{id})^{l+1} = 0,$$

when $(T^\beta - \mathrm{id})^l \neq 0$.

When $\beta = 1$, we call T is **unipotent monodromy**. When $\beta = 1$ and $l = k$, T is **maximal unipotent monodromy**.

There is analytic proof of the lemma depending on Alphors lemma, but we would like give a geometric description for original case.

Example 1.15 (Landman's theorem)**.** Let $g : \mathcal{X} \to \Delta$ be local family having unique degeneration 0 with m-dimensional fiber. The Clemens mapping is the composite map form $\mathcal{X}_t$ to the singular fiber $\mathcal{X}_0 = g^{-1}(0)$,

$$c_t : \mathcal{X}_t \hookrightarrow \mathcal{X} \xrightarrow{r} \mathcal{X}_0$$

where r is the restricted deformation. At a neighborhood (in $\mathcal{X}$) of $p \in \mathcal{X}_0$,the singular fiber can be defined by

$$w_1^{a_1} w_2^{a_2} \cdots w_l^{a_l} = 0,$$

where a_i are positive integer. Let $\beta = \gcd(a_1, a_2, \cdots, a_l)$,there are integers $\{b_i\}_{i=1}^l$ such that $\sum a_i b_i = \beta$.In the neighborhood of p, going along a simple counterclockwise loop around $\mathcal{X}_0$, the transformation can be written as

$$(w_1, \cdots, w_l, \cdots, w_m) \mapsto (\exp(\frac{2\pi b_1\sqrt{-1}}{\beta})w_1, \cdots, \exp(\frac{2\pi b_l\sqrt{-1}}{\beta})w_l, w_{l+1}, \cdots, w_m).$$

Thus T^a acts on $R^* c_{t*}(\mathbb{Q})$ is trivial. Because shown by Deligne

$$E_2^{p,q} = H^q(X_0, R^q c_{t*}(\mathbb{Q})) \Rightarrow H^{p+q}(X_t, \mathbb{Q}),$$

T^β acts trivial on E_∞ which is the graded module of $H^k(\mathcal{X}_t, \mathbb{Q})$.Thus T^β acts unipotently on $H^k(\mathcal{X}_t, \mathbb{Q})$,i.e $(T^\beta - \mathrm{Id})^{k+1} = 0$. Especially,when the family g is semi-stable(i.e.$a_1 = a_2 = \cdots = a_l = 1$),then $\beta = 1$,T is unipotent.

Let $N = \log T_u = (1/\beta) \log T^\beta = (1/\beta) \sum_{j=1}^{k} (1/j)(-1)^{j+1}(T^\beta - \mathrm{Id})^j$, which is a nilpotent.It is clear $N_j N_i = N_i N_j$.

Define a mapping form U^n to $\check{D}$ (The dual space of D and is a compact complex manifold) by

$$\widetilde{\Psi} = \exp(-\sum_{i=1}^{n} \beta_i z_i N_i)\widetilde{\Phi}(\beta_1 z_1, \cdots, \beta_n z_n),$$

it is easy to check $\widetilde{\Psi}$ induce a holomorphic mapping $\Psi : (\Delta^*)^n \longrightarrow \check{D}$

Theorem 1.16 (Nilpotent Orbit Theorem[7])**.** *The map Ψ can extend holomorphiclly to $(\Delta)^n$,especially,$F_0 = \Psi(0, \cdots, 0)$ exists in $\check{D}$.Moreover*

(1) *F_0 is fixed by all T_s where $T \in \{T_1, \cdots T_n\}$*
(2) *The nilpotent orbit $\exp(\sum_{i=1}^{k} z_i N_i)F_0 \in \check{D}$ is horizonal.*
(3) *There exist constants $\alpha, \delta, K \geq 0$,such that the point $\exp(\sum_{i=1}^{k} z_i N_i)F_0 \in D$ when $\mathrm{Im} z_i \geq \alpha$ for all i and satisfies the following inequality*

$$d(\exp(\sum_{i=1}^{k} z_i N_i)F_0, \widetilde{\Psi}(z)) \leq K \sum_{i=1}^{n} (\mathrm{Im} z_i)^\delta \exp(-2\pi(\beta_i)^{-1}\mathrm{Im} z_i),$$

where d denotes a $\mathbb{G}_\mathbb{R}$-invariant Riemannian distance function on D;more over the constants $\alpha, \delta, K \geq 0$ depend only on the choice of d,all β_i and the weight of the Hodge Structure.

Fixing a point $t_0 \in M$,F_t can be seen as a filtration of $V = \mathbb{V}_{\mathbb{C},t_0}$ which corresponds to the point $F_t \in \check{D}$.Not like the filtration of $\Phi(t)$ for any $t \in M$,the filtration of F_0 can not be a Hodge filtration because F_0 may not in D.But all unipotent map N_i will give V a increasing weight filtration W defined over $\mathbb{Q}$ such that

Theorem 1.17 (Schmid[7])**.** *(1) (V, F_0, W) is a mixed Hodge Structure. (2) $N : V \to V$ is a morphism of mixed Hodge Structure of weight -2.*

We do not introduce *Mixed Hodge Structure here*, if one has interest in it. There is good references of Deligne [1],Schmid [7].

1.4. **Canonical extension and application.**

1.4.1. *Deligne canonical extension.* The Nilpotent orbit theorem guarantee that the holomorphic bundle arising from the VHS can extend to D_∞ such that one obtains the theorem 1.9.

Let $\pi : X \to Y$ be a proper surjective morphism with connected fibers between nonsingular algebraic varieties over $\mathbb{C}$.Assume there is open set Y_0 of Y such that $Y - Y_0$ is a divisor with only normal crossings and that $\pi_0 : X_0 \to Y_0$ is smooth,where $X_0 = \pi^{-1}(Y_0)$. The local system $R^i\pi_{0*}\mathbb{C}_{X_0}$ on Y_0 forms a variation of Hodge structures with weight $i \geq 0$. Define the Hodge bundle $\mathcal{H}_0^i = R^i\pi_{0*}\mathbb{C}_{X_0} \otimes_{\mathbb{C}} \mathcal{O}_{Y_0}$ and let $\mathcal{F}_0^i = F^p(\mathcal{H}_0^i)$ be the p-th Hodge filtration of $\mathcal{H}_0^i$ and $\mathcal{Q}_0^p = \mathcal{F}_0^p/\mathcal{F}_0^{p+1}$ be the quotient vector bundles.

Example 1.18. Assume Y^0 be non compact Remiann surface, let $(\mathcal{V}, \nabla)$ be an analytic vector bundle with integrable connection.We obtain the corresponding local system $\mathbb{V}$ defined over $\mathbb{C}$ by from the equation $\nabla v = 0$. Then we have a local monodromy transformation (Picard-Lefschetz) around the puncture $s \in Y - Y_0$(counterclockwise):

$$\mu_s : \mathbb{V}_s \to \mathbb{V}_s$$

Choose a local coordinate z near s, Let $(e_1(z), \cdots, e_n(z))$ be a multivalued flat basis of $\mathbb{V}_s$ over U a small punctured neighborhood of s. One thus gets the monodromy matrix B_s related to such basis.Let $B_s = D_sU_s = U_sD_s$ be the Jordan decomposition where $D_s = \mathrm{diag}(d_k)$ semi-simple (diagonal) and N_s unipotent upper triangular.

$$N_s = \log U_s = \sum_{l>ge1} (-1)^{l+1}\frac{1}{l}(U_s - 1)^l$$

$$\log D_s = \mathrm{diag}(\log d_k)$$

is well defined and $\log B_s = \log D_s + N$, here we pick one value of log of the set d_k. There,there is a matrix

$$M_s = \frac{-1}{2\pi\sqrt{-1}} \log B_s$$

So $\exp(-2\pi\sqrt{-1}M_s) = B_s$.Let

$$h_i(z) = \exp(M_s \log z)e_i(z)$$

such that $h_i(z)$ are single valued sections of $\mathcal{V}$ over U.

Furthermore,$(h_1(z), \cdots, h_n(z))$ provide a frame for a holomorphic extension $\overline{\mathcal{V}}$ of bundle. Because $\nabla(e_i) = 0$,one has

$$\nabla(h_i) = M_s\frac{dz}{z}h_i.$$

The matrix for ∇ is $M_s dz/z$ in this frame,hence $(\mathcal{V},\nabla)$ has regular singularities,that we get the extension algebraic vector bundle $(\overline{\mathcal{V}},\overline{\nabla})$ and different choices of log give the same algebraic structure on $(\mathcal{V},\nabla)$.

For the same reason,when Y is higher dimension,and $D = Y - Y_0 = \cup D_j$ is a divisor with normal crossings only.The VHS $(\mathcal{H}_0^i,\nabla)$ has regular singularities along D and all eigenvalues of B have absolute value one. Moreover,there are two natural canonical extensive locally free sheaves over Y and by GAGA principal the extensions will give $(\mathcal{H}_0^i,\nabla)$ same algebraic structure which is compatible with the intrinsic algebraic structure of the direct image,i.e.,the theorem 1.9 by Deligne-Griffiths-Katz-Schmid.

As the notions in example 1.18, assume that all eigenvalues of B have absolute value one.The two extensions of vector bundle $\mathcal{V}$ to Y depend on the choices of $\log B$,where the values lie on the interval $[0,2\pi\sqrt{-1})$ or $(-2\pi\sqrt{-1},0]$ respectively. The extension given by $[0,2\pi\sqrt{-1})$ will be called the *upper canonical extension* and denote by ${}^u\mathcal{V}$ which is always called **Deligne's canonical extension**, the one given by $(-2\pi\sqrt{-1},0]$ will be called the *lower canonical extension* and denote by ${}^l\mathcal{V}$.Certainly ${}^l\mathcal{V} \subset^u \mathcal{V}, {}^u\mathcal{V}$(resp.${}^l\mathcal{V}$) is the smallest(resp. largest) extension of $\mathcal{V}$ such that along any real analytic aec any flat section $v(z)$ satisfies $|v(z)| \leqslant |\log z|^k$(resp.$|v(z)| \geqslant |\log z|^k$) for some $k > 0$.

For Y is of higher dimension,we also has these extensions.Let U be an open set of Y with coordinate functions $z_1,...,z_m$ such that $U \cap D = \{z_1 \cdots z_e = 0\}$ for some $0 \le e \le m$.Let $B_i(i = 1,...,e)$ be monodromies matrix of $\mathcal{H}_0$ corresponding to loops around z_i-axes(for case of VHS, B_i are all quasi-unipotent). Let $v_1,\cdots,v_r$ be multi-valued flat sections of $\mathcal{H}$ on U which make a basis at each point. Then the expressions

$$h_j = \exp(-\sum_{i=1}^{e} \log B_i \cdot \log z_i/2\pi\sqrt{-1})v_j, \text{ for } j = 1,...,r$$

give single-value holomorphic section of $\mathcal{H}_0$,where the branches of $\log B_i$ are chosen in same interval. The extension $\mathcal{H}$ is generated by the basis h_j such that $\mathcal{H}$ does not depend on the choice of z_i and v_j. Actually,

Theorem 1.19 (Geometric Version of Nilpotent Orbit theorem)**.** *Assume that all the local monodromies of the local system $R^i\pi_{0*}\mathbb{C}_{X_0}$ around $Y - Y_0$ are unipotent,then we have the Deligne's canonical extension $\mathcal{H}^i$ of $\mathcal{H}_0^i$(in this case ${}^u\mathcal{H}^i$ must be ${}^l\mathcal{H}^i$).The Nilpotent Orbit Theorem says*

$\mathcal{F}^p := j_*F^p(\mathcal{H}_0^i) \cap \mathcal{H}^i$ are all holomorphic subbundle (exactly all bundles are algebraic) of $\mathcal{H}^i$ where $j : Y_0 \to Y$ is the inclusion and the all quotients $\mathcal{Q}^p = \mathcal{F}^p/\mathcal{F}^{p+1}$ are locally free sheaves.

1.4.2. *Local freeness of relative dual sheaf and semi-positivity.* Denote relative dual sheaf on X to be $\omega_{X/Y} = \omega_X \otimes \pi^*\omega_Y^{-1}$ and let $d = \dim X - \dim Y$.

Definition 1.20 (Unipotent Reduction Condition)**.** Let $\pi : X \to Y$ be an algebraic fiber space, we say π satisfies the unipotent reduction condition(**URC**) if and only if the following conditions holds:

1. There is a Zariski open dense subset Y_0 of Y such that $D = Y - Y_0$ is a **divisor of normal crossing on** Y(i.e. D is a reduced effective divisor and if $D = \sum_{i=1}^{N} D_i$ is the decomposition to irreducible components,then D_i are non-singular and cross normally).
2. $\pi_0 \triangleq \pi|_{X_0} : X_0 \to Y_0$ is smooth where $X_0 = \pi^{-1}(Y_0)$.
3. The local monodromies of $R^d\pi_{0*}\mathbb{C}_{X_0}$ around D are unipotent where $d = \dim X - \dim Y$.

Remark: If $\pi : X \to Y$ is semi-stable family,then the **URC** holds automatically. When $\dim Y = 1$ the semi-stable reduction always exists,but for higher dimensional Y,the semi- stable reduction theorem is not yet proved.However we have unipotent reduction shown by Kawamata[5].

Theorem 1.21. *[Local freeness of dual sheaf] Let $\pi : X \to Y$ be an algebraic fiber space which satisfies* **URC***, let $R^d\pi_{0*}\mathbb{C}_{X_0} \otimes \mathcal{O}_{Y_0}$ be the polarized VHS over Y_0 and $\mathcal{F}_0^d = \pi_{0,*}\omega_{X_0/Y_0}$ be the bottom of the filtration. $\mathcal{F}_0^d$ have Deligne canonical locally free extension $\mathcal{F}^d$ over Y and $\pi_*\omega_{X/Y}|_{Y_0} = \mathcal{F}_0^d$. Then*

$$\pi_*\omega_{X/Y} = \mathcal{F}^d,$$

so that $\pi_*\omega_{X/Y}$ is a locally free sheaf.

Furthermore,due to $\mathcal{F}^d$ is the bottom filtration and VHS is flat holomorphic vector bundle. Reasonably we have

Theorem 1.22 (Fujita-Kawamata semi-positivity [6, 5])**.** *Let $\pi : X \to Y$ be an algebraic fiber space which satisfies* **URC***. Then $\pi_*\omega_{X/Y}$ is a locally free sheaf and semi-positive.*

Hodge Structure is a very elegant and deep theory,it has powerful applications for algebraic geometry,complex geometry.We can only give a short induction here. There are many beautiful theorems but we have not time to talk: for examples,$SL(2)$-orbit theorem,mixed Hodge theory,mixed variation of Hodge Structure.

2. Moduli space of polarized Calabi-Yau manifolds

2.1. **Calabi-Yau manifolds.**

Theorem 2.1 (Yau's Solution to Calabi Conjecture)**.** *For any compact Kähler Manifold X of complex dimension n with $K_X = \wedge^n \Omega_X^1 = \mathcal{O}_X$, there will be a unique Ricci flat metric on M (i.e. $\exists \mid$ Kähler metric g_{ij} such that $R(g)_{ij} = 0$).*

By this fundamental theorem,one has

Definition 2.2 (Calabi-Yau Manifold). Let X be a compact manifold,the followings are equivalence:

(a) Originally, X admits a Riemannian metric with global holonomy group
$$0 \neq H^* \subseteq \mathrm{SU}(n).$$
(b) X is a compact Kähler manifold with trivial canonical bundle. (Thus $\Theta_X \otimes \Omega_X^n \cong \Omega_X^{n-1}$.)

In algebraic geometry,we consider the case of manifolds with **Holomony Group**
$$H^* = \mathrm{SU}(n),$$
which have to be projective Calabi-Yau manifold.

Let X be CY manifolds of dimension n.Thus,for $0 < i < n$,
$$h^i(X, \mathcal{O}_X) = h^{0,i}(X) = h^0(X, \Omega_X^i) = h^{n,n-i}(X) = 0.$$

Example 2.3 (Complete intersection)**.** Let $X \subset \mathbb{P}^{n+k}$ be a varieties defined by $F_1 = F_2 = \cdots = F_k = 0$ and $\deg F_i = d_i$.For generic choice of F_i X is smooth manifold of dimension n. By adjunction formula,the canonical line bundle
$$K_X = \mathcal{O}_X(\sum_1^k d_i - n - k - 1)$$
is trivial if $\sum di = n + k + 1$.

By Lefschetz's hyperplane theorem, these manifolds satisfy $H^0(\Omega_X^i) = 0$ for $0 < i < n$. Thus one obtains Calabi-Yau manifold of complete intersection type.

Example 2.4. Taking a double cover of $\mathbb{P}^3$ branched over 8 planes in general position,blowing up along the 28 singular lines.Then we will obtain many Calabi-Yau threedfolds.

2.2. **On moduli space.**

Theorem 2.5 (Viehweg[9])**.** *The coarse quasi-projective moduli spaces $\mathcal{M}$ exist for polarized manifolds.In particular for polarized $K3$ surfaces, Calabi-Yau manifolds and Abelian varieties.*

The existence 'Coarse Moduli' will imply every closed point of scheme $\mathcal{M}$ can be represented by a polarized variety.i.e.

$$\mathcal{M}_h(\mathbb{C}) \longleftrightarrow \{\text{CY-folds with fixed Hilbert polynomial h}\}.$$

According to Matsusaka Big theorem,the moduli functor is bounded,i.e. each polarized Calabi-Yau manifold with same Hilbert polynomial h can be embedded into a projective space $\mathbb{P}^N$ where N is only dependent h.

In usually way, we want to choose a "good" compactification $f : \mathfrak{X} \to \overline{M}$ for a given smooth family $f : \mathfrak{X}_0 \to M$, i.e. an extension of f to a morphism of projective manifolds with $S = \overline{M} - U$ and $\Delta = \mathfrak{X} - \mathfrak{X}_0$ normal crossing divisors (NCD).

When the smooth family is over curve, it always has 'good' compactifiction sauch that singular fibers are all reduced and normal crossing,i.e. semi-stable reduction.

Consider deformation theory(in the category of complex analysis).

Let (X, L) be polarized CY manifold,

$$\pi : (\mathfrak{X}, X) \to (\mathfrak{M}_{c_1(L)}, 0)$$

be the polarized Kuranishi family and D be the classifying space of the polarized $\mathbb{Q}$-Hodge structure of $H^n_{prim}(X, \mathbb{C})$.

Theorem 2.6 (Bogomolov-Todorov-Tian)**.** *Let X be a Calabi-Yau manifold,then the Kuranishi family of (X, L) is universal,and the base manifold is a smooth open set with dimension*

$$\dim_{\mathbb{C}} H^1(X, \Theta_X)_{c_1(L)}.$$

Here $H^1(X, \Theta)_{c_1(L)} := \mathrm{Ker}(H^1(X, \Theta_X) \xrightarrow{\wedge c_1(L)} H^2(X, \mathcal{O}_X))$. This theorem is also proven by Ran,Z. and Kawamata with purely algebraic method.

Torelli theorem : (Roughly specking)Whether the Hodge Structure determines the complex structure.

Given any family of polarized CY manifold

$$f : (\mathcal{X}, X) \to (Z, 0).$$

Locally, near $0 \in Z$,one has the commutative diagram of period map

$$\begin{array}{ccc} \mathfrak{M}_{c_1(L)} & \xleftarrow{\iota} & Z \\ & {\scriptstyle \Phi}\searrow \quad \swarrow{\scriptstyle \Phi_Z} & \\ & D & \end{array}$$

Let $\mu_0 = (d\Phi)_0$ and $\lambda_0 = (d\Phi_Z)_0$.Let $\rho = (d\iota)_0$ which is the Kodaira-Spencer map.

Theorem 2.7 (Infinitesimal Torelli Theorem by Griffiths)**.**

$$\mu_0 : T_{\mathfrak{M}_{c_1(L),0}} = H^1(X, \Theta_X)_{c_1(L)} \to \bigoplus_p \mathrm{Hom}(H^{n-p,p}_{prim}, H^{n-p-1,p+1}_{prim})$$

which is injective map in case of Calabi-Yau manifolds.Furthermore,one has

$$\lambda_0 = \mu_0 \circ \rho.$$

Therefore,that the period map Φ be not degenerate at 0 is equivalent to that the Kodaira-Spencer map be injective.T he generic points of the base variety of a non-isotrivial family of Calabi-Yau are in that case.

Using deep Hodge theory(variation of mixed Hodge structure and mixed Higgs bundle), we have

Theorem 2.8 (Zuo [11])**.** *Any coarse moduli with infinitesimal Torelli theorem must be of* log- *general type, especially for Calabi-Yau.*

2.3. **Weil-Petersson Metric.** According to Yau's solution to Calabi's Conjecture,there is unique Ricci flat metric $g(t)$ on $\mathfrak{X}_t$ in the given polarization $[\omega(t)]$.

Then $g(t)$ induces a metric on $\Lambda^{0,1}(T_{\mathfrak{X}_t})$.

We can define **Weil-Petersson** metric G_{WP} on $\mathfrak{M}_{c_1(L)}$ by

$$G_{WP}(v, w) := \int_{\mathfrak{X}_t} < \rho_t(v), \rho_t(w) >_{g(t)}$$

for any $v, w \in T_{\mathfrak{M}_{c_1(L)},t}$.

G_{WP} is a Kähler metric on $\mathfrak{M}_{c_1(L)}$ by infinitesimal Torelli theorem .

Tian and Todorov show that the Kähler form is

$$\omega_{WP}(t) = -\frac{\sqrt{-1}}{2}\partial\overline{\partial}\log h = c_1(H^{n,0}(\mathfrak{X}_t), h) \qquad \text{(WP)}$$

where h is the Hodge metric of the VHS $R^n_{prim}\pi_*(\mathbb{C})$ restring to $\pi_* K_{\mathfrak{X}/\mathfrak{M}}$.

Shown by Chin-Lung Wang,

Theorem 2.9 (Wang[10])**.** *The Weil-Pertsson metric on moduli space of CY is incomplete. Moreover, WP metric will have finite distance near canonical degeneration.*

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