# VERTEX OPERATOR ALGEBRAS AND DIFFERENTIAL GEOMETRY 

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Mathematical beauty is more than its own reward. P.A.M. Dirac

The purpose of the series of the lectures is to introduce some applications of vertex operator algebras to differential geometry. These applications are in the same spirit of the applications of Grassmannian algebras to differentiable manifolds that lead to exterior differential forms and the exterior differential operator, and the applications of Clifford algebras and spinor representations that lead the Dirac operators on spin Riemannian manifolds. In particular, we will explain the relationship with elliptic genera. The vertex operator algebra we exploit include the semi-infinite wedge product and the semi-infinite symmetry power of an infinite dimensional space. We believe such applications will shed some lights on the geometry and topology of infinite dimensional manifolds that naturally arise in string theory.

We will emphasize on supersymmetry and supersymmetric indices, which already appear in the classical setting. Supersymmetry already appears in the Hodge theory of Laplacian operators on Riemannian manifolds and Kähler manifolds. The corresponding indices coincide with the Euler characteristic and Hirzeburch $\chi_{y}$ genus respectively. Elliptic genera, which generalize classical genera, naturally appear in the infinite dimensional setting as one considers the supersymmetric indices of the associated superconformal vertex algebras.

A very important notion in string theory is that of an $N=2$ superconformal field theory (SCFT). Physicists showed that the primary chiral fields of an $N=2$ SCFT form an algebra. The proof of this fact in physics literature share many common features of the Hodge theory of Kähler manifolds. See e.g. [14, 21].

A closely related notion is that of a topological vertex algebra of which one can consider the BRST cohomology. Given an $N=2$ SCFT, there are two ways to twist it to obtain a toplogical vertex algebra. The BRST cohomology groups of these two toplogical vertex algebras correspond to the algebras of the primary chiral and anti-chiral fields respectively of the orignial $N=2$ SCFT. Given a Calabi-Yau manifold $M$, it has been widely discussed in physics literature for many years that there is an $N=2$ SCFT associated to it, with the two twists giving the so-called the A-theory and the B-theory respectively. See e.g. [1].

Malikov, Schechtman, and Vaintrob [17] have constructed for any Calabi-Yau manifold a sheaf of topological vertex algebras. Their theory corresponds to the $A$-theory. In [28] we give a different approach based on standard techniques in differential geometry. We use holomorphic vector bundles of $N=2$ superconformal vertex algebras on a complex manfiold $M$, and the $\bar{\partial}$ operator on such bundles. We show that the corresponding cohomology group has a natural structure of an $N=2$
superconofrmal vertex algebra, whose two twists provide the desired $A$ theory and $B$ theory.

Vafa [20] suggested an approach to quantum cohomology based on vertex algebra constructed via semi-infinite forms on loop space. Recall that a closed string in a manifold $M$ is a smooth map from $S^{1}$ to $M$. The configuration space of all closed string is just the free loop space $L M$. Earlier researches in algebraic topology mostly dealt with the ordinary cohomology of the loop spaces. However the cohomology theory related to semi-infinite forms on the loop space seems to be more interesting. As is well-known in the theory of vertex algebras, the space of such forms has a natural structure of a vertex algebra being the Fock space of a natural infinite dimensional Clifford algebra. One also has to consider the semi-infinite symmetry product which also has a natural structure of a vertex algebra being the bosonic space of a natural infinite dimensional Heisenberg algebra. Superconformal structures naturally arise when the fermionic and bosonic parts are combined.

We begin with the Hodge theory on Riemannian manifolds in $\S 1$. There is a underlying Lie superalgebra which we call the $U(1)$ supersymmetry algebra. The corresponding supersymmetric index is exactly the Euler characteristic. We study an algebraic analogue in $\S 2$. More precisely we study differential operators on the space of differential forms with polynomial coefficients. By taking suitable metric we obtain a formal Hodge theory analogous to the Hodge theory on Riemannian manifold. We also compute the corresponding supersymmetric index.

We then move onto the Hodge theory on Kähler manifolds in $\S 4$ and present some larger Lie superalgebras underlying it. A suitably defined supersymmetric index in this case gives the Hirzebruch $\chi_{y}$ genus. An algebraic analogue is studied in $\S 5$.

The next natural topological invariant to consider is the elliptic genus. This involves the constructions in [28] of an $N=1$ superconformal vertex algebra associated to any Riemannian manifold, and an $N=2$ superconformal vertex algebra associated to any complex manifold. First of all on the algebraic level one needs to take the number of variables to infinity in the examples studied in $\S 2$ and $\S 5$. More precisely, we will study exterior algebras with infinitely many generators in $\S 7$ and polynomial algebras with infinitely many generators in $\S 8$. This leads us naturally to vertex algebras whose definition is presented in $\S 9$. We recall some well-known constructions of vertex algebras in $\S 10$. We present some basics of $N=2$ superconformal vertex algebras in $\S 11$. For applications to differential geometry, the reader can consult [28].

## 1. Hodge Theory for Riemannian Manifolds and $N=1$ Supersymmetric Index

Throughout this section $M$ is a compact oriented manifold of real dimension $n$.
1.1. De Rham cohomology and Euler characteristic. The space $C^{\infty}(M)$ of smooth functions on $M$ is a commutative algebra with unit. In particular,

$$
\begin{aligned}
& (f \cdot g) \cdot h=f \cdot(g \cdot h) \\
& f \cdot g=g \cdot f \\
& 1 \cdot f=f \cdot 1=f
\end{aligned}
$$

for $f, g, h \in C^{\infty}(M)$. Let $x^{1}, \ldots, x^{n}$ be local coordinates, then we regard a smooth function locally as a formal power series:

$$
f=f_{i_{1}, \ldots, i_{k}} x^{i_{1}} \cdots x^{i_{k}}
$$

We have

$$
x^{i} \cdot x^{j}=x^{j} \cdot x^{i} .
$$

On the other hand, the space of differential forms is a graded commutative algebra with unit. In particular,

$$
\begin{aligned}
& (\alpha \wedge \beta) \cdot \gamma=\alpha \wedge(\beta \cdot \gamma) \\
& \alpha \cdot \beta=(-1)^{|\alpha| \cdot|\beta|} \beta \wedge \alpha, \\
& 1 \wedge \alpha=\alpha \cdot 1=\alpha
\end{aligned}
$$

for $\alpha, \beta, \gamma \in \Omega^{*}(M)$. In local coordinates $x^{1}, \ldots, x^{n}$, a smooth differential form can be written as

$$
\alpha=\alpha_{i_{1}, \ldots, i_{k}} d x^{i_{1}} \wedge \cdots \wedge d x^{i_{k}} .
$$

We have

$$
d x^{i} \wedge d x^{j}=-d x^{j} \wedge d x^{i}
$$

The exterior differential operator

$$
d: \Omega^{*}(M) \rightarrow \Omega^{*+1}(M)
$$

satisfies the following properties:

$$
\begin{align*}
& d(\alpha \wedge \beta)=d \alpha \wedge \beta+(-1)^{|\alpha|} \alpha \wedge d \beta  \tag{1}\\
& d^{2}=0 \tag{2}
\end{align*}
$$

In local coordinates we have

$$
d \alpha=d x^{i} \wedge \frac{\partial}{\partial x^{i}} \alpha
$$

There are two kinds of operators involved here: $\left\{\frac{\partial}{\partial x^{i}}: i=1, \ldots, n\right\}$ and $\left\{d x^{i} \wedge\right.$ : $i=1, \ldots, n\}$. Note

$$
\begin{aligned}
& \frac{\partial}{\partial x^{i}} \frac{\partial}{\partial x^{j}}=\frac{\partial}{\partial x^{j}} \frac{\partial}{\partial x^{i}}, \\
& \left(d x^{i} \wedge\right)\left(d x^{j} \wedge\right)=-\left(d x^{j} \wedge\right)\left(d x^{i} \wedge\right)
\end{aligned}
$$

The second property (2) of $d$ implies that $\operatorname{Im} d \subset \operatorname{ker} d$, hence one defines the de Rham cohomology group

$$
H^{p}(M)=\left.\operatorname{ker} d\right|_{\Omega^{p}(M)} /\left.\operatorname{Im} d\right|_{\Omega^{p-1}(M)}
$$

The first property (1) implies that there is an induced structure of a graded commutative algebra with unit on

$$
H^{*}(M)=\oplus_{p=0}^{n} H^{p}(M)
$$

The de Rham theorem states that the de Rham cohomology of $M$ is isomorphic to its singular or simplicial cohomology, hence it is a topological invariant. In particular, $H^{p}(M)$ is finite dimensional for each $p$. The Euler characteristic of $M$ is defined by:

$$
\chi(M)=\sum_{p=0}^{n}(-1)^{p} \operatorname{dim} H^{p}(M) .
$$

1.2. Laplace operator and Hodge theory. Suppose now $M$ is endowed with a Riemannian metric $g$. Then one can define the Hodge star operator

$$
\begin{equation*}
*: \Omega^{p}(M) \rightarrow \Omega^{n-p}(M) . \tag{3}
\end{equation*}
$$

It satisfies the following property:

$$
\left.*^{2}\right|_{\Omega^{p}(M)}=(-1)^{p(n-p)} .
$$

One can define a metric on $\Omega^{*}(M)$ by:

$$
\langle\alpha, \beta\rangle=\int_{M} \alpha \wedge * \beta
$$

Define $d^{*}: \Omega^{p}(M) \rightarrow \Omega^{p-1}(M)$ by

$$
d^{*}=-* d * .
$$

Then by (3) and the Stokes theorem it is straightforward to see that

$$
\langle d \alpha, \beta\rangle=\left\langle\alpha, d^{*} \beta\right\rangle
$$

It follows from (2) and (3) that

$$
\left(d^{*}\right)^{2}=0 .
$$

The Laplace operator is defined by;

$$
\Delta=d d^{*}+d^{*} d=\left(d+d^{*}\right)^{2} .
$$

It is easy to see that
$[\Delta, d]=\left[\Delta, d^{*}\right]=0$,

$$
\begin{equation*}
\langle\Delta \alpha, \beta\rangle=\langle\alpha, \Delta \beta\rangle \tag{4}
\end{equation*}
$$

A differential form $\alpha$ is said to be harmonic if

$$
\Delta \alpha=0
$$

It is easy to see that $\alpha$ is harmonic if and only if

$$
d \alpha=d^{*} \alpha=0
$$

I.e.,

$$
\alpha \in \operatorname{ker} d \cap \operatorname{ker} d^{*}=\operatorname{ker}\left(d+d^{*}\right) .
$$

Denote by $\mathcal{H}^{*}(M)$ the space of harmonic forms on $M$.
Now $\Delta$ is a self-adjoint elliptic operator, by standard theory there are the Hodge decompositions

$$
\begin{equation*}
\Omega^{*}(M)=\mathcal{H}^{*}(M) \oplus \operatorname{Im} d \oplus \operatorname{Im} d^{*}=\mathcal{H}^{*}(M) \oplus \operatorname{Im}\left(d+d^{*}\right) . \tag{6}
\end{equation*}
$$

One can see from the Hodge decomposition

$$
\begin{equation*}
H^{*}(M) \cong \mathcal{H}^{*}(M), \tag{7}
\end{equation*}
$$

i.e., every de Rham cohomology class is represented by a unique harmonic form.
1.3. Heat operator and its trace. There is also an eigenspace decomposition:

$$
\begin{equation*}
\Omega^{*}(M)=\oplus_{\lambda} \Omega^{*}(M)_{\lambda} \tag{8}
\end{equation*}
$$

where the sum is taken over all eigenvalues of $\Delta$. The eigenvalues of $\Delta$ has some nice properties, for example, they form a discrete set that goes to infinity and every eigenspace is finite-dimensional. Furthermore, an operator $e^{-t \Delta}: \Omega^{*}(M) \rightarrow \Omega^{*}(M)$ can be defined such that its action on the eigenspace $\Omega^{*}(M)_{\lambda}$ is multiplication by $e^{-t \lambda}$, and it is of trace class:

$$
\operatorname{tr} e^{-t \Delta}=\sum_{\lambda} e^{-\lambda t} \operatorname{dim} \Omega^{*}(M)_{\lambda} .
$$

Set

$$
\Omega^{\underline{0}}(M)=\oplus_{p \text { even }} \Omega^{p}(M), \quad \Omega^{\underline{1}}(M)=\oplus_{p \text { odd }} \Omega^{p}(M)
$$

Then $Q=d+d^{*}$ maps $\Omega_{\underline{0}}^{\underline{0}}(M)$ to $\Omega \underline{\underline{1}}(M)$ and vice versa. Since $Q$ commutes with $\Delta$, it maps $\Omega^{*}(M)_{\lambda}$ to itself. By the Hodge decomposition it is easy to see that $\left.Q\right|_{\Omega^{*}(M)_{\lambda}}$ is an isomorphism for $\lambda \neq 0$. Define an operator $(-1)^{F}: \Omega^{*}(M) \rightarrow$ $\Omega^{*}(M)$ by

$$
(-1)^{F} \alpha=(-1)^{p} \alpha, \quad \alpha \in \Omega^{p}(M)
$$

It follows that

$$
\begin{equation*}
\operatorname{tr}\left(\left.(-1)^{F} e^{-t \Delta}\right|_{\Omega^{*}(M)_{\lambda}}\right)=0, \quad \lambda \neq 0 . \tag{9}
\end{equation*}
$$

We need the fermionic number operator $J: \Omega^{*}(M) \rightarrow \Omega^{*}(M)$ defined by:

$$
J(\alpha)=p \alpha, \quad \alpha \in \Omega^{p}(M)
$$

It is clearly self-adjoint, and $(-1)^{F}=(-1)^{J}$.
In physics literature, eigenvectors of zero eigenvalues are often referred to as the zero modes, eigenvectors of nonzero eigenvalues are the nonzero modes. Hence the harmonic forms are the zero modes of the Laplacian operator. A straightforward consequence of the isomorphism (7) is

$$
\begin{aligned}
\chi(M) & =\sum_{p=0}^{n}(-1)^{p} \operatorname{dim} \mathcal{H}^{p}(M)=\sum_{p=0}^{n}(-1)^{p} \operatorname{tr}\left(\left.e^{-t \Delta}\right|_{\Omega^{p}(M)_{0}}\right) \\
& =\operatorname{tr}\left(\left.(-1)^{F} e^{-t \Delta}\right|_{\Omega^{*}(M)_{0}}\right)=\operatorname{tr}\left(\left.(-1)^{F} e^{-t \Delta}\right|_{\Omega^{*}(M)}\right),
\end{aligned}
$$

where in the last identity we have used (9). This is the starting point of the proof of the Gauss-Bonnet-Chern theorem by heat kernel method proposed by McKean and Singer. Also note for $\lambda \neq 0$,

$$
\Omega^{p}(M)_{\lambda}=d \Omega^{p-1}(M)_{\lambda} \oplus d^{*} \Omega^{p+1}(M)_{\lambda}
$$

and $d$ induces an isomorphism:

$$
d \Omega^{p-1}(M)_{\lambda} \cong d^{*} \Omega^{p}(M)_{\lambda}
$$

Therefore one has [2]:

$$
\operatorname{tr} y^{J} e^{-t \Delta}=\sum_{p=0}^{n} y^{p} \operatorname{dim} H^{p}(M)+(1+y) q_{e^{-t}}(y)
$$

for some polynomial

$$
\begin{aligned}
\operatorname{tr} y^{J} e^{-t \Delta} & =\sum_{p=0}^{n} y^{p} \operatorname{tr}\left(\left.e^{-t \Delta}\right|_{\Omega^{p}(M)}\right) \\
& =\sum_{p=0}^{n} y^{p} \operatorname{tr}\left(\left.e^{-t \Delta}\right|_{\Omega^{p}(M)_{0}}\right)+\sum_{p=0}^{n} y^{p} \sum_{\lambda \neq 0} \operatorname{tr}\left(\left.e^{-t \Delta}\right|_{\Omega^{p}(M)_{\lambda}}\right) \\
& =\sum_{p=0}^{n} y^{p} \operatorname{dim} H^{p}(M)+\sum_{\lambda \neq 0} e^{-t \lambda} \sum_{p=0}^{n} y^{p}\left(\operatorname{dim} \Omega^{p}(M)_{\lambda}\right) \\
& =\sum_{p=0}^{n} y^{p} \operatorname{dim} H^{p}(M)+\sum_{\lambda \neq 0} e^{-t \lambda} \sum_{p=0}^{n} y^{p}\left(\operatorname{dim} d \Omega^{p-1}(M)_{\lambda}+\operatorname{dim} d^{*} \Omega^{p+1}(M)_{\lambda}\right) \\
& =\sum_{p=0}^{n} y^{p} \operatorname{dim} H^{p}(M)+\sum_{\lambda \neq 0} e^{-t \lambda} \sum_{p=0}^{n} y^{p}\left(\operatorname{dim} d^{*} \Omega^{p}(M)_{\lambda}+\operatorname{dim} d^{*} \Omega^{p+1}(M)_{\lambda}\right) \\
& =\sum_{p=0}^{n} y^{p} \operatorname{dim} H^{p}(M)+(1+y) \sum_{p=1}^{n} y^{p} \sum_{\lambda \neq 0} e^{-t \lambda} \operatorname{dim} d^{*} \Omega^{p}(M)_{\lambda} .
\end{aligned}
$$

1.4. The $U(1)$ supersymmetry algebra. In this subsection we summarized the algebraic structure behind the above discussions.

Let us introduce some terminologies and notations. A $\mathbb{Z}$-graded vector space is a vector space with a decomposition

$$
V=\oplus_{p \in \mathbb{Z}} V^{p} .
$$

An element $v \in V^{p}$ is said to be homogeneous of degree $p$, and we write $|v|=p$. A linear map $A: V \rightarrow V$ is said to be homogeneous of degree $k$ if $A\left(V^{p}\right) \subset V^{p+k}$ for all $p$, and we write $|A|=k$. When $|A|$ is even we say $A$ is an even operator, otherwise an odd operator. Given two homogeneous operators $A, B: V \rightarrow V$, we write

$$
[A, B]=A B-(-1)^{|A| \cdot|B|} B A
$$

Write $Q=d, Q^{\dagger}=d^{*}, H=\Delta$. We have the following commutation relations:

$$
\begin{aligned}
& {[J, J]=0} \\
& {[J, Q]=Q,\left[J, Q^{\dagger}\right]=-Q^{\dagger},[J, H]=0,} \\
& {[Q, Q]=\left[Q^{\dagger}, Q^{\dagger}\right]=[H, H]=0,} \\
& {\left[Q, Q^{\dagger}\right]=H,[H, Q]=\left[H, Q^{\dagger}\right]=0 .}
\end{aligned}
$$

In other words, if

$$
\begin{aligned}
& \mathfrak{g}_{0}=\mathbb{C} J \oplus \mathbb{C} H, \\
& \mathfrak{g}_{\underline{1}}=\mathbb{C} Q \oplus \mathbb{C} Q^{\dagger},
\end{aligned}
$$

then $\mathfrak{g}=\mathfrak{g}_{\underline{0}} \oplus \mathfrak{g}_{\underline{1}}$ is a Lie superalgebra. Note the Lie subalgebra $\mathbb{R} J$ is the Lie algebra of $U(1)$, so $J$ is called in physics literature as the $U(1)$ charge operator; $H$ is a central element; $\mathbb{R} Q$ is a representation of $\mathbb{R} J$, while $\mathbb{R} Q^{\dagger}$ is the dual representation, and the Lie bracket

$$
[\cdot, \cdot]: \mathbb{C} Q \otimes \mathbb{C} Q^{\dagger} \rightarrow \mathbb{C} H
$$

is given the natural pairing. We call this Lie superalgebra the $U(1)$ supersymmetry algebra. Then $J \mapsto J, H \mapsto \Delta, Q \mapsto d$, and $Q^{\dagger} \mapsto d^{*}$ gives a representation of this Lie superalgebra on $\Omega^{*}(M)$.

Now suppose we have a module $\mathcal{M}$ of the $U(1)$ supersymmetry algebra. If there is a Hermitian metric on $A$ such that

$$
J^{*}=J, \quad H^{*}=H, \quad Q^{*}=Q^{\dagger},
$$

where for an operator $P, P^{*}$ denotes its adjoint operator, then we say $\mathcal{M}$ is unitary.
Assume now $\mathcal{M}$ is unitary, and $J$ and $H$ are diagonizable with finite dimensional eigenspaces. Define the character by:

$$
\chi(\mathcal{M})(q, y)=\operatorname{tr}(-y)^{J} q^{H}
$$

and the $N=1$ supersymmetric index by

$$
\chi(\mathcal{M})(q)=\operatorname{tr}(-1)^{J} q^{H} .
$$

For example, when $\mathcal{M}$ is $\Omega^{*}(M), \chi(\mathcal{M})\left(e^{-t}\right)$ is exactly the Euler characteristic of $M$, hence it is topological (i.e., does not depend on the choice of the Riemannian metric). However, $\chi(\mathcal{M})\left(y, e^{-t}\right)$ is not topological and it encodes all the spectral data of the Laplacian operator.
1.5. A variation. One can also write $Q=d+d^{*}, H=Q^{2}$. Then we have

$$
\begin{aligned}
& {[Q, Q]=2 H,} \\
& {\left[(-1)^{F},(-1)^{F}\right]=0,} \\
& {[H, H]=0,} \\
& {\left[(-1)^{F}, Q\right]=-Q,} \\
& {\left[(-1)^{F}, H\right]=0,} \\
& {[Q, H]=0 \text {. }}
\end{aligned}
$$

If one sets

$$
\mathfrak{g}_{0}=\mathbb{R}(-1)^{F} \oplus \mathbb{R} H, \quad \mathfrak{g}_{1}=\mathbb{R} Q,
$$

then $\mathfrak{g}=\mathfrak{g}_{0} \oplus \mathfrak{g}_{\underline{1}}$ is a Lie superalgebra, for which $\Omega^{*}(M)=\Omega \underline{1}(M) \oplus \Omega^{\underline{1}}(M)$ is a representation. The $\operatorname{trace} \operatorname{tr}\left((-1)^{F} q^{H}\right)$ is called the supertrace of the operator $q^{H}$, and is often denoted by $\operatorname{str} q^{H}$.

## 2. Formal Hodge Theory and $U(1)$ Supersymmetry Algebra

In this section we formulate some algebraic analogue of the Hodge theory and obtain representations of the $U(1)$ supersymmetry algebra in the same fashion.
2.1. The one-variable case. Consider operators $\beta, \beta^{\dagger}: \mathbb{C}[z] \rightarrow \mathbb{C}[z]$ defined by:

$$
\beta(f(z))=\frac{d}{d z} f(z), \quad \quad \beta^{\dagger}(f(z))=z f(z)
$$

Then we have

$$
\beta(1)=0,
$$

and $\mathbb{C}[z]$ is linearly generated by $\left\{\left(\beta^{\dagger}\right)^{n}|0\rangle: n \geq 0\right\}$. Hence 1 can be regarded as a vacuum vector and will be denoted by $|0\rangle$. The following commutation relations are satisfied:

$$
\begin{equation*}
\left[\beta, \beta^{\dagger}\right]=\mathrm{id} \tag{10}
\end{equation*}
$$

In other words, $\mathbb{R} \beta \oplus \mathbb{R} \beta^{\dagger} \oplus \mathbb{R}$ id is the Heisenberg Lie algebra.
Introduce a Hermitian metric on $\mathbb{C}[z]$ such that

$$
\begin{equation*}
\beta^{*}=\beta^{\dagger} \tag{11}
\end{equation*}
$$

and $|0\rangle$ has length 1 . Then we must have

$$
\begin{equation*}
\left\langle z^{m}, z^{n}\right\rangle=\delta_{m, n} m!. \tag{12}
\end{equation*}
$$

With this metric, $\mathbb{R}[z]$ is a unitary representation of the Heisenberg algebra.
We also define the exterior differential operator $d: \mathbb{C}[z] \rightarrow \mathbb{C}[z] d z$ by

$$
d f(z)=\frac{d f}{d z} d z .
$$

As usual $d: \mathbb{C}[z] d z \rightarrow 0$ is the zero operator. On the basis we have

$$
d z^{n}=n z^{n-1} d z, \quad d\left(z^{n} d z\right)=0
$$

On the exterior algebra $\Lambda(d z)$ generated by $d z$, define operators $\varphi$ and $\varphi^{\dagger}$ by

$$
\begin{aligned}
\varphi(1) & =0 \\
\varphi^{\dagger}(1) & =d z
\end{aligned}
$$

$$
\begin{aligned}
\varphi(d z) & =1 \\
\varphi^{\dagger}(d z) & =0
\end{aligned}
$$

$$
\varphi^{\dagger}(d z)=0
$$

It is straightforward to check that

$$
\begin{equation*}
\left[\varphi, \varphi^{\dagger}\right]=\mathrm{id} \tag{14}
\end{equation*}
$$

Also define a Hermitian metric on $\Lambda(d z)$ by taking $\{1, d z\}$ as an orthonormal basis. Then we have

$$
\begin{equation*}
\varphi^{*}=\varphi^{\dagger} \tag{15}
\end{equation*}
$$

Let $A=\mathbb{C}[z] \oplus \mathbb{C}[z] d z=\mathbb{C}[z] \otimes \Lambda(d z)$, and naturally extend the metric and the operators $\beta, \beta^{\dagger}, \varphi, \varphi^{\dagger}$ to this space. Then one has

$$
\begin{align*}
{[\beta, \varphi] } & =\left[\beta, \varphi^{\dagger}\right]=0  \tag{16}\\
{\left[\beta^{\dagger}, \varphi\right] } & =\left[\beta^{\dagger}, \varphi^{\dagger}\right]=0 \tag{17}
\end{align*}
$$

One can consider the adjoint operator $d^{*}$ of $d$. It is easy to see that:

$$
\begin{equation*}
d^{*}\left(z^{n} d z\right)=z^{n+1}, \quad d^{*}\left(z^{n}\right)=0 \tag{18}
\end{equation*}
$$

Indeed,

$$
\begin{aligned}
& \left\langle z^{m}, d^{*}\left(z^{n} d z\right)\right\rangle=\left\langle d z^{m}, z^{n} d z\right\rangle=\left\langle m z^{m-1} d z, z^{n} d z\right\rangle \\
= & m \delta_{m-1, n}(m-1)!=m!\delta_{m, n+1}=\left\langle z^{m}, z^{n+1}\right\rangle .
\end{aligned}
$$

Also define $\Delta=d d^{*}+d^{*} d$. One can easily see that

$$
\begin{equation*}
\Delta\left(z^{n}\right)=n z^{n}, \quad \Delta\left(z^{n} d z\right)=(n+1) z^{n} d z \tag{19}
\end{equation*}
$$

It is straightforward to see that

$$
\begin{equation*}
d=\varphi^{\dagger} \beta, \quad \quad d^{*}=\beta^{\dagger} \varphi, \quad \Delta=\beta^{\dagger} \beta+\varphi^{\dagger} \varphi \tag{20}
\end{equation*}
$$

Also define $J: \mathbb{C}[z] \oplus \mathbb{C}[z] d z \rightarrow \mathbb{C}[z] \oplus \mathbb{C}[z] d z$ by

$$
J(f(z))=0, \quad J(f(z) d z)=f(z) d z
$$

We have

$$
\begin{equation*}
J=\varphi^{\dagger} \varphi \tag{21}
\end{equation*}
$$

Now it is straightforward to check that $J, H=\delta, Q=d, Q^{\dagger}=d^{*}$ defines a representation of the $U(1)$ supersymmetry algebra. Furthermore, by (19) we have

$$
\begin{align*}
& \chi(A)(q, y)=\sum_{n \geq 0}\left(q^{n}-y q^{n+1}\right)=\frac{1-y q}{1-q},  \tag{22}\\
& \chi(A)(q)=\frac{1}{1-q} . \tag{23}
\end{align*}
$$

2.2. The $n$-variable case. Suppose now we have $n$ variables $z^{1}, \ldots, z^{n}$. Then we have $n$ annihilators $\left\{\beta^{i}: i=1, \ldots, n\right\}$ and $n$ creators $\left\{\left(\beta_{i}\right)^{\dagger}: i=1, \ldots, n\right\}$ on the space $\mathbb{C}\left[z^{1}, \ldots, z^{n}\right]$, where

$$
\begin{equation*}
\beta^{i}=\frac{\partial}{\partial z^{i}}, \quad\left(\beta^{i}\right)^{\dagger}=z^{i} . \tag{24}
\end{equation*}
$$

One has

$$
\begin{align*}
{\left[\beta^{i}, \beta^{j}\right] } & =\left[\left(\beta^{i}\right)^{\dagger},\left(\varphi^{j}\right)^{\dagger}\right]=0,  \tag{25}\\
{\left[\beta^{i},\left(\beta^{j}\right)^{\dagger}\right] } & =\delta_{i j} . \tag{26}
\end{align*}
$$

There is a Hermitian metric on $\mathbb{C}\left[z^{1}, \ldots, z^{n}\right]$ such that

$$
\begin{equation*}
\left(\beta^{i}\right)^{*}=\left(\beta^{i}\right)^{\dagger}, \tag{27}
\end{equation*}
$$

and $|0\rangle=1$ has length 1 . Indeed one can take

$$
\begin{equation*}
\left\langle z_{1}^{a_{1}} \cdots z_{n}^{a_{n}}, z_{1}^{b_{1}} \cdots z_{n}^{b_{n}}\right\rangle=\prod_{i=1}^{n} \delta_{a_{i}, b_{i}} a_{i}!. \tag{28}
\end{equation*}
$$

On the Grassmannian algebra

$$
\Lambda\left(d z^{1}, \ldots, d z^{n}\right)=\left\{\sum_{k=0}^{n} \sum_{1 \leq i_{1}<\cdots<i_{k} \leq n} \alpha_{i_{1} \cdots i_{k}} d z^{i_{1}} \wedge d z^{i_{k}}\right\}
$$

we have $n$ creators $\left\{\left(\varphi^{i}\right)^{\dagger}: i=1, \ldots, n\right\}$ and $n$ annihilators $\left\{\varphi^{i}: i=1, \ldots, n\right\}$ defined by:

$$
\left(\varphi^{i}\right)^{\dagger}=d z^{i} \wedge, \quad \varphi^{i}=\iota \frac{\partial}{\partial z^{i}}
$$

Then one has

$$
\begin{align*}
{\left[\varphi^{i}, \varphi^{j}\right] } & =\left[\left(\varphi^{i}\right)^{\dagger},\left(\varphi^{j}\right)^{\dagger}\right]=0  \tag{29}\\
{\left[\varphi^{i},\left(\varphi^{j}\right)^{\dagger}\right] } & =\delta_{i j} . \tag{30}
\end{align*}
$$

There is a Hermitian metric on $\Lambda\left(d z^{1}, \ldots, d z^{n}\right)$ such that

$$
\begin{equation*}
\left(\varphi^{i}\right)^{*}=\left(\varphi^{i}\right)^{\dagger} \tag{31}
\end{equation*}
$$

and $|0\rangle=1$ has length 1 . Indeed one can take

$$
\left\{d z^{i_{1}} \wedge \cdots \wedge d z^{i_{k}}: 0 \leq k \leq n, 1 \leq i_{1}<\cdots<i_{k} \leq n\right\}
$$

to be an orthonormal basis.
Extend the above operators and metrics naturally to the tensor product $\mathbb{C}\left[z^{1}, \ldots, z^{n}\right] \otimes$ $\Lambda\left(d z^{1}, \ldots, d z^{n}\right)$. Then one has

$$
\begin{align*}
{\left[\beta^{i}, \varphi^{j}\right] } & =\left[\beta^{i},\left(\varphi^{j}\right)^{\dagger}\right]=0,  \tag{32}\\
{\left[\left(\beta^{i}\right)^{\dagger}, \varphi^{j}\right] } & =\left[\left(\beta^{i}\right)^{\dagger},\left(\varphi^{j}\right)^{\dagger}\right]=0 . \tag{33}
\end{align*}
$$

As usual the exterior differential operator $d: \mathbb{C}\left[z^{1}, \ldots, z^{n}\right] \otimes \Lambda\left(d z^{1}, \ldots, d z^{n}\right) \rightarrow$ $\mathbb{C}\left[z^{1}, \ldots, z^{n}\right] \otimes \Lambda\left(d z^{1}, \ldots, d z^{n}\right)$ is defined by

$$
d\left(\alpha_{i^{1}, \ldots, i_{k}} d z^{i_{1}} \wedge \cdots \wedge d z^{i_{k}}\right)=\frac{\partial f}{\partial z^{i}} d z^{i} \wedge d z^{i_{1}} \wedge \cdots \wedge d z^{i_{k}}
$$

One has

$$
\begin{equation*}
d=\left(\varphi^{i}\right)^{\dagger} \beta_{i} \tag{34}
\end{equation*}
$$

One can consider the adjoint operator $d^{*}$ of $d$. It is easy to see that:

$$
\begin{equation*}
d^{*}=\left(\beta^{i}\right)^{\dagger} \varphi_{i} \tag{35}
\end{equation*}
$$

Also define $\Delta=d d^{*}+d^{*} d$. Then one has

$$
\begin{equation*}
\Delta=\left(\beta^{i}\right)^{\dagger} \beta^{i}+\left(\varphi^{i}\right)^{\dagger} \varphi^{i} \tag{36}
\end{equation*}
$$

Finally define the fermionic number operator $J: \mathbb{C}\left[z^{1}, \ldots, z^{n}\right] \otimes \Lambda\left(d z^{1}, \ldots, d z^{n}\right) \rightarrow$ $\mathbb{C}\left[z^{1}, \ldots, z^{n}\right] \otimes \Lambda\left(d z^{1}, \ldots, d z^{n}\right)$ by

$$
J\left(\alpha_{i^{1}, \ldots, i_{k}} d z^{i_{1}} \wedge \cdots \wedge d z^{i_{k}}\right)=k \alpha_{i^{1}, \ldots, i_{k}} d z^{i_{1}} \wedge \cdots \wedge d z^{i_{k}}
$$

One has

$$
\begin{equation*}
J=\left(\varphi^{i}\right)^{\dagger} \varphi^{i} \tag{37}
\end{equation*}
$$

Now it is straightforward to check that $J, H=\delta, Q=d, Q^{\dagger}=d^{*}$ defines a representation of the $U(1)$ supersymmetry algebra on $A=\mathbb{C}\left[z^{1}, \ldots, z^{n}\right] \otimes \Lambda\left(d z^{1}, \ldots, d z^{n}\right)$, which is a tensor product of $n$ copies of the same representation:

$$
\mathbb{C}\left[z^{1}, \ldots, z^{n}\right] \otimes \Lambda\left(d z^{1}, \ldots, d z^{n}\right)=\otimes_{i=1}^{n}\left(\mathbb{C}\left[z^{i}\right] \otimes \Lambda\left(d z^{i}\right)\right.
$$

Therefore we have

$$
\begin{align*}
& \chi(A)(q, y)=\left(\frac{1-y q}{1-q}\right)^{n}  \tag{38}\\
& \chi(A)(q)=\frac{1}{(1-q)^{n}} \tag{39}
\end{align*}
$$

2.3. The weighted $n$-variable case. Now we present a modification of the above construction. On $\mathbb{C}\left[p_{1}, \ldots, p_{n}\right]$ consider $n$ annihilators $\left\{\beta^{i}: i=1, \ldots, n\right\}$ and $n$ creators $\left\{\left(\beta_{i}\right)^{\dagger}: i=1, \ldots, n\right\}$ on the space $\mathbb{C}\left[p_{1}, \ldots, p_{n}\right]$, where

$$
\begin{equation*}
\beta^{i}=i \frac{\partial}{\partial p_{i}}, \quad \quad\left(\beta^{i}\right)^{\dagger}=p_{i} \tag{40}
\end{equation*}
$$

One has

$$
\begin{align*}
{\left[\beta^{i}, \beta^{j}\right] } & =\left[\left(\beta^{i}\right)^{\dagger},\left(\varphi^{j}\right)^{\dagger}\right]=0  \tag{41}\\
{\left[\beta^{i},\left(\beta^{j}\right)^{\dagger}\right] } & =i \delta_{i j} . \tag{42}
\end{align*}
$$

There is a Hermitian metric on $\mathbb{C}\left[p^{1}, \ldots, p^{n}\right]$ such that

$$
\begin{equation*}
\left(\beta^{i}\right)^{*}=\left(\beta^{i}\right)^{\dagger}, \tag{43}
\end{equation*}
$$

and $|0\rangle=1$ has length 1 . Indeed one can take

$$
\begin{equation*}
\left\langle p_{1}^{a_{1}} \cdots p_{n}^{a_{n}}, p_{1}^{b_{1}} \cdots p_{n}^{b_{n}}\right\rangle=\prod_{i=1}^{n} \delta_{a_{i}, b_{i}} i^{a_{i}} a_{i}!. \tag{44}
\end{equation*}
$$

Remark 2.1. It is well-known that the algebra of symmetric polynomials in $n$ variables $z^{1}, \ldots, z^{n}$ is freely generated by the Newton power sums:

$$
p_{i}=\left(z_{1}\right)^{i}+\cdots+\left(z^{n}\right)^{i}
$$

The natural metric on this space is given by (44).
On the Grassmannian algebra $\Lambda\left(d p_{1}, \ldots, d p_{n}\right)$ we define $n$ creators $\left\{\left(\varphi^{i}\right)^{\dagger}: i=\right.$ $1, \ldots, n\}$ and $n$ annihilators $\left\{\varphi^{i}: i=1, \ldots, n\right\}$ by:

$$
\left(\varphi^{i}\right)^{\dagger}=d p_{i} \wedge, \quad \varphi^{i}=\iota \frac{\partial}{\partial p_{i}}
$$

Then one has

$$
\begin{align*}
{\left[\varphi^{i}, \varphi^{j}\right] } & =\left[\left(\varphi^{i}\right)^{\dagger},\left(\varphi^{j}\right)^{\dagger}\right]=0  \tag{45}\\
{\left[\varphi^{i},\left(\varphi^{j}\right)^{\dagger}\right] } & =\delta_{i j} \tag{46}
\end{align*}
$$

There is a Hermitian metric on $\Lambda\left(d p_{1}, \ldots, d p_{n}\right)$ such that

$$
\begin{equation*}
\left(\varphi^{i}\right)^{*}=\left(\varphi^{i}\right)^{\dagger} \tag{47}
\end{equation*}
$$

and $|0\rangle=1$ has length 1 . Indeed one can take

$$
\left\{d p_{i_{1}} \wedge \cdots \wedge d p_{i_{k}}: 0 \leq k \leq n, 1 \leq i_{1}<\cdots<i_{k} \leq n\right\}
$$

to be an orthonormal basis.
Extend the above operators and metrics naturally to the tensor product $\mathbb{C}\left[p_{1}, \ldots, p_{n}\right] \otimes$ $\Lambda\left(d p_{1}, \ldots, d p_{n}\right)$. Then again the even operators commute with the odd operators. As above the exterior differential operator $d: \mathbb{C}\left[p_{1}, \ldots, p_{n}\right] \otimes \Lambda\left(d p_{1}, \ldots, d p_{n}\right) \rightarrow$ $\mathbb{C}\left[p_{1}, \ldots, p_{n}\right] \otimes \Lambda\left(d p_{1}, \ldots, d p_{n}\right)$ can be written as

$$
\begin{equation*}
d=\left(\varphi^{i}\right)^{\dagger} \beta_{i} \tag{48}
\end{equation*}
$$

and its adjoint operator $d^{*}$ can be written as

$$
\begin{equation*}
d^{*}=\left(\beta^{i}\right)^{\dagger} \varphi_{i} \tag{49}
\end{equation*}
$$

However the expression for $\Delta=d d^{*}+d^{*} d$ now changes to:

$$
\begin{equation*}
\Delta=\left(\beta^{i}\right)^{\dagger} \beta^{i}+i\left(\varphi^{i}\right)^{\dagger} \varphi^{i} \tag{50}
\end{equation*}
$$

Finally define the fermionic number operator $J: \mathbb{C}\left[p_{1}, \ldots, p_{n}\right] \otimes \Lambda\left(d p_{1}, \ldots, d p_{n}\right) \rightarrow$ $\mathbb{C}\left[p_{1}, \ldots, p_{n}\right] \otimes \Lambda\left(d p_{1}, \ldots, d p_{n}\right)$ by

$$
J\left(\alpha_{i^{1}, \ldots, i_{k}} d p_{i_{1}} \wedge \cdots \wedge d p_{i_{k}}\right)=k \alpha_{i^{1}, \ldots, i_{k}} d p_{i_{1}} \wedge \cdots \wedge d p_{i_{k}}
$$

One has

$$
\begin{equation*}
J=\left(\varphi^{i}\right)^{\dagger} \varphi^{i} \tag{51}
\end{equation*}
$$

Now it is straightforward to check that $J, H=\delta, Q=d, Q^{\dagger}=d^{*}$ defines a representation of the $U(1)$ supersymmetry algebra on $A=\mathbb{C}\left[p_{1}, \ldots, p_{n}\right] \otimes \Lambda\left(d p_{1}, \ldots, d p_{n}\right)$. It is not hard to see that

$$
\begin{align*}
& \chi(A)(q, y)=\prod_{i=1}^{n} \frac{1-y q^{i}}{1-q^{i}}  \tag{52}\\
& \chi(A)(q)=\frac{1}{\prod_{i=1}^{n}\left(1-q^{i}\right)} \tag{53}
\end{align*}
$$

## 3. Relationship with Witten deformation

In this section we interpret the algebraic constructions in last section in terms of Witten deformation.
3.1. Witten deformation. Let $f$ be a smooth function on a compact oriented Riemannian manifold. Consider

$$
d_{t} \alpha=e^{-t f} d\left(e^{t f} \alpha\right), \quad d_{t}^{*} \alpha=e^{t f} d^{*}\left(e^{-t f} \alpha\right), \quad \Delta_{t}=d_{t} d_{t}^{*}+d_{t}^{*} d_{t}
$$

One can also consider

$$
J_{t}(\alpha)=e^{-t f} J\left(e^{t f} \alpha\right)
$$

It is clearly that $J_{t}=J$ for all $t$. It is easy to see that $J=J_{t}, H=\Delta_{t}, Q=d_{t}$, $Q^{\dagger}=d_{t}^{*}$ form a $U(1)$ supersymmetry algebra.

Let $H_{t}^{*}(M)$ be the cohomology of $d_{t}$, and let $\mathcal{H}_{t}^{*}(M)$ denote the space of harmonic forms for $\Delta_{t}$. Then we have an isomorphism

$$
H_{t}^{*}(M) \cong \mathcal{H}_{t}^{*}(M)
$$

as vector spaces (by standard theory for elliptic operators), and also an isomorphism

$$
H_{t}^{*}(M) \cong H^{*}(M)
$$

induced by the map $\alpha \mapsto e^{-t f} \alpha$. These facts have been used to prove the Morse inequalities [22, 27]. In particular we have

$$
\begin{equation*}
\operatorname{tr}(-1)^{J_{t}} q^{\Delta_{t}}=\sum_{p=0}^{n}(-1)^{p} \operatorname{dim} H_{t}^{p}(M)=\sum_{p=0}^{n}(-1)^{p} \operatorname{dim} H^{p}(M)=\operatorname{tr}(-1)^{J} q^{\Delta} \tag{54}
\end{equation*}
$$

3.2. Witten deformation and harmonic oscillator. Now let $M$ be the real line $\mathbb{R}$ with the standard metric with linear coordinates $x$, and $f=\frac{1}{2} x^{2}$. Then we have

$$
\begin{equation*}
d_{t} \alpha=d x \wedge\left(\frac{d}{d x}+t x\right) \alpha, \quad \quad d_{t}^{*} \alpha=\iota_{\frac{d}{d x}}\left(-\frac{d}{d x}+t x\right) \alpha \tag{55}
\end{equation*}
$$

Hence

$$
\left.\Delta_{t}\right|_{\Omega^{0}(\mathbb{R})}=\left(-\frac{d}{d x}+t x\right)\left(\frac{d}{d x}+t x\right)=-\frac{d^{2}}{d x^{2}}+t^{2} x^{2}
$$

is the Hamiltonian operator for the harmonic oscillator in quantum mechanics. We will take $t=1$. Note

$$
\left(\frac{d}{d x}+x\right) e^{-\frac{1}{2} x^{2}}=0
$$

and

$$
\left[\frac{d}{d x}+x,-\frac{d}{d x}+x\right]=2 \mathrm{id}
$$

Hence we take

$$
|0\rangle=e^{-\frac{1}{2} x^{2}}
$$

and regard

$$
\beta=\frac{1}{\sqrt{2}}\left(\frac{d}{d x}+x\right)
$$

as the annihilator, and

$$
\beta^{\dagger}=\frac{1}{\sqrt{2}}\left(-\frac{d}{d x}+x\right)
$$

as the creator. Consider the subspace of $C^{\infty}(\mathbb{R})$ with basis:

$$
\left\{\left(\beta^{\dagger}\right)^{n}|0\rangle=\frac{1}{2^{n / 2}}\left(-\frac{d}{d x}+x\right)^{n} e^{-\frac{x^{2}}{2}}: n \geq 0\right\}
$$

It is well-known that

$$
H_{n}(x)=e^{\frac{x^{2}}{2}}\left(-\frac{d}{d x}+x\right)^{n} e^{-\frac{x^{2}}{2}}
$$

are the Hermite polynomials which have the following generating series:

$$
\sum_{n \geq 0} \frac{H_{n}(x)}{n!} t^{n}=\exp \left(2 x t-t^{2}\right)
$$

Furthermore,

$$
\frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} H_{m}(x) H_{n}(x) e^{-x^{2}} d x=\delta_{m, n} 2^{m} m!
$$

It follows that

$$
\left\{\frac{1}{\sqrt{n!}}\left(\beta^{\dagger}\right)^{n}|0\rangle: n \geq 0\right\}
$$

are orthonormal with respect to the $L^{2}$ metric on $\mathbb{R}$. Denote by $\mathcal{H}$ the Hilbert space with this basis. It is straight forward to see that $d_{1}$ maps $\mathcal{H}$ to $\mathcal{H} d x$, and $d_{1}^{*}$ maps $\mathcal{H} d x$ to $\mathcal{H}$. By (55) one sees that

$$
\frac{1}{\sqrt{2}} d_{t}^{*}\left(\left(\beta^{\dagger}\right)^{n}|0\rangle d x\right)=\left(\beta^{\dagger}\right)^{n+1}|0\rangle
$$

Now it is straightforward to see that the following map

$$
F: \mathcal{H} \otimes \Lambda(d x) \rightarrow \mathbb{C}[z] \otimes \Lambda(d z)
$$

defined by

$$
\frac{1}{2^{n / 2}} H_{n}(x) \mapsto z^{n}, \quad \quad d x \mapsto d z
$$

is an isomorphism of graded vector spaces with metrics, such that $d_{1}$ is mapped to $d$, and $d_{1}^{*}$ is mapped to $d^{*}$.
3.3. Generalizations to $\mathbb{R}^{n}$. It is straightforward to generalize to $\mathbb{R}^{n}$. One can take

$$
f(x)=\frac{1}{2} \sum_{i=1}^{n}\left(x^{i}\right)^{2}
$$

to get the $n$-variable case in $\S 2.2$. One can also take

$$
f(x)=\frac{1}{2} \sum_{i=1}^{n} \lambda_{i}\left(x^{i}\right)^{2}
$$

to get weighted cases, e.g., take $\lambda_{i}=i$.

## 4. Hodge Theory for Kähler Manifolds and $N=2$ Supersymmetric Index

Throughout this subsection $M$ is a compact complex manifold of complex dimension $n$.
4.1. Dolbeault cohomology and Hirzebruch $\chi_{y}$ genus. When $M$ is a complex manifold, there is a finer decomposition of the space of differential forms:

$$
\Omega^{*, *}(M)=\oplus_{p, q=0}^{n} \Omega^{p, q}(M)
$$

where $\Omega^{p, q}(M)$ is the space of differential forms of type $(p, q)$. In local complex coordinates $z^{1}, \ldots, z^{n}$, a smooth $(p, q)$-form can be written as

$$
\alpha=\alpha_{i_{1}, \ldots, i_{p} ; \bar{j}_{1}, \ldots, \bar{j}_{q}} d z^{i_{1}} \cdots d z^{i_{p}} \wedge d \bar{z}^{j_{1}} \cdots d \bar{z}^{j_{q}}
$$

We have

$$
d z^{i} \wedge d z^{j}=-d z^{j} \wedge d z^{i}, \quad d z^{i} \wedge d \bar{z}^{j}=-d \bar{z}^{j} \wedge d z^{i}, \quad d \bar{z}^{i} \wedge d \bar{z}^{j}=-d \bar{z}^{j} \wedge d \bar{z}^{i}
$$

The exterior differential operator $d$ can be decomposed into two operators:

$$
d=\partial+\bar{\partial}
$$

satisfying the following properties:

$$
\begin{align*}
& \partial(\alpha \wedge \beta)=\partial \alpha \wedge \beta+(-1)^{|\alpha|} \alpha \wedge \partial \beta  \tag{56}\\
& \bar{\partial}(\alpha \wedge \beta)=\bar{\partial} \alpha \wedge \beta+(-1)^{|\alpha|} \alpha \wedge \bar{\partial} \beta  \tag{57}\\
& \partial^{2}=[\partial, \bar{\partial}]=\bar{\partial}^{2}=0 \tag{58}
\end{align*}
$$

One has $\operatorname{Im} \bar{\partial} \subset$ ker $\bar{\partial}$, and so one defines the Dolbeault cohomology group

$$
H^{p, q}(M)=\left.\operatorname{ker} \bar{\partial}\right|_{\Omega^{p, q}(M)} /\left.\operatorname{Im} \bar{\partial}\right|_{\Omega^{p, q-1}(M)} .
$$

There is an induced structure of a graded commutative algebra with unit on

$$
H^{*, *}(M)=\oplus_{p, q=0}^{n} H^{p, q}(M) .
$$

The Hirzebruch $\chi_{y}$ genus of $M$ is defined by:

$$
\chi_{y}(M)=\sum_{p=0}^{n}(-y)^{p} \sum_{q=0}^{n}(-1)^{q} \operatorname{dim} H^{p, q}(M)
$$

4.2. Hodge theory for $\bar{\partial}$ and $\partial$. Suppose now $M$ is endowed with a Hermitian metric $g$. Then the Hodge star operator maps $\Omega^{p, q}(M)$ to $\Omega^{n-p, n-q}(M)$. One can define a Hermitian metric on $\Omega^{*, *}(M)$ by:

$$
\langle\alpha, \beta\rangle=\int_{M} \alpha \wedge * \bar{\beta}
$$

Define $\bar{\partial}^{*}: \Omega^{p, q}(M) \rightarrow \Omega^{p, q-1}(M)$ and $\partial^{*}: \Omega^{p, q}(M) \rightarrow \Omega^{p-1, q}(M)$ by

$$
\bar{\partial}^{*}=-* \bar{\partial} *, \quad \partial^{*}=-* \partial *
$$

Then by (3) and the Stokes theorem it is straightforward to see that

$$
\langle\bar{\partial} \alpha, \beta\rangle=\left\langle\alpha, \bar{\partial}^{*} \beta\right\rangle, \quad\langle\partial \alpha, \beta\rangle=\left\langle\alpha, \partial^{*} \beta\right\rangle .
$$

It follows that

$$
\left(\bar{\partial}^{*}\right)^{2}=\left[\bar{\partial}, \partial^{*}\right]=\left(\partial^{*}\right)^{2}=0
$$

Define the associated Laplace operators by:

$$
\Delta_{\bar{\partial}}=\bar{\partial} \bar{\partial}^{*}+\bar{\partial}^{*} \bar{\partial}=\left(\bar{\partial}+\bar{\partial}^{*}\right)^{2}, \quad \Delta_{\partial}=\partial \partial^{*}+\partial^{*} \partial=\left(\partial+\partial^{*}\right)^{2}
$$

It is easy to see that

$$
\begin{align*}
{\left[\Delta_{\bar{\partial}}, \bar{\partial}\right] } & =\left[\Delta_{\bar{\partial}}, \bar{\partial}^{*}\right]=0, & {\left[\Delta_{\partial}, \partial\right] } & =\left[\Delta_{\partial}, \partial^{*}\right]=0, \\
\left\langle\Delta_{\bar{\partial}} \alpha, \beta\right\rangle & =\left\langle\alpha, \Delta_{\bar{\partial}} \beta\right\rangle, & \left\langle\Delta_{\partial} \alpha, \beta\right\rangle & =\left\langle\alpha, \Delta_{\partial} \beta\right\rangle \tag{59}
\end{align*}
$$

A differential form $\alpha$ is said to be $\bar{\partial}$-harmonic if

$$
\Delta_{\bar{\partial}} \alpha=0
$$

It is easy to see that $\alpha$ is $\bar{\partial}$-harmonic if and only if

$$
\bar{\partial} \alpha=\bar{\partial}^{*} \alpha=0
$$

I.e.,

$$
\alpha \in \operatorname{ker} \bar{\partial} \cap \operatorname{ker} \bar{\partial}^{*}=\operatorname{ker}\left(\bar{\partial}+\bar{\partial}^{*}\right)
$$

Denote by $\mathcal{H}_{\bar{\partial}}^{*}(M)$ the space of $\bar{\partial}$-harmonic forms on $M$.
Now $\Delta_{\bar{\partial}}$ is a self-adjoint elliptic operator, by standard theory there are decompositions

$$
\begin{equation*}
\Omega^{*, *}(M)=\mathcal{H}_{\bar{\partial}}^{*, *}(M) \oplus \operatorname{Im} \bar{\partial} \oplus \operatorname{Im} \bar{\partial}^{*}=\mathcal{H}^{*, *}(M) \oplus \operatorname{Im}\left(\bar{\partial}+\bar{\partial}^{*}\right) \tag{61}
\end{equation*}
$$

One can see from the Hodge decomposition

$$
\begin{equation*}
H^{*, *}(M) \cong \mathcal{H}_{\bar{\partial}}^{*, *}(M) \tag{62}
\end{equation*}
$$

i.e., every Dolbeault cohomology class is represented by a unique $\bar{\partial}$-harmonic form.

Define two operators $J_{L}, J_{R}: \Omega^{*, *}(M) \rightarrow \Omega^{*, *}(M)$ by

$$
J_{L}(\alpha)=p \alpha, \quad J_{R}(\alpha)=q \alpha, \quad \alpha \in \Omega^{p, q}(M)
$$

Again, an operator $e^{-t \Delta_{\bar{\partial}}}: \Omega^{*, *}(M) \rightarrow \Omega^{*, *}(M)$ can be defined and it is of trace class. Take $Q_{\bar{\partial}}=\bar{\partial}+\bar{\partial}^{*}$. Since $Q_{\bar{\partial}}$ commutes with $\Delta_{\bar{\partial}}$, by the same argument as in $\S 1.3$, one has

$$
\chi_{y}(M)=\sum_{p=0}^{n}(-y)^{p} \sum_{q=0}^{n}(-1)^{q} \operatorname{dim} \mathcal{H}^{p, q}(M)=\operatorname{tr}\left((-y)^{J_{L}}(-1)^{J_{R}} e^{-t \Delta_{\bar{o}}}\right)
$$

More generally, one can consider the Witten index:

$$
\operatorname{tr}\left((-y)^{J_{L}}(-1)^{J_{R}} e^{-t \Delta_{\bar{\partial}}} e^{-t^{\prime} \Delta_{\partial}}\right)
$$

We will see below that this is the same as the $\chi_{y}$ genus when $M$ is Kähler.
4.3. The $S U(2)$ supersymmetry algebra. It is easy to see that for a Hermitian manifold, $J=J_{L}, Q=\partial, Q^{\dagger}=\partial^{*}$, and $H=\Delta_{\partial}$ generate a $U(1) N=1$ supersymmetry algebra, so are $J=J_{R}, Q=\bar{\partial}, Q^{\dagger}=\bar{\partial}^{*}$, and $H=\Delta_{\bar{\partial}}$. When $M$ is Kähler, these two copies of $U(1)$ supersymmetry algebras can be combined into an $S U(2)$ supersymmetry algebra.

Let $L: \Omega^{*, *}(M) \rightarrow \Omega^{*+1, *+1}(M)$ be the multplication by the Kähler form, and let $\Lambda$ be the adjoint operator of $L$, and let $h=n-J_{L}-J_{R}$. Then $\Lambda, L, h$ form a Lie algebra isomorphic to $\mathfrak{s u}(2)$, i.e.,

$$
\begin{equation*}
[\Lambda, L]=h, \quad[h, \Lambda]=2 \Lambda \tag{63}
\end{equation*}
$$

$$
[h, L]=-2 L
$$

Furthermore, by the Kähler identities one has

$$
\begin{aligned}
{[\Lambda, \partial] } & =\sqrt{-1} \bar{\partial}^{*}, & {[L, \partial] } & =0, \\
{\left[L, \bar{\partial}^{*}\right] } & =-\sqrt{-1} \partial, & & {[h, \partial]=-\partial } \\
{\left[\Lambda, \bar{\partial}^{*}\right] } & =0, & &
\end{aligned}
$$

i.e., $\mathbb{C} \partial \oplus \mathbb{C} \bar{\partial}^{*}$ form a spin $1 / 2$ representation of $\mathfrak{s u}(2)$, or equivalent, the spinor representation $S$ of $\mathfrak{s o ( 3 )}$. Similarly, one also has

$$
\begin{array}{rlrl}
{[\Lambda, \bar{\partial}]} & =-\sqrt{-1} \partial^{*}, & {[L, \bar{\partial}]} & =0, \\
& {[h, \bar{\partial}]=-\bar{\partial}} \\
{\left[\Lambda, \partial^{*}\right]} & =0, & & {\left[h, \partial^{*}\right]=\partial^{*}}
\end{array}
$$

i.e., $\mathbb{C} \bar{\partial} \oplus \mathbb{C} \partial^{*}$ form the dual spinor representation $S^{*} \cong S$ of $\mathfrak{s o}(3) \cong \mathfrak{s u}(2)$. Finally, write note

$$
\begin{aligned}
{[\partial, \bar{\partial}] } & =\left[\partial, \bar{\partial}^{*}\right]=0, & & {\left[\partial^{*}, \bar{\partial}\right]=\left[\partial^{*}, \bar{\partial}^{*}\right]=0 } \\
{[\bar{\partial}, \partial] } & =\left[\bar{\partial}, \partial^{*}\right]=0, & & {\left[\bar{\partial}^{*}, \partial\right]=\left[\bar{\partial}^{*}, \partial^{*}\right]=0 } \\
{\left[\partial, \partial^{*}\right] } & =\Delta_{\partial}=\frac{1}{2} \Delta, & & {\left[\bar{\partial}, \bar{\partial}^{*}\right]=\Delta_{\bar{\partial}}=\frac{1}{2} \Delta . }
\end{aligned}
$$

Furthermore, $\Delta$ commutes with all operators $\Lambda, L, h, \partial, \partial^{*}, \bar{\partial}, \bar{\partial}^{*}$.
Definition 4.1. The $S U(2)$ supersymmetry algebra is the Lie superalgebra $\mathfrak{g}$ with $\mathfrak{g}_{0}=\mathfrak{s u}(2) \oplus \mathbb{C} H, \mathfrak{g}_{\underline{1}}=S \oplus S^{*}$, such that $H$ is a central element, and the Lie bracket induces and action of $\mathfrak{s u}(2)$ on $S$ by the spinor representation, and an action on $S^{*}$ by the dual spinor representation. Furthermore,

$$
[a, b]= \begin{cases}0, & a, b \in S \text { or } a, b \in S^{*} \\ \operatorname{tr}(a \otimes b), & a \in S, b \in S^{*}\end{cases}
$$

where $\operatorname{tr}: S \otimes S^{*} \rightarrow \mathbb{C}$ is the trace on $\operatorname{End}(S)$.
Hence we have shown that if

$$
\begin{aligned}
& \mathfrak{g}_{\underline{0}}=(\mathbb{C} \Lambda \oplus \mathbb{C} L \oplus \mathbb{C} h) \oplus \mathbb{C} \Delta \\
& \mathfrak{g}_{\underline{1}}=\left(\mathbb{C} \partial \oplus \mathbb{C} \bar{\partial}^{*}\right) \oplus\left(\mathbb{C} \bar{\partial} \oplus \mathbb{C} \partial^{*}\right)
\end{aligned}
$$

then $\mathfrak{g}=\mathfrak{g}_{0} \oplus \mathfrak{g}_{1}$ is an $S U(2)$ supersymmetry algebra.
Even the $S \bar{U}(2)$ supersymmetry algebra is a nice algebraic structure intrinsic in the Hodge theory of Kähler manifolds, however, to define the supersymmetry index, we need a larger Lie super algebra that encodes the left and right fermionic number operators $J_{L}$ and $J_{R}$.
Definition 4.2. The $N=2$ extended supersymmetry algebra is the Lie superalgebra with even generators $\lambda, h, L, J_{L}, J_{R}, H$, odd generators $Q^{+}, Q^{+\dagger}, Q^{-}, Q^{-\dagger}$, and the following nontrivial commutation relations:

$$
\begin{aligned}
& {[\Lambda, L]=h,} \\
& {[h, \Lambda]=2 \Lambda,} \\
& {[h, L]=-2 L,} \\
& {\left[J_{L}, \Lambda\right]=-\Lambda,} \\
& {\left[J_{L}, L\right]=L,} \\
& {\left[J_{L}, h\right]=0,} \\
& {\left[J_{R}, \Lambda\right]=-\Lambda,} \\
& {\left[J_{R}, L\right]=L,} \\
& {\left[J_{R}, h\right]=0,} \\
& {\left[\Lambda, Q^{ \pm}\right]= \pm \sqrt{-1} Q^{\mp \dagger},} \\
& {\left[L, Q^{ \pm}\right]=0,} \\
& {\left[h, Q^{ \pm}\right]=\mp Q^{+},} \\
& {\left[\Lambda, Q^{ \pm \dagger}\right]=0,} \\
& {\left[L, Q^{ \pm \dagger}\right]= \pm Q^{ \pm},} \\
& {\left[h, Q^{ \pm \dagger}\right]= \pm Q^{ \pm \dagger},} \\
& {\left[J_{L}, Q^{+}\right]=Q^{+},} \\
& {\left[J_{L}, Q^{+\dagger}\right]=-Q^{+\dagger},} \\
& {\left[J_{L}, Q^{-}\right]=\left[J_{L}, Q^{-\dagger}\right]=0,} \\
& {\left[J_{R}, Q^{-}\right]=Q^{-}, \quad\left[J_{R}, Q^{-\dagger}\right]=-Q^{-\dagger}, \quad\left[J_{R}, Q^{+}\right]=\left[J_{R}, Q^{+\dagger}\right]=0,} \\
& {\left[Q^{ \pm}, Q^{ \pm}\right]=0, \quad\left[Q^{ \pm}, Q^{\mp}\right]=0, \quad\left[Q^{ \pm}, Q^{\mp \dagger}\right]=0,} \\
& {\left[Q^{ \pm}, Q^{ \pm \dagger}\right]=\frac{1}{2} H .}
\end{aligned}
$$

Given such an algebra, let $Q=Q^{+}+Q^{-\dagger}, Q^{\dagger}=Q^{+\dagger}+Q^{-}, J=J_{L}-J_{R}$, it is easy to see that $J, H, Q, Q^{\dagger}$ for a $U(1)$ supersymmetry algebra.

Now suppose we have a module $\mathcal{M}$ of the extended $N=2$ supersymmetry algebra. If there is a Hermitian metric on $A$ such that

$$
J_{L}^{*}=J_{L}, \quad J_{R}^{*}=J_{R}, \quad H^{*}=H, \quad\left(Q^{ \pm}\right)^{*}=Q^{ \pm \dagger}
$$

where for an operator $P, P^{*}$ denotes its adjoint operator.
Assume now $\mathcal{M}$ is unitary, and $J$ and $H$ are diagonalizable with finite dimensional eigenspaces. Define the $N=2$ supersymmetric index by:

$$
\chi(\mathcal{M})(q, y)=\operatorname{tr}(-y)^{J_{L}}(-1)^{J_{R}} q^{H}
$$

For example, when $\mathcal{M}$ is $\Omega^{*, *}(M)$ for a Kähler manifold, $\chi(\mathcal{M})\left(y, e^{-t}\right)$ is exactly the $\chi_{y}$ genus of $M$.

One can also consider the Witten genus:

$$
\operatorname{tr}(-y)^{J_{L}}(-1)^{J_{R}} q^{H} \tilde{q}^{H} .
$$

Using the supersymmetry operator $Q^{-}$, an argument similar to that in $\S 1.3$ shows

$$
\operatorname{tr}(-y)^{J_{L}}(-1)^{J_{R}} q^{H} \tilde{q}^{H}=\operatorname{tr}(-y)^{J_{L}}(-1)^{J_{R}} q^{H} .
$$

4.4. Another $U(1)$ supersymmetry algebra in Kähler geometry. Let $Q=$ $\partial+\bar{\partial}^{*}, Q^{\dagger}=\partial^{*}+\bar{\partial}, H=\Delta$, and let $J: \Omega^{*, *}(M) \rightarrow \Omega^{*, *}(M)$ be defined by

$$
J(\alpha)=(p-q) \alpha, \quad \alpha \in \Omega^{p, q}(M) .
$$

Then $J, H, Q, Q^{\dagger}$ generate a $U(1)$ suppersymmetry algebra.

## 5. Formal Kähler Hodge Theory and $N=2$ Extended Supersymmetry Algebra

In this section we formulate some algebraic analogue of the Hodge theory of and obtain representations of the $N=2$ extended supersymmetry algebra in the same fashion. We will keep the notations in $\S 2$.
5.1. The one-variable case. Consider operators $\beta, \beta^{\dagger}, \gamma, \gamma^{\dagger}: \mathbb{C}[z, \bar{z}] \rightarrow \mathbb{C}[z, \bar{z}]$ defined by:

$$
\begin{array}{ll}
\beta(f(z, \bar{z}))=\frac{\partial}{\partial z} f(z, \bar{z}), & \beta^{\dagger}(f(z, \bar{z}))=z f(z, \bar{z}), \\
\gamma(f(z, \bar{z}))=\frac{\partial}{\partial \bar{z}} f(z, \bar{z}), & \gamma^{\dagger}(f(z, \bar{z}))=\bar{z} f(z, \bar{z})
\end{array}
$$

Then we have

$$
\beta(1)=\gamma(1)=0,
$$

and $\mathbb{C}[z, \bar{z}]$ is linearly generated by $\left\{\left(\beta^{\dagger}\right)^{m}\left(\gamma^{\dagger}\right)^{n} 1: n \geq 0\right\}$. Hence 1 can be regarded as a vacuum vector and will be denoted by $|0\rangle$. The following commutation relations are satisfied:

$$
\begin{equation*}
[\beta, \gamma]=\left[\beta, \gamma^{\dagger}\right]=\left[\gamma, \beta^{\dagger}\right]=0, \quad\left[\beta, \beta^{\dagger}\right]=\left[\gamma, \gamma^{\dagger}\right]=\mathrm{id} \tag{64}
\end{equation*}
$$

Introduce a Hermitian metric on $\mathbb{C}[z, \bar{z}]$ such that

$$
\begin{equation*}
\beta^{*}=\beta^{\dagger}, \quad \quad \gamma^{*}=\gamma^{\dagger} \tag{65}
\end{equation*}
$$

and $|0\rangle$ has length 1 . Then we must have

$$
\begin{equation*}
\left\langle z^{m_{1}} \bar{z}^{n_{1}}, z^{m_{2}} \bar{z}^{n_{2}}\right\rangle=\prod_{i=1}^{2} \delta_{m_{i}, n_{i}} m_{i}!. \tag{66}
\end{equation*}
$$

We also define differential operators $\partial, \bar{\partial}, d: \mathbb{C}[z] \rightarrow \mathbb{C}[z] d z$ by

$$
d f(z)=\frac{d f}{d z} d z .
$$

On the exterior algebra $\Lambda(d z, d \bar{z})$ generated by $d z, d \bar{z}$, define operators $\varphi, \psi, \varphi^{\dagger}, \psi^{\dagger}$ by

$$
\begin{aligned}
\varphi & =\iota \frac{\partial}{\partial z}, & \psi & =\iota \frac{\partial}{\partial \bar{z}} \\
\varphi^{\dagger} & =d z \wedge, & \psi^{\dagger} & =d \bar{z} \wedge .
\end{aligned}
$$

It is straightforward to check that

$$
\begin{align*}
& {[\varphi, \psi]=\left[\varphi, \psi^{\dagger}\right]=\left[\psi^{\dagger}, \varphi\right]=0}  \tag{67}\\
& {\left[\varphi, \varphi^{\dagger}\right]=\left[\psi, \psi^{\dagger}\right]=\mathrm{id}} \tag{68}
\end{align*}
$$

Also define a Hermitian metric on $\Lambda(d z, d \bar{z})$ by taking $\{1, d z, d \bar{z}, d z \wedge d \bar{z}\}$ as an orthonormal basis. Then we have

$$
\begin{equation*}
\varphi^{*}=\varphi^{\dagger}, \quad \quad \psi^{*}=\psi^{\dagger} \tag{69}
\end{equation*}
$$

Extend the metric and the operators $\beta, \beta^{\dagger}, \gamma, \gamma^{\dagger}, \varphi, \varphi^{\dagger}, \psi, \psi^{\dagger}$ to the space $\mathbb{C}[z, \bar{z}] \otimes$ $\Lambda(d z, d \bar{z})$. As before, the even operators commute with the odd operators.

Operators $\partial, \bar{\partial}$ can be defined. One can also consider the adjoint operator $\partial^{*}$, $\bar{\partial}^{*}$ and $d^{*}$. It is straightforward to see that

$$
\begin{array}{lll}
\partial=\varphi^{\dagger} \beta, & \partial^{*}=\beta^{\dagger} \varphi, & \Delta_{\partial}=\beta^{\dagger} \beta+\varphi^{\dagger} \varphi, \\
\bar{\partial}=\psi^{\dagger} \gamma, & \bar{\partial}^{*}=\gamma^{\dagger} \psi, & \Delta_{\bar{\partial}}=\gamma^{\dagger} \gamma+\psi^{\dagger} \psi .
\end{array}
$$

Also define $J_{L}, J_{R}, \Lambda, L, h: \mathbb{C}[z, \bar{z}] \otimes \Lambda[d z, d \bar{z}] \rightarrow \mathbb{C}[z, \bar{z}] \otimes \Lambda[d z, d \bar{z}]$ by

$$
\begin{align*}
& J_{L}=\varphi^{\dagger} \varphi, \\
& \Lambda=\sqrt{-1} \varphi \psi,  \tag{73}\\
& J_{R}=\psi^{\dagger} \psi,  \tag{72}\\
& L=\sqrt{-1} \varphi^{\dagger} \psi^{\dagger}, \\
& h=\varphi \varphi^{\dagger}+\psi^{\dagger} \psi .
\end{align*}
$$

Then it is easy to see that we obtain an $S U(2)$ extended supersymmetry algebra. We now compute the Witten index:

$$
\begin{aligned}
\operatorname{tr}\left((-y)^{J_{L}}(-1)^{J_{R}} q^{H_{L}} \tilde{q}^{H_{R}}\right) & =\operatorname{tr}\left(\left.(-y)^{J_{L}} q^{H_{L}}\right|_{\mathbb{C}[z] \otimes \Lambda(d z)}\right) \cdot \operatorname{tr}\left(\left.(-1)^{J_{R}} \tilde{q}^{H_{R}}\right|_{\mathbb{C}[\bar{z}] \otimes \Lambda(d \bar{z})}\right) \\
& =\frac{1-y q}{1-q} .
\end{aligned}
$$

Consider the $U(1)$ symmetry algebra generated by $J=J_{L}-J_{R}, H=d d^{*}+d^{*} d$, $Q=\partial+\bar{\partial}^{*}, Q^{*}=\partial^{*}+\bar{\partial}$. We have

$$
\begin{aligned}
J & =\varphi^{\dagger} \varphi-\psi^{\dagger} \psi, & H & =\beta^{\dagger} \beta+\gamma^{\dagger} \gamma+\varphi^{\dagger} \varphi+\psi^{\dagger} \psi, \\
Q & =\varphi^{\dagger} \beta+\gamma^{\dagger} \psi, & Q^{\dagger} & =\beta^{\dagger} \varphi+\psi^{\dagger} \gamma,
\end{aligned}
$$

and

$$
\begin{equation*}
\operatorname{tr}(-y)^{J} q^{H}=\frac{1-y q}{1-q} \cdot \frac{1-y^{-1} q}{1-q} \tag{74}
\end{equation*}
$$

5.2. The weighted $n$-variable case. One can consider the weighted $n$-variable generalization as in §2.3. We leave the details to the reader. we have

$$
\begin{aligned}
J & =\left(\varphi^{i}\right)^{\dagger} \varphi^{i}-\left(\psi^{i}\right)^{\dagger} \psi^{i}, & H & =\left(\beta^{i}\right)^{\dagger} \beta^{i}+\left(\gamma^{i}\right)^{\dagger} \gamma^{i}+i\left(\varphi^{i}\right)^{\dagger} \varphi+i\left(\psi^{i}\right)^{\dagger} \psi^{i} \\
Q & =\left(\varphi^{i}\right)^{\dagger} \beta^{i}+\left(\gamma^{i}\right)^{\dagger} \psi^{i}, & Q^{\dagger} & =\left(\beta^{i}\right)^{\dagger} \varphi^{i}+\left(\psi^{i}\right)^{\dagger} \gamma^{i}
\end{aligned}
$$

and

$$
\begin{equation*}
\operatorname{tr}(-y)^{J} q^{H}=\prod_{i=1}^{n} \frac{\left(1-y q^{i}\right)\left(1-y^{-1} q^{i}\right)}{\left(1-q^{i}\right)^{2}} \tag{75}
\end{equation*}
$$

## 6. Relationship with Holomorphic Witten deformations

6.1. Holomorphic Witten deformation induced by a function. Let $f$ be a smooth Morse on a compact complex Kähler manifold. Consider

$$
\begin{array}{lll}
\partial_{t} \alpha=e^{-t f} \partial\left(e^{t f} \alpha\right), & \partial_{t}^{*} \alpha=e^{t f} \partial^{*}\left(e^{-t f} \alpha\right), & \Delta_{\partial, t}=\partial_{t} \partial_{t}^{*}+\partial_{t}^{*} \partial_{t} \\
\bar{\partial}_{t} \alpha=e^{-t f} \bar{\partial}\left(e^{t f} \alpha\right), & \bar{\partial}_{t}^{*} \alpha=e^{t f} \bar{\partial}^{*}\left(e^{-t f} \alpha\right), & \Delta_{\bar{\partial}, t}=\bar{\partial}_{t} \bar{\partial}_{t}^{*}+\bar{\partial}_{t}^{*} \bar{\partial}_{t}
\end{array}
$$

It is easy to see that $\Lambda, L, h, J_{L}, J_{R}, H=2 \Delta_{\partial, t}=2 \Delta_{\bar{\partial}, t}, Q_{L}=\partial_{t}, Q_{L}^{\dagger}=\partial_{t}^{*}$, $Q_{R}=\bar{\partial}_{t}, Q_{R}^{\dagger}=\bar{\partial}_{t}^{*}$ form an extended $N=2$ supersymmetry algebra.
6.2. Witten deformation and harmonic oscillator. Now let $M$ be $\mathbb{C}$ with the standard metric and linear coordinate $z$, and $f=\frac{1}{2}|z|^{2}$. Then we have

$$
\begin{align*}
\partial_{t} \alpha & =d z \wedge\left(\frac{\partial}{\partial z}+\frac{t}{2} \bar{z}\right) \alpha, & \partial_{t}^{*} \alpha & =\iota_{\frac{\partial}{\partial z}}\left(-\frac{\partial}{\partial \bar{z}}+\frac{t}{2} z\right) \alpha  \tag{76}\\
\bar{\partial}_{t} \alpha & =d \bar{z} \wedge\left(\frac{\partial}{\partial \bar{z}}+\frac{t}{2} z\right) \alpha, & \bar{\partial}_{t}^{*} \alpha & =\iota_{\frac{\partial}{\partial \bar{z}}}\left(-\frac{\partial}{\partial z}+\frac{t}{2} \bar{z}\right) \alpha
\end{align*}
$$

We will take $t=1$. Note

$$
\left(\frac{\partial}{\partial z}+\frac{1}{2} \bar{z}\right) e^{-\frac{1}{2}|z|^{2}}=\left(\frac{\partial}{\partial \bar{z}}+\frac{1}{2} z\right) e^{-\frac{1}{2}|z|^{2}}=0
$$

and

$$
\begin{aligned}
& {\left[\frac{\partial}{\partial z}+\frac{1}{2} \bar{z},-\frac{\partial}{\partial \bar{z}}+\frac{1}{2} z\right]=\left[\frac{\partial}{\partial \bar{z}}+\frac{1}{2} z,-\frac{\partial}{\partial z}+\frac{1}{2} \bar{z}\right]=\mathrm{id}} \\
& {\left[\frac{\partial}{\partial z}+\frac{1}{2} \bar{z}, \frac{\partial}{\partial \bar{z}}+\frac{1}{2} z\right]=\left[-\frac{\partial}{\partial \bar{z}}+\frac{1}{2} z,-\frac{\partial}{\partial z}+\frac{1}{2} \bar{z}\right]=0} \\
& {\left[\frac{\partial}{\partial z}+\frac{1}{2} \bar{z},-\frac{\partial}{\partial z}+\frac{1}{2} \bar{z}\right]=\left[\frac{\partial}{\partial \bar{z}}+\frac{1}{2} z,-\frac{\partial}{\partial \bar{z}}+\frac{1}{2} z\right]=0 .}
\end{aligned}
$$

Hence we take the vacuum to be

$$
|0\rangle=e^{-\frac{1}{2}|z|^{2}}
$$

and regard

$$
\beta=\frac{\partial}{\partial z}+\frac{1}{2} \bar{z}, \quad \gamma=\frac{\partial}{\partial \bar{z}}+\frac{1}{2} z
$$

as the annihilators, and

$$
\beta^{\dagger}=-\frac{\partial}{\partial \bar{z}}+\frac{1}{2} z, \quad \gamma^{\dagger}=-\frac{\partial}{\partial z}+\frac{1}{2} \bar{z}
$$

as the creators. Consider the subspace $\mathcal{H}$ of $C^{\infty}(\mathbb{C})$ with basis:

$$
\left\{\left(\beta^{\dagger}\right)^{m}\left(\gamma^{\dagger}\right)^{n}|0\rangle=\left(-\frac{\partial}{\partial z}+\frac{1}{2} \bar{z}\right)^{m}\left(-\frac{\partial}{\partial \bar{z}}+\frac{1}{2} z\right)^{n} e^{-\frac{|z|^{2}}{2}}\right\}
$$

It is straightforward to see that the following map

$$
F: \mathcal{H} \otimes \Lambda(d z, d \bar{z}) \rightarrow \mathbb{C}[z, \bar{z}] \otimes \Lambda(d z, d \bar{z})
$$

defined by

$$
\left(\beta^{\dagger}\right)^{m}\left(\gamma^{\dagger}\right)^{n}|0\rangle \mapsto z^{m} \bar{z}^{n}, \quad d z \mapsto d z, \quad d \bar{z} \mapsto d \bar{z}
$$

is an isomorphism of graded vector spaces with metrics, such that $\partial_{1}$ is mapped to $\partial, \bar{\partial}$ is mapped to $\bar{\partial}_{1}, \partial^{*}$ is mapped to $\bar{\partial}_{1}^{*}$, and $\bar{\partial}^{*}$ is mapped to $\bar{\partial}_{1}^{*}$.

It is straightforward to generalize to $\mathbb{C}^{n}$. One can take

$$
f(x)=\frac{1}{2} \sum_{i=1}^{n}\left|z^{i}\right|^{2}
$$

to get the $n$-variable case in $\S 2.2$. One can also take

$$
f(x)=\frac{1}{2} \sum_{i=1}^{n} \lambda_{i}\left|z^{i}\right|^{2}
$$

to get weighted cases, e.g., take $\lambda_{i}=i$.

## 7. Semi-Infinite Wedge Algebra and Free Fermions

7.1. Semi-infinite wedge. Let $H$ be a Hilbert space with a complete orthonormal basis $\left\{e_{r}\right\}_{r \in \mathbb{Z}+\frac{1}{2}}$. Suppose $S=s_{1}<s_{2}<\ldots \subset \mathbb{Z}+\frac{1}{2}$ satisfy the following conditions:
(i) $S_{-}=\mathbb{Z}_{<0}+\frac{1}{2}-S$ is finite, and
(ii) $S_{+}=S-\left(\mathbb{Z}_{<0}+\frac{1}{2}\right)$ is finite.

For each such $S$, let

$$
e_{S}:=e_{s_{1}} \wedge e_{s_{2}} \wedge \cdots
$$

Denote by $\Lambda^{\frac{\infty}{2}}(H)$ the vector space with $\left\{e_{S}\right\}$ as an orthonormal basis. The vector

$$
e_{-1 / 2} \wedge e_{-3 / 2} \wedge e_{-5 / 2} \wedge \cdots
$$

is called the vacuum vector.
Define the charge operator $J$ and the energy operator $H$ on $\Lambda^{\frac{\infty}{2}}(H)$ as follows:

$$
\begin{aligned}
& J\left(e_{S}\right)=\left(\left|S_{-}\right|-\left|S_{+}\right|\right) e_{S} \\
& H\left(e_{S}\right)=\left(\sum_{s \in S_{+}} s-\sum_{t \in S_{-}} t\right) e_{S}
\end{aligned}
$$

7.2. Physical Interpretation: Dirac sea of electrons. The semi-infinite wedge provides a perfect mathematical formulation for Dirac's quantum theory containing both electrons and positrons. The wedge produce is required by Pauli's Exclusion Principle as usual. Each $e_{k}$ denotes an electron at energy level $k$. For each $e_{S}$ above, almost all negative levels are filled, forming an infinitely deep sea of electrons; those unfilled negative levels (holes) are regarded as positrons, each having charge 1, hence the total contribution to the charge by the holes is $\left|S_{-}\right|$, and the total contribution to the energy is

$$
-\sum_{t \in S_{-}} t
$$

On the other hand, there are only finitely many $e_{s_{i}}$ with positive $s_{i}$. They corresponds to electrons at energy level $s_{i}$, hence their total contribution to the energy is

$$
\sum_{s \in S_{+}} s
$$

On the other hand, since each electron has charge -1 , their contribution to the total charge is $-\left|S_{+}\right|$.
7.3. Wedging operators and their adjoints. For each $r \in \mathbb{Z}+\frac{1}{2}$, denote by $\psi_{r}$ the wedge product by $e_{r}$ on $\Lambda^{\frac{\infty}{2}}(H)$, and by $\psi_{r}^{*}$ its adjoint. The following relations are easy to verify:

$$
\left[\psi_{r}, \psi_{s}\right]=\left[\psi_{r}^{*}, \psi_{s}^{*}\right]=0, \quad\left[\psi_{r}, \psi_{s}^{*}\right]=\delta_{r, s}
$$

Let us consider their actions on the vacuum vector. When $r>0$,

$$
\begin{aligned}
& \psi_{r}|0\rangle=e_{r} \wedge e_{-1 / 2} \wedge e_{-3 / 2} \wedge \cdots \\
& \psi_{-r}^{*}|0\rangle=(-1)^{r-1 / 2} e_{-1 / 2} \wedge \cdots \wedge \hat{e}_{-r} \wedge \cdots
\end{aligned}
$$

In physical terminology, $\psi_{r}$ "creates" an electron of energy $r$ while $\psi_{-r}^{*}$ "creates" a positron of energy $r$, hence both $\psi_{r}$ and $\psi_{-r}^{*}$ are creators for $r>0$. Dually, both $\psi_{r}^{*}$ and $\psi_{-r}$ are annihilators for $r>0$.

We consider the generating series of the above operators as follows:

$$
\begin{aligned}
& b(z):=\sum_{r \in \mathbb{Z}+1 / 2} b_{r} z^{-r+\frac{1}{2}}, \\
& c(z):=\sum_{r \in \mathbb{Z}+1 / 2} c_{r} z^{-r+\frac{1}{2}},
\end{aligned}
$$

where $b_{r}=\psi_{r}^{*}$ and $c_{r}=\psi_{-r}$. The series $b(z)$ and $c(z)$ are formally regarded as meromorphic fields of operators on $\mathbb{C}^{*}=\mathbb{C}-\{0\}$, acting on $\Lambda^{\frac{\infty}{2}}(H)$. The operators $b_{r}$ and $c_{r}$ are called the modes of $b(z)$ and $c(z)$, respectively. Note $b_{r}|0\rangle=c_{r}|0\rangle=0$ for $r>0$. More generally, for any $v \in \Lambda \frac{\infty}{2}(H)$,

$$
b_{r} v=0, \quad c_{r} v=0
$$

for $r$ sufficiently large. This motivates the following:
Definition 7.1. A field on a vector space $V$ is a formal power series

$$
a(z)=\sum_{n \in \mathbb{Z}} a_{n} z^{-n-1}, \quad a_{n} \in \operatorname{End} V,
$$

such that for any $v \in V, a_{n} v=0$ for $n$ sufficiently large.
Denote by $F$ the Grassmannian algebra generated by $\left\{b_{r}, c_{r}\right\}_{r<0}$.
Lemma 7.1. There is a natural isomorphism $F \rightarrow \Lambda^{\frac{\infty}{2}}(H)$ defined by

$$
b_{r_{1}} \cdots b_{r_{m}} c_{s_{1}} \cdots c_{s_{n}} \mapsto b_{r_{1}} \cdots b_{r_{m}} c_{s_{1}} \cdots c_{s_{n}}|0\rangle
$$

$r_{1}, \ldots, r_{m}, s_{1}, \ldots, s_{n} \in 1 / 2+\mathbb{Z}_{<0}$.
Under this isomorphism, one can also regard $b(z)$ and $c(z)$ as fields on $F$.
7.4. Normally ordered product. A natural question is to obtain more fields on $\Lambda^{\frac{\infty}{2}}(H)$, or equivalently, on $F$. There are basically two methods to do it. The first is based on the following is an easy observation:

Lemma 7.2. If $a(z)=\sum_{n \in Z} a_{n} z^{-n-1}$ is a field on $V$, then so is $\partial_{z} a(z)=$ $-\sum_{n \in \mathbb{Z}}(n+1) a_{-n} z^{-n-2}$.

Hence all derivatives of $b(z)$ and $c(z)$ are fields. Notice that

$$
\frac{1}{n!} \partial_{z}^{n} b(z)|0\rangle=\sum_{r \leq 1 / 2-n}\binom{-r+1 / 2}{n} b_{r} z^{-r+1 / 2-n}
$$

It follows that

$$
\left.\frac{1}{n!} \partial_{z}^{n} b(z)|0\rangle\right|_{z=0}=b_{1 / 2-n}|0\rangle .
$$

Introduce the following notation:

$$
\partial_{z}^{(n)}=\frac{1}{n!} \partial_{z}^{n}
$$

Another way to obtain new fields is by taking the product of two fields. Unfortunately, there are some problems with the naive product defined as follows:

$$
a(z) \tilde{a}(z)=\sum_{k \in \mathbb{Z}} a_{k} z^{-k-1} \sum_{l \in \mathbb{Z}} \tilde{a}_{l} z^{-l-1}=\sum_{n \in \mathbb{Z}}\left(\sum_{k+l=n-1} a_{k} \tilde{a}_{l}\right) z^{-n-1},
$$

which involves infinite sum. For example,

$$
b(z) c(z)|0\rangle=\sum_{n \in \mathbb{Z}}\left(\sum_{r+s=n} b_{r} c_{s}|0\rangle\right) z^{-n-1},
$$

when $n=0$,

$$
\sum_{r \in \mathbb{Z}+\frac{1}{2}} b_{r} c_{-r}|0\rangle=\sum_{r \in \frac{1}{2}+\mathbb{Z}_{\geq 0}}|0\rangle=\infty|0\rangle .
$$

To solve this problem, one defines the normally ordered product by

$$
: a_{k} \tilde{a}_{l}:= \begin{cases}a_{k} \tilde{a}_{l}, & l \geq 0 \\ (-1)^{|a||\tilde{a}|} \tilde{a}_{l} a_{k}, & l<0\end{cases}
$$

or equivalently,

$$
: a(z) \tilde{a}(w):=a(z)_{+} \tilde{a}(w)+(-1)^{|a||\tilde{a}|} \tilde{a}(w) a(z)_{-},
$$

where $|a|$ is the order of $a$,

$$
a(z)_{+}:=\sum_{n<0} a_{n} z^{-n-1}, \quad a(z)_{-}:=\sum_{n \geq 0} a_{n} z^{-n-1} .
$$

Similarly define : $a(z) b(z)$ :.
Proposition 7.1. If $a(z)$ and $\tilde{a}(z)$ are two fields on $V$, then so is: $a(z) \tilde{a}(z)$ :.
Remark 7.1. In general, the normally ordered product is neither graded commutative nor associative.
7.5. State-field correspondence for free fermion space. There is a one-toone correspondence between the vectors in $F$ and some fields on $F$ as follows. For $I=\left(0 \leq i_{1}<\cdots<i_{m}\right), J=\left(0 \leq j_{1}<\cdots<j_{n}\right)$, set

$$
v_{I J}=b_{-i_{1}-1 / 2} \cdots b_{-i_{m}-1 / 2} c_{-j_{1}-1 / 2} \cdots c_{-j_{n}-1 / 2}
$$

They form a basis of $F$. Also set

$$
v_{I J}(z)=: \partial^{\left(i_{1}\right)} b(z) \cdots \partial^{\left(i_{m}\right)} b(z) \partial^{\left(j_{1}\right)} c(z) \cdots \partial^{\left(j_{n}\right)} c(z): .
$$

It is straightforward to check that

$$
\left.v_{I J}(z)|0\rangle\right|_{z=0}=v_{I J}
$$

7.6. Operator product expansion. The formal power series

$$
a(z) \tilde{a}(w)=\sum_{k \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} \tilde{a}_{l} a_{k} z^{-k-1} w^{-l-1}
$$

always makes sense, however its limit when $z \rightarrow w$ may not exist. To analyze the singularity, first note the limit of : $a(z) \tilde{a}(w):$ is : $a(z) \tilde{a}(z):$. Then we note:

$$
a(z) \tilde{a}(w)-: a(z) \tilde{a}(w):=\left[a(z)_{-}, \tilde{a}(w)\right] .
$$

For example,

$$
\begin{aligned}
& b(z) c(w)-: b(z) c(w):=\left[b(z)_{-}, c(w)\right] \\
= & \sum_{r \in 1 / 2+\mathbb{Z}} \sum_{\geq 0}\left[b_{r}, c_{s}\right] z^{-r-1 / 2} w^{-s-1 / 2} \\
= & \sum_{r \in 1 / 2+\mathbb{Z} \geq 0} \sum_{s \in 1 / 2+\mathbb{Z}} \delta_{r,-s} z^{-r-1 / 2} w^{-s-1 / 2} \\
= & \sum_{n=0}^{\infty} z^{-n-1} w^{n}=i_{z, w} \frac{1}{z-w} .
\end{aligned}
$$

Here $i_{z, w}$ means the power series expansion in the region $|z|>|w|$. In the same fashion,

$$
\begin{aligned}
& -c(w) b(z)-: b(z) c(w):=-\left[b(z)_{+}, c(w)\right] \\
= & -\sum_{r \in 1 / 2+\mathbb{Z}_{<0}} \sum_{s \in 1 / 2+\mathbb{Z}}\left[b_{r}, c_{s}\right] z^{-r-1 / 2} w^{-s-1 / 2} \\
= & -\sum_{r \in 1 / 2+\mathbb{Z}_{<0}} \sum_{s \in 1 / 2+\mathbb{Z}} \delta_{r,-s} z^{-r-1 / 2} w^{-s-1 / 2} \\
= & -\sum_{n=0}^{\infty} z^{n} w^{-n-1}=i_{w, z} \frac{1}{z-w} .
\end{aligned}
$$

We will often omit $i_{z, w}$ and $i_{w, z}$ when there are no confusions.
Definition 7.2. Let $a(z)$ and $\tilde{a}(z)$ be two fields on a vector space $V$. An equality of the form

$$
\begin{equation*}
a(z) \tilde{a}(w)=\sum_{k=0}^{N-1} \frac{c^{k}(w)}{(z-w)^{k+1}}+: a(z) \tilde{a}(w): \tag{78}
\end{equation*}
$$

or simply

$$
a(z) \tilde{a}(w) \sim \sum_{k=0}^{N-1} \frac{c^{k}(w)}{(z-w)^{k+1}}
$$

is called the operator product expansion of the fields $a(z)$ and $\tilde{a}(z)$. Since

$$
\lim _{z \rightarrow w}: a(z) \tilde{a}(w):=: a(w) \tilde{a}(w):
$$

$: a(z) \tilde{a}(w):$ is called the regular part of the OPE, and the rest of the OPE is called the singular part.

### 7.7. OPE and commutation relations. Recall the OPE

$$
b(z) c(w)=i_{z, w} \frac{1}{z-w}+: b(z) c(w):
$$

is equivalent to

$$
\left[b_{r}, c_{s}\right]=\delta_{r,-s} \mathrm{id}
$$

for $r \in \frac{1}{2}+\mathbb{Z}_{\geq 0}, s \in \frac{1}{2}+\mathbb{Z}$, and the OPE

$$
-c(w) b(z)=i_{w, z} \frac{1}{z-w}+: b(z) c(w):
$$

is equivalent to

$$
\left[b_{r}, c_{s}\right]=\delta_{r,-s} \mathrm{id}
$$

for $r \in \frac{1}{2}+\mathbb{Z}_{<0}, s \in \frac{1}{2}+\mathbb{Z}$. Therefore,

$$
b(z) c(w)+c(w) b(z)=i_{z, w} \frac{1}{z-w}-i_{w, z} \frac{1}{z-w}=\sum_{n \in \mathbb{Z}} z^{-n-1} w^{n}
$$

and it is equivalent to

$$
\left[b_{r}, c_{s}\right]=\delta_{r,-s} \mathrm{id}
$$

for $r \in \frac{1}{2}+\mathbb{Z}, s \in \frac{1}{2}+\mathbb{Z}$. Introduce

$$
\delta(z, w)=\sum_{n \in \mathbb{Z}} z^{-n-1} w^{n}
$$

Note

$$
\begin{aligned}
& i_{z, w} \frac{1}{(z-w)^{j+1}}=\sum_{m=0}^{\infty}\binom{m}{j} z^{-m-1} w^{m-j} \\
& i_{w, z} \frac{1}{(z-w)^{j+1}}=-\sum_{m=-1}^{-\infty}\binom{m}{j} z^{-m-1} w^{m-j} \\
& \partial_{w}^{(j)} \delta(z, w)=i_{z, w} \frac{1}{(z-w)^{j+1}}-i_{w, z} \frac{1}{(z-w)^{j+1}}=\sum_{m=-\infty}^{\infty}\binom{m}{j} z^{-m-1} w^{m-j} .
\end{aligned}
$$

Now we can state the following
Proposition 7.2. The following properties are equivalent:
(i) $[a(z), \tilde{a}(w)]=\sum_{j=0}^{N-1} \partial_{w}^{(j)} \delta(z, w) c^{j}(w)$ for some fields $c^{j}(w)$.
(ii)

$$
\begin{aligned}
& a(z) \tilde{a}(w)=\sum_{j=0}^{N-1}\left(i_{z, w} \frac{1}{(z-w)^{j+1}}\right) c^{j}(w)+: a(z) \tilde{a}(w):, \\
& (-1)^{|a||\tilde{a}|} \tilde{a}(w) a(z)=\sum_{j=0}^{N-1}\left(i_{w, z} \frac{1}{(z-w)^{j+1}}\right) c^{j}(w)+: a(z) \tilde{a}(w):, \\
& \text { for some fields } c^{j}(w)
\end{aligned}
$$

(iii) $\left[a_{m}, \tilde{a}_{n}\right]=\sum_{j=0}^{N-1}\binom{m}{j} c_{m+n-j}^{j}$.
(iv) $\left[a_{m}, \tilde{a}(w)\right]=\sum_{j=0}^{N-1}\binom{m}{j} c^{j}(w) w^{m-j}$.

For a proof, see [11], pp. 20-21.
Definition 7.3. Two fields $a(z)$ and $\tilde{a}(z)$ are said to be mutually local if they satisfy one of the conditions in the above Proposition.

### 7.8. Free fields and Wick's Theorem.

Definition 7.4. A collection of fields $\left\{a^{\alpha}(z)\right\}$ is called a free field theory if all these fields are mutually local and all the singular parts of the OPE are multiples of the identity.

The normally ordered product of more than two fields $a^{1}(z), a^{2}(z), \ldots, a^{N}(z)$ is defined inductively by

$$
: a^{1}(z) \cdots a^{N}(z):=: a^{1}(z): a^{2}(z) \cdots a^{N}(z)::
$$

The following simple result is one of the main tools for calculations of OPE:
Theorem 7.1. (Wick's theorem) Let $\left\{a^{1}(z), \ldots, a^{M}(z), b^{1}(z), \ldots, b^{N}(z)\right\}$ be a free field theory. Then we have the following OPE:

$$
: a^{1}(z) \cdots a^{M}(z):: b^{1}(w) \cdots b^{N}(w):
$$

Write $\left[a^{i} b^{j}\right]=\left[a^{i}(z)_{-}, b^{j}(w)\right]$. Then one has:

$$
\begin{aligned}
& : a^{1}(z) \cdots a^{M}(z):: b^{1}(w) \cdots b^{N}(w): \\
= & \sum_{s=0}^{\min (M, N)} \pm\left[a^{i_{1}} b^{j_{1}}\right] \cdots\left[a^{i_{s}} b^{j_{s}}\right]: a^{1}(z) \cdots a^{M}(z) b^{1}(w) \cdots b^{N}(w):\left(i_{1}, \cdots, i_{s} ; j_{1}, \cdots, j_{s}\right)
\end{aligned}
$$

where the subscript $\left(i_{1}, \cdots, i_{s} ; j_{1}, \cdots, j_{s}\right)$ means that the fields $a^{i_{1}}(z), \cdots, a^{i_{s}}(z)$, $b^{j_{1}}(w), \cdots, b^{j_{s}}(w)$ are removed, and the $\pm$ sign is determined by the Koszul convention: each interchange of two adjacent odd fields changes the sign.
7.9. Virasoro fields. It is straightforward to see that the energy operator can be expressed in terms of normally ordered products as follows:

$$
H=\sum_{r \in 1 / 2+\mathbb{Z}_{\geq 0}} r c_{-r} b_{r}-\sum_{r \in 1 / 2+\mathbb{Z}_{<0}} r b_{r} c_{-r}=\sum_{r \in 1 / 2+\mathbb{Z}} r: c_{-r} b_{r}:
$$

This can be generalized to the following field

$$
L(z)=\frac{1}{2}: \partial_{z} b(z) c(z):+\frac{1}{2}: \partial_{z} c(z) b(z): .
$$

Indeed,

$$
\begin{aligned}
L(z) & =\frac{1}{2} \sum_{r, s \in 1 / 2+\mathbb{Z}}\left(\left(-r-\frac{1}{2}\right): b_{r} c_{s}:+\left(-s-\frac{1}{2}\right): c_{s} b_{r}:\right) z^{-r-s-2} \\
& =\frac{1}{2} \sum_{n \in \mathbb{Z}} \sum_{r+s=n}(r-s): c_{s} b_{r}: z^{-n-2}
\end{aligned}
$$

If we write

$$
L(z)=\sum_{n \in \mathbb{Z}} L_{n} z^{-n-2}
$$

then

$$
L_{n}=\frac{1}{2} \sum_{r+s=n}(r-s): c_{s} b_{r}:
$$

In particular, $H=L_{0}$.

Proposition 7.3. For $\lambda \in \mathbb{C}$, set

$$
L^{\lambda}(z)=(1-\lambda): b^{\prime}(z) c(z):+\lambda: c^{\prime}(z) b(z):
$$

We have the following OPEs:

$$
\begin{align*}
L^{\lambda}(z) b(w) & \sim \frac{\lambda b(w)}{(z-w)^{2}}+\frac{b^{\prime}(w)}{z-w},  \tag{79}\\
L^{\lambda}(z) c(w) & \sim \frac{(1-\lambda) c(w)}{(z-w)^{2}}+\frac{c^{\prime}(w)}{z-w},  \tag{80}\\
L^{\lambda}(z) L^{\lambda}(w) & \sim \frac{\partial_{w} L^{\lambda}(w)}{z-w}+\frac{2 L^{\lambda}(w)}{(z-w)^{2}}+\frac{-1+6 \lambda-6 \lambda^{2}}{(z-w)^{4}} . \tag{81}
\end{align*}
$$

Proof. By Wick Theorem we have

$$
\begin{aligned}
L^{\lambda}(z) b(w) & =\left((1-\lambda): b^{\prime}(z) c(z):+\lambda: c^{\prime}(z) b(z):\right) b(w) \\
& \sim(1-\lambda) \frac{b^{\prime}(z)}{z-w}-\lambda b(z) \partial_{z} \frac{1}{z-w} \\
& \sim(1-\lambda) \frac{b^{\prime}(w)}{z-w}+\lambda \frac{b(w)+b^{\prime}(w)(z-w)}{(z-w)^{2}} \\
& \sim \frac{\lambda b(w)}{(z-w)^{2}}+\frac{b^{\prime}(w)}{z-w}
\end{aligned}
$$

This proves (79). The other two OPE's are proved in the same fashion.
Definition 7.5. A field $L(z)$ is called a Virasoro field if it satisfies the following OPE:

$$
\begin{equation*}
L(z) L(w) \quad \sim \frac{L^{\prime}(w)}{z-w}+\frac{2 L(w)}{(z-w)^{2}}+\frac{c / 2}{(z-w)^{4}} \tag{82}
\end{equation*}
$$

where the constant $c$ is called the central charge of the Virasoro field. Given a Virasoro field, if a field $a(z)$ satisfies

$$
L(z) a(w) \sim \frac{h a(w)}{z-w}+\frac{\partial_{w} a(w)}{(z-w)^{2}}+O\left(\frac{1}{(z-w)^{3}}\right)
$$

then we say $a$ has conformal weight $h$. If

$$
L(z) a(w) \sim \frac{h a(w)}{z-w}+\frac{\partial_{w} a(w)}{(z-w)^{2}},
$$

then we say $a$ is primary of conformal weight $h$.
From the above definition, one sees that $L^{\lambda}(z)$ is a Virasoro field of central charge $-\left(12 \lambda^{2}-12 \lambda+2\right)$ on $F$, for which $b(z)$ and $c(z)$ are primary fields of weights $1-\lambda$ and $\lambda$, respectively. The case of $\lambda=\frac{1}{2}$ corresponds to the Neveu-Schwarz sector in physic s literature, while the case of $\lambda=0$ or 1 corresponds to the Ramond sector.

It is straightforward to verify the following:
Lemma 7.3. Suppose $L(z)$ is a Virasoro field of central charge $c$, and

$$
L(z)=\sum_{n \in \mathbb{Z}} L_{n} z^{-n-2}
$$

then we have

$$
\left[L_{m}, L_{n}\right]=(m-n) L_{m+n}+\frac{m^{3}-m}{12} \delta_{m,-n} c
$$

Hence if there is a Virasoro field of central charge $c$ on $V$, then $V$ is a representation of Virasoro algebra of central charge c.
7.10. Charge field. Similarly, the charge operator can be rewritten as

$$
J=-\sum_{r \in 1 / 2+\mathbb{Z}_{\geq 0}} c_{-r} b_{r}+\sum_{r \in 1 / 2+\mathbb{Z}_{<0}} b_{r} c_{-r}=\sum_{r \in 1 / 2+\mathbb{Z}}: b_{r} c_{-r}:
$$

which can be generalized to a field

$$
\alpha(z)=: b(z) c(z): .
$$

Indeed,

$$
\alpha(z)=\sum_{r, s \in 1 / 2+\mathbb{Z}}: b_{r} c_{s}: z^{-r-s-1}=\sum_{n \in \mathbb{Z}}\left(\sum_{r+s=n}: b_{r} c_{s}:\right) z^{-n-1} .
$$

If one writes $\alpha(z)=\sum_{n \in \mathbb{Z}} \alpha_{n} z^{-n-1}$, then

$$
\alpha_{n}=\sum_{r+s=n}: b_{r} c_{s}:
$$

in particular, $J=\alpha_{0}$. We have the following OPE's:

$$
\begin{align*}
& \alpha(z) \alpha(w) \sim \frac{1}{(z-w)^{2}},  \tag{83}\\
& \alpha(z) b(w) \sim \frac{b(w)}{z-w},  \tag{84}\\
& \alpha(z) c(w) \sim-\frac{c(w)}{z-w},  \tag{85}\\
& L^{\lambda}(z) \alpha(w) \sim \frac{\alpha^{\prime}(w)}{z-w}+\frac{\alpha(w)}{(z-w)^{2}}+\frac{2 \lambda-1}{(z-w)^{3}} . \tag{86}
\end{align*}
$$

## 8. Semi-Infinite Symmetric Algebra and Free bosons

8.1. Oscillator algebra representation on free fermion space. The OPE (83) is equivalent to the following commutation relations:

$$
\begin{equation*}
\left[\alpha_{m}, \alpha_{n}\right]=m \delta_{m,-n} \tag{87}
\end{equation*}
$$

Recall the oscillator algebra is spanned by $\left\{\alpha_{m}\right\}_{m \in \mathbb{Z}}$ and central element $h$ satisfying:

$$
\left[\alpha_{m}, \alpha_{n}\right]=m \delta_{m,-n} h
$$

Hence $F$ is a representation of the oscillator algebra. Since $\left[\alpha_{0}, \alpha_{m}\right]=0$, this representation preserves the charge decomposition.
8.2. The bosonic Fock space. Consider the space $B=\mathbb{C}\left[\alpha_{-1}, \alpha_{-2}, \ldots\right]$. The the oscillator algebra acts on $B$ as follows. The central element $h$ acts as multiplication by a constant $\hbar, \alpha_{0}$ acts as 0 . For $n>0, \alpha_{-n}$ acts as multiplication by $\alpha_{-n}, \alpha_{n}$ acts as $n \hbar \frac{\partial}{\partial \alpha_{-n}}$. Introduce

$$
\alpha(z)=\sum_{n \in \mathbb{Z}} \alpha_{n} z^{-n-1}
$$

then $\alpha(z)$ is a field on $B$. Furthermore,

$$
\alpha(z) \alpha(w) \sim \frac{1}{(z-w)^{2}} .
$$

Remark 8.1. The famous Boson-Fermion correspondence is a natural isomorphism between $F$ and $B\left[\left[q, q^{-1}\right]\right]$. It has remarkable applications to soliton theory.
8.3. The state-field correspondence on the bosonic Fock space. For $I=$ $\left(i_{1} \geq \cdots \geq i_{n} \geq 0\right)$, set

$$
\alpha_{I}=\alpha_{-i_{1}-1} \cdots \alpha_{-i_{n}-1}
$$

Also set

$$
\alpha_{I}(z)=: \partial^{\left(i_{1}\right)} \alpha(z) \cdots \partial^{\left(i_{n}\right)} \alpha(z): .
$$

It is straightforward to see that

$$
\left.\alpha_{I}(z)|0\rangle\right|_{z=0}=\alpha_{I}
$$

8.4. The Virasoro field on the bosonic Fock space. Let

$$
L(z)=\frac{1}{2}: \alpha(z) \alpha(z):
$$

on $B$. Then by Wick's theorem, it is straightforward to see that

$$
L(z) L(w) \sim \frac{L^{\prime}(w)}{z-w}+\frac{2 L(w)}{(z-w)^{2}}+\frac{1}{(z-w)^{4}}
$$

I.e., $L(z)$ is a Virasoro field of central charge 2 on $B$.

## 9. Vertex Algebras and Conformal Vertex Algebras

9.1. Definition of a vertex algebra. We can now give the definition of a vertex algebra as in [11].
Definition 9.1. A vertex algebra consists of following data:
(i) A graded vector space $V=\oplus_{n \in \mathbb{Z}} V_{n}$, called the state space;
(ii) A vector $|0\rangle \in V_{0}$, called the vacuum vector;
(iii) A map $Y: V \rightarrow \operatorname{End}(V)\left[\left[z, z^{-1}\right]\right]$, whose image lies in the set of fields, called the state-field correspondence.
Write $Y(a, z)=\sum_{n \in \mathbb{Z}} a_{n} z^{-n-1}, a \in V$. Define $T a=a_{-2}|0\rangle$. The following axioms are required:

- (translation coinvariance): $[T, Y(a, z)]=\partial Y(a, z)$,
- (vacuum) $Y(|0\rangle, z)=\operatorname{id}_{V},\left.Y(a, z)|0\rangle\right|_{z=0}=a$,
- (locality) $Y(a, z)$ and $Y(b, z)$ are mutually local, $a, b, \in V$.

The following important theorem enables one to construct vertex algebras.
Theorem 9.1. Let $V$ be a graded vector space, let $|0\rangle \in V_{0}$ and let $T$ be an endomorphism of $V$ of degree 0 . Let $\left\{a^{\alpha}(z)\right\}_{\alpha \in I}$ be a collection of fields on $V$ such that
(i) $\left[T, a^{\alpha}(z)\right]=\partial_{z} a^{\alpha}(z)(\alpha \in I)$,
(ii) $T|0\rangle=0,\left.a^{\alpha}(z)|0\rangle\right|_{z=0}=a^{\alpha}(\alpha \in I)$, where $a^{\alpha}$ are linear independent,
(iii) $a^{\alpha}(z)$ and $a^{\beta}(z)$ are mutually local $(\alpha, \beta \in I)$,
(iv) the vectors $a_{\left(-j_{1}-1\right)}^{\alpha_{1}} \cdots a_{\left(-j_{n}-1\right)}^{\alpha_{n}}|0\rangle$ with $j_{k} \geq 0$ span $V$.

Then the formula

$$
Y\left(a_{\left(-j_{1}-1\right)}^{\alpha_{1}} \cdots a_{\left(-j_{n}-1\right)}^{\alpha_{n}}|0\rangle, z\right)=: \partial^{\left(j_{1}\right)} a^{\alpha_{1}}(z) \cdots \partial^{\left(j_{n}\right)} a^{\alpha_{n}}(z):
$$

defines a unique structure of a vertex algebra on $V$ such that $|0\rangle$ is the vacuum vector, $T$ is the infinitesimal translation operator, and

$$
Y\left(a^{\alpha}, z\right)=a^{\alpha}(z), \quad \alpha \in I
$$

As an example, one easily sees that the fermionic Fock space $F$ is a vertex algebra.
9.2. Borcherds OPE formula and Borcherds commutation relations. The Borcherds OPE formula is

$$
\begin{align*}
Y(a, z) Y(b, w) & =\sum_{n \in \mathbb{Z}} i_{z, w} \frac{Y\left(a_{(n)} b, w\right)}{(z-w)^{n+1}} \\
& =\sum_{n \geq 0} i_{z, w} \frac{Y\left(a_{(n)} b, w\right)}{(z-w)^{n+1}}+: Y(a, z) Y(b, w): \tag{88}
\end{align*}
$$

Equivalently,

$$
\begin{equation*}
[Y(a, z), Y(b, w)]=\sum_{j \geq 0} \partial_{w}^{(j)} \delta(z-w) \cdot Y\left(a_{(j)} b, w\right) \tag{89}
\end{equation*}
$$

Comparing the coefficients of $z^{-n-1} w^{-m-1}$ on both sides of (89), one obtains

$$
\begin{equation*}
\left[a_{(n)}, b_{(m)}\right]=\sum_{j \geq 0}\binom{n}{j}\left(a_{(j)} b\right)_{(m+n-j)}, \tag{90}
\end{equation*}
$$

which is a special case of the following Borcherds identity:

$$
\begin{align*}
& \sum_{j \geq 0}\binom{m}{j}\left(a_{(n+j)} b\right)_{(m+k-j)}  \tag{91}\\
= & \sum_{j \geq 0}(-1)^{j}\binom{n}{j} a_{(m+n-j)} b_{(k+j)}-(-1)^{|a||b|} \sum_{j \geq 0}(-1)^{j+n}\binom{n}{j} b_{(n+k-j)} a_{(m+j)} .
\end{align*}
$$

9.3. The quasi-commutativity and quasi-associativity. The OPE has the following "commutativity". If $a(z)$ and $b(w)$ satisfies

$$
a(z) b(w)=\sum_{j=0}^{N-1}\left(i_{z, w} \frac{1}{(z-w)^{j+1}}\right) c^{j}(w)+: a(z) b(w):
$$

then one has (cf. Kac [11], Theorem 2.3):

$$
(-1)^{|a||b|} b(w) a(z)=\sum_{j=0}^{N-1}\left(i_{w, z} \frac{1}{(z-w)^{j+1}}\right) c^{j}(w)+: a(z) b(w):
$$

By abuse of notations,

$$
a(z) b(w) \sim(-1)^{|a||b|} b(w) a(z)
$$

In particular, if all $c^{j}=0$, then we have

$$
: a(z) b(z):=(-1)^{|a||b|}: b(z) a(z): .
$$

Define the $n$-th product of two mutually local fields $a(z)$ and $b(z)$ by:

$$
a(w)_{(n)} b(w)= \begin{cases}\operatorname{Res}_{z}[a(z), b(z)](z-w)^{n}, & n \geq 0 \\ : \partial_{w}^{(-n-1)} a(w) b(w) ;, & n<0\end{cases}
$$

Alternatively (cf. [11], §3.2),
$a(w)_{(n)} b(w)=\operatorname{Res}_{z}\left(a(z) b(w) i_{z, w}(z-w)^{n}-(-1)^{|a||b|} b(w) a(z) i_{w, z}(z-w)^{n}\right)$.
Then for any three mutually local fields $a(z), b(z)$ and $c(z)$ the following Borcherds identity holds:

$$
\begin{aligned}
& \sum_{j=0}^{\infty}\binom{m}{j}\left(a(z)_{(n+j)} b(z)\right)_{(m+k-j)} c(z) \\
= & \sum_{j=0}^{\infty}(-1)^{j}\binom{n}{j}\left(a(z)_{(m+n-j)}\left(b(z)_{(k+j)} c(z)\right)\right. \\
& \left.-(-1)^{n+|a||b|} b(z)_{(n+k-j)}\left(a(z)_{(m+j)} c(z)\right)\right),
\end{aligned}
$$

for any $m, n, k \in \mathbb{Z}$. Now assume that the $n$-th products among $a(z), b(z)$ and $c(z)$ are all zero for $n \geq 0$, then for $m=0, n=k=-1$, we get

$$
\left(a(z)_{(-1)} b(z)\right)_{(-1)} c(z)=a(z)_{(-1)}\left(b(z)_{(-1)} c(z)\right)
$$

Since the $(-1)$-th product is the normally ordered product, in this case, the normally ordered product is associative.

### 9.4. Conformal structures and conformal vertex algebras.

Definition 9.2. Let $V$ be a vertex algebra. A conformal vector of $V$ is an even vector $\nu$ such that $Y(\nu, z)=\sum_{n \in \mathbb{Z}} L_{n} z^{-n-2}$ is a Virasora field of central charge $c$, and has the following properties:
(1) $L_{-1}=T$, and
(2) $L_{0}$ is diagonalizable on $V$.

The field $L(z)=Y(\nu, z)$ is called the energy-momentum field of the algebra $V$. A vertex algebra with a conformal vector is called a conformal vertex algebra of rank $c$, or a vertex operator algebra (VOA).

Example 9.1. By Proposition 7.3 on the fermionic Fock space,

$$
\nu_{\lambda}=(1-\lambda) b_{-3 / 2} c_{-1 / 2}+\lambda c_{-3 / 2} b_{-1 / 2}
$$

is a conformal vector of central charge

$$
c_{\lambda}=-\left(12 \lambda^{2}-12 \lambda+2\right) .
$$

Given a conformal vertex algebra $V$, we say a nonzero vector $a \in V$ has conformal weight $h \in \mathbb{C}$ if $L_{0} a=h a$. (In physics literature, conformal weight is called conformal dimension or scaling dimension or simply dimension.) There is a similar definition for End $V$. We say a field $a(z)=\sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1}$ has conformal weight $h \in \mathbb{C}$ if

$$
\left[L_{0}, a(z)\right]=\left(h+z \partial_{z}\right) a(z)
$$

or equivalently,

$$
\left[L_{0}, a_{(n)}\right]=(h-n-1) a_{(n)}
$$

for $n \in \mathbb{Z}$. In other words, the endomorphism $a_{(n)}$ has conformal weight $h-n-1$, so formally $z$ has conformal weight -1 .

Lemma 9.1. If $a$ is a vector in a conformal vertex algebra that has conformal weight $h$, then the field $Y(a, z)$ has conformal weight $h$. In other words, the statefield correspondence does not change the conformal weight.

Proof. By Borcherds commutation relations (89), we have

$$
\left[L_{m}, Y(a, z)\right]=\sum_{j \geq-1}\binom{m+1}{j+1} Y\left(L_{j} a, z\right) z^{m-j}
$$

In particular, when $j=-1$, we have

$$
\left[L_{-1}, Y(a, z)\right]=Y\left(L_{-1} a, z\right)
$$

Since $L_{-1}=T$, we have

$$
Y\left(L_{-1} a, z\right)(=Y(T a, z))=[T, Y(a, z)]=\partial_{z} Y(a, z)
$$

On the other hand, when $m=0$, we get

$$
\left[L_{0}, Y(a, z)\right]=Y\left(L_{-1} a, z\right) z+Y\left(L_{0} a, z\right)=\left(z \partial_{z}+h\right) Y(a, z)
$$

Remark 9.1. It is useful to write a field $a(z)$ of conformal weight $h$ as

$$
a(z)=\sum_{n \in-h+\mathbb{Z}} a_{n} z^{-n-h}
$$

where each $a_{n}$ has conformal weight $-n$.
Lemma 9.2. Let $a(z)$ and $b(z)$ be two fields in a conformal vertex algebra of conformal weights $h_{1}$ and $h_{2}$ respectively, then : $a(z) b(z):$ has conformal weight $h_{1}+h_{2}$. If

$$
a(z) b(w) \sim \sum_{j=0}^{N-1} \frac{c^{j}(w)}{(z-w)^{j+1}}
$$

then $c^{j}(w)$ has conformal weight $h_{1}+h_{2}-j-1$.
Proof. We use the fact that if two elements $A$ and $B$ in End $V$ have conformal weights $h_{A}$ and $h_{B}$ respectively, then $A B$ has conformal weight $h_{A}+h_{B}$. Then the first statement follows from the definition of the normally ordered product. For the second statement, recall that

$$
c^{j}(w)=\operatorname{Res}_{z}[a(z), b(w)](z-w)^{j}
$$

where $\operatorname{Res}_{z}$ means the coefficient of $z^{-1}$. Now $z$ and $w$ are given conformal weight -1 , then $[a(z), b(w)]$ has conformal weight $h_{1}+h_{2}$, and $(z-w)^{j}$ has conformal weight $-j$. Hence $c^{j}(w)$ has conformal weight $h_{1}+h_{2}-j-1$.

In the same fashion, one can prove the following two Lemmas.
Lemma 9.3. Let $V$ be a vertex algebra with $U(1)$ current $J(z)$. If $a \in I$ has $U(1)$ charge $q$, then so does the field $Y(a, z)$.
Lemma 9.4. Let $V$ be a vertex algebra with $U(1)$ charge, nd let $a(z)$ and $b(z)$ be two fields that have $U(1)$ charges $q_{1}$ and $q_{2}$ respectively. Then : $a(z) b(z)$ : has charge $q_{1}+q_{2}$. If

$$
a(z) b(w) \sim \sum_{j=0}^{N-1} \frac{c^{j}(w)}{(z-w)^{j+1}},
$$

then $c^{j}(w)$ has charge $q_{1}+q_{2}$.
9.5. Primary fields. From the proof of Lemma 9.1, one sees that if $a$ has conformal weight $h$, then we have

$$
Y(\nu, z) Y(a, w) \sim \frac{\partial_{w} Y(a, w)}{z-w}+\frac{h Y(a, w)}{(z-w)^{2}}+\cdots
$$

(In particular, by (82), $\nu$ has conformal weight 2 . That explains why $L(z)$ is written as $\sum_{n \geq \mathbb{Z}} L_{n} z^{-n-2}$ by Remark 9.1.) This motivates the following:

Definition 9.3. A field $a(w)$ is said to be primary of conformal weight $h$ with repect to a Virasoro field $L(z)$ if

$$
\begin{equation*}
L(z) a(w) \sim \frac{\partial_{w} a(w)}{z-w}+\frac{h a(w)}{(z-w)^{2}} \tag{92}
\end{equation*}
$$

Equivalently,

$$
\left[L_{n}, a(z)\right]=\left(z^{n+1} \partial_{z}+h(n+1) z^{n}\right) a(z)
$$

A vector $a$ in a conformal vertex algebra $(V, \nu)$ is said to be primary of conformal weight $h$ if $Y(a, z)$ is a primary field of conformal weight $h$.
9.6. Characters of conformal vertex algebras. Suppose $V$ is a VOA such that the eigenvalues of $L_{0}$ form a countable set $\left\{h_{1}, h_{2}, \ldots\right\}$ on $\mathbb{C}$, and all eigenspaces are finite dimensional. Let

$$
V=\oplus_{n \geq 1} V^{h_{n}}
$$

be the eigenspace decomposition of $L_{0}$ on $V$. Then the character of $V$ is by definition

$$
\operatorname{ch}(V ; q)=q^{-\frac{c}{24}} \operatorname{str} q^{L_{0}}=q^{-\frac{c}{24}} \sum_{n \geq 1} \operatorname{str}\left(\left.\mathrm{id}\right|_{V^{h n}}\right) q^{h_{n}}
$$

where str is the supertrace which is just the ordinary trace on the even subspace, and negative the ordinary trace on the odd subspace. We will use the following auxiliary notation:

$$
G(V ; q)=q^{-\frac{c}{24}} \sum_{n \geq 1} V^{h_{n}} q^{h_{n}}
$$

Then $\operatorname{ch}(V ; q)$ is obtained by taking supertrace of the identity map on $G(V ; q)$ term by term.
9.7. Charged character and index. A $U(1)$ current on a conformal vertex algebra $V$ is a field $J(z)=Y(j, z)$ for some even vector $j \in V$ such that

$$
\begin{aligned}
& J(z) J(w) \sim \frac{\hat{c}}{(z-w)^{2}}, \\
& L(z) J(w) \sim \frac{\partial_{w} J(w)}{z-w}+\frac{J(w)}{(z-w)^{2}}+\frac{d}{(z-w)^{3}},
\end{aligned}
$$

for some numbers $\hat{c}$ and $d$, and $J_{0}$ is diagonizable. The above OPEs are equivalent to the following commutation relations:

$$
\begin{aligned}
& {\left[J_{m}, J_{n}\right]=m \hat{c} \delta_{m,-n}} \\
& {\left[L_{m}, J_{n}\right]=-n J_{m+n}+\frac{m(m+1) d}{2} \delta_{m+n-1,-1}}
\end{aligned}
$$

In particular, $\left[L_{0}, J_{0}\right]=0$ and so $L_{0}$ and $J_{0}$ have common eigenspaces.

Now suppose $V$ is a VOA with a $U(1)$ current. Suppose the eigenvalues of $L_{0}$ and $J_{0}$ are two countable subsets of $\mathbb{C},\left\{h_{1}, h_{2}, \ldots\right\}$ and $\left\{j_{1}, j_{2}, \ldots\right\}$ respectively, and let

$$
V=\oplus_{n, m \geq 1} V^{h_{n}, j_{m}}
$$

are the decomposition of $V$ into common eigenspaces of $L_{0}$ and $J_{0}$. Assume each $V^{h, j_{m}}$ is finite dimensional. Then the character with $U(1)$ charge of a conformal vertex algebra $V$ with $U(1)$ current is defined by

$$
\operatorname{ch}(V ; q, y)=\operatorname{tr} q^{L_{0}-\frac{c}{24}}(-y)^{J_{0}}=\sum_{n, m} \operatorname{tr}\left(\left.\mathrm{id}\right|_{V^{h n, j m}}\right) q^{h_{n}-\frac{c}{24}}(-y)^{j_{m}}
$$

We also write

$$
G(V ; q, y)=\sum_{n, m} V^{h_{n}, j_{m}} q^{h_{n}-\frac{c}{24}}(-y)^{j_{m}} .
$$

When the $\mathbb{Z}$-grading given by eigenspace decomposition of $J_{0}$ coincides with the $\mathbb{Z}$-grading on $V$, one clearly has

$$
\operatorname{ch}(V ; q, 1)=\operatorname{ch}(V ; q)
$$

Example 9.2. The charged character for the fermionic Fock space is

$$
\operatorname{ch}(V ; q, y)=q^{-c_{\lambda} / 24} \prod_{j=1}^{\infty}\left(1-y q^{j+1-\lambda}\right)\left(1-y^{-1} q^{j+\lambda}\right)
$$

with respect to the conformal vector $\nu_{\lambda}$.
9.8. Some modular functions. For use below we collect some functions which often appear in the calculations of the characters of conformal vertex algebras (with $U(1)$ charge).

Dedekind eta function:

$$
\begin{equation*}
\eta(q)=q^{\frac{1}{24}} \prod_{n>0}\left(1-q^{n}\right) \tag{93}
\end{equation*}
$$

Jacobi theta functions:

$$
\begin{align*}
& \theta_{3}(v, \tau)=\prod_{n>0}\left(1-q^{n}\right)\left(1+q^{n-\frac{1}{2}} e^{2 \pi i v}\right)\left(1+q^{n-\frac{1}{2}} e^{-2 \pi i v}\right),  \tag{94}\\
& \theta_{2}(v, \tau)=\prod_{n>0}\left(1-q^{n}\right)\left(1-q^{n-\frac{1}{2}} e^{2 \pi i v}\right)\left(1-q^{n-\frac{1}{2}} e^{-2 \pi i v}\right),  \tag{95}\\
& \theta_{1}(v, \tau)=q^{\frac{1}{8}} e^{2 \pi i v} \prod_{n>0}\left(1-q^{n}\right)\left(1+q^{n} e^{2 \pi i v}\right)\left(1+q^{n-1} e^{-2 \pi i v}\right),  \tag{96}\\
& \theta(v, \tau)=q^{\frac{1}{8}} \cdot 2 \sin \pi v \prod_{n>0}\left(1-q^{n}\right)\left(1-q^{n} e^{2 \pi i v}\right)\left(1-q^{n} e^{-2 \pi i v}\right), \tag{97}
\end{align*}
$$

where $q=e^{\pi i \tau}$. By abuse of notations, we write:

$$
\begin{align*}
& \theta_{3}(q)=\theta_{3}(0, \tau)=\prod_{n>0}\left(1-q^{n}\right)\left(1+q^{n-\frac{1}{2}}\right)^{2}  \tag{98}\\
& \theta_{2}(q)=\theta_{2}(0, \tau)=\prod_{n>0}\left(1-q^{n}\right)\left(1-q^{n-\frac{1}{2}}\right)^{2}  \tag{99}\\
& \theta_{1}(q)=\theta_{1}(0, \tau)=2 q^{\frac{1}{8}} \prod_{n>0}\left(1-q^{n}\right)\left(1+q^{n}\right)^{2} . \tag{100}
\end{align*}
$$

## 10. Some Constructions of Vertex Algebras

10.1. Highest weight representations and vertex algebras. Many vertex algebras are constructed from highest weight representations of infinite dimensional Lie algebras. We collect some of them in this section.

Suppose $\mathfrak{g}$ is a complex Lie algebra with symmetric bilinear form $(\cdot \mid \cdot)$, invariant in the sense that

$$
([a, b] \mid c)+(b \mid[a, c])=0, \quad a, b, c \in \mathfrak{g}
$$

The affinization of $(\mathfrak{g},(\cdot \mid \cdot))$ is the Lie algebra

$$
\hat{\mathfrak{g}}=\mathbb{C}\left[t, t^{-1}\right] \otimes \mathbb{C} \mathfrak{g} \oplus \mathbb{C} K
$$

with the following nonzero commutation relations:

$$
\left[a_{m}, b_{n}\right]=[a, b]_{m+n}+m(a \mid b) \delta_{m,-n} K,
$$

where for each $a \in \mathfrak{g}$ and $n \in \mathbb{Z}, a_{m}$ stands for $t^{m} \otimes a$. This is called the current algebra.
Example 10.1. (1) When $\mathfrak{g}$ is a simple Lie algebra with the Killing form, then $\hat{\mathfrak{g}}$ is the affine Kac-Moody algebra.
(2) When $\mathfrak{g}$ is the one-dimensional Lie algebra with a nondegenerate bilinear form, then $\hat{\mathfrak{g}}$ is the oscillator algebra.

Consider the decomposition

$$
\hat{\mathfrak{g}}=\hat{\mathfrak{g}}^{+} \oplus \hat{\mathfrak{g}}^{0} \oplus \hat{\mathfrak{g}}^{-}
$$

where $\hat{\mathfrak{g}}^{+}=t \mathfrak{g}[t], \hat{\mathfrak{g}}^{-}=t^{-1} \mathfrak{g}\left[t^{-1}\right]$, and $\hat{\mathfrak{g}}^{0}=\mathfrak{g} \oplus \mathbb{C} K$. It is easy to see that $\hat{\mathfrak{g}}^{\geq 0}=$ $\mathfrak{g}[t] \oplus \mathbb{C} K$, is a Lie sublalgebra of $\hat{\mathfrak{g}}$. Suppose $\pi: \hat{\mathfrak{g}}^{\geq} \rightarrow \operatorname{End} W$ is a representation. The induced $\hat{\mathfrak{g}}$-module is defined by

$$
\operatorname{Ind}_{\hat{\mathfrak{g}} \geq}^{\hat{\mathfrak{g}}}=U(\hat{\mathfrak{g}}) \otimes_{U(\hat{\mathfrak{g}} \geq)} W .
$$

It is easy to see that

$$
\operatorname{Ind}_{\mathfrak{\mathfrak { g }} \geq}^{\mathfrak{\hat { y }}} \cong S\left(\hat{\mathfrak{g}}^{-}\right) W
$$

as vector spaces. Each $a_{n}$ acts on $\operatorname{Ind}_{\hat{\mathfrak{g}} \geq}^{\hat{\mathfrak{g}}} \geq$. It is easy to see that

$$
a(z)=\sum_{n \in \mathbb{Z}} a_{n} z^{-n-1}
$$

is a field on $\operatorname{Ind} d_{\hat{\mathfrak{g}} \geq}^{\hat{\mathfrak{g}}}$. Furthermore,

$$
a(z) b(w) \sim \frac{[a, b](w)}{z-w}+\frac{(a \mid b) k}{(z-w)^{2}}
$$

To get a vertex algebra, one considers the following special induced module. Let $\pi$ be the one-dimensional representation such that $\mathfrak{g}$ acts as zero operator and $h$ acts as multiplication by a constant $k$. Denote by $\tilde{V}^{k}(\mathfrak{g})$ the induced representation. Then as a vector space, $\tilde{V}^{k}(\mathfrak{g})$ is spanned by elements of the form

$$
a_{I}^{J}=a_{-i_{1}-1}^{j_{1}} \cdots a_{-i_{n}-1}^{j_{n}},
$$

where $a^{j_{1}}, \ldots, a^{j_{n}} \in \mathfrak{g}, i_{1}, \ldots, i_{n} \geq 0$. Then $\tilde{V}^{k}(\mathfrak{g})$ is a vertex algebra with

$$
Y\left(a_{I}^{J}, z\right)=: \partial^{\left(i_{1}\right)} a^{j_{1}}(z) \cdots \partial^{\left(i_{n}\right)} a^{j_{n}}(z): .
$$

Example 10.2. (Free bosons) Let $T$ be a finite dimensional vector space with a symmetric bilinear form $(\cdot \mid \cdot)$. One can regard $T$ as an abelian Lie algebra, and obtain the vertex algebra $\tilde{V}^{k}(T)$ defined above. In particular, when $T$ is onedimensional, one recovers the free bosonic vertex algebra discussed in $\S 2$.
Remark 10.1. The above construction can be generalized to Lie superalgebras and superaffinization. The free fermionic vertex algebra in $\S 1$ can be realized as a special case.
Remark 10.2. One can also replace $\hat{\mathfrak{g}}$ by Virasoro algebra and its various generalizations, such as $W_{1+\infty}$ algebra, Neveu-Schwarz algebra, etc.
10.2. Sugawara construction. Let $\mathfrak{g}$ be a finite dimensional simple or commutative Lie algebra, with an invariant symmetric bilinear form $(\cdot \mid \cdot)$. Suppose $\left\{a^{i}\right\}$ and $\left\{b^{i}\right\}$ are dual basis of $\mathfrak{g}$ with respect to $(\cdot \mid \cdot)$, i.e.,

$$
\left(a^{i} \mid b^{j}\right)=\delta_{i j}
$$

It is well-known that

$$
\sum_{i}\left[a^{i},\left[b^{i}, X\right]\right]=2 h^{\vee} X
$$

for all $X \in \mathfrak{g}$, where $h^{\vee}$ is called the dual Coxeter number of $\mathfrak{g}$. Then for $k \neq-h^{\vee}$, the vector

$$
\nu=\frac{1}{2\left(k+h^{\vee}\right)} \sum_{i} a_{-1}^{i} b_{-1}^{i}
$$

is a conformal vector of the vertex algebra $\tilde{V}^{k}(\mathfrak{g})$ with central charge

$$
c_{k}=\frac{k \operatorname{dim} \mathfrak{g}}{k+h^{\vee}}
$$

Furthermore, all fields $a(z), a \in \mathfrak{g}$, are primary of conformal weight 1 with respect to $Y(\nu, z)$. See e.g. [11], Theorem 5.7.
10.3. Coset models. Given a subspace $U$ of a vertex algebra $V$, its centralizer

$$
\{b \in V \mid[Y(a, z), Y(b, w)]=0, \forall b \in U\}
$$

is a subalgebra of $V$, called by physicists a coset model.
For example, let $(\mathfrak{g},(\cdot \mid \cdot))$ be as above. Suppose $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$ which is itself a direct sum of simple of commutative Lie algebras, $\left.(\cdot \mid \cdot)\right|_{\mathfrak{h}}$ is nondegenerate. The fields $h(z)$ with $h \in f h$ generates a subalgebra of $\tilde{V}^{k}(\mathfrak{g})$. Then its centralizer is a vertex algebra with

$$
\nu=\nu=\nu_{\mathfrak{g}}-\nu_{\mathfrak{h}}
$$

as a conformal vector with central charge the difference between the central charges of $\nu_{\mathfrak{g}}$ and $\nu_{\mathfrak{h}}$. This is called the Goddard-Kent-Olive construction.
10.4. Cohomological models. An ideal of a vertex algebra $V$ is a $T$-invariant subspace not containing $|0\rangle$, such that

$$
a_{(n)} J \subset J, \forall a \in V
$$

By quasi-symmetry

$$
Y(a, z) v=(-1)^{|a||v|} e^{z T} Y(v,-z) a
$$

hence

$$
a_{(n)} V \subset J, \forall a \in J
$$

Hence the quotient space $V / J$ has an induced structure of a vertex algebra.
A derivation of degree $k$ on a vertex algebra $V$ is a linear map $\delta: V \rightarrow V$ of degree $k$ such that

$$
\delta\left(a_{(n)} b\right)=(\delta a)_{(n)} b+(-1)^{k|a|} a_{(n)} \delta b
$$

for all $a, b \in V, n \in \mathbb{Z}$. A derivation $\delta$ of degree 1 is a differential if $\delta^{2}=0$. As usual, the cohomology of a differential is by definition

$$
H(V, \delta)=\operatorname{ker} \delta / \operatorname{Im} \delta
$$

It is easy to prove the following:
Lemma 10.1. Suppose $\delta$ is a differential on a vertex algebra $V$. Then $\operatorname{ker} \delta$ is a subalgebra of $V$, and $\operatorname{Im} \delta$ is an ideal ker $\delta$. Hence $H(V, \delta)$ has an induced structure of a vertex algebra.

## 11. $N=2$ Superconformal Vertex Algebras

11.1. $N=2$ SCVA. An $N=2$ superconformal vertex algebra (SCVA) is a vertex algebra $V$ with two odd vectors $\tau^{ \pm}$and two even vectors $\nu$ and $j$, such that the fields $G^{ \pm}(z)=Y\left(\tau^{ \pm}, z\right)$ and $J(z)=Y(j, z)$ satisfy the following OPE's:

$$
\begin{aligned}
& G^{+}(z) G^{-}(w) \sim \frac{\frac{1}{3} c}{(z-w)^{3}}+\frac{J(w)}{(z-w)^{2}}+\frac{L(w)+\frac{1}{2} \partial_{w} J(w)}{z-w}, \\
& G^{ \pm}(z) G^{ \pm}(w) \sim 0, \\
& J(z) J(w) \sim \frac{\frac{1}{3} c}{(z-w)^{3}}, \\
& J(z) G^{ \pm}(w) \sim \pm \frac{G^{ \pm}(w)}{z-w}, \\
& L(z) L(w) \sim \frac{\partial_{w} L(w)}{z-w}+\frac{2 L(w)}{(z-w)^{2}}+\frac{\frac{1}{2} c}{(z-w)^{4}} \\
& L(z) G^{ \pm}(w) \sim \frac{\frac{3}{2} G^{ \pm}(w)}{(z-w)^{2}}+\frac{\partial_{w} G^{ \pm}(w)}{z-w} \\
& L(z) J(w) \sim \frac{J(w)}{(z-w)^{2}}+\frac{\partial_{w} J(w)}{z-w}
\end{aligned}
$$

with $L_{-1}\left(=\left[G_{-\frac{1}{2}}^{+}, G_{-\frac{1}{2}}^{-}\right]\right)=T$, and $L_{0}$ and $J_{0}$ are diagonizable.
11.2. Primary fields in an $N=2$ superconformal vertex algebra. We now generalize Definition 9.3 to $N=2$ SCVA. By Borcherds formula (88),

$$
\begin{aligned}
L(z) Y(a, w) & \sim \frac{\partial_{w} Y(a, w)}{z-w}+\frac{Y\left(L_{0} a, w\right)}{(z-w)^{2}}+\cdots \\
J(z) Y(a, w) & \sim \frac{Y\left(J_{0} a, w\right)}{z-w}+\cdots \\
G^{ \pm}(z) Y(a, w) & \sim \frac{Y\left(G_{-\frac{1}{2}}^{ \pm} a, w\right)}{z-w}+\cdots
\end{aligned}
$$

Definition 11.1. A state $a$ in an $N=2$ superconformal vertex algebra is called primary of conformal weight $h$ and $U(1)$ charge $q$ if it satisfies

$$
\begin{aligned}
L(z) Y(a, w) & \sim \frac{\partial_{w} Y(a, w)}{z-w}+\frac{h Y(a, w)}{(z-w)^{2}} \\
J(z) Y(a, w) & \sim \frac{q Y(a, w)}{z-w}, \\
G^{ \pm}(z) Y(a, w) & \sim \frac{Y\left(G_{-\frac{1}{2}}^{ \pm} a, w\right)}{z-w} .
\end{aligned}
$$

Similarly, a field $a(z)$ is called primary of conformal weight $h$ and $U(1)$ charge $q$ if it satisifies

$$
\begin{aligned}
& L(z) a(w) \sim \frac{\partial_{w} a(w)}{z-w}+\frac{h a(w)}{(z-w)^{2}}, \\
& J(z) a(w) \sim \frac{q a(w)}{z-w}, \\
& G^{ \pm}(z) a(w) \sim \frac{\left[G_{-\frac{1}{2}}^{ \pm}, a(w)\right]}{z-w} .
\end{aligned}
$$

By Borcherds formula (88), a state $a$ is primary of conformal weight $h$ and $U(1)$ charge $q$ iff

$$
\begin{aligned}
& L_{n} a=J_{n} a=0, \quad n \geq 1, \\
& L_{0} a=h a, \quad J_{0} a=q a, \\
& G_{r}^{ \pm} a=0, \quad r \geq \frac{1}{2} .
\end{aligned}
$$

11.3. Primary chiral algebra of an $N=2$ SCFT. We now recall some important results from Lerche-Vafa-Warner [14]. In an $N=2$ SCVA, a state $a$ is called chiral if it satisfies:

$$
G_{-\frac{1}{2}}^{+} a=0
$$

Similarly, a field $a(z)$ is called chiral if it satisfies:

$$
\left[G_{-\frac{1}{2}}^{+}, a(z)\right]=0
$$

Anti-chiral states and fields are defined with + replaced by - .
Definition 11.2. An $N=2$ SCVA $V$ is said to be unitary if there a positive definite Hermitian metric $\langle\cdot \mid \cdot\rangle$ on $V$ such that $\left(G_{r}^{+}\right)^{*}=G_{-r}^{-}$.

Lemma 11.1. Let $(V,\langle\cdot \mid \cdot\rangle)$ be a unitary $N=2 S C V A$, then we have

$$
L_{n}^{*}=L_{-n}, \quad J_{n}^{*}=J_{-n}
$$

Definition 11.3. An $N=2$ SCVA is said to be nondegenerate if it is nondegenerate as a conformal vertex algebra.

Lemma 11.2. In a unitary $N=2 S C V A V$, if $a$ is $a$ vector of conformal weight $h$ and $U(1)$ charge $q$, then

$$
h \geq|q| / 2
$$

with $h=q / 2$ (resp. $h=-q / 2$ ) iff $a$ is a primary chiral (resp. anti-chiral) state.

Proof. Applying the commutation relation

$$
\left[G_{\frac{1}{2}}^{-}, G_{-\frac{1}{2}}^{+}\right]=2 L_{0}-J_{0}
$$

to $a$ and then taking inner product with $a$, one gets

$$
\left|G_{-\frac{1}{2}}^{+} a\right|^{2}+\left|G_{\frac{1}{2}}^{-} a\right|^{2}=(2 h-q)|a|^{2}
$$

Here we have used the fact that $G_{\frac{1}{2}}^{-}=\left(G_{-\frac{1}{2}}^{+}\right)^{\dagger}$. Therefore, $h \geq q / 2$. Now equality holds iff

$$
\begin{equation*}
G_{-\frac{1}{2}}^{+} a=G_{\frac{1}{2}}^{-} a=0 \tag{101}
\end{equation*}
$$

So if $a$ is primary and chiral, then (101) holds and hence $h=q / 2$. Conversely, assume that $h=q / 4$ and hence (101) holds. Since $G^{ \pm}(z)$ has conformal weight $\frac{3}{2}$ and $U(1)$ charge $\pm 1, G_{r}^{ \pm} a$ has conformal weight $h-r$ and $U(1)$ charge $q \pm 1$ by conformal weight conservation (Lemma 9.2) and $U(1)$ charge conservation (Lemma 9.4). If $G_{r}^{+} a \neq 0$ for $r \geq \frac{1}{2}$ or $G_{r}^{-} a \neq 0$ for $r \geq \frac{3}{2}$, then we would have $h-r \geq \frac{1}{2}(q \pm 1)$ and hence $r \leq \pm \frac{1}{2}$, a contradiction.

The other half of the lemma can be proved in the same fashion by using

$$
\left[G_{\frac{1}{2}}^{+}, G_{-\frac{1}{2}}^{-}\right]=2 L_{0}+J_{0}
$$

Corollary 11.1. In a unitary $N=2 S C V A V$, the primary chiral states form an graded commuative associative algebra induced by the normally ordered product: the product between states $a$ and $b$ is given by $a_{(-1)} b$. Similarly for the primary anti-chiral states. (These algebras will be refered to as the primary chiral algebra and primary anti-chiral algebra of $V$ respectively.)
Proof. Let $a$ and $b$ be two primary chiral states, then $h_{a}=q_{a} / 2$ and $h_{b}=q_{b} / 2$. Let

$$
a(z) b(w)=\sum_{j=0}^{N} \frac{c^{j}(w)}{(z-w)^{j+1}}+: a(z) b(w):
$$

for some fields $c^{j}$. Then by Lemma 9.2 and Lemma 9.4, we have

$$
h\left(c^{j}\right)=h_{a}+h_{b}-j-1, \quad q\left(c^{j}\right)=q_{a}+q_{b}
$$

By Lemma 11.2, $h\left(c^{j}\right) \geq q\left(c^{j}\right)$, hence $c^{j}=0$. Hence by Borcherds identity, the set of fields $\{a(z) \mid a$ is primary chiral $\}$ is graded commutative and associative with identity under the normally ordered product (cf. §9.3). Recall that

$$
: a(z) b(z):=: Y(a, z) Y(b, z):=Y\left(a_{(-1) b}, z\right),
$$

hence the set of primary chiral states fowm a graded commutative asssociative algebra with identity. The proof for primary anti-chiral states is similar.

Lemma 11.3. Let $V$ be a unitary $N=2 S C V A$ of central charge $c$. If $a \in V$ is a primary chiral state of conformal weight $h$ and $U(1)$ charge $q$, then one has

$$
h \leq c / 6
$$

Proof. From the commutation relation:

$$
\left[G_{\frac{3}{2}}^{-}, G_{-\frac{3}{2}}^{+}\right]=2 L_{0}-3 J_{0}+2 c / 3
$$

one see that

$$
2 h-3 q+2 c / 3 \geq 0
$$

Now use the fact that $q=2 h$ for primary chiral states.
Corollary 11.2. The dimension of the space of primary chiral states in a nondegenerate unitary $N=2 S C V A$ is finite.
11.4. Topological vertex algebras and BRST cohomology. Closely related to $N=2$ SCVA are the toplogical vertex algebras. Recall thpt a toplogical vertex algebra of rank $d$ is a conformal vertex algebra of central charge 0 , equipped with an even element $J$ of conformal weight 1 , an odd element $Q$ of conformal weight 1 , and an odd element $G$ of conformal weight 2 , such that their fields satisfy the following OPE's:

$$
\begin{align*}
& T(z) T(w) \sim \frac{2 T(w)}{(z-w)^{2}}+\frac{\partial_{W} T(w)}{z-w},  \tag{102}\\
& J(z) J(w) \sim \frac{d}{(z-w)^{2}},  \tag{103}\\
& T(z) J(w) \sim-\frac{d}{(z-w)^{3}}+\frac{J(w)}{(z-w)^{2}}+\frac{\partial_{w} J(w)}{z-w},  \tag{104}\\
& G(z) G(w) \sim 0,  \tag{105}\\
& T(z) G(w) \sim \frac{2 G(w)}{(z-w)^{2}}+\frac{\partial_{w} G(w)}{z-w},  \tag{106}\\
& J(z) G(w) \sim-\frac{G(w)}{z-w},  \tag{107}\\
& Q(z) Q(w) \sim 0,  \tag{108}\\
& T(z) Q(w) \sim \frac{Q(w)}{(z-w)^{2}}+\frac{\partial_{w} Q(w)}{z-w},  \tag{109}\\
& J(z) Q(w) \sim \frac{Q(w)}{z-w},  \tag{110}\\
& Q(z) G(w) \sim \frac{d}{(z-w)^{3}}+\frac{J(w)}{(z-w)^{2}}+\frac{T(w)}{z-w} . \tag{111}
\end{align*}
$$

Here we have written the Virasor field associated with the conformal vector as $T(z)=\sum_{n \in \mathbb{Z}} T_{n} z^{-n-2}$. As a consequence of (108),

$$
Q_{0}^{2}=\frac{1}{2}\left[Q_{0}, Q_{0}\right]=0
$$

The operator $Q_{0}$ is called the $B R S T$ operator, and the cohomology

$$
H^{*}\left(V, Q_{0}\right)=\operatorname{Ker} Q_{0} / \operatorname{Im} Q_{0}
$$

is called the BRST cohomology. From (108), one gets

$$
T(z)=\left[Q_{0}, G(z)\right]
$$

Hence for any $v \in \operatorname{Ker} Q_{0}$, we have

$$
T_{n} v=\left[Q_{0}, G_{n}\right] v=Q_{0} G_{n} v \in \operatorname{Im} Q_{0}
$$

In other words, the map $[v] \mapsto\left[T_{n} v\right]$ induces a trivial representation of the Virasoro algebra on the BRST cohomology.

Inspired by Witten [24, 25], Eguchi and Yang [5] discovered the following important twisting construction:

Proposition 11.1. Given an $N=2 S C V A V$ with Virasoro field $L(z)$, supercurrents $G^{ \pm}(z)$ and $U(1)$ current $J(z)$, one obtains a topological vertex algebra by taking:

$$
\begin{array}{ll}
T(z)=L(z)+\frac{1}{2} \partial_{z} J(z), & J_{\text {top }}(z)=J(z) \\
Q(z)=G^{+}(z), & G(z)=G^{-}(z)
\end{array}
$$

or

$$
\begin{array}{ll}
T(z)=L(z)-\frac{1}{2} \partial_{z} J(z), & J_{t o p}(z)=-J(z), \\
Q(z)=G^{-}(z), & G(z)=G^{+}(z) .
\end{array}
$$

Conversely, given a topological vertex algebra, one can obtain an $N=2$ SCVA structure on it by

$$
\begin{array}{ll}
L(z)=T(z)-\frac{1}{2} \partial_{z} J_{t o p}(z), & J(z)=J_{t o p}(z) \\
G^{+}(z)=Q(z), & G^{-}(z)=G(z)
\end{array}
$$

or

$$
\begin{array}{ll}
L(z)=T(z)+\frac{1}{2} \partial_{z} J_{t o p}(z), & J(z)=-J_{t o p}(z) \\
G^{+}(z)=G(z), & G^{-}(z)=Q(z)
\end{array}
$$

In the above, we have used $J_{\text {top }}$ to dentoe the $U(1)$ charge for the topological vertex algebra.

Definition 11.4. The two twists in Proposition 11.1 will be referered to as the $A$ twist and the $B$ twist respectively.

As remarked in Lian-Zuckerman [15], §3.9.4, the BRST cohomology of a topological vertex algebra is graded commuatative and associative. The following results from Lerche-Vafa-Warner [14] provide an alternative explanation.

Lemma 11.4. (Hodge decomposition) In a unitary $N=2 S C V A V$ of central charge $c$, any state of conformal weight $h$ and $U(1)$ charge $q$ can be uniquely written as

$$
a=a_{0}+G_{-\frac{1}{2}}^{+} a_{+}+G_{\frac{1}{2}}^{-} a_{-},
$$

for some primary chiral state $a_{0}$ and some states $a_{+}$and $a_{-}$. Furthermore, when $a$ is chiral, then one can take $a_{-}=0$.
Proposition 11.2. For a nondegenerate unitary $N=2 S C V A$, the primary chiral (resp. anti-chiral) algebra is isomorphic to the BRST cohomology of the A twist (resp. the $B$ twist).
11.5. Elliptic genus. Let $V$ be an $N=2$ SCVA, then $V$ is a VOA with a $U(1)$ current. Its character with $U(1)$ charge (if it can be defined) is called the elliptic genus of $V$ in physics literature.

## 12. Free Bosons from a Vector Space with Inner Product

In this and the next few sections we recall the constructions of vertex algebras from a vector space with an inner product.
12.1. Vector spaces with inner products. Let $T$ be a finite dimensional real (or complex) vector space with an inner product $g: T \otimes T \rightarrow \mathbb{R}$ (or $\mathbb{C}$ ). In other words,

$$
g(a, b)=g(b, a)
$$

for $a, b \in T$, and if $a \neq 0$, then $g(a, \cdot): T \rightarrow \mathbb{R}$ (or $\mathbb{C}$ ) is a nontrivial linear function on $T$. Denote by $O(T, g)$ the group of $\mathbb{R}$ (or $\mathbb{C}$ )-linear transformations of $T$ that preserve $g$.

A polarization of a vector space $T$ with an inner product $g$ is a decomposition $T=T^{\prime} \oplus T^{\prime \prime}$, such that

$$
g\left(a_{1}, a_{2}\right)=g\left(b_{1}, b_{2}\right)=0,
$$

for $a_{1}, a_{2} \in T^{\prime}, b_{1}, b_{2} \in T^{\prime \prime}$. Given a polorization $T=T^{\prime} \oplus T^{\prime \prime}$ of $(T, g)$, it is easy to see that $g$ induces an isomorphism $T^{\prime \prime} \cong\left(T^{\prime}\right)^{*}$. (In particular, this implies that a vector space with an inner product is even dimensional if it admits a polarization.) Conversely, let $W$ be a finite dimensional real or complex vector space. Denote by $W^{*}$ the dual space of $W$. Introduce an inner product on $W \oplus W^{*}$ as follows:

$$
g(a, b)=g(b, a)=b(a), \quad g\left(a_{1}, a_{2}\right)=g\left(b_{1}, b_{2}\right)=0
$$

where $a, a_{1}, a_{2} \in W, b, b_{1}, b_{2} \in W^{*}$. (We refer to $g$ as the canonical inner product on $W \oplus W^{*}$.) Then $W$ and $W^{*}$ give a polarization of $W \oplus W^{*}$ with respect to the above inner product.

Let $T_{\mathbb{R}}$ be a real vector space with a real inner product $g_{\mathbb{R}}$ and an almost complex structure $J: T_{\mathbb{R}} \rightarrow T_{\mathbb{R}}$ (i.e. $J^{2}=-\mathrm{id}$ ) such that

$$
g_{\mathbb{R}}(J v, J w)=g_{\mathbb{R}}(v, w)
$$

or $v, w \in V_{\mathbb{R}}$. Let $T=T_{\mathbb{R}} \otimes \mathbb{C}$ and $g=g_{\mathbb{R}} \otimes \mathbb{C}$. Then $(T, g)$ has a polarization given by

$$
T^{\prime}=\left\{\left.\frac{1}{2}(v-\sqrt{-1} J v) \right\rvert\, v \in T\right\}, \quad T^{\prime \prime}=\left\{\left.\frac{1}{2}(v+\sqrt{-1} J v) \right\rvert\, v \in T\right\} .
$$

Reagrd $(T, J)$ as a complex vector space $T_{c}$. Then we have

$$
T^{\prime} \cong T_{c}, \quad T^{\prime \prime} \cong \overline{T_{c}}
$$

Denote by $U(T, J, g)$ the group of $\mathbb{R}$-linear transformations of $T_{\mathbb{R}}$ that preserve both $J$ and $g_{\mathbb{R}}$. Then the induced action of $U(T, J, g)$ on $T$ preserves the polarization.
12.2. Free bosons from vector spaces with inner products. Consider the Lie algebras

$$
H(T, g)=\widehat{T}=T\left[t, t^{-1}\right] \oplus \mathbb{C} K
$$

with commutation relations

$$
\begin{equation*}
\left[a_{m}, b_{n}\right]=m g(a, b) \delta_{m,-n} K, \quad[K, \widehat{T}]=0 \tag{112}
\end{equation*}
$$

$a, b \in T$, where $a_{m}$ stands for $a t^{m}$.
Let $V$ be a representation of the Lie algebra $H(T, g)$, such that

$$
a(z)=\sum_{n \in \mathbb{Z}} a_{n} z^{-n-1}
$$

is a field for any $a \in T$. (Such a representation is called a field representation.) Then $\{a(z)\}$ is a collection of mutually local fields with OPE:

$$
\begin{equation*}
a(z) b(w) \sim \frac{k g(a, b)}{(z-w)^{2}} . \tag{113}
\end{equation*}
$$

There is a well-known field representation of $H(T, g)$ on

$$
B^{k}(T, g)=S\left(\oplus_{n<0} t^{n} T\right)=\otimes_{n>0} S\left(t^{-n} T\right)
$$

More precisely, $K$ acts as multiplication by $k$, and for any element $a \in T, a_{n}$ acts as symmetric product by $a_{n}$ if $n<0$, and as $k n$ times the contraction by $a_{-n}$ if $n \geq 0$. Denote the element 1 in $B^{k}(T, g)$ by $|0\rangle$, then $B^{k}(T, g)_{R}$ is spanned by elements of the form:

$$
a_{-j_{1}-1}^{1} \cdots a_{-j_{m}-1}^{m}=a_{-j_{1}-1}^{1} \cdots a_{-j_{m}-1}^{m}|0\rangle
$$

where $a^{1}, \cdots, a^{m} \in T$ and $j_{1}, \cdots, j_{m} \geq 0$. The space $B^{k}(T, g)$ is called the (bosonic) Fock space, where $a_{n}$ is called a creation operator if $n<0$ and an annihilation operator if $n \geq 0$.

Proposition 12.1. There is a structure of vertex algebra on $B^{k}(T, g)$ defined by $Y(|0\rangle, z)=$ id and

$$
Y\left(a_{-j_{1}-1}^{1} \cdots a_{-j_{m}-1}^{m}, z\right)=: \partial^{\left(j_{1}\right)} a^{1}(z) \cdots \partial^{\left(j_{m}\right)} a^{m}(z):
$$

for $a^{1}, \cdots, a^{m} \in T$ and $j_{1}, \cdots, j_{m} \geq 0$. Let $\left\{b^{i}\right\}$ and $\left\{c^{j}\right\}$ be two bases of $T$ such that $g\left(b^{i}, c^{j}\right)=\delta_{i j}$. Assume that $k \neq 0$. Then

$$
\nu=\frac{1}{2 k} \sum_{i} b_{-1}^{i} c_{-1}^{i}
$$

is a conformal vector of central charge $\operatorname{dim} T$.
Proof. We verify the axioms for a vertex algebra. First, $Y(|0\rangle, z)=$ id by definition. It is obvious that

$$
\begin{aligned}
& Y\left(a_{-j_{1}-1}^{1} \cdots a_{-j_{m}-1}^{m}, z\right)|0\rangle=: Y\left(a_{-j_{1}-1}^{1}, z\right) \cdots Y\left(a_{-j_{m}-1}^{m}, z\right):|0\rangle \\
= & Y\left(a_{-j_{1}-1}^{1}, z\right)_{+} \cdots Y\left(a_{-j_{m}-1}^{m}, z\right)_{+}|0\rangle \\
= & \left(a_{-j_{1}-1}^{1}+z\left(j_{1}+1\right) a_{-j_{1}-2}^{1}+o\left(z^{2}\right)\right) \cdots\left(a_{-j_{m}-1}^{m}+z\left(j_{m}+1\right) a_{-j_{m}-2}^{1}+o\left(z^{2}\right)\right)|0\rangle \\
= & a_{-j_{1}-1}^{1} \cdots a_{-j_{m}-1}^{m}+z \sum_{1 \leq k \leq m}\left(j_{k}+1\right) a_{-j_{1}-1}^{1} \cdots a_{j_{k}-2}^{k} \cdots a_{-j_{m}-1}^{m}+o\left(z^{0}\right) .
\end{aligned}
$$

The degree 0 terms gives

$$
\left.Y\left(a_{-j_{1}-1}^{1} \cdots a_{-j_{m}-1}^{m}, z\right)|0\rangle\right|_{z=0}=a_{-j_{1}-1}^{1} \cdots a_{-j_{m}-1}^{m}
$$

while the first degree term is

$$
T\left(a_{-j_{1}-1}^{1} \cdots a_{-j_{m}-1}^{m}\right)=\sum_{1 \leq k \leq m}\left(j_{k}+1\right) a_{-j_{1}-1}^{1} \cdots a_{j_{k}-2}^{k} \cdots a_{-j_{m}-1}^{m} .
$$

The translation invariance can be reduced to the condition that

$$
\left[T, a_{-n}\right]=n a_{-n-1}
$$

for $n \in \mathbb{Z}$, which can be checked directly. Differentiating (113), we get

$$
\begin{equation*}
a_{-m-1}(z) b_{-n-1}(w) \sim(-1)^{m} \frac{(m+n+1)!}{m!n!} \frac{k g(a, b)}{(z-w)^{m+n+2}}, \tag{114}
\end{equation*}
$$

where $a_{-m-1}(z)=Y\left(a_{-m-1}, z\right)$. Hence we can apply Wick's theorem to the collection $\left\{a_{-m-1}(z) \mid a \in T, m \in \mathbb{Z}\right\}$ to establish the locality axiom, and prove the last statement on conformal vectors (cf. Kac [11], pp. 96-96).

Remark 12.1. It is clear that $\nu$ is independent of the choice of the basis. Furthermore, $O(T, g)$ acts on $B^{k}(T, g)_{R}$ as automorphisms of vertex algebra, and $\nu$ is preserved by this action.
12.2.1. Conformal Weight decomposition. By Wick's theorem, one gets

$$
\begin{equation*}
L(z) Y\left(b_{-1}, w\right) \sim \frac{\partial_{w} Y\left(b_{-1}, w\right)}{z-w}+\frac{Y\left(b_{-1}, w\right)}{(z-w)^{2}} \tag{115}
\end{equation*}
$$

So $b_{-1}$ is primary of conformal weight 1 . It follows that $a_{-j_{1}-1}^{1} \cdots a_{-j_{m}-1}^{m}$ has conformal weight

$$
\sum_{i=1}^{m}\left(j_{i}+1\right) .
$$

It is then straightforward to get:

$$
\begin{align*}
G_{q}\left(B^{k}(T, g)\right) & =q^{-\operatorname{dim} T / 24} \otimes_{n>0} S_{q^{n}} T  \tag{116}\\
\operatorname{ch}_{q}\left(B^{k}(T, g)\right) & =\frac{1}{\eta(q)^{\operatorname{dim} T}} . \tag{117}
\end{align*}
$$

## 13. Free fermions from vector spaces with inner products

13.1. Infinite dimensional Clifford algebras. Let $T$ be finite dimensional complex vector space with an inner product $g(\cdot, \cdot)$. Denote by $\widehat{T}_{N S}$ and $\widehat{T}_{R}$ the Lie superalgebras with even parts $\mathbb{C} K$, and odd parts $\oplus_{r \in \frac{1}{2}+\mathbb{Z}} \mathbb{C} \varphi_{r}$ and $\oplus_{r \in \mathbb{Z}} \mathbb{C} \varphi_{r}$ respectively, which satisfy the commutation relations:

$$
\begin{equation*}
\left[\varphi_{r}, \psi_{s}\right]=g(\varphi, \psi) \delta_{r,-s} K, \quad\left[K, \varphi_{r}\right]=0 \tag{118}
\end{equation*}
$$

for $\varphi, \psi \in T, r, s \in \frac{1}{2}+\mathbb{Z}$ for $\widehat{T}_{N S}$, and $r, s \in \mathbb{Z}$ for $\widehat{T}_{R}$. These are infinite dimensional Clifford algebras. We denote them by $C(T, g)_{N S}$ and $C(T, g)_{R}$ respectively.
13.2. Free fermions: Neveu-Schwarz sector. A field representation of $C(T, g)_{N S}$ is a representation $V$ such that for all $\varphi \in T, \varphi(z)=\sum_{r \in \frac{1}{2}+\mathbb{Z}} \varphi_{r} z^{-r-\frac{1}{2}}$ is an odd field. The commutation relations (118) is equivalent to $\{\varphi(z) \mid \varphi \in T\}$ being a collection of mutually local field with the following OPE:

$$
\begin{equation*}
\varphi(z) \psi(w) \sim \frac{k g(\varphi, \psi)}{z-w} \tag{119}
\end{equation*}
$$

There is a field representation of $C(T, g)_{N S}$ on

$$
F^{k}(T, g)=\Lambda\left(\oplus_{n>0} t^{-n+\frac{1}{2}} T\right)=\otimes_{n>0} \Lambda\left(t^{-\left(n-\frac{1}{2}\right)} T\right),
$$

where $n \in \mathbb{Z}$. More precisely, $K$ acts as $k$ id, and for any element $\varphi \in T, \varphi_{r}$ acts as exterior product by $\varphi_{r}$ if $r<0$, and as $k$ times the contraction by $\varphi_{-r}$ if $r>0$. Denote the element 1 in $F^{k}(T, g)$ as $|0\rangle$, then $F^{k}(T, g)$ is spanned by elements of the form:

$$
\varphi_{-j_{1}-\frac{1}{2}}^{1} \cdots \varphi_{-j_{m}-\frac{1}{2}}^{m}=\varphi_{-j_{1}-\frac{1}{2}}^{1} \cdots \varphi_{-j_{m}-\frac{1}{2}}^{m}|0\rangle
$$

where $\varphi^{1}, \cdots, \varphi^{m} \in T$ and $j_{1}, \cdots, j_{m} \geq 0$. The space $F^{k}(T, g)$ is called the fermionic Fock space, where $\left\{\varphi_{-r} \mid \varphi \in T, r \in \frac{1}{2}+\mathbb{Z}, r>0\right\}$ are the creation operators, and $\left\{\varphi_{r} \mid \varphi \in T, r \in \frac{1}{2}+\mathbb{Z}, r>0\right\}$ are the annihilation operators.

Similar to Proposition 12.1, we have the following (cf. Kac [11], pp. 98-100):
Proposition 13.1. There is $v$ structure of a vertex algebra on $F^{k}(T, g)$ defined by

$$
Y\left(\phi_{-j_{1}-\frac{1}{2}}^{1} \cdots \phi_{-j_{m}-\frac{1}{2}}^{m}, z\right)=: \partial^{\left(j_{1}\right)} \phi^{1}(z) \cdots \partial^{\left(j_{n}\right)} \phi^{m}(z):
$$

for $\phi^{1}, \cdots, \phi^{m} \in T$ and integers $j_{1}, \cdots, j_{m} \geq 0$. Assume that $\left\{\varphi^{i}\right\}$ and $\left\{\psi^{j}\right\}$ are two bases of $T$ such that $g\left(\psi^{i}, \varphi^{j}\right)=\delta_{i j}$. Then for any $k \neq 0$,

$$
\nu=\frac{1}{2 k} \sum_{i} \varphi_{-\frac{3}{2}}^{i} \psi_{-\frac{1}{2}}^{i}
$$

is a conformal vector of central charge

$$
c=\frac{1}{4} \operatorname{dim} T
$$

Remark 13.1. It is clear that $O(T, g)$ acts on $F^{k}(T, g)_{N S}$ by vertex algebra automorphisms. Furthermore, $\nu$ is independent of the choice of the basis and is preserved by this action.
13.2.1. Conformal weight decomposition. Let $L(z)=Y(\nu, z)$. By Wick's theorem, it is easy to see that

$$
L(z) \varphi(w) \sim \frac{\frac{1}{2} \varphi(w)}{(z-w)^{2}}+\frac{\partial_{w} \varphi_{-1}(w)}{z-w}
$$

I.e., $\varphi(z)$ is a primary field of conformal weight $\frac{1}{2}$. Furthermore, an element of the form $\varphi_{-j_{1}-\frac{1}{2}}^{1} \cdots \varphi_{-j_{m}-\frac{1}{2}}^{m}$ has conformal weight $\sum_{i=1}^{m}\left(\frac{1}{2}+j_{i}\right)$ with respect to $Y(\nu, z)$. From this it is clear that

$$
\begin{align*}
G_{q}\left(F^{k}(T, g)_{N S}\right) & =q^{-\operatorname{dim} T / 48} \otimes_{n>0} \Lambda_{q^{n-\frac{1}{2}}}(T)  \tag{120}\\
\operatorname{ch}_{q}\left(F^{k}(T, g)_{N S}\right) & =\left(q^{-\frac{1}{48}} \prod_{n>0}\left(1+q^{n-\frac{1}{2}}\right)\right)^{\operatorname{dim} T}=\left(\frac{\theta_{3}(q)}{\eta(q)}\right)^{\operatorname{dim} T / 2} \tag{121}
\end{align*}
$$

We define an operator $F: F^{k}(T, g) \rightarrow \mathbb{Z}_{+}$called fermionic number by

$$
F\left(\phi_{-j_{1}-\frac{1}{2}}^{1} \cdots \phi_{-j_{m}-\frac{1}{2}}^{m}\right)=m,
$$

and define Witten's operator $(-1)^{F}: F^{k}(T, g) \rightarrow F^{k}(T, g)$ by

$$
(-1)^{F}\left(\phi_{-j_{1}-\frac{1}{2}}^{1} \cdots \phi_{-j_{m}-\frac{1}{2}}^{m}\right)=(-1)^{m}\left(\phi_{-j_{1}-\frac{1}{2}}^{1} \cdots \phi_{-j_{m}-\frac{1}{2}}^{m}\right)
$$

Since $F$ commutes with $L_{0}$, we treat $F$ as the $U(1)$ charge $J_{0}$. We have

$$
\operatorname{ch}_{q, y}\left(F^{k}(T, g)_{N S}\right)=\left(q^{-\frac{1}{48}} \prod_{n>0}\left(1+y q^{n-\frac{1}{2}}\right)\right)^{\operatorname{dim} T} .
$$

Taking $y=-1$, we get

$$
\begin{equation*}
\operatorname{ch}_{q,-1}\left(F^{k}(T, g)_{N S}\right)=\left(q^{-\frac{1}{48}} \prod_{n>0}\left(1-q^{n-\frac{1}{2}}\right)\right)^{\operatorname{dim} T}=\left(\frac{\theta_{2}(q)}{\eta(q)}\right)^{\operatorname{dim} T / 2} \tag{122}
\end{equation*}
$$

13.3. Free fermions from a vector space with an inner product and a polarization. From now on we assume that $(T, g)$ admits a polarization $T=$ $T^{\prime} \oplus T^{\prime \prime}$. Then

$$
F^{k}(T, g)_{N S}=\otimes_{n>0} \Lambda\left(t^{-n+\frac{1}{2}} T^{\prime}\right) \otimes_{n>0} \Lambda\left(t^{-n+\frac{1}{2}} T^{\prime \prime}\right)
$$

Consider also the space

$$
\begin{aligned}
F^{k}(T, g)_{R} & =\Lambda\left(\oplus_{n \geq 0} t^{-n} T^{\prime}\right) \otimes \Lambda\left(\oplus_{n>0} t^{-n} T^{\prime \prime}\right) \\
& =\Lambda\left(T^{\prime}\right) \otimes_{n>0} \Lambda\left(t^{-n} T^{\prime}\right) \otimes_{n>0} \Lambda\left(t^{-n} T^{\prime \prime}\right)
\end{aligned}
$$

Notice that $\Lambda\left(T^{\prime}\right)$ is isomorphic to the space $\Delta(T)$ of spinors of $(T, g)$. So we can also write

$$
F^{k}(T, g)_{R}=\Delta(T) \otimes_{n>0} \Lambda\left(t^{-n} T\right)
$$

There is a field representation of $C(T, g)_{R}$ on this space: $K$ acts as multiplication by $k$, when $n>0, \varphi \in T, \varphi_{-n}$ acts as exterior product, while $\varphi_{n}$ acts as contraction, furthermore, $\varphi_{0}$ acts as exterior product when $\varphi \in T^{\prime}$, and as $k$ times the contraction when $\varphi \in T^{\prime \prime}$. Let $\left\{\varphi^{i}\right\}$ be a basis of $T^{\prime},\left\{\psi^{i}\right\}$ a basis of $T^{\prime \prime}$, such that $g\left(\varphi^{i}, \psi^{j}\right)=\delta_{i j}$. Then $F^{k}(T, g)_{R}$ has a basis consists of elements of the form:

$$
\varphi_{-k_{1}}^{i_{1}} \cdots, \varphi_{-k_{m}}^{i_{m}} \psi_{-l_{1}-1}^{j_{1}} \cdots \psi_{-l_{n}-1}^{j_{n}},
$$

where $k_{1}, \cdots, k_{m}, l_{1}, \cdots, l_{n} \geq 0$. For the NS case, set

$$
\varphi^{i}(z)=\sum_{r \in \frac{1}{2}+\mathbb{Z}} \varphi_{r}^{i} z^{-r-\frac{7}{2}}, \quad \psi^{i}(z)=\sum_{r \in \frac{1}{2}+\mathbb{Z}} \psi_{r}^{i} z^{-r-\frac{1}{2}}
$$

for the R case, set

$$
\varphi^{i}(z)=\sum_{n \in \mathbb{Z}} \varphi_{n}^{i} z^{-n}, \quad \quad \psi^{i}(z)=\sum_{n \in \mathbb{Z}} \psi_{n}^{i} z^{-n-1}
$$

Then in both cases $\left\{\varphi^{i}(z), \psi^{i}(z)\right\}$ form a collection of mutually local odd fields with the following OPE's:

$$
\begin{aligned}
\varphi^{i}(z) \psi^{j}(w) & \sim \frac{\delta_{i j} k}{z-w}, \\
\varphi^{i}(z) \varphi^{j}(w) & \sim 0 \\
\psi^{i}(z) \psi^{j}(w) & \sim 0 .
\end{aligned}
$$

Similar to Proposition 12.1, we have the following the following:
Proposition 13.2. Let $(T, g)$ be a finite dimensional complex vector space with a polarization $T=T^{\prime} \oplus T^{\prime \prime}$. Also let $\left\{\varphi^{i}\right\}$ and $\left\{\psi^{w}\right\}$ be basis of $T^{\prime}$ and $T^{\prime \prime}$ respectively such that $g\left(\varphi^{i}, \psi^{j}\right)=\delta_{i j}$. Then for any $k \neq 0$, there is a structure of a conformal vertex algebra on $F^{k}(T, g)_{N S}$ (resp. $F^{k}(T, g)_{R}$ ) such that for any $\lambda \in \mathbb{C}$,

$$
\nu_{\lambda}=\frac{1}{k}\left((1-\lambda) \sum_{i} \varphi_{-\frac{3}{2}}^{i} \psi_{-\frac{1}{2}}^{i}+\lambda \sum_{i} \psi_{-\frac{2}{2}}^{i} \varphi_{-\frac{1}{2}}^{i}\right)
$$

(resp.

$$
\left.\nu_{\lambda}=\frac{1}{k}\left((1-\lambda) \sum_{i} \varphi_{-1}^{i} \psi_{-1}^{i}+\lambda \sum_{i} \psi_{-2}^{i} \varphi_{0}^{i}\right)\right)
$$

is a conformal vector of central charge

$$
-\left(6 \lambda^{2}-6 \lambda+1\right) \operatorname{dim} T
$$

Remark 13.2. Denote by $G L\left(T^{\prime}\right)$ the group of linear transformation on $T^{\prime}$. Since $T^{\prime \prime} \cong\left(T^{\prime}\right)^{*}$, there is an induced action of $G L\left(T^{\prime}\right)$ on $T^{\prime \prime}$, hence $G L\left(T^{\prime}\right)$ acts on $T$, preserving $g$. This action extends to actions on $F^{k}(T, g)_{N S}$ and $F^{k}(T, g)_{R}$ as automorphisms of vertex algebras. Furthermore, $\nu_{\lambda}$ is independent of the choice of the basis and is preserved by the action mentioned above.

Let $L_{\lambda}(z)=Y\left(\nu_{\lambda}, z\right)$. Then by Wick's theorem,

$$
\begin{aligned}
L_{\lambda}(z) \varphi^{i}(w) & \sim \frac{\partial \varphi^{i}(w)}{z-w}+\frac{\lambda \varphi^{i}(w)}{(z-w)^{2}} \\
L_{\lambda}(z) \psi^{i}(w) & \sim \frac{\partial \psi^{i}(w)}{z-w}+\frac{(1-\lambda) \psi^{i}(w)}{(z-w)^{2}}
\end{aligned}
$$

Hence $\varphi^{i}(z)$ and $\psi^{i}(z)$ are primary fields of conformal weights $\lambda$ and $1-\lambda$ respectively. It follows that

$$
\begin{aligned}
& G_{q}\left(F^{k}(T, g)\right)=q^{\left(\lambda^{2}-\lambda+\frac{1}{6}\right) \operatorname{dim} T / 4} \otimes_{n>0} \Lambda_{q^{n-1+\lambda}}\left(T^{\prime}\right) \otimes_{n>0} \Lambda_{q^{n-\lambda}}\left(T^{\prime \prime}\right) \\
& \operatorname{ch}_{q}\left(F^{k}(T, g)\right)=\left(q^{\left(\lambda^{2}-\lambda+\frac{1}{6}\right) / 2} \prod_{n>0}\left(1+q^{n-1+\lambda}\right)\left(1+q^{n-\lambda}\right)\right)^{\operatorname{dim} T / 2}
\end{aligned}
$$

Here we have omitted the subscript $N S$ and $R$ since there is no difference. When $\lambda=0$ or 1 , the character is given by

$$
\left(q^{\frac{1}{12}} \prod_{n>0}\left(1+q^{n-3}\right)\left(1+q^{n}\right)\right)^{\operatorname{dim} T / 2}=\left(\frac{\theta_{1}(0, \tau)}{\eta(q)}\right)^{\operatorname{dim} T / 2}
$$

When $\lambda=1 / 2$, the character is given by

$$
\left.\left(q^{-\frac{1}{24}} \prod_{n>0} 1+q^{n-\frac{1}{2}}\right)^{4}\right)^{\operatorname{dim} T / 4}=\left(\frac{\theta_{3}(0, \tau)}{\eta(q)}\right)^{\operatorname{dim} T / 2}
$$

We define a $U(1)$ current by

$$
J(z)=\frac{1}{k} \sum_{i}: \varphi^{i}(z) \psi^{i}(z):
$$

By Wick's theorem it is easy to see that

$$
\begin{aligned}
& J(z) J(w) \sim \frac{\frac{\operatorname{dim} T}{2}}{(z-w)^{0}} \\
& L_{\lambda}(z) J(w) \sim \frac{\partial_{w} J(w)}{z-w}+\frac{J(w)}{(z-w)^{2}}+\frac{(0 \lambda-1) \operatorname{dim} T / 2}{(z-w)^{3}}
\end{aligned}
$$

In the $N S$ case,

$$
J_{0}=\sum_{n>0} \sum_{i}\left(\varphi_{-n+\frac{1}{2}}^{i} \psi_{n-\frac{1}{2}}^{i}-\psi_{-n+\frac{1}{2}}^{i} \varphi_{n-\frac{1}{2}}^{i}\right)
$$

and so $\varphi_{-n+\frac{1}{2}}^{i}$ has $U(1)$ charge 1 and $\psi_{n-\frac{1}{2}}$ has $U(1)$ charge -1 ; in the $R$ case,

$$
J_{0}=\frac{1}{k} \sum_{i} \varphi_{0}^{i} \psi_{0}^{i}+\sum_{n>0} \sum_{i}\left(\varphi_{-n}^{i} \psi_{n}^{i}-\psi_{-n}^{i} \varphi_{n}^{i}\right)
$$

and so $\varphi_{-n}(n \geq 0)$ has $U(1)$ charge 1 and $\psi_{-n}(n>0)$ has $U(1)$ charge -1 . Therefore,

$$
\begin{aligned}
& G_{q, y}\left(F^{k}(T, g)\right)=q^{\left(\lambda^{8}-\lambda+\frac{1}{6}\right) \operatorname{dim} T / 4} \otimes_{n>0} \Lambda_{q^{n-1+\lambda} y}\left(T^{\prime}\right) \otimes_{n>0} \Lambda_{q^{n-\lambda} y^{-1}}\left(T^{\prime \prime}\right), \\
& \operatorname{ch}_{q}\left(F^{k}(T, g)\right)=\left(q^{\left(\lambda^{2}-\lambda+\frac{1}{6}\right) / 2} \prod_{n>0}\left(1+q^{n-1+\lambda} y\right)\left(1+q^{n-\lambda} y^{-1}\right)\right)^{\operatorname{dim} T / 0}
\end{aligned}
$$

We will need some interesting special cases. Taking $\lambda=\frac{1}{2}$, we get

$$
\begin{aligned}
G_{q, y}\left(F^{k}(T, g)\right) & =q^{-\operatorname{dim} Q / 48} \otimes_{n>0} \Lambda_{q^{n-\frac{1}{2}} y}\left(T^{\prime}\right) \otimes_{n>0} \Lambda_{q^{n-\frac{1}{2}} y^{-1}}\left(T^{\prime \prime}\right) \\
\operatorname{ch}_{q, y} F^{k}((T, g)) & =\left(q^{-1 / 87} \prod_{n>0}\left(1+q^{n-\frac{3}{7}} y\right)\left(1+q^{n-\frac{1}{2}} y^{-1}\right)\right)^{\operatorname{dim} T / 2} \\
& =\left(\frac{\theta_{3}(v, \tau)}{\eta(q)}\right)^{\operatorname{dim} T / 2}
\end{aligned}
$$

Taking $y=1$ and $y=-1$, we get

$$
\begin{align*}
G_{q, 1}\left(F^{k}(T, g)\right) & =q^{-\operatorname{dim} T / 48} \otimes_{n>0} \Lambda_{q^{n-\frac{1}{2}}}(T)  \tag{123}\\
G_{q,-1}\left(F^{k}(T, g)\right) & =q^{-\operatorname{dim} T / 48} \otimes_{n>0} \Lambda_{-q^{n-\frac{1}{2}}}(T)  \tag{124}\\
\operatorname{ch}_{q, 1} F^{k}((T, g)) & =\left(\frac{\theta_{3}(q)}{\eta(q)}\right)^{\operatorname{dim} T / 2}  \tag{125}\\
\operatorname{ch}_{q,-1} F^{k}((T, g)) & =\left(\frac{\theta_{3}(q)}{\eta(q)}\right)^{\operatorname{dim} T / 2} \tag{126}
\end{align*}
$$

We can also take $\lambda=0$. Then we have

$$
\begin{align*}
G_{q, y}\left(F^{k}(T, g)\right) & =q^{\operatorname{dim} T / 24} \otimes_{n>0} \Lambda_{q^{n-1} y}\left(T^{\prime}\right) \otimes_{n>0} \Lambda_{q^{n} y^{-1}}\left(T^{\prime \prime}\right)  \tag{127}\\
\operatorname{ch}_{q, y}\left(F^{k}(T, g)\right) & =\left(q^{\frac{1}{12}} \prod_{n>0}\left(1+q^{n-1} y\right)\left(1+q^{n} y^{-1}\right)\right)^{\operatorname{dim} T / 2}  \tag{128}\\
& =\left(\frac{y \theta_{1}(-v, \tau)}{\eta(q)}\right)^{\operatorname{dim} T / 2}
\end{align*}
$$

where $q=e^{\pi \sqrt{-1} \tau}, y=e^{2 \pi \sqrt{-8} v}$. Taking $y=1$ and $y=-1$, we get

$$
\begin{align*}
G_{q, 1}\left(F^{k}(T, g)\right) & =q^{\operatorname{dim} T / 24} \Delta(T) \otimes \otimes_{n>0} \Lambda_{q^{n}}(T)  \tag{129}\\
G_{q,-1}\left(F^{k}(T, g)\right) & =q^{\operatorname{dim} T / 24}\left(\Delta^{+}(T)-\Delta^{-}(T)\right) \otimes_{n>0} \Lambda_{-q^{n}}(T)  \tag{130}\\
\operatorname{ch}_{q, 1}\left(F^{k}(T, g)\right) & =\left(\frac{2 \theta_{4}(q)}{\eta(q)}\right)^{\operatorname{dim} T / 2} \tag{131}
\end{align*}
$$

Here we have used the fact that $\Lambda\left(T^{\prime}\right)$ is isomorphic to the spinor space $\Delta(T)$ of $(T, g)$, and $\Lambda^{\text {even }}\left(T^{\prime}\right) \cong \Delta^{+}(T)$ and $\Lambda^{\text {odd }}\left(T^{\prime}\right) \cong \Delta^{-}(T)$.

Remark 13.3. It is easy to see that the charge $J(z)$ is independent of the choice of the basis and is preserved by the $G L\left(T^{\prime}\right)$-action. Special case of $F^{k}(T, g)$ with $U(1)$ charge $J(z)$ is the charged free fermions (cf. Kac [11], §1.5 and §5.1).

Remark 13.4. Formulas (123) - (132) are closely related to elliptic genera of spin manifolds and their modular properties.

## 14. $N=1$ SCVA from a Vector Space with Inner Product

14.1. $N=1$ superconformal vertex algebras. An $N=1$ superconformal vertex algebra a vertex algebra $V$ of order $c$ with an odd vector $\tau$ (called $N=1$ superconformal vector), such that the field $G(z)=Y(\tau, z)=\sum_{n \in \frac{1}{2}+\mathbb{Z}} G_{m} z^{-n-\frac{3}{2}}$ satisfies

$$
\begin{aligned}
& {\left[G_{m}, G_{n}\right]=2 L_{m+n}+\frac{1}{3}\left(m^{2}-\frac{1}{4}\right) \delta_{m+n} c} \\
& {\left[G_{m}, L_{n}\right]=\left(m-\frac{n}{2}\right) G_{m+n}}
\end{aligned}
$$

14.2. $N=1$ SCVA from a vector space with inner product. Suppose that $(T, g)$ is a finite dimensional complex vector space with an inner product. Let $\left\{e^{i}\right\}$ be a basis of $T$, such that $g\left(e^{i}, \bar{e}^{j}\right)=\delta_{i j}$. As above, these bases will be written as $\left\{b^{i}\right\}$ and $\left\{c^{i}\right\}$ respectively For the copy of $T$ in the bosonic sector, write the elements in the basis as $a^{i}$; for the copy of $T$ in the fermionic sector, write them as $\phi^{i}$. Calculations by Wick's theorem yields the following:

Proposition 14.1. There is a natural structure of an $N=1 S C V A$ on

$$
V(T, g)_{N S}=F^{1}(T, g)_{N S} \otimes B^{1}(T, g)_{R}=\otimes_{n>0} \Lambda\left(t^{-n+\frac{1}{2}} T\right) \otimes_{n>0} S\left(t^{-n} T\right)
$$

given by

$$
\tau=\frac{1}{k} \sum_{i} a_{-1}^{i} \phi_{-\frac{1}{2}}^{i}, \quad \nu=\frac{1}{2 k} \sum_{i} a_{-1}^{i} a_{-1}^{i}+\frac{1}{2 k} \sum_{i} \phi_{-\frac{3}{2}}^{i} \phi_{-\frac{1}{2}}^{i},
$$

with central charge $\frac{3}{2} \operatorname{dim} T$.
Remark 14.1. It is clear that the conformal vector $\nu$ and the superconformal vector $\tau$ in Proposition 14.1 is independent of the choice of the basis. Furthermore, $O(T, g)$ acts as automorphisms of the $N=1$ SCVA structure.

It is easy to see that

$$
G_{q}\left(V(T, g)_{N S}=q^{-\frac{\operatorname{dim} T}{16}} \otimes_{n>0} S_{q^{n}}(T C) \otimes_{n>0} \Lambda_{q^{n-\frac{1}{2}}}(T)\right.
$$

A related formal power series

$$
q^{-\frac{\operatorname{dim} T}{16}} \otimes_{n>0} S_{q^{n}}(T) \otimes_{n>0} \Lambda_{-q^{n-\frac{1}{2}}}(T)
$$

can be obtained by introducing an operator $(-1)^{F}$.

## 15. $N=2$ SCVA FRom a VEctor Space with inner product

Suppose that $(T, g)$ is a finite dimensional complex vector space with an inner product and a polarization $T=T^{\prime} \oplus T^{\prime \prime}$. Let $\left\{e^{i}\right\}$ be a basis of $T^{\prime}$ and let $\left\{\bar{e}^{i}\right\}$ be a basis of $T^{\prime \prime}$, such that $g\left(e^{i}, \bar{e}^{j}\right)=\delta_{i j}$. As above, these bases will be written as $\left\{b^{i}\right\}$ and $\left\{c^{i}\right\}$ respectively for the copy of $T$ in the bosonic sector, and as $\left\{\varphi^{i}\right\}$ and $\left\{\psi^{i}\right\}$ respectively for the copy of $T$ in the fermionic sector. Set

$$
\begin{aligned}
& V(T, g)_{N S}=F^{1}(T, g)_{N S} \otimes B^{1}(T, g)_{R}=\otimes_{n>0} \Lambda\left(t^{-n+\frac{1}{2}} T\right) \otimes_{n>0} S\left(t^{-n} T\right) \\
& V(T, g)_{R}=F^{1}(T, g)_{R} \otimes B^{1}(T, g)_{R}=\Delta(T) \otimes_{n>0} \Lambda\left(t^{-n} T\right) \otimes_{n>0} S\left(t^{-n} T\right)
\end{aligned}
$$

Using the polarization, we can also write:

$$
\begin{aligned}
& V_{N S}(T, g)=\bigotimes_{n>0} \Lambda\left(t^{-n+\frac{1}{2}} T^{\prime}\right) \bigotimes_{n>0} \Lambda\left(t^{-n+\frac{1}{2}} T^{\prime \prime}\right) \bigotimes_{n>0} S\left(t^{-n} T^{\prime}\right) \bigotimes_{n>0} S\left(t^{-n} T^{\prime \prime}\right), \\
& V_{R}(T, g)=\bigotimes_{n \geq 0} \Lambda\left(t^{-n} T^{\prime}\right) \bigotimes_{n>0} \Lambda\left(t^{-n} T^{\prime \prime}\right) \bigotimes_{n>0} S\left(t^{-n} T^{\prime}\right) \bigotimes_{n>0} S\left(t^{-n} T^{\prime \prime}\right)
\end{aligned}
$$

Calculations by Wick's theorem yield the following:
Proposition 15.1. (a) $V(T, g)_{N S}$ is an $N=2 S C V A$ with superconformal structures given by the following vectors:

$$
\begin{aligned}
\tau^{+} & =\sum_{i} b_{-1}^{i} \psi_{-\frac{1}{2}}^{i}, & \tau^{-} & =\sum_{i} c_{-1}^{i} \varphi_{-\frac{1}{2}}^{i} \\
j & =\sum_{i} \psi_{-\frac{1}{2}}^{i} \phi_{-\frac{1}{2}}^{i}, & \nu & =\sum_{i}\left(b_{-1}^{i} c_{-1}^{i}+\frac{1}{2} \varphi_{-\frac{3}{2}}^{i} \psi_{-\frac{1}{2}}^{i}+\frac{1}{2} \psi_{-\frac{3}{2}}^{i} \varphi_{-\frac{1}{2}}^{i}\right) .
\end{aligned}
$$

(b) $V(T, g)_{R}$ is an $N=2$ SCVA with superconformal structures given by the following vectors:

$$
\begin{aligned}
\tau^{+} & =\sum_{i} b_{-1}^{i} \psi_{-1}^{i}, & \tau^{-} & =\sum_{i} c_{-1}^{i} \varphi_{0}^{i} \\
j & =\sum_{i} \psi_{-1}^{i} \phi_{0}^{i}, & \nu & =\sum_{i}\left(b_{-1}^{i} c_{-1}^{i}+\frac{1}{2} \varphi_{-1}^{i} \psi_{-1}^{i}+\frac{1}{2} \psi_{-2}^{i} \varphi_{0}^{i}\right) .
\end{aligned}
$$

(c) In all cases, the Virasoro field $Y(\nu, z)$ has central charge

$$
c=\frac{3}{2} \operatorname{dim} T=3 \operatorname{dim} T^{\prime} .
$$

Remark 15.1. It is clear that all of the vectors in Proposition 15.1 are independent of the choice of the basis. Furthermore, $G L\left(T^{\prime}\right)$ or $U(T, J, g)$ acts as automorphisms.

Theorem 15.1. (a) For the $N=2 S C V A$ in Proposition 15.1 (a)and (b), the BRST cohomology of the topological vertex algebras obtained by $A$ twist and $B$ twist (cf. Proposition 11.1) are isomorphic to $\Lambda\left(T^{\prime \prime}\right)$ and $\Lambda\left(T^{\prime}\right)$ as graded commutative algebras respectively.

Proof. We will only prove the Neveu-Schwarz case. The Ramond case is similar. Let $Q_{0}$ be the zero mode of $Q(z)=G^{+}(z)$. We have

$$
Q_{0}=\sum_{n<0} \sum_{i} b_{n}^{i} \psi_{-n-\frac{1}{2}}^{i}+\sum_{n \geq 0} \sum_{i} \psi_{-n-\frac{1}{2}}^{i} b_{n}^{i}
$$

Since $b_{0}^{i}$ acts as 0 , we actually have $Q_{0}=Q_{-}+Q_{+}$, where

$$
Q_{-}=\sum_{n<0} \sum_{i} b_{n}^{i} \psi_{-n-\frac{1}{2}}^{i}, \quad Q_{+}=\sum_{n>0} \sum_{i} \psi_{-n-\frac{1}{2}}^{i} b_{n}^{i}
$$

It is easy to see that $Q_{-}^{2}=\left[Q_{-}, Q_{+}\right]=Q_{+}^{2}=0$, hence we get a double complex and two spectral sequences with $E_{1}$ term the $Q_{+}$-cohomology and the $Q_{-}$-cohomology respectively (cf. Bott-Tu [3]). Now

$$
=\bigotimes_{n>0}\left(\Lambda\left(t^{-n+\frac{1}{2}} T^{\prime}\right) \otimes S\left(t^{-n} T^{\prime \prime}\right)\right) \otimes \Lambda\left(t^{-\frac{1}{2}} T^{\prime \prime}\right) \otimes \bigotimes_{n>0}\left(\Lambda\left(t^{-\left(n+\frac{1}{2}\right)} T^{\prime \prime}\right) \otimes S\left(t^{-n} T^{\prime}\right)\right) .
$$

On the first factor, $Q_{-}$acts as the differential in the tensor product of infinitely many copies of Koszul complexes, while $Q_{+}$acts trivially; on the third factor, $Q_{-}$ acts trivially, while $Q_{+}$acts as the differential in the tensor product of infinitely many copies of algebraic de Rham complexes of $V$. By taking cohomology first in $Q_{-}$then in $Q_{+}$. one sees that cohomology in $Q$ is isomorphic to $\Lambda\left(t^{-1} T^{\prime \prime}\right)$ as a vector space. Now any element of $\Lambda\left(t^{-1} T^{\prime \prime}\right)$ is of the form

$$
\psi_{-\frac{1}{2}}^{j_{1}} \cdots \psi_{-\frac{1}{2}}^{j_{n}}
$$

for some $j_{1}, \cdots, j_{n}$. It corresponds to the field

$$
: \psi^{j_{1}}(z) \cdots \psi^{j_{n}}(z): .
$$

Given two elements $\psi_{-1}^{j_{1}} \cdots \psi_{-1}^{j_{n}}$ and $\psi_{-\frac{1}{2}}^{k_{1}} \cdots \psi_{-\frac{1}{2}}^{k_{m}}$, by Wick's theorem, we have

$$
\begin{aligned}
& :\left(: \psi^{j_{1}}(z) \cdots \psi^{j_{n}}(z):\right)\left(: \psi^{k_{1}}(z) \cdots \psi^{k_{n}}(z):\right): \\
= & : \psi^{j_{1}}(z) \cdots \psi^{j_{n}}(z) \psi^{k_{1}}(z) \cdots \psi^{k_{n}}(z):
\end{aligned}
$$

Hence on the $Q_{0}$-cohomology, the product induced from the normally ordered product is isomorphic to the ordinary exterior product on $\Lambda\left(t^{-1} T^{\prime \prime}\right)$. The case of $Q(z)=G^{-}(z)$ is similar.

## 16. Superconformal Vertex Algebras in Differential Geometry

In this section we apply some of the constructions to differentiable manifolds. In particular, we establish a natural relationship with the elliptic genera.
16.1. Motivations. Vafa [20] suggested an approach to quantum cohomology based on vertex algebra constructed via semi-infinite forms on loop space. Recall that a closed string in a manifold $M$ is a smooth map from $S^{1}$ to $M$. The space of all closed string is just the free loop space $L M$. He suggested to study look at the cohomology theory of semi-infinite forms on the loop space.

Recall that there is a natural action of $S^{1}$ on $L M$ given by rotations on $S^{1}$ the fixed point set is exactly the set $M$ of constant loops. Using Fourier series expansion, one sees that the complexified tangent space of $L M$ restricted to $M$ has the following decomposition;

$$
\left.T L M\right|_{M} \otimes \mathbb{C} \cong \bigoplus_{n \in \mathbb{Z}} t^{n} T M \otimes \mathbb{C}
$$

The bundle of semi-infinite form on $L M$ restricted to $M$ is

$$
\left.\Lambda^{\frac{\infty}{2}+*}(L M)\right|_{M} \cong \Lambda\left(\oplus_{n \leq 0} t^{n} T^{*} M\right)
$$

When $M$ is endowed with a Riemannian metric, $\left.\Lambda^{\frac{\infty}{2}+*}(L M)\right|_{M}$ is a bundle of conformal vertex algebras that contains $\Lambda\left(T^{*} M\right)$ as a subbundle.

So far we have only talked about the fermionic part. To get the bosonic part hence the supersymmetry, we use the language of supermanifolds (Kostant [13]). Recall that a supermanifold is an ordinary manifold $M$ together with a $\mathbb{Z}_{2}$-graded structure sheaf. The even part of the structure sheaf is the sheaf of $C^{\infty}$ function on $M$, while the odd part is the sheaf of sections to the exterior bundle $\Lambda(E)$ of some vector bundle $E$ on $M$. The super tangent bundle of $(M, E)$ is a upper vector bundle

$$
T(M, E)=T M \oplus E^{*}
$$

where $T M$ is the even part, $E^{*}$ is the odd part. And the differential forms on ( $M, E$ ) are just sections to $\Lambda\left(T^{*} M\right) \otimes S(E)$. A canonical choice for $E$ is the cotangent bundle $T^{*} M$, then we get a supermanifold which corresponds to $\Lambda\left(T^{*} M\right)$. The supertangent bundle of $\left(M, T^{*} M\right)$ is just

$$
T\left(M, T^{*} M\right)=T M \oplus \Pi T M,
$$

where $\Pi T M$ means a copy of $T M$ regarded as an odd vector bundle. We now consider the super loop space $L^{s} M=M a p\left(S^{1},\left(M, T^{*} M\right)\right.$ ) and regard ( $M, T^{*} M$ ) as the fixed point set of the natural circle action. We have

$$
\left.T L^{s} M\right|_{\left(M, T^{*} M\right)} \otimes \mathbb{C} \cong \bigoplus_{n \in \mathbb{Z}} t^{n} T M \bigoplus_{n \in \mathbb{Z}} t^{n} \Pi T M
$$

The bundle of semi-infinite form on $L^{s} M$ restricted to $\left(M, T^{*} M\right)$ is

$$
\left.\Lambda^{\frac{\infty}{2}+*}\left(L^{s} M\right)\right|_{\left(M, T^{*} M\right)} \cong \Lambda\left(\oplus_{n \leq 0} t^{n} T^{*} M\right) \otimes S\left(\oplus_{n<0} t^{n} T^{*} M\right)
$$

This gives us a bundle of superconformal vertex algebras.
16.2. VOA bundles. Let $V$ be a vertex algebra, denote by $\operatorname{Aut}(V)$ the automorphism group of $V$. Let $M$ be a smooth topological space, a vertex algebra bundle with fiber $V$ over $M$ is a vector bundle $\pi: E \rightarrow M$ with fiber $V$ such that the transition functions lie in $\operatorname{Aut}(V)$. Similarly define conformal vertex algebra bundles and superconformal vertex algebra bundles. When $M$ is a smooth manifold or a complex manifold, one can also define smooth or holomorphic vertex algebra bundles.

Lemma 16.1. Given a vertex algebra bundle $E \rightarrow M$, the space $E(M)$ of sections has an induced structure of a vertex algebra. Similarly for (charged) conformal vertex algebra bundles and superconformal vertex algebra bundles.

Proof. Since the vacuum $|0\rangle$ is preserved by the automorphisms, it defines a section which we denote by $|0\rangle_{M}$. Given two sections $A$ and $B$, the assignment

$$
x \in M \mapsto A(x)_{(n)} B(x)
$$

defines a section denoted by $A_{(n)} B$. It is straightforward to check that

$$
Y(A, z) B=\sum_{n \in \mathbb{Z}} A_{(n)} B z^{-n-1}
$$

then defines a structure of a vertex algebra on $E(M)$.
16.3. $N=1$ SCVA bundles from Riemannian manifolds. For any Riemannian manifold $(M, g)$, we consider the principal bundle $O(M, g)$ of orthonormal frames. Pick a point $x \in M$. The structure group of $O(T, g)$ is $O\left(T_{x} M, g_{x}\right)$, which acts on $V\left(T_{x} M \otimes \mathbb{C}, g \otimes \mathbb{C}\right)$ by automorphisms. Applying Proposition 14.1, Remark 14.1 and Lemma 16.1, we get the following:

Theorem 16.1. Let $(M, g)$ be a Riemannian manifold. Then

$$
V(T M \otimes \mathbb{C}, g \otimes \mathbb{C})_{N S}=\otimes_{n>0} \Lambda\left(t^{-n+\frac{1}{2}} T M \otimes \mathbb{C}\right) \otimes_{n>0} S\left(t^{-n} T M \otimes \mathbb{C}\right)
$$

is an $N=1 S C V A$ bundle, hence $\Gamma\left(M, V(T M, g)_{N S}\right)$ is an $N=1 S C V A$.

The bundle $V(T M \otimes \mathbb{C}, g \otimes \mathbb{C})_{N S}$ has appeared in the theory of elliptic genera. It is easy to see that

$$
G_{q}\left(V(T M \otimes \mathbb{C}, g \otimes \mathbb{C})_{N S}=q^{-\frac{\operatorname{dim} T}{16}} \otimes_{n>0} S_{q^{n}}(T M \otimes \mathbb{C}) \otimes_{n>0} \Lambda_{q^{n-\frac{1}{2}}}(T M \otimes \mathbb{C})\right.
$$

(cf. Witten [23], (27)). A related formal power series

$$
q^{-\frac{\operatorname{dim} T}{16}} \otimes_{n>0} S_{q^{n}}(T M \otimes \mathbb{C}) \otimes_{n>0} \Lambda_{-q^{n-\frac{1}{2}}}(T M \otimes \mathbb{C})
$$

(cf. Liu [16]) can be obtained by introducing an operator $(-1)^{F}$.
16.4. $N=2$ SCVA bundles from complex manifolds. Let $(M, J)$ be a complex manifold. Denote by $T_{c} M$ the holomorphic tangent bundle. The fiberwise pairing between $T_{c} M$ and $T_{c}^{*} M$ induces a canonical complex inner product $\eta$ on the holomorphic vector bundle $T_{c} M \oplus T_{c}^{*} M$ with a manifest polarization $T^{\prime}=T_{c} M$, $T^{\prime \prime}=T_{c}^{*} M$. By construction of Proposition 15.1 and Remark 15.1, we obtain an $N=2$ SCVA bundle $V\left(T_{c} M \oplus T_{c}^{*} M, \eta\right)_{R}$. Since this bundle is holomorphic, one can consider the $\bar{\partial}$ operator on it:

$$
\bar{\partial}: \Omega^{0, *}\left(V\left(T_{c} M \oplus T_{c}^{*} M, \eta\right)_{R}\right) \rightarrow \Omega^{0, *+1}\left(V\left(T_{c} M \oplus T_{c}^{*} M, \eta\right)_{R}\right)
$$

Theorem 16.2. For any complex manifold $M, \Omega^{0, *}\left(V\left(T_{c} M \oplus T_{c}^{*} M, \eta\right)_{R}\right)$ has a natural structure of an $N=2 S C V A$ such that $\bar{\partial}$ is a differential. Consequently, the Dolbeault cohomology

$$
H^{*}\left(M, V\left(T_{c} M \oplus T_{c}^{*} M, \eta\right)_{R}\right)
$$

is an $N=2$ SCVA; furthermore, the BRST cohomology of its associated topological vertex algebras (cf. Proposition 11.1) is isomorphic to $H^{*}\left(M, \Lambda\left(T_{c} M\right)\right.$ or $H^{*}\left(M, \Lambda\left(T_{c}^{*} M\right)\right)$ depending on whether we take $Q(z)=G^{+}(z)$ or $G^{-}(z)$. Similar results can be obtained for $V\left(T_{c} M \oplus T_{c}^{*} M, \eta\right)_{N S}$. However the BRST cohomologies are trivial for the corresponding Dolbeault cohomology.

Proof. We regard $\Lambda\left({\overline{T_{c}}}^{*} M\right)$ as a bundle of holomorphic vertex algebra, therefore, $V\left(T_{c} M \oplus T_{c}^{*} M, \eta\right)_{R} \otimes \Lambda\left({\overline{T_{c}}}^{*} M\right)$ has a natural structure of an $N=2$ SCVA. By Lemma 16.1, the section space

$$
\Gamma\left(M, V\left(T_{c} M \oplus T_{c}^{*} M, \eta\right)_{R} \otimes \Lambda\left({\overline{T_{c}}}^{*} M\right)\right)
$$

is an $N=2$ SCVA. One can easily verify that $\bar{\partial}$ is a differential by choosing a local holomorphic frame of $T_{c} M$. It follows from Lemma 10.1 that $H^{*}\left(M, V\left(T_{c} M \oplus\right.\right.$ $\left.\left.T_{c}^{*} M, \eta\right)_{R}\right)$ is an $N=2$ SCVA. Notice that on $\Gamma\left(M, V\left(T_{c} M \oplus T_{c}^{*} M, \eta\right)_{R} \otimes \Lambda\left({\overline{T_{c}}}^{*} M\right)\right)$ two operators $\bar{\partial}$ and $Q_{0}$ act such that

$$
\bar{\partial}^{2}=\left[\bar{\partial}, Q_{0}\right]=Q_{0}^{2}=0 .
$$

In the above we have taken the $\bar{\partial}$-cohomology first, then take the $Q_{0}$-cohomology. We can also do it in a different order. By Theorem 15.1, the $Q_{0}$-cohomology is $\Lambda\left(T_{c} M\right)$ or $\Lambda\left(T_{c}^{*} M\right)$, its $\bar{\partial}$-cohomology is the Dolbeault cohomology. This completes the proof.

The vertex algebra obtained in the above theorem will be called the vertex cohomology.

As in $N=1$ vertex algebra bundle in the Riemannian case (cf. §16.3), $V\left(T_{c} M \oplus\right.$ $\left.T_{c}^{*} M, \eta\right)_{R}$ is related to the elliptic genera. One has

$$
\begin{aligned}
& G_{q, y}\left(V\left(T_{c} M \oplus T_{c}^{*} M, \eta\right)_{R}\right) \\
= & \otimes_{n>0} \Lambda_{y^{-1} q^{n-\frac{1}{2}}}\left(T_{c} M\right) \otimes_{n>0} \Lambda_{y q^{n-\frac{1}{2}}}\left(T_{c}^{*} M\right) \otimes_{n>0} S_{q^{n}}\left(T_{c} M\right) \otimes_{n>0} S_{q^{n}}\left(T_{c}^{*} M\right) .
\end{aligned}
$$

One sees that in the $A$ twist we have

$$
\begin{aligned}
& G_{q, y}\left(V\left(T_{c} M \oplus T_{c}^{*} M, \eta\right)_{R}\right) \\
= & \otimes_{n>0} \Lambda_{y^{-1} q^{n}}\left(T_{c} M\right) \otimes_{n>0} \Lambda_{y q^{n-1}}\left(T_{c}^{*} M\right) \otimes_{n>0} S_{q^{n}}\left(T_{c} M\right) \otimes_{n>0} S_{q^{n}}\left(T_{c}^{*} M\right)
\end{aligned}
$$

(cf. Hirzebruch [10], (16)), while in the $B$ twist we have

$$
\begin{aligned}
& G_{q, y}\left(V\left(T_{c} M \oplus T_{c}^{*} M, \eta\right)_{R}\right) \\
= & \otimes_{n>0} \Lambda_{y q^{n-1}}\left(T_{c} M\right) \otimes_{n>0} \Lambda_{y^{-1} q^{n}}\left(T_{c}^{*} M\right) \otimes_{n>0} S_{q^{n}}\left(T_{c} M\right) \otimes_{n>0} S_{q^{n}}\left(T_{c}^{*} M\right)
\end{aligned}
$$

(cf. Dijkgraaf et. al. [4], (A.8)). It is clear that taking $\operatorname{tr}(-y)^{J_{0}} q^{L_{0}-\frac{c}{24}}$ on the vertex cohomology naturally leads one to the two-varible ellitpic genera.

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