Topology of 3-Manifolds

Lecture 1. Introduction

Let us begin w/ surfaces. X: Hausdorff, countable basis (conn.), locally $\mathbb{R}^2$

Orientable surface : contains no Möbius band.

Thm. M^2 compact, $\partial M = \phi$ orientable $\Rightarrow$ M homeomorphic to

S^2, $T^2 = S^1 \times S^1$ or Σ_g $(g \geq 2)$

These are topological results. completely classified by the genus g

Relationship to geometry is also interesting

Example 2-Dimensional Geometries. (X, G) G Lie group acting on X.

1) spherical : underlying sets (Points), lines (geodesics), isometries$^+$

1) Spherical	$+1$	$S^2 = \{x \in \mathbb{R}^3 \mid \lvert x\rvert = 1\}$,	$S^2 \cap P$.	$SO(3)$.
2) Euclidean	0	$\mathbb{E}^2 = \mathbb{R}^2$	lines	$SO(2) \ltimes \mathbb{R}^2$
3) Hyperbolic	-1	$\mathbb{H}^2 = \{ \operatorname{Im} z > 0 \}$	circles, lines $\perp \mathbb{R}$	$PSL(2, \mathbb{R})$
		$\mathbb{H}^3 = \{ (z, t) \in \mathbb{C} \times \mathbb{R} \mid t > 0 \}$.		$PSL(2, \mathbb{C})$.

An interesting statement:

Σ^2 closed orientable $\Rightarrow$ $\Sigma^2 = S^2$, or T^2 (admitting S^1 action) or can be cut into 3-holed spheres

 , $\Sigma_{0,3}$

where interior has a complete hyperbolic metric of finite area

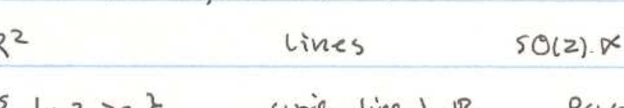
$$\mathring{\Sigma}_{0,3} \cong \mathbb{H}/\Gamma(2) \qquad \Gamma(2) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in PSL(2, \mathbb{Z}) \;\middle|\; \equiv I \bmod 2 \right\}$$

$\cong$ $\mathbb{C} - \{0, 1\}$.

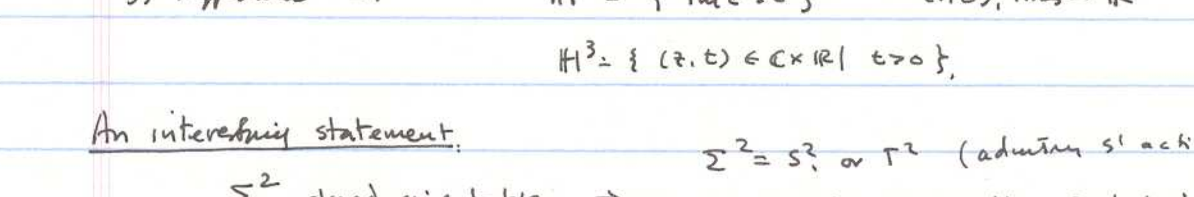
$\boxed{\text{connected sum}}$

homeo

$\boxed{\text{gluing construction.}}$



Torus.



Topology of 3-Manifolds

3-manifolds

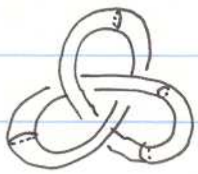
Examples. 1 S^3, $T^3 = S^1\times S^1\times S^1$, $\Sigma_g \times S^1$, $UT\Sigma_g$ (unit tangent bundles)

$UT(S^2) = \{ (x, v) \in \mathbb{R}^3\times\mathbb{R}^3 \mid |x|=1,\ |v|=1,\ x\cdot v = 0 \}$ $(\cong SO(3) \cong \mathbb{R}P^3)$

Construction out of gluing:

Take a knot $K \subset S^3$ (smooth), remove a smooth small nbhd $N(K)$

$M_K' = S^3 - \mathring{N}(K) \cong SL(2,\mathbb{R})/SL(2,\mathbb{Z})$.



$M_K = M_K' \cup_{\partial\, id} M_K'$ is a closed orientable 3-mfd.

$M_K \cong M_{K'} \Leftrightarrow K \cong K'$ knots are the same $\Leftrightarrow \exists$ homeo $h: S^3 \to S^3$

$\partial M_K' \cong S^1\times S^1 \cong$  $h(K) = K'$

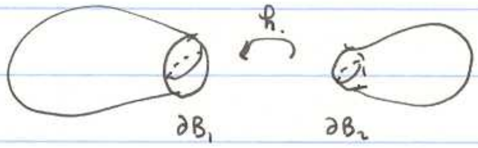
Example SFS: M^3 with a non-trivial S^1 action. or  glued by $SL(2,\mathbb{Z})$

No known classification of knots in S^3 as in theorem 1 (by a finite set of computable invariants).

Thurston's Geometrization Conjecture + Basic Results on 3-manifolds.

Connected Sum Operation. Suppose M_1, M_2 are two closed 3-manifolds

$$M = M_1 \# M_2 = (M_1 - \mathring{B}_1^3) \cup_h (M_2 - \mathring{B}_2^3) \qquad h: \partial B_1 \to \partial B_2 \text{ homeo}$$



There is a 2-sphere $S^2 \subset M$, bounding both $M_i - \mathring{B}_i^3$.

Def. 1) A 2-sphere $S^2 \subset M$ is essential if it does not bound a 3-ball.

2) A 3-manifold M is irreducible if it contains no essential 2-sphere (or every topological 2-sphere bounds a 3-ball).
smooth

M^3 closed, orientable.

<u>Prop</u>. If M contains an essential 2-sphere $\Rightarrow$
either $M = M_1 \# M_2$, $M_1, M_2 \neq S^3$ or
$M = S^2 \times S^1$.

(Ricci flow: $S^2 \Rightarrow$ singularity of $|Rm| \to +\infty$)

Thus, we will focus on irreducible 3-manifolds.

<u>Def</u>. A surface $\Sigma \neq S^2$ in M is <u>incompressible</u> if $\pi_1(\Sigma) \xrightarrow{i_*} \pi_1(M^3)$ is injective. (Each essential loop in Σ is essential in M).

<u>Example</u> $\Sigma \subset$ ball $\subset M^3$ always compressible.

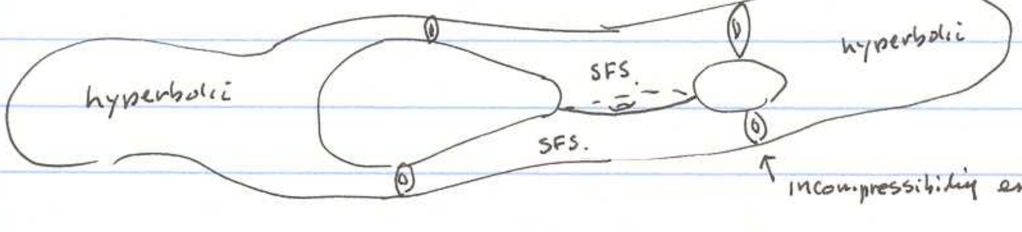
<u>Thurston's Geometrization Conjecture</u> (orientable) G.C.

M^3 closed irreducible 3-manifold. Then there exists a finite collection of disjoint incompressible tori (could be empty) $T_1 \cup \cdots \cup T_N$ so that each component of $M^3 - (T_1 \cup \cdots \cup T_N)$
is (1) either has a complete hyperbolic metric of finite volume
(2) has a non-trivial S^1 action.

<u>Thurston's Theorem</u>. The conjecture holds if $N \geq 1$. (Perelman, Hamilton).

G.C $\Rightarrow$ Poincaré

<u>Picture</u>: M^3 irreducible



<u>My Ref</u>: list of Eight Geometries $\mathbb{H}^3$, S^3, $\mathbb{E}^3$, $\mathbb{H}^2 \times \mathbb{R}$, $\widetilde{SL(2,\mathbb{R})}$, Nil, Sol, $S^2 \times \mathbb{R}$.

$\longleftrightarrow$ SFS admitting S^1-action

Topology of 3-Manifolds

Goal: introduce some known topological theories of 3-manifolds.

Jordan curve theorem.

Thm (Schoenflies Problem) If $S^2 \subset \mathbb{R}^3$ smooth embedded 2-sphere, it bounds a 3-ball.

(Alexander's theorem)

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Thm (The Sphere Thm) If $\pi_2(M) \neq 0$, ($\exists\ \varphi: S^2 \to M$ not null-homotopic), then there exists an essential embedded 2-sphere in M.

Thm (The Loop Thm) M 3-manifold w/ $\partial M \neq \phi$, S connected surface in ∂M so that $i_* : \pi_1(S) \to \pi_1(M)$ induced by inclusion is not 1-1. Then there exists an embedded disk $D \subset M$, $\partial D \subset S$ and ∂D does not bound a disk in S.

Thm (Kneser). For any closed M^3, $\exists\ n \in \mathbb{Z}_{>0}$ so that if

$$M^3 = M_1 \# \cdots \# M_k \quad k \geq n \Rightarrow \text{one of } M_i = S^3.$$

- Haken's normal surface theory
- How Thurston discovered the hyperbolic metric on  which motivates G.C.
- How to prove Mostow Rigidity thm.