Anti Kupiainen 6/8. ①

Intro. to Turbulence ( )

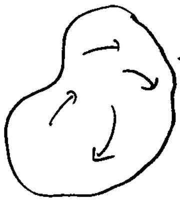
Lecture 1, 2 : Navier-Stokes Eqs. / Turbulence

Lecture 3. 4 : Advection by ~~Turbulence~~ Velocities.

---

1. Fluid (Gas, Liquid)

Velocity Field. $u(t,x) \in \mathbb{R}^n$.



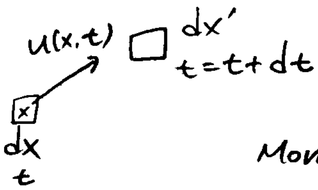
$\Omega \subset \mathbb{R}^n$, $n=2,3$

NSE.

$$\begin{cases} \dfrac{\partial u}{\partial t} + u\cdot\nabla u = \nu\Delta u - \nabla p + f & \cdots (0) \\ \nabla\cdot u = 0 & \cdots\cdots (00) \end{cases} \qquad u=(u_1,\cdots,u_n)$$

ν: viscosity, p: pressure , f: external force.

"Physicist Derivation"



$$X' = X + u(t,X)\,dt$$

$$dx' = (1+\nabla\cdot u\,dt)\,dx$$

$$\text{Momentum} = \underset{\substack{\uparrow \\ \text{density}}}{\rho}\underbrace{(t,x)\,dx}_{\text{mass}}\cdot u(t,x)$$

$$\overset{t=t+dt}{=} \rho(t+dt, x')\,dx'\,u(t',x')$$

$$\text{Force} = \frac{d}{dt}\text{Momentum}$$

$$f_i = \frac{\partial}{\partial t}(\rho u_i) + \nabla\cdot(u\rho u_i) \cdots (1).$$

$\parallel$

$$f + \nu\Delta u - \nabla p.$$

$$\text{MASS} = \rho(t,x)\,dx.$$

$$\frac{\partial\rho}{\partial t} + \nabla(u\rho) = 0 \cdots (2)$$

Incompressible Fluid $\rho(x,t) = \text{const} = \rho_0$.

(2) $\Rightarrow$ $\nabla\cdot u = 0$.

(1) $\Rightarrow$ (0) if $\rho_0 = 1$.

---

u, p : unknown

$\Omega = (\mathbb{R}/2\pi L)^n$ or $\mathbb{R}^n$.

p can be computed from u.

$$\nabla\cdot(0) \Rightarrow \nabla\cdot(u\cdot\nabla u) = -\Delta p + \nabla\cdot f$$

$$p = -\Delta^{-1}(\nabla\cdot(u\cdot\nabla u)) + \Delta^{-1}\nabla\cdot f$$

Δ^{-1} : Green Function in Ω

Define P on $\mathrm{Vect}(\Omega)$ $\qquad \partial_i = \frac{\partial}{\partial x_i}$

$$(Pu)_i(x) = u_i(x) - \partial_i(\Delta^{-1}\nabla\cdot u)(x).$$

$P^2 = P$ projects on divergenceless vector fields.

$$\dot u + P(\underbrace{\cancel{\quad}}_{u\nabla u}) = \nu\Delta u + Pf$$

Integro. PDE. $\quad \Delta^{-1} \propto \frac{1}{|x-y|} \quad n=3.$

p: non-local function.
instantaneous change on whole p.
→ makes NSE hard.

$$\partial_i P u_i = \partial_i u_i - \underbrace{\Delta\Delta^{-1}}_{=1}\nabla\cdot u = 0.$$

$$\hat u(k) = \int_\Omega e^{ikx} u(x)\,dx. \qquad \widehat{Pu_i}(k) = \left(\delta_{ij} - \frac{k_i k_j}{k^2}\right)\hat u_j(k).$$

orthogonal in $L^2(\Omega)$.

---

2. $\exists !$ solutions.

$\Omega = (\mathbb{R}/2\pi L)^d, \quad d = 2, 3$

Energy $\quad E(t) = \frac{1}{2}\|u(t)\|_2^2 = \int_\Omega e(t,x)\,dx \quad , \quad e(t,x) = \frac{1}{2}u(t,x)^2$

Let u be a smooth solution of NSE.

$$\dot e = u\cdot\dot u = u\cdot(\nu\Delta u - u\cdot\nabla u - \nabla p + f)$$

$$\overset{\text{check.}}{=} \nabla\cdot j - \nu(\nabla u)^2 + u\cdot f \qquad \text{current } j = \tfrac{1}{2}(\nu\nabla - u)u^2 - up$$

$$\dot E = -\nu\underbrace{\int_\Omega (\nabla u)^2}_{\substack{\text{dissipation}\\ \text{due to viscosity}}} + \underbrace{\int u\cdot f}_{\substack{\text{pumping}\\ \text{(energy injection)}}}$$

simplify : $\begin{cases} \int_\Omega u(0,x)\,dx = 0. \\ \int_\Omega f(t,x)\,dx = 0. \end{cases}$ $\qquad \boxed{HW}$ Do general case!

$$\Rightarrow \int_\Omega u(t,x)\,dx = 0 \quad \forall t. \longrightarrow \text{Poincaré} \quad \|\nabla u\|_2^2 \geq L^{-2}\|u\|^2$$

$$\dot{E} = -\nu\int_\Omega (\nabla u)^2 + \int u\cdot f \Rightarrow \frac{1}{2}\frac{d}{dt}\|u(t)\|_2^2 + \nu\|\nabla u\|_2^2 \le \|u\|_2\|f\|_2$$ ③

$$\Rightarrow \forall \varepsilon>0 : \quad \|u\|_2\|f\|_2 \le \frac{1}{2}\left(\varepsilon\|u\|^2 + \frac{1}{\varepsilon}\|f\|^2\right)$$

$$\le \frac{1}{2}\left(L^2\varepsilon\|\nabla u\|^2 + \frac{1}{\varepsilon}\|f\|^2\right)$$

$\boxed{\varepsilon L^2 = \nu}$ $$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \underbrace{\frac{\nu}{2}\|\nabla u\|^2}_{\ge \frac{\nu}{2L^2}\|u\|^2} \le \frac{L^2}{2\nu}\|f\|^2$$

$$\frac{d}{dt}\|u\|^2 + \frac{\nu}{L^2}\|u\|^2 \le \frac{L^2}{\nu}\|f\|^2$$

$$\left.\begin{array}{l}\dot{y} + \alpha y \le \beta\\ y\ge 0\end{array}\right\} \overset{\text{Gronwall}}{\Rightarrow} y(t) \le e^{-\alpha t}y(0) + \frac{\beta}{\alpha}(1-e^{-\alpha t})$$

$$E(t) \le e^{-\frac{\nu}{L^2}t}E(0) + \frac{L^4}{\nu}\left(1-e^{-\frac{\nu}{L^2}t}\right)\cdot \sup_{s\le t}\|f(s)\|_2^2$$

Weak Solutions

Let u be a smooth NS sol., v smooth s.t. $v(t,x)=0$ $\forall t > \exists T$

ϕ

$$0 = \int_{\Omega\times[0,\infty)} v\cdot(\overset{\text{NSE}}{0})\,dt\,dx = -\int u(\dot{v} + \nu\Delta v + u\cdot\nabla v + fv) - \int p\nabla\cdot v \cdots (*)$$

$$(\circ\circ) \Rightarrow \quad 0 = \int \phi\nabla\cdot u = -\int u\cdot\nabla\phi \cdots (**)$$

$u(t)\in L^2(\Omega)$ is a weak solution to N.S. if $(*),(**)$ holds $\forall v,\phi$.

Leray. '30s: $\exists$ weak solutions.

proof). Energy conservation + compactness.

Uniqueness & Smoothness of Weak Solo $= \$1{,}000{,}000$ $\boxed{n=3}$

$\boxed{n=2}$ (Easy) $\because$ Vorticity conservation. $\omega = \nabla\times u = \partial_1 u_2 - \partial_2 u_1$.

NS $\Rightarrow$ $\dot{\omega} + u\cdot\nabla\omega = \nu\Delta\omega + g$, $\quad g = \partial_1 f_2 - \partial_2 f_1$.

"Advection / Transport Eq"

$$\nabla\cdot u = 0 \Rightarrow u = \nabla^\perp\Delta^{-1}\omega, \quad \nabla^\perp = (\partial_2, -\partial_1)$$

Enstrophy $= \Phi(t) = \frac{1}{2}\|\omega(t)\|_2^2 = \int_\Omega \varphi(t,x)\,dx$. $\quad \varphi(t,x) = \omega(t,x)^2$.

$$\dot{\varphi} = -\nu(\nabla\omega)^2 + \omega g + \nabla\cdot j_\omega, \qquad j_\omega = \frac{1}{2}(\nu\nabla - u)\omega^2.$$

$$\dot{\Phi} = -\nu\int_\Omega(\nabla\omega)^2 + \int_\Omega \omega g.$$

$$\|\omega(t)\|_2^2 \le C^{-\alpha t}\|\omega(0)\|_2^2 + \frac{1}{\alpha^2}(1-e^{-\alpha t})\cdot\sup_{s\le t}\|g(s)\|_2^2, \qquad \alpha = \frac{\nu}{L^2}$$

$\boxed{HW}$ $\int_\Omega(\nabla u)^2 = \int(\partial_i u_j)^2 = \int_\Omega \omega^2 \Rightarrow u \in H^1 \Rightarrow \exists!$ sol.

$\boxed{n=3}$ $\quad \dot{\omega} = \nu\Delta\omega + [\omega,u] + g. \qquad [\omega,u] = \underbrace{\omega\cdot\nabla u}_{\text{vorticity stretching}} - \underbrace{u\cdot\nabla\omega}_{\text{transport}}$.

Let $u(0,x) \in H^1$, then $u(t,x) \in H^1$ for $t \le T(\|\nabla u(0)\|_2^2)$

$T \to 0$ as $\|\nabla u(0)\|_2^2 \to \infty$. $\qquad$ Reynold's number

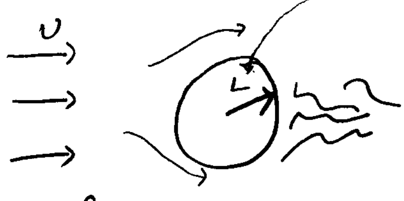
---

Turbulence Problem.

Force of fixed spatial scale.

There are 3 dim'l parameters.

L	m	$L' = L/\ell$
v	m/s	$v' = \frac{\tau}{\ell}v$
ν	m²/s	$\nu' = \frac{\tau}{\ell^2}\nu$



$v = \lim_{x_1\to-\infty} u(t,x)$

length $\qquad$ time

NS is scale covariant: pick ℓ, τ

$$u(t,x) = \frac{\ell}{\tau}u'\left(\frac{t}{\tau},\frac{x}{\ell}\right)$$

$$f(t,x) = \frac{\ell}{\tau^2}f'\left(\frac{t}{\tau},\frac{x}{\ell}\right) \qquad \nu = \frac{\ell^2}{\tau}\nu'$$

NS(u', f', ν') holds.

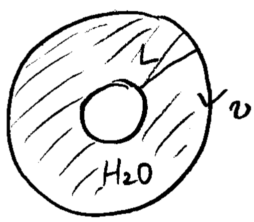
Reynold's number $\quad R = \frac{Lv}{\nu} = R'$

observations:

$R < O(1)$, Regular Flow

$R \in [O(1), O(10^2)]$ Transition to Chaos

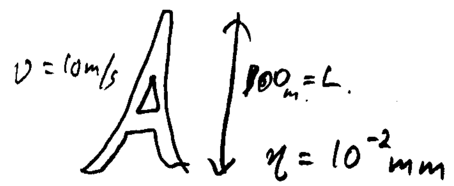
$R > O(10^3)$ Turbulence



Experimental Facts : For large R.

Universality

Suppose $\exists 1$ sol. to NS $\forall u_0 \in \underset{\mathcal{M}}{\underbrace{L^2(\Omega)}}$.



Let $\varphi_t : \mathcal{M} \to \mathcal{M}$ Flow of NS.

$\varphi_t u_0 = u(t)$.

$F : \mathcal{M} \longrightarrow \mathbb{R}$ "observable."

e.g. $F(u) = \prod_{j=1}^{N} u_{i_j}(x_j) \ldots\ldots (*)$

Time average of F: $\langle F \rangle = \lim_{T\to\infty} \frac{1}{T} \int_0^T dt\, F(\underset{u(t)}{\underbrace{\varphi_t u_0}})$

Facts. 1, $\langle F \rangle = G_N(x_1, \cdots, x_N)$, $|x_i - x_j| \ll L$.

$\Rightarrow G_N$ is translation inv. $\quad G_2(x,y) = g(x-y)$

2, $\exists$ length scale

η "Kolmogorov Scale" s.t. $\frac{\eta}{L} \to 0$ as $R \to \infty$.

s.t. self-similarity holds for scales l. $\eta \ll l \ll L$

$$\langle f((u(x)-u(y))) \rangle \approx \langle f(\lambda^{-a}(u(\lambda x) - u(\lambda y))) \rangle$$

if $\eta \ll |x-y|, \lambda|x-y| \ll L$.

Structure Function $\quad S_n(x,y) = \langle [(u(x)-u(y))\cdot\hat{n}]^n \rangle \quad \hat{n} = \frac{x-y}{|x-y|}$

$\approx \lambda^{-an} S_n(\lambda x, \lambda y) \qquad a \approx 1/3$.

$\Rightarrow \quad S_n(x,y) \approx |x-y|^{J_n}. \quad J_n \approx n/3$.

$\eta \ll |x-y| \ll L. \qquad \eta/L \to 0$ as $R \to \infty$.

L: fixed, $R \to \infty \Rightarrow \eta \to 0$.

$u(x)$ is only Hölder continuous w/ exponent $1/3$. $|u(x)-u(y)| \sim |x-y|^{1/3}$

e.g. Brownian Motion $\beta(x) \in \mathbb{R}$, $x \in \mathbb{R}$.

$\beta(x) - \beta(y) \approx \lambda^{-a}(\beta(\lambda x)) - \beta(\lambda y))$, $a = 1/2$.

Anti Kupiainen 6/12

- Model of isotropic turbulence.

$$\begin{cases} \dot{u} = \nu\Delta u - u\cdot\nabla u - \nabla p + f \\ \nabla\cdot u = 0 \end{cases}$$

$f(t,x)$: • translation invariant in x, t.
• scale L.

f: Random. $f(t,x)$: $\underset{\text{Gaussian}}{\text{random}}$ variables, E = expectation value.

$$Ef(t,x) = 0.$$
$$E f_i(t,x) f_j(s,y) = B_{ij}(x-y)\,\delta(t-s).$$

Scale L : $B(x-y) = C\left(\frac{x-y}{L}\right)$, C: smooth, decay at ∞.

$\nabla\cdot u = 0 \Leftrightarrow \sum_i \partial_i B_{ij} = 0$

Aside on Brownian motion.

1. $\beta(t) \in \mathbb{R}$ is BM.
Gaussian random variable w/ $E\beta(t)=0$, $E\beta(s)\beta(t) = \min(s,t)$
$E\beta(t)^2 = t$, independent increments.

see Oksendal

$$E(\beta(t)-\beta(s))(\beta(s)-\beta(\tau)) = 0 \quad t>s>\tau$$
$$E(\beta(t)-\beta(s))^2 = t-s, \quad t>s.$$

$$\beta(t)-\beta(s) = \int_s^t d\beta(\tau) \qquad E\,d\beta(\tau)d\beta(\tau') = \delta(\tau-\tau')\,d\tau\,d\tau'$$

$$d\beta(\tau) = \dot\beta(\tau)\,d\tau \qquad \dot\beta : \text{white noise}, \quad \dot\beta(\tau)\dot\beta(\tau') = \delta(\tau-\tau')$$

$$d\beta(t)^2 = dt.$$

2. $\beta(t) \in \mathbb{R}^N$. $E\beta_i(t)\beta_j(s) = C_{ij}\min(t,s)$, $C \geq 0$.
3. BM in a function space.

$\beta(t,x)$, $x \in \Omega \subset \mathbb{R}^n$.

$$E\beta(t,x)\beta(s,y) = C(x,y)\min(t,s).$$

Stochastic Diff. Eqn.

ODE $\quad t \longrightarrow u(t) \in \mathcal{M}$.

$f(t)$, $v(t) \in \mathrm{Vect}\,\mathcal{M}$.

$$\dot{u} = v(t,u) + f(t).$$

SDE. $\quad du = v(t,u)dt + d\beta(t)$.

$$u(t) = u(0) + \int_0^t v(t', u(t'))dt' + \int_0^t d\beta(t')$$

NS. $\quad \mathcal{M} = \mathrm{Vect}\,\Omega$

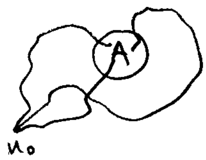
$$v(u,t,x) = \nu\Delta u(t,x) - u\cdot\nabla u - \nabla p \quad , \quad \beta \text{ with } C(x,y) = C\left(\frac{x-y}{L}\right).$$

Solution of SDE is (Markov process) — whatever happens in the future depends on the past through present.

Transition Probabilities:

$$P_t(u_0, A) = \mathrm{Prob}\,(\,u(t)\in A \mid u(0) = u_0\,)$$

$$= \text{probability measure on } \mathcal{M} \quad (\text{on } L^2(\Omega)).$$



$$\mu_t(A) = \int_{\mathcal{M}} \mu_0(du_0)\cdot P_t(u_0, A) \qquad \mu_0 : \text{probability measure on } \mathcal{M}.$$

Ergodicity

Stationary measure μ^*, $\quad \mu_t = \mu_0 = \mu^* \quad \forall t$.

$$P_t(u_0, \cdot) \xrightarrow[t\to\infty]{} \mu^* \quad \text{a.e. } \mu_0.$$

$\boxed{\text{THM}}$ $\quad n = 2$. For a "reasonable" C, including arbitrary big (Re)?
the NS SDE is ergodic $\quad$ (B,K), (E, SINAI), (KS)

Properties of μ^*

- Notation: $u_0 \longrightarrow u(t)$ depend on $\{\beta(s) \mid s\in[0,t]\}$.

$$E_{\{\beta(s) \mid s\le t\}} F = \langle F\rangle_t.$$

$$E_{\mu^*} F = \int \mu^*(du)\, F(u) = \langle F\rangle = \lim_{t\to\infty} \langle F\rangle_t.$$

Energy Balance

(Aside) $\quad$ Ito formula 1. $\beta(t)\in\mathbb{R}$, $\quad du = v(u,t)dt + d\beta(t)$. $\qquad (d\beta)^2 = dt$.

$$F: \mathcal{M} \longrightarrow \mathbb{R}, \quad F(u), \quad dF = (v\,dt + d\beta)\,F'(u) + \frac{1}{2}\underbrace{d\beta(t)^2}_{dt} F''(u)$$

2. $\beta_1, \cdots, \beta_N \in \mathbb{R}^N$

$$dF = (v_i\,dt + d\beta_i)\,\partial_i F + \frac{1}{2} C_{ij} \frac{\partial^2 F}{\partial u_i \partial u_j}, \quad d\beta_i\, d\beta_j = C_{ij}\,dt$$

3. $\beta(t,x)$. $dF = \int_\Omega (v_i(t,x)dt + d\beta_i(t,x)) \frac{\delta F}{\delta u_i(x)} + \frac{1}{2} \iint_{\Omega\times\Omega} C_{ij}\left(\frac{x-y}{L}\right) \frac{\delta^2 F}{\delta u_i(x)\delta u_j(y)}$ (3)

Ex. $F = \frac{1}{2} u(t,x)^2 = e(t,x)$

$$de = \left(-\nu(\nabla u)^2 dt + u\cdot d\beta + \nabla\cdot j\, dt\right) + \frac{1}{2}\mathrm{Tr}\, C(0)$$

$\langle de\rangle_t$: $\langle u\, d\beta\rangle = 0$. $u(t)$: $\beta(s)$, $s \le t$.

$d\beta = \beta(t+\varepsilon) - \beta(t)$.

$u_0 = 0 \Rightarrow \langle\ \rangle_t$ is translation invariant.

$\langle j(x)\rangle_t$: x-independent.

$\langle \nabla\cdot j\rangle_t = 0$. $\varepsilon = \frac{1}{2}\mathrm{Tr}\, C(0)$ = rate of injection of energy by F to the system.

$$\frac{d}{dt}\langle e(t,x)\rangle_t = -\nu\langle \nabla u(t,x)^2\rangle_t + \varepsilon$$

stationary state

$$\underbrace{\nu\langle(\nabla u(t,x))^2\rangle}_{\text{dissipation}} = \underbrace{\varepsilon}_{\text{injection}} \leftarrow \text{Energy balance.}$$

Hopf equations

$$G_N(t,x) = \langle \prod_{i=1}^{N} u(t,x_i)\rangle_t$$

Ito $\dot{G}_N = \sum_{i=1}^{N} \langle(-\nu\Delta_i u(t,x_i) - P(u\cdot\nabla u)(t,x_i))\cdot \prod_{j\ne i} u(t,x_j)\rangle$

$$+ \frac{1}{2}\sum_{i<j} C\left(\frac{x_i - x_j}{L}\right)\langle \prod_{k\ne i,j} u(t,x_k)\rangle$$

$$= -\nu\sum_{i=1}^{N}\Delta_i G_N + D G_{N+1} + \sum C\left(\frac{x_i-x_j}{L}\right) G_{N-2}(\hat{i}\hat{j})$$

Prop. (Kolmogorov 4/5-Law)

Under suitable smoothness assumptions, $n=3$.

$$\lim_{L\to\infty}\lim_{\nu\to 0} S_3(x) = -\frac{4}{5}\varepsilon r.$$

where $S_3(x) = \left\langle\left(\frac{x}{|x|}(u(x) - u(0))\right)^3\right\rangle$ $r = |x|$.

$\boxed{\text{Idea}}$. $N=2$ Hopf equation.

G_3 in terms of G_2

Symmetry allows to solve S_3

Step 1. $(\text{Hopf})_2$

$$\mathrm{Tr}\, C\left(\frac{x-y}{L}\right) = +\nu\overbrace{(\Delta_x+\Delta_y)\langle u(x)\cdot u(y)\rangle}^{-2\nu\langle \nabla u(x)\nabla u(y)\rangle}$$
$$+\underbrace{\langle (u(x)\cdot\nabla_x)u(x)\cdot u(y)\rangle + \langle (u(y)\cdot\nabla_y)u(x)\cdot u(y)\rangle}_{b}$$
$$+\underbrace{\langle \nabla p(x)\cdot u(y)\rangle + \langle \nabla p(y)\cdot u(x)\rangle}_{=0 \quad \nabla\cdot u=0}$$

Assume
1. $\nu>0 \Rightarrow G_N(x)$ are smooth (& translation invariant)
2. $\lim_{\nu\to 0}\langle \nabla u(x)\nabla u(y)\rangle$ exists if $x\neq y$.

HW $\nabla\cdot u=0 \Rightarrow b = -\frac{1}{2}\nabla_x\cdot\langle \delta u(\delta u)^2\rangle$

$$\nu\langle \nabla u(x)\cdot\nabla u(y)\rangle - \frac{1}{4}\nabla_x\langle \delta u(\delta u)^2\rangle = \frac{1}{2}\mathrm{Tr}\, C\left(\frac{x-y}{L}\right)$$

1. $\nu>0$, $x\to y$. $\nu\langle(\nabla u(x))^2\rangle = \varepsilon$. Energy Balance.
2. $x\neq y$, $y=0$. $\nu\to 0$ $\quad -\frac{1}{4}\nabla\langle \delta u(\delta u)^2\rangle = \frac{1}{2}\mathrm{Tr}\, C\left(\frac{x-y}{L}\right) \to$ relation (A,B,C)
$$= \varepsilon + O\left(\frac{x}{L}\right)$$

$$S_3(r) = \langle(\hat{x}\cdot\delta u)^3\rangle$$

$$b_{\alpha\beta,\gamma}(x-y) = \langle u_\alpha(x)u_\beta(y)u_\gamma(y)\rangle$$
$$\overset{O(3)}{=} A(r)\,\delta_{\alpha\beta}\hat{x}_\gamma + B(r)(\delta_{\alpha\gamma}\hat{x}_\beta + \delta_{\beta\gamma}\hat{x}_\alpha) + C(r)\hat{x}_\alpha\hat{x}_\beta\hat{x}_\gamma$$

$$\to -6\,\hat{x}_\alpha\hat{x}_\beta\hat{x}_\gamma b_{\alpha\beta,\gamma}(x) = -6C-12B-6A$$

2. $\partial_\gamma b_{\alpha\beta,\gamma}=0 \to$ divergence ...
$$-\left(\frac{2(A+B)}{r}+A'\right)\delta_{\alpha\beta} + \left(\frac{2(B+C)}{r}-C'-2B'\right)\hat{x}_\alpha\hat{x}_\beta$$
$\Rightarrow$ 2 relations (A,B,C) $\to$ Can be solved!

$$\underline{rC''+7C'+\frac{8}{r}C = \varepsilon + O\left(\frac{r}{L}\right)}$$ regular at $r=0$.
$$\Rightarrow C = -\frac{\varepsilon}{15}r + O\left(\frac{r^2}{L}\right)$$

$$\lim_{L\to\infty}\lim_{\nu\to 0} S_3 = -\frac{4}{5}\varepsilon r + \cancel{O\left(\frac{r^2}{L}\right)}$$

Kolmogorov 41 theory.

$C = C(x)$. $\nabla\cdot C = 0$

$$\lim_{L\to\infty}\lim_{\nu\to 0} S_3(r,\nu,L,C) = -\frac{4}{5}\varepsilon r \quad , \quad \varepsilon = \frac{1}{2}\mathrm{Tr}\, C(0).$$

$$u'(t,x) = \frac{\tau}{\ell} u(\tau t, \ell x)$$

$$f'(t,x) = \frac{\tau^2}{\ell} f(\tau t, \ell x)$$

$$E f'(t,x) f'(s,y) = \frac{\tau^4}{\ell^2} C\left(\frac{\ell(x-y)}{L}\right) \overbrace{\delta(\tau t - \tau s)}^{\frac{1}{\tau}\delta(t-s)}$$

u', f, NS: $L' = L/\ell$, $\nu' = \frac{\tau}{\ell^2}\nu$. , $C' = \frac{\tau^3}{\ell^2} C$. $\varepsilon' = \frac{\tau^3}{\ell^2}\varepsilon$.

$$S_n(r,\nu,L,C) = \langle(\hat{x}\cdot\delta u)^n\rangle_{\nu,L,C}$$

$$= \left(\frac{\ell}{\tau}\right)^n S_n\left(\frac{x}{\ell}, \frac{\tau}{\ell^2}\nu, \frac{L}{\ell}, \frac{\tau^3}{\ell^2}C\right)$$

$\boxed{K41}$ $\exists \lim_{L\to\infty}\lim_{\nu\to 0} S_n(r,\nu,L,C)$ exists.

$$S_n(r,C) = \left(\frac{\ell}{\tau}\right)^n S_n\left(\frac{r}{\ell}, \frac{\tau^3}{\ell^2}C\right) \quad , \quad \ell = r, \ \frac{\tau^3}{\ell^2} = \varepsilon^{-1}.$$

$$= (\varepsilon r)^{n/3} \underbrace{S_n\left(1, \frac{C}{\varepsilon}\right)}_{\equiv A_n} \quad , \quad \frac{C}{\varepsilon} = \frac{C(x)}{\mathrm{Tr}\, C(0)}.$$

unit injection

$\boxed{\nu \neq 0,\ L \neq \infty}$ $S_n(r,\nu,L,C) = (\varepsilon r)^{n/3} S_n\left(1, \left(\frac{\eta}{r}\right)^{4/3}, \frac{L}{r}, \varepsilon^{-1}C\right)$, $\eta = \varepsilon^{-1/4}\nu^{3/4}$

Inertial range, $\underset{\text{dissipation}}{\eta} \ll r \ll \underset{\text{injection}}{L}$. Kolmogorov scale

$$R = \frac{L^{4/3}\varepsilon^{1/3}}{\nu} = \frac{UL}{\nu} \qquad \eta = R^{-3/4}L. \qquad R = 10^8$$

$S_n(r) \sim r^{\zeta_n}$, $\zeta_n = n/3$.

$\zeta_2 \approx 0.7$ (0.66), $\zeta_3 = 1$. $\zeta_4 \approx 1.28$ (1.33) $\zeta_7 \sim 2.0$ (2.33) gets worse.

$S_n = \langle(\ )^n\rangle$. ζ_n : convex or concave in n. ← Hölder.

$$S_n(r,\nu,L,C) = (\varepsilon r)^{n/3}\left(\frac{L}{r}\right)^{\frac{n}{3}-\zeta_n}\left(1 + O\left(\frac{r}{L}, \frac{\eta}{r}\right)\right)$$

$\left(\frac{L}{r}\right)^{\gamma_n - \frac{n}{2}\gamma_2}$ concave, convex $\left(\frac{r}{\eta}\right)^{\zeta_n - n/3}$

Calculate this #.

Intermittence. $S_n(r)/S_2(r)^{n/2}$

if $\hat{r}\cdot\delta u$ Gaussian $\Rightarrow$ CONSTANT