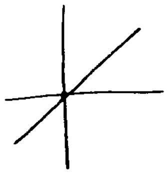
Lecture Note [Jun Li at CMS 6/14 ~ 18] by Kim, Yon Seo ①

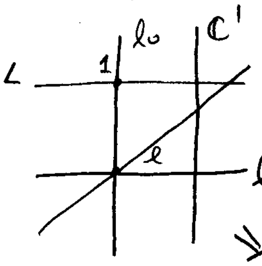
- Moduli spaces

<u>Ex</u>: Moduli space of lines in $\mathbb{C}^{n+1}$ through 0



$l \subset \mathbb{C}^{n+1}$

- all lines in $\mathbb{C}^{n+1} = \mathbb{CP}^n$
- compact, has a canonical complex structure.



$\mathbb{C}^2$ $\mathbb{C}^2$ $(n=1)$

$\mathbb{CP}^1 \setminus \{l_\infty\} \cong \mathbb{C}^1$

$\mathbb{CP}^1 \setminus \{l_0\} \cong \mathbb{C}^1$

$\Rightarrow$ $\mathbb{CP}^1$ is a compact complex manifold.

* Grassmannian :

moduli space of all $\mathbb{C}^r \subset \mathbb{C}^n$, denoted by $Gr(r,n)$

<u>e.g.</u> $r=1$. $Gr(1,n) \cong \mathbb{CP}^{n-1}$

- Four important classes of moduli spaces.

1. $\mathcal{M}_{g,n}$ = moduli space of equiv. classes of n-pointed genus g smooth curves
2. $\mathcal{M}_C(r,d)$ = moduli space of rank r, degree d stable vector bundles on C
3. $\mathcal{M}_{g,n}(X,A)$ = moduli of stable maps to X.
4. $\mathcal{M}_X(r,c_1,c_2)$ = moduli of stable sheaves on X.

* $\mathcal{M}_{g,n} = \{ (C, x_1, \cdots, x_n) \mid C$ is a smooth curve (cpt Riemann surface with hol. structure) of genus g, $x_1, \cdots, x_n \in C$ are n distinct marked points $\} / \sim$

$(C_1, x_1, \cdots, x_n) \sim (C_2, y_1, \cdots, y_n) \iff \boxed{\begin{matrix} (C_1, & x_1, \cdots, x_n) \\ \updownarrow \cong & \text{biholomorphism.} \\ (C_2, & y_1, \cdots, y_n) \end{matrix}}$

* $\mathcal{M}_C(r,d) = \left\{ \begin{matrix} V \\ \downarrow \\ C \end{matrix} \;\middle|\; \begin{matrix} V: \text{holomorphic vector bundle on } C \\ \operatorname{rk} V = r, \quad \deg V = d, \quad V: \text{stable} \end{matrix} \right\} / \sim$

C: smooth curve.

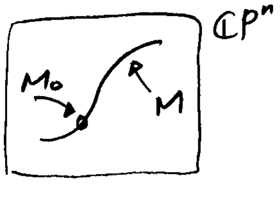
$$\begin{array}{ccc} V_1 & \xrightarrow{\cong} & V_2 \\ \downarrow & & \downarrow \\ C & \xrightarrow{id} & C \end{array} \iff V_1 \overset{\text{iso.}}{\cong} V_2 \text{ as holomorphic vector bundle.}$$

<u>Rmk</u>. Intuitively, $\mathcal{M}_{g,n}$ and $\mathcal{M}_C(r,d)$ are topological spaces.

$\boxed{Q1}$ Do $\mathcal{M}_{g,n}$ & $\mathcal{M}_C(r,d)$ admit canonical algebraic (complex) structures? If yes, how to construct them?

$\boxed{A}$ Both $\mathcal{M}_{g,n}$ & $\mathcal{M}_C(r,d)$ are quasi-projective varieties.

(smooth projective variety $M \subset \mathbb{CP}^n$ cpt. submfd
quasi- " " $M \setminus \{\text{lower dimensional}\}$



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Main Steps to prove these existence theorems: $(n=0,\ g\geq 1)$.

0. Speak of the moduli space of all degree d curves in $\mathbb{C}P^2$

$C \triangleq p^{-1}(0)$, where $p(z_0, z_1, z_2)$ is homogeneous polynomial of degree d.

$\{$ space of all degree d homogeneous polynomials $\}/\mathbb{C}^*$ ← $p_1^{-1}(0) = p_2^{-1}(0) \Leftrightarrow p_1 = c\cdot p_2$

1. $[C] \in \mathcal{M}_g$, $K_C^{\otimes n}$ ← fixed $\gg 1$. K_C ← canonical line bundle

$\exists$ n-canonical embedding $C \hookrightarrow \mathbb{P}H^0(K_C^{\otimes n})$

By Riemann-Roch, $\dim H^0(K_C^{\otimes n}) = N+1$ ← independent of choice of C

2. Want to keep the background space same.

We pick $\mathbb{P}H^0(K_C^{\otimes n}) \cong \mathbb{P}^N$, this way we get each $[C] \in \mathcal{M}_{g,n}$ is associated to a set of n-canonical curves that comes from $C \hookrightarrow \mathbb{C}P^N$ w/ different choices of $\mathbb{P}H^0(K_C^{\otimes n}) \overset{\varphi}{\cong} \mathbb{C}P^N$

$$\begin{array}{cc} \bullet[C] & \bullet[C'] \\ \downarrow & \downarrow \\ C \subset \mathbb{P}H^0(K_C^{\otimes n}) & C' \subset \mathbb{P}H^0(K_{C'}^{\otimes n}) \\ \text{SII} & \text{SII} \\ \mathbb{C}P^N & \mathbb{C}P^N \end{array}$$

NOT canonical up to certain choice

<u>Key</u>
- Two φ's differ by an element of $PGL(N+1, \mathbb{C})$
- $[C] \rightsquigarrow \{ C \underset{\text{Canonical}}{\subset} \mathbb{C}P^N \}$ ← $\cong PGL(N+1, \mathbb{C})$
- First take all $\cong$s and then mod out later.

3. We let $\mathcal{H}ilb_{can} = \{ C \overset{\cdot}{\subset} \mathbb{C}P^N \mid C \text{ smooth},\ g(C) = g,\ i^*\mathcal{O}(1) \equiv K_C^{\otimes n} \}$

↻ ↳ Hilbert scheme, is a quasi-projective variety.

$\sigma \in GL(N+1, \mathbb{C})$

$$\begin{array}{ccc} C \overset{i}{\subset} \mathbb{P}^N & & \\ \cong \downarrow \sigma & & /\mathbb{C}^* \\ C \overset{\sigma\circ i}{\subset} \mathbb{P}^N & & PGL(N, \mathbb{C}) \end{array}
\qquad
\begin{array}{ccc} \mathcal{H}ilb_{can} & \ni & \{ C \subset \mathbb{C}P^N \} \\ \pi \downarrow & & \downarrow \\ \mathcal{M}_g & \ni & [C] \end{array}$$

<u>Rmk</u>. $\pi^{-1}([C]) = PGL(N+1, \mathbb{C}) \cdot \{ C \subset \mathbb{C}P^N \}$

<u>Claim</u> $\mathcal{M}_g = \mathcal{H}ilb_{can} / PGL(N+1, \mathbb{C})$.

<u>Note</u> ① We can show that $\mathcal{H}ilb_{can}$ is quasi-projective by reducing the problem to Grassmannian.

② The quotient $\mathcal{H}ilb / PGL(N+1)$ could be very bad because $PGL(N+1)$ is not a compact group.

$\boxed{Q}$ $G \curvearrowright W$ ← nice variety. When does W/G exist as a nice algebraic variety

$\boxed{A}$ This is answered by Mumford's GIT theory.

① There is a notion of G-stable points in W
$W^s \underset{\text{open}}{\subsetneq} W$.

$$\begin{array}{ccc} W & \subset & \mathbb{P}^M \\ \circlearrowleft & ! & \circlearrowleft \\ G & \subset & SL(M+1) \end{array}$$

② W^s/G exists as a quasi-projective variety.

$$\begin{array}{c} \mathbb{P}^M \\ \cup \text{ open} \end{array}$$

semi-stable.

$$\begin{tikzcd} W^s \arrow[r, phantom, "\subset"] \arrow[d, "\text{ordinary}"'] & (\mathbb{P}^M)^s \arrow[r, phantom, "\underset{\text{open}}{\subset}"] \arrow[d, "\text{quotient}"'] & (\mathbb{P}^M)^{ss} \arrow[r, phantom, "\supset"] \arrow[d] & W^{ss} \arrow[d] \\ W^s/G \arrow[r, phantom, "\subset"] & (\mathbb{P}^M)^s/G \arrow[r, phantom, "\underset{\text{open}}{\subset}"] & (\mathbb{P}^M)^{ss}/\!/G \arrow[r, phantom, "\underset{\text{closed}}{\supset}"] & W^{ss}/G \end{tikzcd}$$

$(\mathbb{P}^M)^s/G$: quasi-proj. $(\mathbb{P}^M)^{ss}/\!/G$: proj. ← GIT-quotient. //

• <u>Local structure of the moduli spaces.</u>

$$\begin{array}{ccc} \mathcal{X} & \supset & \mathcal{X}_{s_0} = [\xi] \in \text{Moduli} \\ \downarrow & & \downarrow \\ S & \ni & s_0 \end{array}$$

I. Construct a local semi-universal family.

II. Such construction is based on deformation theory
Usually, $T_{[s_0]}S$ = given by a cohomology theory $\mathbb{H}^1(-)$

Ex $\mathcal{M}_g \ni [C]$
$T_{[s_0]}S$ = "1st order deformation of C" $= H^1(C, T_C) \overset{\text{open}}{\supset} S$
$(\mathcal{M}_C(r,d) \ni [V]$, $T_{[V]}S = H^1(C, V^\vee \otimes V) \overset{\text{open}}{\supset} S\ )$

• $S \underset{\text{open}}{\subset} \mathbb{C}^R$ is <u>NOT true</u> in general
↳ To deal with these situation, we need to use <u>obstruction theory</u>

III. There is an obstruction space, usually $\mathbb{H}^2(-)$, and a Kuranishi map.

$D \xrightarrow{\exists K} \mathbb{H}^2$ ← holo. $\quad 0 \in D \underset{\text{open}}{\subset} \mathbb{H}^1 \quad$ $K(0) = 0$, $dK(0) = 0$ such that $S = K^{-1}(0)$.

IV. $S/\mathrm{Aut}\,\xi \xrightarrow{\text{open}} \mathcal{M}$, $\xi \in \mathcal{M}$.

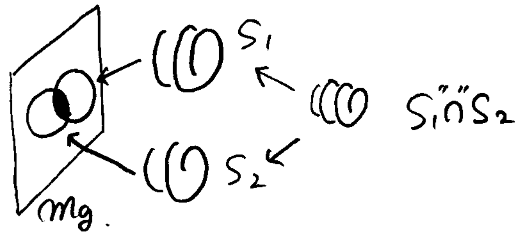
• <u>Global geometry of the moduli spaces.</u>

* "Universal family of $\mathcal{M}$" is essential to study $\mathcal{M}$.
<u>Problem</u>: If $\mathrm{Aut}\,\xi \neq \{1\}$, then universal family may not exist.

$\boxed{\text{Compromise}}$ I. Deligne-Mumford stack ← $\overline{\mathcal{M}}_{g,n}$

$[C] \in \mathcal{M}_g$
$\mathrm{Aut}(C)$: finite

$$\begin{array}{ccc} \mathcal{X} & \supset & \mathcal{X}_{s_0} \cong C \\ \downarrow & & \downarrow \\ \mathrm{Aut}(C) \circlearrowright S & \ni & s_0 \end{array}$$



$$\begin{tikzcd} & S \arrow[dl, "/\mathrm{Aut}(C)"'] \arrow[dr, "i"] & \\ S/\mathrm{Aut}(C) \arrow[rr, hook, "\text{open}"'] & & \mathcal{M}_g \end{tikzcd}$$

local universal family exists everywhere over finite branched coverings.

<u>New language</u>: stack, we view $S \xrightarrow{i} \mathcal{M}_g$ as an <u>open stack</u>

$$\begin{tikzcd} & x_1 \arrow[r] & S_1 \arrow[dr] & \\ x_{12} \arrow[r] & S_1 \cap S_2 \arrow[ur] \arrow[dr] & & \mathcal{M} \\ & x_2 \arrow[r] & S_2 \arrow[ur] & \end{tikzcd}$$

$\mathcal{M}_g$ is called a <u>Stack</u> (treat like a manifold), topological space.
$\boxed{\text{lose}}$ intuition. $\boxed{\text{gain}}$ universal family.

II. Descent method — $\mathcal{M}_C(r,d)$

$\left[\begin{smallmatrix} V \\ \downarrow \\ C \end{smallmatrix}\right] \in \mathcal{M}_C(2,0)$ $\qquad \mathrm{Aut}(V) = \begin{cases} \{1\} & \text{if } V\text{: stable} \\ \mathbb{C}^* & \text{if } V = I_1 \oplus I_2 \\ PGL(2,\mathbb{C}) & \text{if } V = I \oplus I \end{cases}$ } infinite group.

$\left[\begin{smallmatrix} \mathcal{O}_C^{\oplus N} \to V \\ \downarrow \\ C \end{smallmatrix}\right] \ni \mathrm{Quot}$ ← quotient vector bundle, $\mathcal{O}_C^{\oplus N}$ ← <u>fixed</u>.

$$\begin{array}{c} \mathrm{Quot} /\!/ SL(N) \\ \downarrow \\ \mathcal{M}_C(2,0) \end{array}$$

There is a universal family over Quot:

$$\begin{array}{c} \mathcal{O}_{\mathrm{Quot}\times C}^{\oplus N} \longrightarrow \mathcal{F} \circlearrowleft SL(N) \\ \downarrow \\ \mathrm{Quot}\times C \circlearrowleft SL(N) \end{array}$$

<u>We have</u>
- $\mathcal{M}_C(2,0) = \mathrm{Quot} /\!/ SL(N)$
- $\begin{array}{c} \mathcal{E} \\ \downarrow \\ \mathrm{Quot}\times C \end{array}$ ← $SL(N)$-equivariant.
- Use $\mathcal{E}$ to construct $SL(N)$-equivariant geometric objects of Quot.
- Investigate when these objects descends to $\mathcal{M}_C(r,d)$ (descent theory)

[$\boxed{\text{Application}}$ when $(r,d)=1$, the moduli space $\mathcal{M}_C(r,d)$ admits a universal family.]

- Some examples on the utility of the universal families.

$\mathcal{M}_C(2,1) \cong \mathcal{M}_C(2,3) \cdots \cong \mathcal{M}_C(2,d)$, d: odd. smooth.

$\begin{array}{c} V \\ \downarrow \\ C \end{array}$ $\quad V \otimes \mathcal{O}_C(1)$ degree 3.

$$\begin{tikzcd} & \mathcal{E} \arrow[d] & \\ & \mathcal{M}_C(2,1)\times C \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathcal{M}_C(2,1) & & C \end{tikzcd}$$

← rank 2.

$\pi_1^*\mathcal{L} \otimes \mathcal{E}$ ← also universal. twist of $c_1(\mathcal{E})$
$\mathcal{L} \in \mathrm{Pic}(\mathcal{M}_C(2,1))$
Hence $c_1(\mathcal{E})$ doesn't contain much information.

$c_1(\mathcal{E})$ normalize

$$c_2(\mathcal{E}) \in H^4(\mathcal{M}_C(2,1)\times C) = \bigoplus_{k=0}^{2} H^{4-k}(\mathcal{M}_C(2,1)) \otimes H^k(C)$$

$$\hookrightarrow c_2(\mathcal{E}) = f \boxtimes 1 + \sum_{j=1}^{2g} \psi_j \boxtimes \zeta_j + \eta \boxtimes \omega$$

$f, \psi_1, \dots, \psi_{2g}, \eta \in H^*(\mathcal{M}_C(2,d))$, $1 \in H^0(C)$, $\tau_j \in H^1(C)$, $\omega \in H^2(C)$.

$$c_1(\mathcal{E}) = h \otimes 1 + \sum \varphi_j \otimes \tau_j + \alpha \cdot 1 \otimes \omega .$$

$\boxed{\text{Thm}}$ $f, \psi_j, \eta, h, \varphi_j$ generate $H^*(\mathcal{M}_C(2,1))$ as a ring.

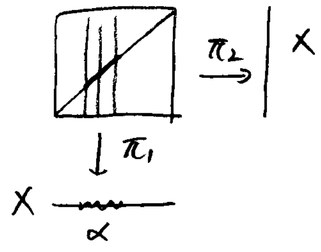
proof). "Diagonal technique"

(X: smooth variety, $\Delta \subset X \times X$: diagonal

$[\Delta]^\vee \in H^*(X \times X)$, $[\Delta]^\vee = \sum_i a_i \otimes \tilde{a}_i$, $a_i, \tilde{a}_i \in H^*(X)$)

Poincaré dual

$\Rightarrow H^*(X)$ is generated by $\{a_i\}$ as a vector space.



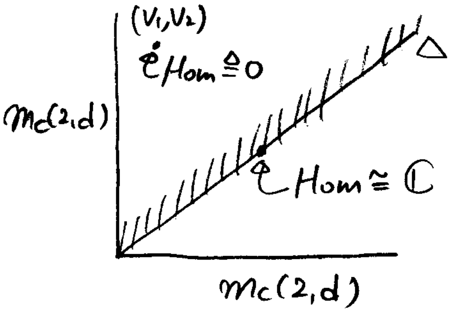
$$\pi_{2*}(\pi_1^* \alpha \cup [\Delta]^\vee) = \alpha .$$

In $\mathcal{M}_C(2,d)$ case, $\Delta \subset \mathcal{M}_C(2,d) \times \mathcal{M}_C(2,d)$

$$[\Delta]^\vee \in H^*(\mathcal{M}_C(2,1) \times \mathcal{M}_C(2,1))$$

Trick $[V_1], [V_2] \in \mathcal{M}_C(2,d)$ with d: odd

$$\mathrm{Hom}_C(V_1, V_2) \triangleq H^0(C, V_1^\vee \otimes V_2) = \begin{cases} 0 & \text{if } [V_1] \neq [V_2] \\ \mathbb{C} & \text{if } [V_1] = [V_2] . \end{cases}$$



$\exists\, \mathcal{E}$ universal

$$\mathcal{E} \downarrow \mathcal{M}_C(2,1) \times C \qquad \mathcal{M}_C(2,1) \times \mathcal{M}_C(2,1) \times C .$$

$$\begin{array}{ccc} \pi_{12*}(\pi_{13}^* \mathcal{E}^\vee \otimes \pi_{23}^* C) & & \pi_{12*}(\pi_{13}^* \mathcal{E}^\vee \otimes \pi_{23}^* \mathcal{E}) \xrightarrow{\text{restrict}} \mathrm{Hom}(V_1, V_2) \\ \downarrow & & \downarrow \qquad\qquad\qquad \downarrow \\ \mathcal{M}_C(2,1) \times \mathcal{M}_C(2,1) \times C & & \mathcal{M}_C(2,d) \times \mathcal{M}_C(2,d) \ni (V_1, V_2) \end{array}$$

$$[\Delta]^\vee = c_m(\pi_{12*}(\pi_{13}^* \mathcal{E}^\vee \otimes \pi_{23}^* \mathcal{E})) .$$

Use Grothendieck–Riemann–Roch Thm. to express Chern classes in terms of Chern classes of $\mathcal{E}$.

$\Rightarrow H^2(\mathcal{M}_C(2,d))$ is generated as a ring by the Künneth components of c_i(universal).