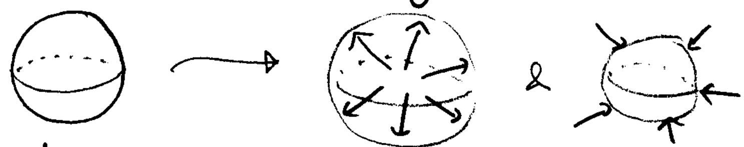
Lecture note [H. Nicolai at CMS 6/7 ~ 6/14] by Kim, Yon Seo

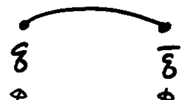
1) Mini - Introduction to String Theory

Q Are basic constituents of matter

- pointlike $\xrightarrow[\text{lead to}]{}$ ✓ (relativistic) Quantum Field Theory.
  ✓ "Standard Model of Elementary Particle Physics"
  ↳ extremely well-tested framework ($\approx 10^{-18}$m)
- membrane ⟶ Dirac's model of electron to solve 'divergent' self-energy
  balance between electric charge and tension



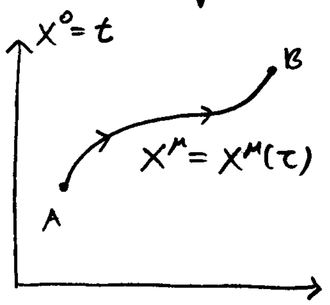
- string-like ⟶ motivated by "meson" (



)
  (c.f. Veneziano formula '69)
  ↳ * Superstring Theory (1978)
  - 1-dim'l extended object
  - (perturbatively) finite quantum gravity ?
  * Supermembranes
  - non-perturbative ? ⟶ "M-theory" (look W. Taylor's talk

Rmk. "super-" means putting Bosons & Fermions in a way they make symmetry

2) Warm-up, Relativistic Point Particle



$\mathbb{R}^d$ = Minkowski space w/ $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$, $(-+\cdots+)$

$X^\mu(\tau)$: invariant under reparametrization of τ.

$$S = -m\int_A^B ds = -m\int_{\tau_0}^{\tau_1} \sqrt{-\eta_{\mu\nu}\dot{X}^\mu\dot{X}^\nu}\, d\tau = \int \mathcal{L}\, d\tau.$$

(write $\dot{X}^2 = \eta_{\mu\nu}\dot{X}^\mu\dot{X}^\nu = \dot{X}^\mu\dot{X}_\mu$)

Momentum : $$p^\mu \triangleq \frac{\partial\mathcal{L}}{\partial\dot{X}_\mu} = \frac{m\dot{X}^\mu}{\sqrt{-\dot{X}^2}}$$

↳ Hence $p^2 + m^2 = p^\mu p_\mu + m^2 = 0$ ⟵ constraint $\phi = \phi(X^\mu, p^\mu)$

Canonical Hamiltonian : $H_{can} = \dot{X}^\mu P_\mu - \mathcal{L} = 0$

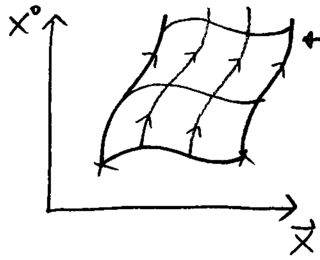
① **Dirac** : constrained Hamiltonian Systems.

- $\underline{\mathcal{H} = H_{can} + \frac{N}{2m}\phi}$, $N = N(x^0) \leftarrow$ depends on parametrization of τ

$$\rightarrow \begin{cases} \dot{x}^\mu = \{x^\mu, \mathcal{H}\} = \frac{N}{m} p^\mu = N\dot{x}^\mu / \sqrt{-\dot{x}^2} \\ \dot{p}^\mu = \{p^\mu, \mathcal{H}\} = 0 . \end{cases}$$

Rmk.. Appearance of constraints usually means local symmetry, gauge invariance

② Nambu-Goto action. (~1970)



$\leftarrow$ World-sheet Σ

$X^\mu = X^\mu(\tau, \sigma) = X^\mu(\sigma^\alpha)$, $\begin{pmatrix} \sigma : \text{coordinate in the String} \\ \sigma^0 = \tau \end{pmatrix}$

$\begin{cases} 0 \le \sigma \le \pi & \text{for open string} \\ 0 \le \sigma \le 2\pi & \text{for closed string} , \quad X^\mu(\tau, \sigma + 2\pi) = X^\mu(\tau, \sigma) \end{cases}$

$$S_{NG} = -T \int dA = -T \int d^2\sigma \sqrt{- \det\left(\frac{\partial X^\mu}{\partial \sigma^\alpha}\frac{\partial X^\nu}{\partial \sigma^\beta}\eta_{\mu\nu}\right)} = -T\int d^2\sigma \sqrt{-\det \partial_\alpha X \cdot \partial_\beta X}$$

T: "String tension" $[cm^{-2}]$

slope parameter $\alpha' = \frac{1}{2\pi T}$

$-$: causal propagation

$\det(\ldots)$: considering world-sheet as embedded surface

Symmetry : reparametrization of coordinate.

③ Polyakov action

$$\begin{cases} X^\mu = X^\mu(\tau, \sigma) \leftarrow \text{consider as "scalar fields" on the world sheet } \Sigma \\ h_{\alpha\beta} = h_{\alpha\beta}(\tau, \sigma) \leftarrow \text{introducing auxiliary metric on } \Sigma . \end{cases}$$

$$S_p = -\frac{T}{2}\int d^2\sigma \underbrace{\sqrt{h}}_{\text{coordinate invariance}} \underbrace{h^{\alpha\beta}\partial_\alpha X \cdot \partial_\beta X}_{\text{intrinsic area of scalar fields}} \qquad \text{where } h \equiv -\det h_{\alpha\beta} .$$

Rmk. Easy to work on since there is no $\sqrt{-\det \partial_\alpha X \partial_\beta X}$ term.

- Equations of motion

$$\delta X^\mu \longrightarrow \underbrace{\frac{1}{\sqrt{h}}\partial_\alpha\left(\sqrt{h}\, h^{\alpha\beta}\partial_\beta\right.}_{\text{Laplace-Beltrami operator on } \Sigma}\left. X^\mu\right) = 0 .$$

$$\delta h_{\alpha\beta} \longrightarrow T_{\alpha\beta} = \partial_\alpha X \cdot \partial_\beta X - \frac{1}{2} h_{\alpha\beta} h^{\gamma\delta} \partial_\gamma X \cdot \partial_\delta X = 0 .$$

Rmk · S_p is identical "on shell" to S_{NG}

$$\det \partial_\alpha X \cdot \partial_\beta X = -\frac{1}{4} h \left(h^{\gamma\delta}\partial_\gamma X \cdot \partial_\delta X\right)^2 \longrightarrow \text{NG-action}$$

ex] Show that $S_p \ge S_{NG}$ and "=" holds when they satisfy Equations of Motion. (Hint : Show $(\mathrm{Tr} A)^2 \ge 4 \cdot \det A \leftarrow$ Symmetric

Symmetries of S_P

* Reparametrizations $\quad \sigma^\alpha \longrightarrow \sigma'^\alpha(\sigma^\alpha) \quad \leftarrow$ 2 parameters.
* Weyl invariance $\quad h_{\alpha\beta}(\sigma) \longrightarrow h'_{\alpha\beta}(\sigma) = e^{2\Lambda(\sigma)} h_{\alpha\beta}(\sigma) \leftarrow$ 1 param

  ↳ only true for strings.
* Global Poincaré Invariance $\quad X^\mu \longrightarrow X'^\mu = \Lambda^\mu{}_\nu X^\nu + a^\mu$,

  ↳ only true when target space = Minkowski space $(\Lambda^\mu{}_\nu \in SO(1,d-1))$

→ gauge-invariance

- get rid of $\underbrace{h_{\alpha\beta}}_{3 \text{ param.} = 2+1} \xrightarrow{\text{dift}} \underbrace{\Omega(\sigma)\eta_{\alpha\beta}}_{\text{"conformal gauge"}} \xrightarrow{\text{Weyl}} \eta_{\alpha\beta}$

Rmk. globally, we're left w/ finite moduli and hence path-integral can be done in finite step.

- Conformal gauge : $\sqrt{h}\, h^{\alpha\beta} = \eta^{\alpha\beta}$

$$\rightarrow S_P = -\frac{T}{2}\int_\Sigma d^2\sigma\, \eta^{\alpha\beta}\partial_\alpha X \cdot \partial_\beta X = \frac{T}{2}\int_\Sigma d^2\sigma (\dot{X}^2 - X'^2)$$

$$= 2T\int_\Sigma d^2\sigma\, \partial_+ X \cdot \partial_- X \quad \text{where } \cdot = \frac{\partial}{\partial \tau},\ ' = \frac{\partial}{\partial \sigma},\ \sigma^\pm = \tau \pm \sigma$$

$$\rightarrow \delta S_P = T\int_\Sigma d^2\sigma\, \delta X^\mu (\partial_\tau^2 - \partial_\sigma^2) X_\mu - T\left[\int d\tau\, X'_\mu \delta X^\mu\right]\Bigg|_{\sigma=0}^{\sigma=\pi,\, 2\pi}$$

Boundary term : closed $\quad X^\mu(\tau, \sigma+2\pi) = X^\mu(\tau,\sigma)$

open $\quad X'_\mu \delta X^\mu \big|_{\sigma=0,\pi} \overset{!}{=} 0$

$$\Rightarrow \partial^\alpha \partial_\alpha X^\mu(\tau,\sigma) = 0 \quad \leftarrow \text{free wave equation} \cdots\cdots (*)$$

✓ 2 possibilities for open boundary condition:

(i) $X'_\mu(\tau,\sigma)\big|_{\sigma=0,\pi} = 0$: Neumann boundary condition

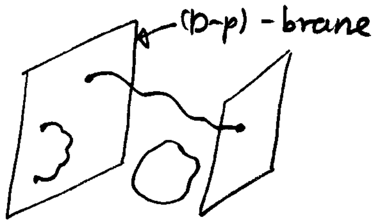
↳ No momentum flowing out of string.

(ii) $\delta X^\mu\big|_{\sigma=0,\pi} = 0$: Dirichlet boundary condition.

↳ fixing boundary.

- It seemed to violate Lorentz invariance, but now physicists think that "restricting boundary is part of theory"
- D-brane.

(iii) Mixed (choosing (i) or (ii) w.r.t. μ) ⟶ (D-p)-brane.



In any case, solutions of (*) are vibrating strings. Assuming Minkowski-space, we can "carry out the program" as follows:

- Another equation of motion

 * $\partial_\alpha \partial^\alpha X^\mu = 0 \Longrightarrow X^\mu(\tau,\sigma) = X^\mu_R(\sigma^-) + X^\mu_L(\sigma^+)$
 * Constraint Equation : $T_{\alpha\beta} = 0$ (check : $h^{\alpha\beta} T_{\alpha\beta} = 0$)

$$\left.\begin{cases} T_{01} = T_{10} = \dot{X}^\mu X'_\mu = 0 \\ T_{00} = T_{11} = \frac{1}{2}(\dot{X}^\mu \dot{X}_\mu + X'^\mu X'_\mu) = 0 \end{cases}\right\} \Rightarrow (\dot{X}^\mu \pm X'^\mu)^2 = 0 .$$

In terms of light-cone coord. $\sigma^\pm$,

$$T_{++} = \partial_+ X \cdot \partial_+ X = 0 , \quad T_{--} = \partial_- X \cdot \partial_- X = 0$$

Energy-Momentum tensor.

Conserved under $\partial_- T_{++} = 0 = \partial_+ T_{--} \Rightarrow T_{\pm\pm} = T_{\pm\pm}(\sigma^\pm)$

- Poisson Brackets

$X^\mu(\tau,\sigma)$, $p^\mu(\tau,\sigma)$

$$\{X^\mu(\tau,\sigma), X^\nu(\tau,\sigma')\} = \{\dot{X}^\mu(\tau,\sigma), \dot{X}^\nu(\tau,\sigma')\} = 0$$

$$\{X^\mu(\tau,\sigma), \dot{X}^\nu(\tau,\sigma')\} = \eta^{\mu\nu}\delta(\sigma-\sigma')$$

spatial direction.

$\boxed{\text{Ex}}$. $f_\pm = f_\pm(\sigma^\pm)$, $L^\pm[f_\pm] = 2\int d\sigma\, f_\pm(\sigma^\pm) T_{\pm\pm}(\sigma^\pm)$

$\rightarrow \{ L^\pm[f_\pm], X\} = -f_\pm(\sigma^\pm)\partial_\pm X$

$$P_\mu = \int d\sigma\, \dot{X}_\mu(\tau,\sigma) , \quad M_{\mu\nu} = \int d\sigma\, (X^\mu \dot{X}^\nu - X^\nu \dot{X}^\mu)$$

Rmk. • Combined momentum of string & D-brane are conserved.

• d=10. target space $= \begin{cases} \mathbb{R}^d \leftarrow \text{Poincaré symmetry.} \\ AdS_5 \times S^5 \ (\text{Isometry } SO(2,4) \times SO(6)) \end{cases}$

Anti de Sitter ← particular solution of Einstein eq.
a lot of symmetries

$p_\mu = \int d\sigma\, \dot{X}^\mu$, $J^{\mu\nu} = \int d\sigma\, (X^\mu \dot{X}^\nu - X^\nu \dot{X}^\mu)$

$X^\mu(\tau, \sigma) = X^\mu_R(\tau-\sigma) + X^\mu_L(\tau+\sigma)$.

• <u>Fourier expansion</u> (or Oscillator expansion)

$$X^\mu_R = \underbrace{q^\mu}_{\text{coord. of CM}} + \underbrace{p^\mu(\tau-\sigma)}_{\text{momentum of CM}} + i \sum_{n\neq 0} \frac{1}{n} \alpha^\mu_n e^{-in(\tau-\sigma)}$$

$$X^\mu_L = q^\mu + p^\mu(\tau+\sigma) + i \sum_{n\neq 0} \frac{1}{n} \bar{\alpha}^\mu_n e^{-in(\tau+\sigma)}$$

coord. of CM, momentum of CM: CMS modes

, $\alpha_0^\mu = \bar{\alpha}_0^\mu = p^\mu$

$X \in \mathbb{R} \Rightarrow (\alpha^\mu_n)^* = \alpha^\mu_{-n}$

we're in Minkowski space. τ, σ.

$\Rightarrow J^{\mu\nu} = e^{\mu\nu} + E^{\mu\nu} + \bar{E}^{\mu\nu}$ with $e^{\mu\nu} = q^\mu p^\nu - q^\nu p^\mu$, $E^{\mu\nu} = \sum_{n=1}^{\infty} \frac{1}{n} (\alpha^\mu_{-n} \alpha^\nu_n - \alpha^\nu_{-n} \alpha^\mu_n)$

"orbital" angular momentum "spin" angular momentum

$\boxed{\text{Ex}}$ Set $X^0 = t$, $J^2 \leq \frac{1}{2}\alpha' M^2$.

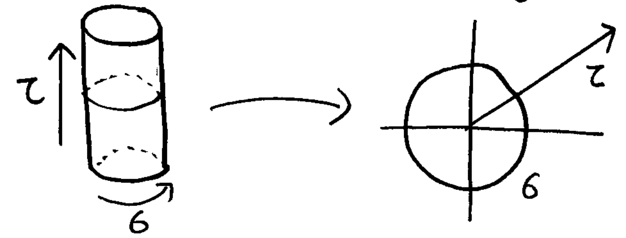
angular momentum $\leq$ internal energy "energy stored in rotation"

<u>Rmk</u>. • Conformal Field Theory.

Wick rotation $\tau \to -i\tau$ makes $-d\tau^2 + d\sigma^2 \to d\tau^2 + d\sigma^2$

Define $z := e^{\tau - i\sigma}$, $\bar{z} = e^{\tau + i\sigma}$

$\Rightarrow$ Left (Right) moving $\to$ Laurent expansion in z, $\bar{z}$



easier to do <u>perturbation theory</u>.

<u>Poisson brackets</u>

$\{q^\mu, p^\nu\} = \eta^{\mu\nu} = (- + \cdots +)$

$\{\alpha^\mu_m, \bar{\alpha}^\nu_n\} = 0$

$\{\alpha^\mu_m, \alpha^\nu_n\} = \{\bar{\alpha}^\mu_m, \bar{\alpha}^\nu_n\} = -i m \eta^{\mu\nu} \delta_{m+n,0}$

From the constraints $T_{\pm\pm} = \frac{1}{2} \partial_\pm X \cdot \partial_\pm X = 0$, it follows that

$\{L_m, L_n\} = -i(m-n) L_{m+n}$ where $L_m = \frac{1}{2}\sum_n \alpha^\mu_{m-n} \alpha_{n\mu}$, $\bar{L}_m = \frac{1}{2}\sum_n \bar{\alpha}^\mu_{m-n} \bar{\alpha}_{n\mu}$

Virasoro relation. "Virasoro generator"

<u>Rmk</u>. • Witt-Virasoro algebra (1939)

• classical constraint $L_m = \bar{L}_m = 0$ for all m.

3) <u>Quantization</u> : 2 possibilities to deal with constraints (as for any gauge thy, e.g. QED, ...)

(i) "covariant" quantization :
quantize first (i.e. simply substitute operators) and impose constraints as operator constraints on physical states.
$\rightsquigarrow$ BRST quantization $\rightsquigarrow$ ghost degrees of freedom.

(ii) "light cone" (or "reduced phase space") quantization :
Solve constraints first, and then quantize.

	advantage	disadvantage
(i)	keep Lorentz invariance	non-physical states
(ii)	only keep physical states	break Lorentz invariance

Equivalence of (i) and (ii) is <u>not</u> gauranteed.
But in String theory, we have (i) $\Leftrightarrow$ (ii) !!

✓ quantization : classical functions $\longrightarrow$ operators
poisson bracket $\{\ ,\ \} \longrightarrow -i[\ ,\ ]$ (let $\hbar=1$)

✓ Canonical commutation relations :

$$[q^\mu, p^\nu] = i\eta^{\mu\nu}, \quad [\alpha_m^\mu, \bar{\alpha}_n^\nu] = 0, \quad [\alpha_m^\mu, \alpha_n^\nu] = [\bar{\alpha}_m^\mu, \bar{\alpha}_n^\nu] = m\eta^{\mu\nu}\delta_{m+n,0}$$

✓ Harmonic oscillators :

$\alpha_{-m}^\mu = (\alpha_m^\mu)^\dagger \leftarrow$ Hermitian conjugate. let $\alpha_m^\mu = \sqrt{m}\, a_m^\mu$

$[a, a^\dagger] = 1$.
$$\begin{cases} \alpha_m^\mu = \text{annihilation operator} & (m \geq 1) \\ \alpha_{-m}^\mu = \text{creation operator} & (m \geq 1) \end{cases}$$

Bosonic string = ∞-ly many harmonic oscillators with $\omega_m = m$.
Now we need to deal with constraints $L_m = \bar{L}_m = 0$.

✓ States : groundstate $|0, p^\mu\rangle \xrightarrow{\text{excitation}} \alpha_{-m_1}^{\mu_1} \cdots \alpha_{-m_N}^{\mu_N} |0, p^\mu\rangle$

* Negative Norm states

$$\| \alpha_{-1}^0 |0\rangle \|^2 = \langle 0| \alpha_1^0 \alpha_{-1}^0 |0\rangle = \langle 0| [\alpha_1^0, \alpha_{-1}^0] |0\rangle = \eta^{00}\langle 0|0\rangle = -1 <$$

$\hookrightarrow$ negative probability ? $\leftarrow$ this is <u>unacceptable</u>.

Way out to this problem : Still need to impose <u>constraints</u> !

$T_{\pm\pm}|\ \rangle \overset{!}{=} 0 \leftarrow$ this will hopefully eliminate unphysical negative norm states.

$\hookrightarrow$ critical dimension, etc. ...

$L_m = \frac{1}{2}\sum_n \alpha^\mu_{m-n}\alpha_{n\mu} \rightsquigarrow \langle 0|L_0|0\rangle = \frac{1}{2}\langle 0|\sum_n \alpha_{-n}\alpha_n|0\rangle$

$= \frac{1}{2}\langle 0|\sum_{n\geq 1}\alpha_n\alpha_{-n}|0\rangle \sim \sum_{n=1}^{\infty} n = \infty \quad (\Rightarrow\!\!\!\!\times)$

Introduce normal ordering to make L_m well-defined.

$$:\alpha_{-n}\alpha_n: \; = \begin{cases} \alpha_{-n}\alpha_n & \text{if } n\geq 1 \\ \alpha_n\alpha_{-n} & \text{if } n<1 \end{cases}$$

$L_m^{\text{quantum}} := \frac{1}{2}\sum_n :\alpha^\mu_{m-n}\alpha_{n\mu}:$ has well-defined matrix elements.

$$\boxed{[L_m, L_n] = (m-n)L_{m+n} + \frac{d}{12}m(m^2-1)\delta_{m+n,0}}$$ "Virasoro algebra"

"anomaly" term   square term is crucial.

Rmk. · this is an example of quantum theory modifies classical relation.
- compatible with Jacobi identity.
- Witt (1939), $L_m = -z^{m+1}(d/dz)$

Requiring $L_m|\,\rangle = 0$ for all m is not consistent

since $0 = \langle *|[L_m, L_{-m}]|*\rangle = \frac{d}{12}m(m^2-1) \neq 0$

It is sufficient to demand $\begin{cases} L_m|*\rangle = 0 & \forall m\geq 1 \\ (L_0 - a)|*\rangle = 0 \end{cases}$

Rmk. · checking for L_1, L_2 is enough.
- detailed investigation

→ No Ghost Theorem (~1972)

① There is always negative norm states for $d>26$.

② No neg. norm (but zero norm) states for $d_{crit}=26$, $a=1$.

③ No neg. norm states for $d<26$, $a=\cdots$

Extra degree of freedom (= Liouville degree of freedom)

$h_{\alpha\beta} \xrightarrow{\text{dift}} \Omega^2\eta_{\alpha\beta} \xrightarrow{\text{Weyl}} \eta_{\alpha\beta}$

NOT allowed for $d<26$
Works for $d=26$

Polyakov theory

• Lightcone Quantization

Introduce lightcone coordinate $X^{\pm} = X^0 \pm X^{d-1}$

By matching $X^+ = p^+ \tau$, we have all α^+ oscillation mode is zero

and $0 = T_{\pm\pm} = -2\partial_\pm X^+ \cdot \partial_\pm X^- + \partial_\pm \vec{X} \cdot \partial_\pm \vec{X}$

$$\Rightarrow \partial_\pm X^- = \frac{1}{2p^+} \partial_\pm \vec{X} \cdot \partial_\pm \vec{X}$$

$$\Rightarrow \alpha_m^- = \frac{1}{p^+} \sum_n : \vec{\alpha}_{m-n} \vec{\alpha}_n : - a\,\delta_{m,0} \quad , \quad \bar{\alpha}_m^- = \cdots\cdots$$

Hence, all physics are now encoded in α_m^i $(i = 1, \cdots, d-2)$ which are transverse oscillators. (No gauge invariance)

<u>Mass</u>
$$m^2 = \begin{cases} \frac{2}{\alpha'} \left[\sum_{n\ge1} (\alpha_{-n}^i \alpha_n^i + \bar{\alpha}_{-n}^i \bar{\alpha}_n^i) - 2a \right] & \text{, if closed string} \\ \frac{1}{\alpha'} \left[\sum_{n\ge1} \alpha_{-n}^i \alpha_n^i - a \right] & \text{, if open string.} \end{cases}$$

<u>Physical States</u> $|0,p\rangle \to \alpha' m^2 = -a$, $\alpha_{-1}^i |0,p\rangle \to \alpha' m^2 = 1-a$.

d-2 transverse oscillators

$\boxed{\text{Fact}}$ There is no massive field with only transverse degree of freedom

Hence, $m = 0$ and $\underline{\underline{a = 1}}$.

① $\sum_{n\ne0} \alpha_{-n}^i \alpha_n^i = 2\sum_{n=1}^{\infty} : \alpha_{-n}^i \alpha_n^i : + (d-2) \sum_{n=1}^{\infty} n$

Recall zeta-function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$, we can extend this function by analytic continuation to get $\zeta(-1) = -\frac{1}{12}$.

② regulate : $\sum_{n=1}^{\infty} n \longrightarrow \sum_{n=1}^{\infty} n e^{-\epsilon n} = \frac{1}{\epsilon^2} - \frac{1}{12} + O(\epsilon)$

regularization parameter

finite contribution

∞-quantity we removed by renormalization

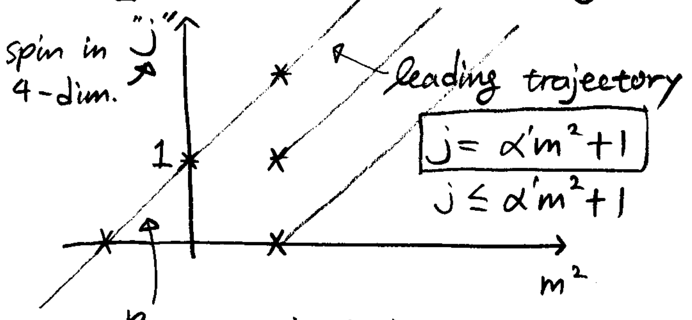
So. from $\frac{d-2}{12} = 2$, we get $d = 26$.

$M^{i-} \sim \alpha^i \alpha^- \sim \alpha^i : \alpha^j \alpha^j :$

$[M^{i-}, M^{j-}] \propto d-26$ <u>J. Schwart</u>

- $|0,p\rangle$ $\alpha' m^2 = -1$ ← Tachyon
- $\alpha_{-1}^i |0,p\rangle$ $\alpha' m^2 = 0$ ← Photon.
- $\begin{cases} \alpha_{-1}^i \alpha_{-1}^j |0,p\rangle & \square\square + \cdot \\ \alpha_{-2}^i |0,p\rangle & \square \end{cases}$

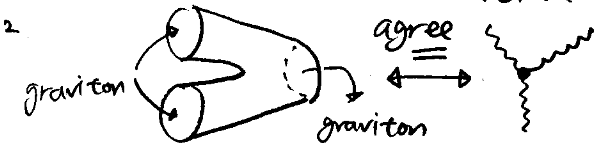
Spectrum (open string) $\qquad J^2 \le \frac{1}{2}\alpha' M^4$



Regge trajectories

Scherk, Schwarz, Yoneya (1975)

$\square\square \overset{!}{=}$ Graviton (massless spin 2 particle
$\in \sqrt{G}\, R(G)$)



4) What about membranes ?

D0 - brane $\leftarrow$ point particle , $\quad$ D1 - brane $\leftarrow$ string.

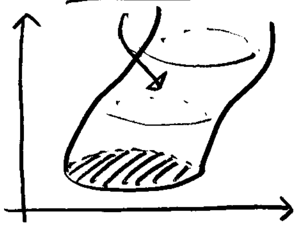
* supersymmetric extended objects. (p-branes in D space-time dim.)

- exists only for $D \le 11$
  - 11 - dim $\rightsquigarrow$ p = 2. unique theory $\exists!$ super-gravity.
  - 10 - dim $\rightsquigarrow$ many theories, (M-thy $\supset$ superstring / supergravity $\supset$ Einstein thy.)

* Non-perturbative superstring = M-theory.

M(atrix) Theory $\equiv$ M(embrane) Theory !

World Volume



- In flat target space $\mathbb{R}^{1,d-1}$

$$\zeta^i = (\zeta^0, \zeta^r) \equiv (\tau, \sigma^r) \quad \text{for } r = 1, \cdots, p$$

$$\zeta \longrightarrow (X^\mu(\zeta), \Theta_\alpha(\zeta)) \qquad \text{GS}$$

Spinor in target space, scalar on the world volume.

$$(\; (X^\mu(\xi), \psi^\mu(\xi)) \qquad \text{NSR} \;)$$

- Nambu-Goto action

Induced metric $\quad g_{ij}(X, \theta) = E_i^{\ \mu} E_j^{\ \nu} \eta_{\mu\nu}$

$$E_i^{\ \mu} = \partial_i X^\mu + \underbrace{\bar{\Theta}\, \Gamma^\mu \partial_i \Theta} \leftarrow \text{term added by SUSY}, \quad \{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}$$

$$\mathcal{L}_{NG} = -\sqrt{-g(X,\theta)} + \underline{\text{WZW-term in superspace}} \leftarrow \text{equal to zero for } \theta = 0$$

$\hookrightarrow$ of order $O(\theta^6)$ for flat target space.

$$\mathcal{L}_P = -\frac{1}{2}\sqrt{-g}\, g^{ij} E_i^{\ \mu} E_j^{\ \nu} + \underbrace{\frac{1}{2}(p-1)\sqrt{-g}}$$

$\hookrightarrow$ cosmological constant,
equal to zero for string case p = 1.

✓ Even flat superspace has torsion

Symmetries

① Global Super-Poincaré $\delta X^\mu = a^\mu + \Lambda^\mu{}_\nu X^\nu - \bar{\varepsilon}\Gamma^\mu\theta$
$$\delta\theta = \frac{1}{4}\Lambda_{\mu\nu}\Gamma^{\mu\nu}\theta + \varepsilon$$

② Diffeomorphism on World-Volume $\# = p+1$ ($=2$ in string case)

③ Local SUSY (κ-symmetry) ← (kind of) simplifies WZW-terms.

$$\delta\theta(\xi) = (\mathbb{1} - \Gamma)\,\kappa(\xi), \quad \Gamma\cdot\Gamma = \mathbb{1}.$$

$$\delta X^\mu = \kappa(1-\Gamma)\Gamma^\mu\theta \qquad \text{removes half of components.}$$

- Rough counting of degree of freedom (for $p=2$ case).

d	4	5	7	11
X^μ	1	2	4	8
θ	$4\to2$	$8\to4$	$16\to8$	$32\to16$

↑ maximally SUSY. 11-dim'l supergravity (1978)

X^μ $\#=d \longrightarrow d-3$ ← $\#(\text{diff})=3$.

$\#(\text{spinors}) \sim 2^{d/2}$

$\#(\text{vectors}) \sim d$.

- Light cone gauge

$\partial_i X^+ = \delta_{i0}$, $\Gamma^+\theta = 0$ (Recall: $X^\pm = \frac{1}{\sqrt{2}}(\pm X^0 + X^{10})$, X^i with $i=1,\dots,9$)

$g_{rs} = \bar{g}_{rs} = \partial_r\vec{X}\cdot\partial_s\vec{X}$, $g_{0r} = u_r = \partial_r X^- + \partial_0\vec{X}\cdot\partial_r\vec{X}$, $g_{00} = 2\partial_0 X^- + (\partial_0\vec{X})^2$

$$\det\begin{pmatrix} g_{00} & u_s \\ u_r & g_{rs}\end{pmatrix} = (\underbrace{\det \bar{g}_{rs}}_{\bar{g}})\cdot(\underbrace{g_{00} - u_r\bar{g}^{rs}u_s}_{-\Delta}) \equiv \bar{g}(-\Delta)$$

$$\mathcal{L} = -\sqrt{-g} = -\sqrt{\bar{g}\cdot\Delta}$$

$$\vec{P} = \frac{\partial\mathcal{L}}{\partial(\partial_0\vec{X})} = \sqrt{\frac{\bar{g}}{\Delta}}\left(\partial_0\vec{X} - u_r\bar{g}^{rs}\partial_s\vec{X}\right), \quad p^+ = \frac{\partial\mathcal{L}}{\partial(\partial_0 X^-)} = \sqrt{\frac{\bar{g}}{\Delta}}$$

after some algebra $\longrightarrow$ $\mathcal{H} = \vec{P}\cdot\partial_0\vec{X} + p^+\partial_0 X^- - \mathcal{L}$
$$= \frac{1}{2p^+}\left(\vec{P}^2 + \bar{g}\right)$$

Two further simplifications:

(i) still some diffeos. $\sigma^r \longrightarrow \sigma^r + \xi^r(\tau, \sigma)$

↳ gauge $u_r = \partial_r X^- + \partial_0\vec{X}\cdot\partial_r\vec{X} = 0$

↳ can only be solved if $\varepsilon^{rs}\partial_r\vec{P}\cdot\partial_s\vec{X} = 0$

↳ another constraint which was missing.

X^+, X^- ← only 2, but $\#(\text{diffeo}) = 3$.

(ii) $\partial_0 p^+ = 0 \Rightarrow p^+ = p_0^+\sqrt{w(\sigma)}$ such that $\int_M d^2\sigma\sqrt{w(\sigma)} = 1$.

$$\begin{cases} \mathcal{H} = \frac{1}{2p_0^+}\left(\vec{P}^2(\sigma^r) + \det(\partial_r\vec{X}\cdot\partial_s\vec{X})\right) \\ \phi = \varepsilon^{rs}\partial_r\vec{P}\cdot\partial_s\vec{X} = 0 \end{cases}$$

Properties of $\mathcal{H}$

① independent of $\vec{X}_0$.

② center of mass motion $\vec{p}_0$, $(\tilde{\vec{p}} = \vec{p} - \vec{p}_0)$

③ $\mathcal{M}^2 = \frac{1}{2p_0^+}\left(\tilde{\vec{p}}^{\,2}(\sigma) + \det(\partial_r \vec{X}\cdot\partial_s\vec{X})\right)$

relate this to IIA-thy. (double dimensional reduction)

$\mathbb{R}^{1,10} \longrightarrow [0, R_{11}] \times \mathbb{R}^{1,9}$ and identify $X^9 \equiv \sigma^2$, $X^{1,\cdots,8} = X^{1,\cdots,8}(\sigma^1)$

$\rightsquigarrow \mathcal{H} = \frac{1}{2p_0^+}\left(\vec{p}^{\,2}(\sigma^1) + \vec{X}'^{\,2}(\sigma)\right) \longleftarrow$ infinite superposition of harmonic oscillators.

(If $p>1$, we don't know how to quantize $\vec{p}^{\,2} + (\vec{x}^{\cdots})^{2p}$)

✓ Degeneracy : potential has valleys.

$\longleftrightarrow$ "collapse" of p-brane

✓ "Membrane number" is not well-defined.

✓ "Membrane topology"?

does NOT contribute to energy

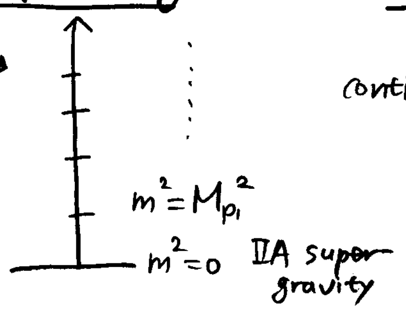
Spectrum

Superstring

✓ 1st quantize

✓ 2nd quantize

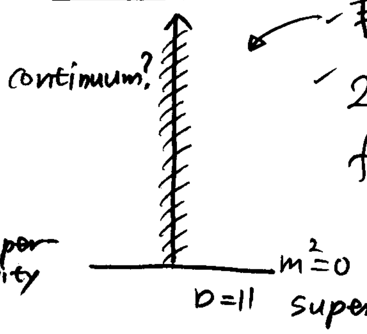
$\rightsquigarrow$ SFT (W. Taylor)



Supermembrane

$\not\exists$ 1st quantize

✓ 2nd quantized from the beginning.



$\boxed{HW}$ $\det(\partial_r\vec{X}\cdot\partial_s\vec{X}) = \{X^a, X^b\}\{X^a, X^b\}$ $\quad 1 \le r, s \le 2$.

$\{X^a, X^b\} := \varepsilon^{rs}\partial_r X^a \partial_s X^b$

$\{A, B\} = -\{B, A\}$ w/ cyclic condition

$\mathcal{H} = \frac{1}{2p_0^+}\int_M d^2\sigma \left(\vec{p}^{\,2} + \{X^a, X^b\}^2\right)$

↳ Lie bracket of A(rea) P(reserving) D(iffeomorphisms)

look almost like YM-Hamiltonian "$\vec{E}^2$" + "$\vec{B}^2$"

Diff $\delta f = -\xi^r\partial_r f \propto \{\xi, f\}$

APD = "incompressible" : $\partial_r\xi^r = 0 \overset{p=2}{\Longrightarrow} \xi^r = \varepsilon^{rs}\partial_s\xi$

$\boxed{Ex}$ $[\xi_1^r\partial_r, \xi_2^s\partial_s] = \xi_3^t\partial_t$

$\varepsilon^{rs}\partial_s\xi_3$ with $\xi_3 = \{\xi_1, \xi_2\}$.

$\mathcal{H} = \frac{1}{2p_0^+}(\vec{p}^2 + \{X^a, X^b\}^2)$. Gauge theory Hamiltonian, Gauge group = APD

Canonical generator of APD. $\phi = \varepsilon^{rs}\partial_r\vec{p}\,\partial_s\vec{X} = \{\vec{p}, \vec{X}\} = 0$ (12)

- ✓ Closed string analogue
  length preserving diffeo. $\sigma \to \sigma + \text{const.} \longrightarrow N_L = N_R \leftarrow 1$ constraint
- ✓ Membrane $\phi_0 = 0 \leftarrow$ infinitely many constraints.

APD "=" $\lim_{N\to\infty} SU(N)$

true for all membrane topology.

✓ Goldstone, Hoppe $\mathcal{M} = S^2$

$\mathrm{Diff}(S^2) \neq \mathrm{Diff}(T^2) \neq \mathrm{Diff}(\Sigma_g)$

depends on how to take limit.

- $\mathcal{M} = T^2$, $0 \le \sigma^1, \sigma^2 \le 2\pi$

  Basis : $Y_{\vec{m}}(\vec{\sigma}) = \frac{1}{\sqrt{4\pi}} e^{i\vec{m}\cdot\vec{\sigma}}$

  check : $\{Y_{\vec{m}}, Y_{\vec{n}}\} \propto (m_1 n_2 - m_2 n_1)\, Y_{\vec{m}+\vec{n}}$

✓ 't Hooft SU(N) matrices

$\exists\, U, V$ s.t. $UV = \omega VU$, $\omega = e^{2\pi i/N}$ (non-commutative torus ?)

$[U^{m_1}V^{m_2}, U^{n_1}V^{n_2}] = (\omega^{m_2 n_1} - \omega^{m_1 n_2})\, U^{m_1+n_1} V^{m_2+n_2}$

$\left(\frac{2\pi i}{N}(m_2 n_1 - m_1 n_2)\, U^{\cdots} V^{\cdots} + O\left(\frac{1}{N^2}\right) \right)$

- $\mathcal{H} = p_a^A p_a^A + (f^A{}_{BC} X_b^B X_c^C)^2$, $a = 1, \cdots, 9$, $A \in SU(N)$.

  $\phi^A = f^{ABC} p_a^B p_b^C = 0$. same model as D0-particles.

  $\mathcal{M}^2 = \{Y^A\}$. $f^{ABC} = \int d^2\sigma\, Y^A \{Y^B, Y^C\}$.

References

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