ALGEBRAIC HYPERBOLICITY AND KOBAYASHI CONJECTURE

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1. Basic Notions

1.1. **Poincare Metric on Δ.** Let $\Delta = \{|z| < 1\}$ be the unit disk. The *Poincare Metric* ρ_Δ is a complete Riemannian metric on Δ defined by

$$ds^2 = \frac{dzd\bar{z}}{(1 - |z|^2)^2} \tag{1.1}$$

The following is so-called *distance decreasing* property of Poincare metric.

Proposition 1.1 (Schwartz-Alhfors). *For any holomorphic map $f : \Delta \to \Delta$, $f^* ds^2 \le ds^2$, i.e., $\rho_\Delta(f(p), f(q)) \le \rho_\Delta(p, q)$ for any two points $p, q \in \Delta$.*

1.2. **Kobayashi-Royden Pseudo-Metric.** Let X be a complex manifold (not necessarily compact). The Kobayashi-Royden pseudo-metric ρ_X is defined in the following way.

Let $p, q \in X$. Choose a sequence of points $p_0 = p$, p_1, ..., $p_n = q$ and holomorphic maps $f_i : \Delta \to X$ with $p_{i-1}, p_i \in f_i(\Delta)$. Then

$$\rho_X(p, q) = \inf_{\{p_i\},\{f_i\}} \sum_{i=1}^{n} \rho_\Delta(f_i^{-1}(p_{i-1}), f_i^{-1}(p_i)) \tag{1.2}$$

An alternative way to define ρ_X is to define the norm $|| \cdot || : T_X \to \mathbb{R}$ on the holomorphic tangent space T_X of X.

Let $p \in X$ and let $v \in T_{X,p}$ be a holomorphic tangent vector at p. We consider all the holomorphic maps from $\Delta_R = \{|z| < R\}$ to X satisfying $f(0) = p$ and $f_*(\partial/\partial z) = v$. Then

$$||v|| = \inf_f \frac{1}{R} \tag{1.3}$$

Geometrically, we are trying to "squeeze" a disk as large as possible into X.

The pseudo-metric induced by $|| \cdot ||$ is exactly ρ_X defined above.

Proposition 1.2. *ρ_X satisfies*

1. **Triangle Inequality***: $\rho_X(p, q) + \rho_X(q, r) \ge \rho_X(p, r)$ for any $p, q, r \in X$.*
2. **Distance Decreasing***: Let $f : X \to Y$ be holomorphic. Then $\rho_Y(f(p), f(q)) \ge \rho_X(p, q)$.*

1.3. **Kobayashi Hyperbolicity.** In general, Kobayashi-Royden pseudometric is not a metric, i.e., it might degenerate ($\rho_X(p, q) = 0$ for some $p \neq q$).

Example 1.3. Let $X = \mathbb{C}$. For any point $z_0 \in \mathbb{C}$ and any number $R > 0$, we have the map $f : \Delta_R \to \mathbb{C}$ which simply sends z to $z + z_0$. Using the second definition of ρ_X, we see $||v|| = 0$ for any $v \in T_{X,z_0}$.

Example 1.4. Indeed, as long as there is a nonconstant holomorphic map $f : \mathbb{C} \to X$, ρ_X degenerates along $f(\mathbb{C})$. For example, ρ_X degenerates everywhere for any complex torus $X = C^n/\Lambda$.

We call a complex manifold hyperbolic in the sense of Kobayashi if ρ_X is a metric. If ρ_X is metric, it is obviously Riemannian.

We call a complex manifold X Brody hyperbolic (B-hyperbolic) if there does not exist any nonconstant holomorphic map $f : \mathbb{C} \to X$. Obviously, hyperbolic implies B-hyperbolic. The converse is true for compact complex manifolds [Br].

Theorem 1.5 (R. Brody)**.** *A compact complex manifold is hyperbolic if and only if it is B-hyperbolic.*

1.4. **Examples of Hyperbolic Manifolds.** Usually, it is difficult to construct interesting examples of hyperbolic manifolds and even more difficult to prove a certain manifold to be hyperbolic. But at least in $\dim X = 1$, we know exactly which X is hyperbolic. Let X be a Riemann surface (not necessarily compact). Let $\pi : Y \to X$ be the universal cover of X. Then Y must be one of $\mathbb{P}^1$, $\mathbb{C}$ and Δ. If $Y = \mathbb{P}^1$ or $\mathbb{C}$, there are obviously nonconstant holomorphic maps $f : \mathbb{C} \to Y \to X$ and hence X is not hyperbolic; if $Y = \Delta$, it is not hard to see that $\rho_X = \pi_* \rho_Y$ does not degenerate. Hence X is hyperbolic if and only if the universal cover of X is Δ.

In the case that X is quasi-projective and $\dim X = 1$, let $X = C - \{p_1, p_2, ..., p_n\}$, where C is a compact Riemann surface and $\{p_i\}$ are n points on C. Then X is hyperbolic if and only if

$$2g(C) - 2 + n > 0 \tag{1.4}$$

In particular,

Proposition 1.6. $\mathbb{P}^1 - \{3$ *points*$\}$ *is hyperbolic.*

Indeed this is equivalent to the classical Little Picard Theorem (LPT).

Example 1.7. Consider the solutions of $x^n + y^n = z^n$ over the field of meromorphic functions over $\mathbb{C}$. Every such solution $x = x(t)$, $y = y(t)$ and $z = z(t)$ gives a holomorphic map $f : \mathbb{C} \to C = \{x^n + y^n = z^n\} \subset \mathbb{P}^2$. The solution is trivial iff f is constant. Therefore, it has nontrivial solutions if and only if $g(C) \geq 2$, i.e., $n = 4$.

Let X be a smooth quasi-projective variety. We call X *Weakly Algebraic Hyperbolic* (WAH) if there does not exist an algebraic subvariety $Y \subset X$

such that the normalization $\widetilde{Y}$ of Y is either $\mathbb{P}^1 - \{2 \text{ points}\} \cong \mathbb{C}^*$ or an elliptic curve $\mathbb{C}/\Lambda$. Obviously, B-hyperbolic implies WAH.

Very little is known in higher dimension. The first important result in higher dimension is the following:

Theorem 1.8 (Generalized LPT by A. Bloch)**.** *$\mathbb{P}^n - (H_1 \cup H_2 \cup ... \cup H_{2n+1})$ is B-hyperbolic for $2n+1$ hyperplanes $H_1, H_2, ..., H_{2n+1} \subset \mathbb{P}^n$ in general position, i.e., no $n+1$ of them have nonempty intersection.*

This is proved using value distribution theory [B]. It is easy to see the number of hyperplanes, $2n+1$, is optimal. Suppose that we only remove $2n$ hyperplanes $H_1, H_2, ..., H_{2n}$ from $\mathbb{P}^n$. Let $p = H_1 \cap H_2 \cap ... \cap H_n$, $q = H_{n+1} \cap H_{n+2} \cap ... \cap H_{2n}$ and $L = \overline{pq}$ be the line passing through p and q. Obviously, $L - \{p, q\} \cong \mathbb{C}^* \subset X = \mathbb{P}^n - (H_1 \cup H_2 \cup ... \cup H_{2n})$. Therefore, removing $2n$ hyperplanes is not enough.

B-hyperbolic does not imply hyperbolic in this case since the manifold is not compact. Hyperberlicity of $\mathbb{P}^n - (H_1 \cup H_2 \cup ... \cup H_{2n+1})$ is proved by M. Green using the techniques of Brody.

Theorem 1.9 (M. Green)**.** *$\mathbb{P}^n - (H_1 \cup H_2 \cup ... \cup H_{2n+1})$ is hyperbolic for $2n+1$ hyperplanes $H_1, H_2, ..., H_{2n+1} \subset \mathbb{P}^n$ in general position, i.e., no $n+1$ of them have nonempty intersection.*

Although most research in hyperbolicity centered around hypersurfaces and their complements in the projective space such as Theorem 1.8 and 1.9. There are some other interesting spaces which has been proved to be hyperbolic or B-hyperbolic.

Theorem 1.10 (Siu-Yeung, Dethloff-Lu, Demailly)**.** *$X - D$ is B-hyperbolic for any abelian variety X and any ample divisor $D \subset X$.*

Theorem 1.11 (Viehweg-Zuo)**.** *The moduli space $\mathcal{M}_g$ of curves of genus g is B-hyperbolic for $g \geq 2$.*

1.5. **Notions of "being general" and "very general".** Many statements of algebraic geometry contain the terms such as "general", "generic", "very/sufficiently general". I want to clarify their meanings here.

Usually, a proposition in algebraic geometry deals with a class of algebraic varieties with certain properties. More often than not, there is a space $\mathcal{M}$ which parameterizes such varieties and is an algebraic vareity itself. This is quite a unique phenomenon of algebraic geometry. Just think of the space parameterizing compact real manifolds of dimension one; it does not even have finite dimension.

We say proposition A holds for a general member $[C] \in \mathcal{M}$ if there exists a closed subvariety $Y \subsetneq \mathcal{M}$ such that A holds for every $[C] \in \mathcal{M} - Y$.

We say proposition A holds for a very/sufficiently general member $[C] \in \mathcal{M}$ if there exists a sequence of closed subvarieties $Y_i \subsetneq \mathcal{M}$ such that A holds for every $[C] \in \mathcal{M} - \cup_{i=1}^{\infty} Y_i$. Note that this only makes sense over an uncountable field such as $\mathbb{C}$.

Example 1.12. A general curve C of genus ≥ 3 has no nontrivial automorphism. Here the parameter space is $\mathcal{M}_g$.

Example 1.13 (Noether-Lefshetz). The Picard group of a very general surface $X \subset \mathbb{P}^3$ of degree $d \geq 5$ is $\mathrm{Pic}(X) = \mathbb{Z}$. Here the parameter space is the linear series $|\mathcal{O}_{\mathbb{P}^3}(d)| = \mathbb{P}^N$ where $N = \binom{d+3}{3} - 1$.

2. Kobayashi Conjecture

Conjecture 2.1 (S. Kobayshi)**.**
(1) *X is hyperbolic for a very general hypersurface $X \subset \mathbb{P}^n$ of sufficiently high degree.*
(2) *$\mathbb{P}^n - D$ is hyperbolic for a very general hypersurface $D \subset \mathbb{P}^n$ of sufficiently high degree.*

In its original form, Kobayashi conjecture did not give the lower bound for $\deg X$ and $\deg D$. Zaidenberg proposed the bound $\deg X \geq 2n - 1$ in part (1) based on the following results.

Theorem 2.2 (H. Clemens)**.** *X is WAH for a very general hypersurface $X \subset \mathbb{P}^n$ of $\deg X \geq 2n$.*

Theorem 2.3 (G. Xu)**.** *X is WAH for a very general quintic surface $X \subset \mathbb{P}^3$.*

Theorem 2.4 (C. Voisin)**.** *X is WAH for a very general hypersurface $X \subset \mathbb{P}^n$ of $\deg X \geq 2n - 1$.*

He also proposed the bound $\deg D \geq 2n + 1$ for part (2) based on the observation that there are lines meeting D at only two points if $\deg D = 2n$. Here is the refined version of Kobayashi conjecture.

Conjecture 2.5 (Kobayashi-Zaidenberg)**.**
(1) *X is hyperbolic for a very general hypersurface $X \subset \mathbb{P}^n$ of $\deg X \geq 2n - 1$.*
(2) *$\mathbb{P}^n - D$ is hyperbolic for a very general hypersurface $D \subset \mathbb{P}^n$ of $\deg D \geq 2n + 1$.*

There are some major breakthroughs recently on the conjecture in dimension 2, although it still looks like intractible in any higher dimensions.

Theorem 2.6 (Siu-Yeung)**.** *$\mathbb{P}^2 - D$ is hyperbolic for a very general curve $D \subset \mathbb{P}^2$ of $\deg D \geq 80$.*

Theorem 2.7 (McQuillan, Demailly-El Goul)**.** *X is hyperbolic for a very general surface $X \subset \mathbb{P}^3$ of $\deg X \geq 24$.*

Theorem 2.8 (El Goul)**.** *$\mathbb{P}^2 - D$ is hyperbolic for a very general curve $D \subset \mathbb{P}^2$ of $\deg D \geq 15$.*

It usually involves two steps in proving X hyperbolic:

(1) Show that every holomorphic map $f : \mathbb{C} \to X$ is algebraically degenerated, i.e., $f(\mathbb{C})$ is contained in an algebraic curve $C \subset X$.
(2) Show that X is WAH.

The first step is analytic in nature and by far the more difficult part, while the second step is purely algebro-geometrical. The first part is essentially a version of Lang's conjecture.

Conjecture 2.9 (S. Lang). *Let X be a projective variety of general type. Then there exists a closed subvariety $Y \subsetneq X$ such that $f(\mathbb{C}) \subset Y$ for every holomorphic map $f : \mathbb{C} \to X$.*

3. Algebraic Hyperbolicity

Question 3.1. *Is $\mathbb{P}^n - D$ WAH for a very general hypersurface $D \subset \mathbb{P}^n$ of $\deg D \geq 2n + 1$?*

The answer is yes. It is known in $n = 2$.

Theorem 3.2 (G. Xu). *Let $D \subset \mathbb{P}^2$ be a very general curve of degree d. Then $|C \cap D| \geq d - 2$ for every curve $C \subset \mathbb{P}^2$ with $\dim(C \cap D) = 0$.*

Another proof of the above theorem was given in [C1].

The weak algebraic hyperbolicity of $\mathbb{P}^n - D$ for $n > 2$ was proved in [C3]. Actually, a stronger statement was proved there.

Theorem 3.3 (X. Chen). *$\mathbb{P}^n - D$ is algebraic hyperbolic for a very general hypersurface $D \subset \mathbb{P}^n$ of degree $d \geq 2n + 1$. More precisely,*

$$2g(C) - 2 + i_{\mathbb{P}^n}(C, D) \geq (d - 2n) \deg C \tag{3.1}$$

for every $C \subset \mathbb{P}^n$ with $\dim(C \cap D) = 0$.

Of course, we need to define algebraic hyperbolicity first.

Proposition 3.4 (J.P. Demailly). *Let X be a hyperbolic compact complex manifold and L be an ample line on X. Then there exists $\varepsilon_L > 0$, depending only on L, such that for any curve $C \subset X$,*

$$2g(C) - 2 \geq \varepsilon_L (C \cdot L) = \varepsilon_L \deg_L C \tag{3.2}$$

Proof. Let $\nu : \widetilde{C} \to C \subset X$ be the normalization of C. Let ω_X and $\omega_{\widetilde{C}}$ be the $(1, 1)$ forms associated to the hyperbolic metrics. Due to the distance decreasing property of hyperbolic metric,

$$\nu^* \omega_X \leq \omega_{\widetilde{C}} \tag{3.3}$$

Integrating both sides over $\widetilde{C}$, we obtain

$$\int_{\widetilde{C}} \omega_X \leq \int_{\widetilde{C}} \omega_{\widetilde{C}} \tag{3.4}$$

There exists a positive constant $\epsilon > 0$ such that $\omega_X \geq \epsilon c_1(L)$. Therefore,

$$\int_{\widetilde{C}} \omega_X \geq \int_{\widetilde{C}} \epsilon c_1(L) = \epsilon (C \cdot L) \tag{3.5}$$

Let $\Phi_{\widetilde{C}}$ be the curvature form associated to the hyperbolic metric of $\widetilde{C}$. Then $\Phi_{\widetilde{C}} = -4\omega_{\widetilde{C}}$. By Guass-Bonett,

$$\int_{\widetilde{C}} \omega_{\widetilde{C}} = -\frac{1}{4}\int_{\widetilde{C}} \Phi_{\widetilde{C}} = -\frac{\pi}{4}(2 - 2g(C)) \tag{3.6}$$

Then (3.2) follows from (3.5) and (3.6). □

We call X algebraic hyperbolic (AH) if (3.2) holds for every curve $C \subset X$. For X a projective variety,

$$X \text{ hyperbolic} \Leftrightarrow X \text{ B-hyperbolic} \Rightarrow X \text{ AH} \Rightarrow X \text{ WAH} \tag{3.7}$$

Theorem 3.5 (H. Clemens)**.** *Let $X \subset \mathbb{P}^n$ be a very general hypersurface of degree d. Then*

$$2g(C) - 2 \geq (d - 2n + 1)\deg C \tag{3.8}$$

for every curve $C \subset X$. Hence X is AH if $d \geq 2n$.

The concept of algebraic hyperbolicity can be generalized to quasi-projective varieties. Every quasi-projective variety can be realized as a projective variety with an effective divisor removed. Let X be a smooth projective variety and $D \subset X$ an effective divisor. We call (X, D) algebraic hyperbolic if there exists $\varepsilon > 0$ such that

$$2g(C) - 2 + i_X(C, D) \geq \varepsilon \deg C \tag{3.9}$$

for every curve $C \subset X$ with $\dim(C \cap D) = 0$. Here $i_X(C, D)$ is defined as follows. Let $\nu : \widetilde{C} \to C \subset X$ be the normalization of C. Then $i_X(C, D) = |\nu^{-1}(D)|$.

Obviously, AH implies WAH. However, the argument of Proposition 3.4 does not go through.

Question 3.6. *Does $X - D$ hyperbolic imply (X, D) AH?*

On the other hand, the algebraic hyperbolicity of log varieties is a natural generalization of that of projective varieties. In particular, it behaves well under deformation.

Proposition 3.7. *Let $X \subset \mathbb{P}^N \times \Delta$ be a flat family of projective varieties over Δ. Suppose that X is smooth and $X_0 = \sum D_j$ is a divisor of normal crossing. If there exists $\varepsilon \in \mathbb{R}$ such that*

$$2g(C) - 2 + i_{D_k}(C, D_k \cap (\cup_{j \neq k} D_j)) \geq \varepsilon \deg C \tag{3.10}$$

for every k and every curve $C \subset D_k$ with $\dim(C \cap (\cup_{j\neq k} D_j)) = 0$, then for a very general $t \in \Delta$

$$2g(C) - 2 \geq \varepsilon \deg C \tag{3.11}$$

for every curve $C \subset X_t$. Hence X_t is AH if $(D_k, D_k \cap (\cup_{j\neq k} D_j))$ is for each k.

Theorem 3.3 is proved via a degeneration argument. First we prove it for D a union of hyperplanes:

Proposition 3.8. *Let $D \subset \mathbb{P}^n$ be a union of d hyperplanes in very general position. Then* (3.1) *holds.*

Then we degenerate a hypersurface to a union of hyperplanes.

Here is a sketch of the proof of Proposition 3.8. Let $D = H_1 \cup H_2 \cup ... \cup H_d$. Using a deformation-theoretic argument, one can show that there are only countably many curves $C \subset \mathbb{P}^n$ satisfying

$$2g(C) - 2 + i_{\mathbb{P}^n}(C, H_1 \cup H_2 \cup ... \cup H_{d-1}) < (d - 2n) \deg C \tag{3.12}$$

If (3.12) fails, we are done. Otherwise, C meets H_d transversely and hence

$$|C \cap H_d| = \deg C \tag{3.13}$$

Induction hypothesis gives us

$$2g(C) - 2 + i_{\mathbb{P}^n}(C, H_1 \cup H_2 \cup ... \cup H_{d-1}) \geq (d - 2n - 1) \deg C \tag{3.14}$$

Also for a very general choice of H_d, $C \cap H_d \cap H_k = \emptyset$ for each $k \neq d$. Then (3.1).

We can also give a proof for Clemens' theorem 3.5 using Proposition 3.7 and 3.8.

Let $X \subset \mathbb{P}^n \times \Delta$ be a pencil of degree d hypersurfaces whose central fiber $X_0 = H_1 \cup ... \cup H_d$ is a union of d hyperplanes. On each $H_k \cong \mathbb{P}^{n-1}$, $\cup_{j \neq k} H_j$ is a union of $d - 1$ hyperplanes. Then by Proposition 3.8,

$$2g(C) - 2 + i_{H_k}(C, H_k \cap (\cup_{j \neq k} H_j)) \geq (d - 2n + 1) \deg C \tag{3.15}$$

Then

$$2g(C) - 2 \geq (d - 2n + 1) \deg C \tag{3.16}$$

for $C \subset X_t$. Careful readers may point out that X is not smooth in this case and Proposition 3.7 cannot be applied directly. This can be resolved by desingularizing X. But I do not want to go through the extra technicalities involved. Interested readers might refer [C3] for details.

Note that the bound $\deg X \geq 2n$ for $X \subset \mathbb{P}^n$ to be AH is not likely optimal, while the bound $\deg D \geq 2n + 1$ for $\mathbb{P}^n - D$ is.

Question 3.9. *Is a very general hypersurface $X \subset \mathbb{P}^n$ of $\deg X = 2n - 1$ AH? Especially, is a very general quintic surface $X \subset \mathbb{P}^3$ AH?*

Finally, I want to point out that this is not the only approach to prove $\mathbb{P}^n - D$ WAH. Another way to prove it is to directly generalize Xu's theorem 3.2.

Question 3.10. *Let D be a very general hypersurface in $\mathbb{P}^n$ of degree d. Is it true that $|C \cap D| \geq d - 2n$ for every curve $C \subset \mathbb{P}^n$ with $\dim(C \cap D) = 0$?*

C. Voisin told me that she could prove $|C \cap D| \geq d - 2n - 2$.

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