Some Maximal Operators related to Families of Singular Integral Operators*

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Abstract

In this paper, we shall study L^p-boundedness of two kinds of maximal operators related to some families of singular integrals.

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1 Introduction

Let y' be a point on the unit sphere S^{n-1} in R^n $(n \geq 2)$ and $d\sigma$ be the induced Lebesgue measure on S^{n-1}. The Calderón-Zygmund singular integral, initially defined on $S(R^n)$, is defined by

$$p.v. \int_{R^n} b(|y|)\frac{\Omega(y')}{|y|^n} f(x-y)dy,$$

where b is a measurable function and $\Omega \in L^1(S^{n-1})$ is a homogeneous function of degree 0 and satisfies $\int_{S^{n-1}} \Omega(y')d\sigma(y') = 0$. The study and applications of this operator have a long history, for example, see [1], [8]-[11] and their references, etc. In this paper, we are interested in two maximal operators related to this integral. The first operator is defined in [5] by

$$T(f)(x) = \sup_h \left| \int_{R^n} h(|y|)\frac{\Omega(y')}{|y|^n} f(x-y)dy \right|$$

where the supreme is over the set of all radial function h satisfying

$$\|h\|_{L^2(R^+, \frac{dr}{r})} = (\int_{R^+} |h(r)|^2 \frac{dr}{r})^{1/2} \leq 1.$$

In [5], Chen and Lin proved that if $\Omega \in C(S^{n-1})$, then $\|T(f)\|_p \leq C_{n,T}\|f\|_p$ for $p > \frac{2n}{2n-1}$. Also, they pointed out that the range $p > \frac{2n}{2n-1}$ is the best possible. However, in this paper, we will prove that the condition $\Omega \in C(S^{n-1})$ can be greatly weakened. We will prove the following theorem.

Theorem 1 *(i) If $\Omega \in H^1(S^{n-1})$ and satisfies $\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0$ where $H^1(S^{n-1})$ is the Hardy space on S^{n-1}, then $\|T(f)\|_p \leq C_{n,p,\Omega}\|f\|_p$ for all $2 \leq p < \infty$. (ii) If $\Omega \in L^q(S^{n-1})$ satisfies $\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0$, $q \in (1,2]$ and $\frac{2nq'}{nq'+2n-2} < p < \infty$, then $\|T(f)\|_p \leq C_{n,p,\Omega}\|f\|_p$, where q' is the conjugate index of q.*

Theorem 1 can be extended to the product space. We define

$$T(f)(x,y) = \sup_{h\in\Delta^2} \left| \int_{R^n\times R^m} \frac{h(|u|,|v|)\Omega(u',v')}{|u|^n |v|^m} f(x-u, y-v)dudv \right|$$

where Δ^2 consists of all h satisfying

$$\|h\|_{L^2(R^+\times R^+, \frac{drds}{rs})} = (\int_{R^+\times R^+} |h(r,s)|^2 \frac{drds}{rs})^{1/2} \leq 1,$$

and $\Omega \in L^1(S^{n-1} \times S^{m-1})$ satisfies the cancellation condition

$$\int_{S^{n-1}} \Omega(u', v') d\sigma(u') = \int_{S^{m-1}} \Omega(u', v') d\sigma(v') = 0 \ (\forall (u', v') \in S^{n-1} \times S^{m-1}). \qquad (1)$$

We have

Theorem 2 *(i) If* $\Omega \in L(\log^+ L)^2(S^{n-1} \times S^{m-1})$ *and satisfies (1), then* $\|T(f)\|_p \leq C_{n,m,p,\Omega} \|f\|_p$ *for all* $2 \leq p < \infty$*. (ii) If* $\Omega \in L^q(S^{n-1} \times S^{m-1})$ *and satisfies (1) and* $q \in (1, 2]$*, then* $\|T(f)\|_\gamma \leq C_{n,m,\gamma,\Omega} \|f\|_\gamma$ *for* $\frac{2\sigma q'}{\sigma q' + 2\sigma - 2} < \gamma < \infty$*, where* $\sigma = \min(n, m)$*.*

Theorem 2(ii) has the following generalization. For $\alpha \geq 1$, consider the maximal operator

$$T^\alpha(f)(x, y) = \sup_{h \in \Delta^\alpha} \left| \int_{R^n \times R^m} \frac{h(|u|, |v|)\Omega(u', v')}{|u|^n |v|^m} f(x - u, y - v) du dv \right|,$$

where Δ^α consists of all h satisfying

$$\|h\|_{L^\alpha(R^+ \times R^+, \frac{drds}{rs})} = (\int_{R^+ \times R^+} |h(r, s)|^\alpha \frac{drds}{rs})^{1/\alpha} \leq 1.$$

We have

Theorem 3 *For* $\alpha \in [1, 2]$*,* $q \in (1, 2]$*,* $\Omega \in L^q(S^{n-1} \times S^{m-1})$ *satisfying (1), we have* $\|T^\alpha(f)\|_p \leq C_{n,m,p,\alpha,\Omega} \|f\|_p$ *for all* $p \in (\frac{\alpha' \sigma q'}{\sigma q' + \alpha' \sigma - \alpha'}, \infty)$*, where* $\sigma = \min(n, m)$*.*

The second operator we shall consider is

$$S(f)(x, y) = \sup_{\Omega} \left| \int_{R^n \times R^m} \frac{h(|u|, |v|)\Omega(u', v')}{|u|^n |v|^m} f(x - u, y - v) du dv \right|$$

where the supreme is over the set of all Ω satisfying $\|\Omega\|_{L^q(S^{n-1} \times S^{m-1})} \leq 1$ and the cancellation condition (1). We have

Theorem 4 *For* $q > 1$ *and* $q' \leq p < \infty$*,* S *is* L^p-*bounded. It also works for the one parameter case.*

2 Proofs of the Theorems

2.1 Proof of Theorem 1

We follow the argument in [5]. By duality,

$$T(f)(x) = \left(\sum_{k=-\infty}^{+\infty} \int_1^2 \left| \int_{S^{n-1}} \Omega(y') f(x - 2^k t y') d\sigma(y') \right|^2 \frac{dt}{t} \right)^{1/2}.$$

Recalling that $\Omega \in H^1(S^{n-1})$ has atomic decomposition (see [6]), we may assume that $\Omega(y') = a(y')$ is an H^1-atom which means that $a(y')$ is an L^∞-function satisfying

$$supp(a) \subset \{y' \in S^{n-1} : |y' - x_0'| < \rho \text{ for some } x_0' \in S^{n-1} \text{ and } \rho \in (0,1]\};$$
$$\textstyle\int_{S^{n-1}} a(y') d\sigma(y') = 0; \qquad \|a\|_\infty \le \rho^{-(n-1)}.$$

Also, without loss of generality, we assume $x_0' = \mathbf{1} = (1, 0, \cdots, 0)$. Now to prove (i) of the theorem, it suffices to show that

$$\|T(f)\|_p \le C_{n,p} \|f\|_p \text{ for } p \ge 2,$$

where $C_{n,p}$ is a constant independent of atoms $a(y')$.

Let $\{\Phi_j\}_{-\infty}^{+\infty}$ be a smooth partition of the unit in $(0, \infty)$ adapted to the interval $(2^{j-1}, 2^{j+1})$. To be precise, we require the following:

$$\Phi_j \in C^\infty(0, \infty), 0 \le \Phi_j \le 1, \sum_{j=-\infty}^{+\infty} \Phi_j(t) = 1 \text{ for all } t \in (0, \infty),$$
$$supp(\Phi_j) \subset (2^{-j-1}, 2^{j+1}).$$

Define the multiplier operators S_j on R^n by

$$(S_j(f))^\wedge(\xi) = \widehat{f}(\xi) \Phi_j(|A_\rho \xi|)$$

where A_ρ is the linear transform such that $A_\rho \xi = (\rho^2 \xi_1, \rho \xi_2, \cdots, \rho \xi_n)$.

Following the argument on page 123 of [5], we have

$$T(f)(x) = \sum_j T_j(f)(x),$$

where

$$T_j(f)(x) = \left(\sum_{k=-\infty}^{+\infty} \int_1^2 \left| \int_{S^{n-1}} \Omega(y') (S_{k+j} f)(x - 2^k t y') d\sigma(y') \right|^2 \frac{dt}{t} \right)^{1/2}.$$

By Plancherel's theorem and Fubini's theorem,

$$\|T_j(f)\|_2^2 = \sum_{k=-\infty}^{+\infty} \int_{2^{-(k+j)-1}\le |A_\rho \xi| \le 2^{-(k+j)+1}} \left|\widehat{f}(\xi)\right|^2 \cdot \left\{ \int_1^2 \left| \int_{S^{n-1}} \Omega(y') e^{-i2^k t y'\cdot\xi} d\sigma(y') \right|^2 \frac{dt}{t} \right\}^2 d\xi.$$

By page 327 in [8], we know

$$\begin{aligned}\int_1^2 \left| \int_{S^{n-1}} \Omega(y') e^{-i2^k t y'\cdot\xi} d\sigma(y') \right|^2 \frac{dt}{t} &= \int_1^2 \left| \int_R F_a(s) e^{-i2^k t s|\xi|} ds \right|^2 \frac{dt}{t} \\ &= \int_{2^k|\xi|}^{2^{k+1}|\xi|} \left| \int_R F_a(s) e^{-its} ds \right|^2 \frac{dt}{t}\end{aligned}$$

where $F_a(s)$ satisfies

$$\begin{aligned}&supp(F_a) \subset (\xi_1' - 2|A_\rho\xi'|, \xi_1' + 2|A_\rho\xi'|) \\ &\|F_a\|_\infty \le C|A_\rho\xi'|^{-1}, \int_R F_a(s)ds = 0, \\ &\text{and } \xi' = \tfrac{\xi}{|\xi|}, A_\rho\xi' = (\rho^2\xi_1', \rho\xi_2', \cdots, \rho\xi_n').\end{aligned}$$

Thus

$$\begin{aligned}\int_{2^k|\xi|}^{2^{k+1}|\xi|} \left| \int_R F_a(s) e^{-its} ds \right|^2 \frac{dt}{t} &= \int_{2^k|\xi|}^{2^{k+1}|\xi|} \left| \int_R F_a(s) \left(e^{-its} - e^{-it\xi_1'}\right) ds \right|^2 \frac{dt}{t} \\ &\le \int_{2^k|\xi|}^{2^{k+1}|\xi|} |A_\rho\xi'|^2 dt \le C\left(2^k|\xi||A_\rho\xi'|\right)^2 = C\left(2^k|A_\rho\xi|\right)^2\end{aligned}$$

if $2^k|A_\rho\xi| \le 1$. For $2^k|A_\rho\xi| \ge 1$, we have

$$\begin{aligned}\int_{2^k|\xi|}^{2^{k+1}|\xi|} \left| \int_R F_a(s) e^{-its} ds \right|^2 \frac{dt}{t} &\le C\left(2^k|\xi|\right)^{-1} \left\|\widehat{F_a}\right\|_2^2 \simeq C\left(2^k|\xi|\right)^{-1} \|F_a\|_2^2 \\ &= C\left(2^k|\xi|\right)^{-1} |A_\rho\xi'|^{-1} = C\left(2^k|A_\rho\xi|\right)^{-1}.\end{aligned}$$

Thus

$$\begin{aligned}\|T_j(f)\|_2^2 \le\ & C\sum_{k=-\infty}^{+\infty} \int_{2^{-(k+j)-1}\le |A_\rho\xi| \le 2^{-(k+j)+1}} \left|\widehat{f}(\xi)\right|^2 \\ & \cdot \min\left\{ \left(2^k|A_\rho\xi|\right)^2, \left(2^k|A_\rho\xi|\right)^{-1} \right\} d\xi \\ \le\ & C2^{-|j|}\|f\|_2^2.\end{aligned}$$

By page 124 in [5], for $p \ge 2$, there is a function g in $L^{(p/2)'}$ such that $\|g\|_{L^{(p/2)'}} = 1$ and

$$\begin{aligned}\|T_j(f)\|_p^2 =\ & \sum_{k=-\infty}^{+\infty} \int_{R^n} \int_1^2 \int_{S^{n-1}} \left|(S_{k+j}f)(x - 2^k t y')\right|^2 \\ & \cdot |\Omega(y')| d\sigma(y') \frac{dt}{t} |g(x)| dx \|\Omega\|_1 \\ \le\ & C\|\Omega\|_1 \sum_{k=-\infty}^{+\infty} \int_{R^n} |(S_{k+j}f)(x)|^2 \\ & \cdot \int_1^2 \int_{S^{n-1}} |\Omega(y')| \left|g(x + 2^k t y')\right| d\sigma(y') \frac{dt}{t} dx \\ \le\ & C\|\Omega\|_1 \left\| \sum_{k=-\infty}^{+\infty} |(S_{k+j}f)|^2 \right\|_{p/2} \|M_\Omega(g)\|_{(p/2)'}\end{aligned}$$

where

$$\begin{aligned} M_\Omega(g)(x) &= \sup_k \tfrac{1}{2^k} \int_{2^k}^{2^{k+1}} \int_{S^{n-1}} |\Omega(y')|\, |g(x+ty')|\, d\sigma(y')dx \\ &\le \int_{S^{n-1}} |\Omega(y')| \left(\sup_k \tfrac{1}{2^k} \int_{2^k}^{2^{k+1}} |g(x+ty')|\, dx \right) d\sigma(y'). \end{aligned}$$

Thus

$$\|M_\Omega(g)\|_{(p/2)'} \le C \|g\|_{(p/2)'} \le C.$$

By Littlewood-Paley theorem, we have

$$\|T_j(f)\|_p \le C \|f\|_p .$$

By interpolation, we obtain (i) of Theorem 1.

To prove (ii) of Theorem 1. First, by checking the proof on p.123 in [5], it is easy to see that for $\frac{2n}{2n-1} < p \le 2$, if $\Omega \in L^2(S^{n-1})$, then

$$\|T(f)\|_p \le C \|\Omega\|_2 \|f\|_p .$$

However, in the proof of (i), we obtained

$$\|T(f)\|_2 \le C \|\Omega\|_{H^1} \|f\|_2 .$$

So, (ii) follows by interpolation.

2.2 Proof of Theorem 2

We will adapt some standard ideas in the one parameter case. By duality, we have

$$\begin{aligned} T(f)(x,y) = &\Bigg(\sum_{k=-\infty}^{+\infty} \sum_{j=-\infty}^{+\infty} \int_1^2 \int_1^2 | \int_{S^{n-1}\times S^{m-1}} \Omega(u',v') \\ &\quad \cdot f(x-2^k r u', y - 2^j s v') d\sigma(u') d\sigma(v')|^2 \tfrac{drds}{rs} \Bigg)^{1/2} . \end{aligned}$$

Take Schwartz functions $p_1 \in S(R^n)$ and $p_2 \in S(R^m)$ such that $supp(\widehat{p_1}) \subset \left\{ \frac{1}{2} \le |x| \le 2 \right\}$, $supp(\widehat{p_2}) \subset \left\{ \frac{1}{2} \le |y| \le 2 \right\}$, $0 \le \widehat{p_i} \le 1$ $(i = 1, 2)$ and

$$\sum_{k=-\infty}^{+\infty} \left| \widehat{p_1}(2^k \xi) \right|^2 = \sum_{j=-\infty}^{+\infty} \left| \widehat{p_2}(2^j \eta) \right|^2 = 1.$$

Set $\widehat{p_{1,k}}(\xi) = \widehat{p_1}(2^k \xi)$, $\widehat{p_{2,j}}(\eta) = \widehat{p_2}(2^j \eta)$, and $f_{\mu,\nu} = (p_{1,\mu} \otimes p_{2,\nu}) * (p_{1,\mu} \otimes p_{2,\nu}) * f$ where $p_{1,\mu} \otimes p_{2,\nu}(x,y) = p_{1,\mu}(x) p_{2,\nu}(y)$. Then, $f = \sum_{(\mu,\nu)\in \mathbf{Z}^2} f_{k+\mu,j+\nu}$ for any $(k,j) \in \mathbf{Z}^2$ where

$\mathbf{Z}$ is the set of all integers. So, by Minkowski's inequality, we have

$$\begin{aligned} T(f)(x,y) = & \Bigg(\sum_{k=-\infty}^{+\infty} \sum_{j=-\infty}^{+\infty} \int_1^2 \int_1^2 \Big| \sum_{(\mu,\nu)\in\mathbf{Z}^2} \int_{S^{n-1}\times S^{m-1}} \Omega(u',v') \\ & \cdot f_{k+\mu,j+\nu}(x-2^k ru', y-2^j sv') d\sigma(u')d\sigma(v') \Big|^2 \frac{drds}{rs} \Bigg)^{1/2} \\ \leq & \sum_{(\mu,\nu)\in\mathbf{Z}^2} \Bigg(\sum_{k=-\infty}^{+\infty} \sum_{j=-\infty}^{+\infty} \int_1^2 \int_1^2 \Big| \int_{S^{n-1}\times S^{m-1}} \Omega(u',v') \\ & \cdot f_{k+\mu,j+\nu}(x-2^k ru', y-2^j sv') d\sigma(u')d\sigma(v') \Big|^2 \frac{drds}{rs} \Bigg)^{1/2} \\ \overset{def}{=} & \sum_{(\mu,\nu)\in\mathbf{Z}^2} T_{\mu,\nu}(f)(x,y). \end{aligned}$$

Now, let

$$\begin{aligned} &\Omega_l(x',y') = \Omega(x',y')\chi_{\Theta_l}(x',y') \\ &\Theta_0 = \{(x',y') \in S^{n-1}\times S^{m-1} : |\Omega(x',y')| \leq 1\} \\ &\Theta_l = \left\{(x',y') \in S^{n-1}\times S^{m-1} : 2^{l-1} < |\Omega(x',y')| \leq 2^l\right\} \text{ for } l \geq 1, \end{aligned}$$

and

$$\begin{aligned} \widetilde{\Omega}_l(x',y') = & \Omega_l(x',y') + \frac{1}{\sigma(S^{n-1})\sigma(S^{m-1})} \int_{S^{n-1}\times S^{m-1}} \Omega_l(u',v')d\sigma(u')d\sigma(v') \\ & - \frac{1}{\sigma(S^{n-1})} \int_{S^{n-1}} \Omega_l(u',y')d\sigma(u') - \frac{1}{\sigma(S^{m-1})} \int_{S^{m-1}} \Omega_l(x',v')d\sigma(v'). \end{aligned}$$

It is easy to see that

$$\begin{aligned} &\widetilde{\Omega}_l \text{ satisfies (1) and } \sum_{l\geq 0} \widetilde{\Omega}_l = \Omega \\ &\left\|\widetilde{\Omega}_l\right\|_{L^1(S^{n-1}\times S^{m-1})} \leq 2^{l+2}\,|\Theta_l| \\ &\left\|\widetilde{\Omega}_l\right\|_{L^\infty(S^{n-1}\times S^{m-1})} \leq 2^{l+2}. \end{aligned} \tag{2}$$

Thus

$$T(f)(x,y) \leq \sum_{l\geq 0} \sum_{(\mu,\nu)\in\mathbf{Z}^2} T^l_{\mu,\nu}(f)(x,y) \tag{3}$$

where

$$\begin{aligned} T^l_{\mu,\nu}(f)(x,y) = & \Bigg(\sum_{k,j=-\infty}^{+\infty} \int_{[1,2]^2} \Big| \int_{S^{n-1}\times S^{m-1}} \widetilde{\Omega}_l(u',v') \\ & \cdot f_{k+\mu,j+\nu}(x-2^k ru', y-2^j sv') d\sigma(u')d\sigma(v') \Big|^2 \frac{drds}{rs} \Bigg)^{1/2}. \end{aligned} \tag{4}$$

Before continuing the proof, we first give two lemmas.

Lemma 5 *For*

$$\begin{aligned} I^l_{r,s}(k,j,\xi,\eta) = & \int_{[1,2]^2} \Big| \int_{S^{n-1}\times S^{m-1}} \widetilde{\Omega}_l(u',v') e^{2\pi i 2^k ru'\cdot\xi} \\ & \cdot e^{2\pi i 2^j sv'\cdot\eta} d\sigma(u')d\sigma(v') \Big|^2 \frac{drds}{rs}, \end{aligned} \tag{5}$$

$\exists\alpha\in(0,1)$ *such that*

$$I_{r,s}^{l}(k,j,\xi,\eta)\leq C_{n,m,\alpha}\min\begin{pmatrix} 2^{l}\left|\Theta_{l}\right|\left|2^{k}\xi\right|,2^{l}\left|\Theta_{l}\right|\left|2^{j}\eta\right|,2^{l}\left|\Theta_{l}\right|\left|2^{k}\xi\right|\left|2^{j}\eta\right|,\\ 2^{l}\left|\Theta_{l}\right|,2^{l}\left|2^{k}\xi\right|^{-\frac{\alpha}{2}}\left|2^{j}\eta\right|,2^{l}\left|2^{k}\xi\right|\left|2^{j}\eta\right|^{-\frac{\alpha}{2}},\\ 2^{l}\left|2^{k}\xi\right|^{-\frac{\alpha}{2}},2^{l}\left|2^{j}\eta\right|^{-\frac{\alpha}{2}},2^{l}\left|2^{k}\xi\right|^{-\frac{\alpha}{2}}\left|2^{j}\eta\right|^{-\frac{\alpha}{2}}\end{pmatrix}. \tag{6}$$

Proof. Similar to the proof of Lemma 10 of [3].

Lemma 6 *For* $\Omega\in L^{2}(S^{n-1}\times S^{m-1})$ *and*

$$G=\{g_{k,j}\}_{k,j}\in L^{p}(R^{n}\times R^{m}\rightarrow l^{2}(\mathbf{Z}^{2}\rightarrow L^{2}([1,2]^{2},\frac{drds}{rs})),dxdy),$$

which means that G *defines on* $R^{n}\times R^{m}$ *and takes values in Hilbert space* $l^{2}(\mathbf{Z}^{2}\rightarrow L^{2}([1,2]^{2},\frac{drds}{rs}))$ *and the Hilbert space consists of all* $l^{2}-$*sequences defined on* $\mathbf{Z}^{2}$ *and taking values in* $L^{2}([1,2]^{2},\frac{drds}{rs})$*, there holds*

$$\begin{aligned}&\left\|\left\|\left\|\mathcal{I}(G)\right\|_{L^{2}([1,2]^{2},\frac{drds}{rs})}\right\|_{l^{2}(\mathbf{Z}^{2})}\right\|_{L^{p}(R^{n}\times R^{m},dxdy)}\\ &\qquad\leq C_{n,m,p}\left\|\Omega\right\|_{2}\left\|\left\|\left\|G\right\|_{L^{2}([1,2]^{2},\frac{drds}{rs})}\right\|_{l^{2}(\mathbf{Z}^{2})}\right\|_{L^{p}(R^{n}\times R^{m},dxdy)},\end{aligned}$$

where $p\in(2,2\min(n,m))$ *and*

$$\mathcal{I}(G)(x,y;r,s)\overset{def}{=}\left\{\int_{S^{n-1}\times S^{m-1}}\Omega(u',v')g_{k,j}(x+2^{k}ru',y+2^{j}sv';r,s)d\sigma(u')d\sigma(v')\right\}_{(k,j)\in\mathbf{Z}^{2}}.$$

Proof. We shall use duality method and spherical maximal function. For $p>2$, taking an $f\in L^{(p/2)'}(R^{n}\times R^{m})$ such that $\|f\|_{(p/2)'}=1$ and

$$\begin{aligned}A&\overset{def}{=}\left\|\left\|\left\|\mathcal{I}(G)\right\|_{L^{2}([1,2]^{2},\frac{drds}{rs})}\right\|_{l^{2}(\mathbf{Z}^{2})}\right\|_{L^{p/2}(R^{n}\times R^{m},dxdy)}\\ &=\int_{R^{n}\times R^{m}}\left\|\left\|\mathcal{I}(G)\right\|_{L^{2}([1,2]^{2},\frac{drds}{rs})}\right\|_{l^{2}(\mathbf{Z}^{2})}f(x,y)dxdy.\end{aligned}$$

Thus, by Hölder's inequality and changes of variables x and y, we get

$$\begin{aligned}A&=\int_{R^{n}\times R^{m}}f(x,y)\sum_{k,j}\int_{[1,2]^{2}}(\int_{S^{n-1}\times S^{m-1}}\Omega(u',v')\\ &\qquad\cdot g_{k,j}(x+2^{k}ru',y+2^{j}sv';r,s)d\sigma(u')d\sigma(v'))^{2}\tfrac{drds}{rs}dxdy\\ &\leq\left\|\Omega\right\|_{2}^{2}\int_{R^{n}\times R^{m}}\sum_{k,j}\int_{[1,2]^{2}}(\int_{S^{n-1}\times S^{m-1}}f(x,y)\\ &\qquad\cdot\left|g_{k,j}(x+2^{k}ru',y+2^{j}sv';r,s)d\sigma(u')d\sigma(v')\right|^{2}d\sigma(u')d\sigma(v')\tfrac{drds}{rs}dxdy\end{aligned}$$

$$= \|\Omega\|_2^2 \int_{R^n\times R^m} \sum_{k,j} \int_{[1,2]^2} |g_{k,j}(x,y;r,s)|^2$$
$$\cdot \int_{S^{n-1}\times S^{m-1}} f(x-2^k ru', y-2^j sv') d\sigma(u')d\sigma(v') \tfrac{drds}{rs} dxdy$$
$$\leq \|\Omega\|_2^2 \int_{R^n\times R^m} M_S' M_S''(f)(x,y) \sum_{k,j} \int_{[1,2]^2} |g_{k,j}(x,y;r,s)|^2 \tfrac{drds}{rs} dxdy$$
$$\leq \|\Omega\|_2^2 \|M_S' M_S''(f)\|_{(p/2)'} \left\| \sum_{k,j} \int_{[1,2]^2} |g_{k,j}(\cdot,\cdot;r,s)|^2 \tfrac{drds}{rs} \right\|_{p/2}$$

where

$$M_s'(f)(x,y) = \sup_{r>0} \int_{S^{n-1}} |f(x-ru',y)|\, d\sigma(u')$$
$$M_s''(f)(x,y) = \sup_{r>0} \int_{S^{m-1}} |f(x,y-sv')|\, d\sigma(v').$$

So, by boundedness of spherical maximal function operators, we get

$$A \leq C_{n,m,p} \|\Omega\|_2^2 \left\| \left\| \|G\|_{L^2([1,2]^2,\frac{drds}{rs})} \right\|_{l^2(\mathbf{Z}^2)} \right\|^2_{L^p(R^n\times R^m,dxdy)}$$

for $(p/2)' > \max(\frac{n}{n-1}, \frac{m}{m-1})$, i.e. $p < 2\min(n,m)$. The last inequality gives the desired result.

Now, we shall prove Theorem 2(i).

By Plancherel's identity, we have

$$\begin{aligned} \left\| T^l_{\mu,\nu}(f) \right\|_2^2 &= \int_{R^n\times R^m} \sum_{k,j} \int_{[1,2]^2} \left| \int_{S^{n-1}\times S^{m-1}} \widetilde{\Omega}_l(u',v') \right. \\ &\quad \left. \cdot f_{k+\mu,j+\nu}(x-2^k ru', y-2^j sv') d\sigma(u')d\sigma(v') \right|^2 \tfrac{drds}{rs} \\ &= \sum_{k,j} \int_{[1,2]^2} \int_{R^n\times R^m} |(f_{k+\mu,j+\nu})^\wedge(\xi,\eta)|^2 \left| \int_{S^{n-1}\times S^{m-1}} \widetilde{\Omega}_l(u',v') \right. \\ &\quad \left. \cdot e^{2\pi i 2^k ru'\cdot\xi} e^{2\pi i 2^j sv'\cdot\eta} d\sigma(u')d\sigma(v') \right|^2 d\xi d\eta \tfrac{drds}{rs}. \end{aligned}$$

Thus, by the definition of $f_{k+\mu,j+\nu}$, we get

$$\begin{aligned} \left\| T^l_{\mu,\nu}(f) \right\|_2^2 \leq & \sum_{k,j} \int_{2^{-(k+\mu)-1}\leq|\xi|\leq 2^{(k+\mu)+1},\, 2^{-(j+\nu)-1}\leq|\eta|\leq 2^{-(j+\nu)+1}} \\ & \cdot \left| I^l_{r,s}(k,j,\xi,\eta) \right|^2 \left| \widehat{f}(\xi,\eta) \right|^2 d\xi d\eta \end{aligned} \tag{7}$$

where $I^l_{r,s}$ is defined in (5).

For $p > 2$, taking a $g \in L^{(p/2)'}(R^n\times R^m)$ such that $\|g\|_{(p/2)'} = 1$ and

$$\left\| (T^l_{\mu,\nu}(f))^2 \right\|_{p/2} = \int_{R^n\times R^m} (T^l_{\mu,\nu}(f)(x,y))^2 g(x,y) dxdy,$$

by Hölder's inequality, we have

$$\begin{aligned}\left\|T^l_{\mu,\nu}(f)\right\|_p^2 &= \left\|(T^l_{\mu,\nu}(f))^2\right\|_{p/2} \le C_{n,m}2^l\left|\Theta_l\right|\sum_{k,j}\int_{R^n\times R^m}\left|f_{k+\mu,j+\nu}(x,y)\right|^2\\ &\cdot\int_{[1,2]^2}\int_{S^{n-1}\times S^{m-1}}\left|\widetilde{\Omega}_l(u',v')g(x+2^k ru',y+2^j sv')\right|\\ &\cdot d\sigma(u')d\sigma(v')\tfrac{drds}{rs}dxdy\\ &\le C_{n,m}2^l\left|\Theta_l\right|\int_{R^n\times R^m}\sum_{k,j}\left|f_{k+\mu,j+\nu}(x,y)\right|^2 M_{\widetilde{\Omega}_l}(g)(x,y)dxdy\end{aligned}$$

where

$$\begin{aligned}M_{\widetilde{\Omega}_l}(g)(x,y) &= \sup_{k,j}\int_{[1,2]^2}\int_{S^{n-1}\times S^{m-1}}\left|\widetilde{\Omega}_l(u',v')g(x+2^k ru',y+2^j sv')\right|d\sigma(u')d\sigma(v')\tfrac{drds}{rs}\\ &\le C_{n,m}\sup_{r,s>0}\tfrac{1}{r^n s^m}\int_{\{|x|\le r\}\times\{|y|\le s\}}\left|\widetilde{\Omega}_l(u',v')g(x+u,y+v)\right|dudv.\end{aligned}$$

This implies

$$\left\|M_{\widetilde{\Omega}_l}(g)\right\|_{(p/2)'} \le C_{n,m,p}\left\|\widetilde{\Omega}_l\right\|_1\|g\|_{(p/2)'} \le C_{n,m,p}2^l\left|\Theta_l\right|.$$

Thus, by Hölder's inequality and Littlewood-Paley theory, we have

$$\left\|T^l_{\mu,\nu}(f)\right\|_p^2 \le C_{n,m,p}2^{2l}\left|\Theta_l\right|^2\left\|\sum_{k,j}\left|f_{k+\mu,j+\nu}(x,y)\right|^2\right\|_{p/2} \le C_{n,m,p}2^{2l}\left|\Theta_l\right|^2\|f\|_p^2,$$

which shows

$$\left\|T^l_{\mu,\nu}(f)\right\|_p \le C_{n,m,p}2^l\left|\Theta_l\right|\|f\|_p \qquad (p>2). \tag{8}$$

Now, let

$$T^l(f)(x,y) = \sum_{(\mu,\nu)\in\mathbf{Z}^2} T^l_{\mu,\nu}(f)(x,y),$$

we have

$$\begin{aligned}\left\|T^l(f)\right\|_p &\le \sum_{(\mu,\nu)\in\mathbf{Z}^2}\left\|T^l_{\mu,\nu}(f)\right\|_p\\ &\le \sum_{(\gamma,\delta)\in\Lambda}\sum_{(\mu,\nu)\in E^N_\gamma\times E^N_\delta}\left\|T^l_{\mu,\nu}(f)\right\|_p \overset{def}{=} \sum_{(\gamma,\delta)\in\Lambda}\Pi^l_{\gamma,\delta}(N),\end{aligned}$$

where N will be chosen later and large enough, $\Lambda = \{0,1,-1\}$, $E^N_0 = \{0,\pm1,\cdots,\pm Nl\}$, $E^N_{-1} = \{-\infty,\cdots,-Nl\}$, $E^N_1 = \{Nl,\cdots,+\infty\}$. For $(\gamma,\delta)=(1,1)$, we have that $\forall(\mu,\nu)\in E^N_1\times E^N_1$,

$$\left\|T^l_{\mu,\nu}(f)\right\|_2 \le C_{n,m}2^{l-\mu-\nu}\left|\Theta_l\right|\|f\|_2$$

by (7) and (6), thus, by (8) and interpolation, we get

$$\begin{aligned}&\left\|T^l_{\mu,\nu}(f)\right\|_p \le C_{n,m,p}2^{l-(\mu+\nu)\theta_p}\left\|f\right\|_p,\\&\Pi^l_{1,1}(N)\le C_{n,m,p}2^{l(1-N\theta_p)}\left\|f\right\|_p\end{aligned}$$

for some $\theta_p\in(0,1)$. Therefore, for sufficiently large N, we have

$$\sum_{l\ge 0}\Pi^l_{1,1}(N)\le C_{n,m,p}\left\|f\right\|_p. \tag{9}$$

Similarly, we have that for sufficiently large N (or see proof in [3]),

$$\left.\begin{aligned}&\textstyle\sum_{l\ge0}\Pi^l_{-1,-1}(N)\le C_{n,m,p}\left\|f\right\|_p\\&\textstyle\sum_{l\ge0}\Pi^l_{-1,1}(N)\le C_{n,m,p}\left\|f\right\|_p\\&\textstyle\sum_{l\ge0}\Pi^l_{1,-1}(N)\le C_{n,m,p}\left\|f\right\|_p\\&\textstyle\sum_{l\ge0}\Pi^l_{0,1}(N)\le C_{n,m,p}N\left\|\Omega\right\|_{L\log(2+L)}\left\|f\right\|_p\\&\textstyle\sum_{l\ge0}\Pi^l_{1,0}(N)\le C_{n,m,p}N\left\|\Omega\right\|_{L\log(2+L)}\left\|f\right\|_p\\&\textstyle\sum_{l\ge0}\Pi^l_{0,-1}(N)\le C_{n,m,p}N\left\|\Omega\right\|_{L\log(2+L)}\left\|f\right\|_p\\&\textstyle\sum_{l\ge0}\Pi^l_{-1,0}(N)\le C_{n,m,p}N\left\|\Omega\right\|_{L\log(2+L)}\left\|f\right\|_p\\&\textstyle\sum_{l\ge0}\Pi^l_{0,0}(N)\le C_{n,m,p}N^2\left\|\Omega\right\|_{L\log^2(2+L)}\left\|f\right\|_p.\end{aligned}\right\} \tag{10}$$

By (3), (9) and (10), we get

$$\left\|T(f)\right\|_p\le C_{n,m,p}N^2\left\|\Omega\right\|_{L\log^2(2+L)}\left\|f\right\|_p \qquad (p>2) \tag{11}$$

which proves Theorem 2(i).

Now, we shall prove Theorem 2(ii).

By duality, $\exists G=\{g_{k,j}\}_{(k,j)\in\mathbf{Z}^2}$ such that

$$\left\|\left\|\left\|G\right\|_{L^2([1,2]^2,\frac{drds}{rs})}\right\|_{l^2(\mathbf{Z}^2)}\right\|_{L^{p'}(R^n\times R^m,dxdy)}=1$$

and

$$\begin{aligned}\left\|T_{\mu,\nu}(f)\right\|_p\quad&\le\log 2\textstyle\int_{R^n\times R^m}\sum_{k,j}\{f_{k+\mu,j+\nu}(x,y)\\&\qquad\cdot(\textstyle\int_{[1,2]^2}\left|(\mathcal{I}(G))_{k,j}(x,y;r,s)\right|^2\frac{drds}{rs})^{1/2}\}dxdy\end{aligned}$$

$$\leq \log 2 \int_{R^n\times R^m} (\sum_{k,j} |f_{k+\mu,j+\nu}(x,y)|^2)^{1/2}$$
$$\cdot (\sum_{k,j} \int_{[1,2]^2} |(\mathcal{I}(G))_{k,j}(x,y;r,s)|^2 \tfrac{drds}{rs})^{1/2} dxdy$$
$$\leq C_{n,m,p} \|\Omega\|_2 \left\| (\sum_{k,j} |f_{k+\mu,j+\nu}(x,y)|^2)^{1/2} \right\|_p ,$$

which means that

$$\|T_{\mu,\nu}(f)\|_p \leq C_{n,m,p} \|\Omega\|_2 \|f\|_p \qquad \text{for } p \in (\frac{2\sigma}{2\sigma-1}, 2) \tag{12}$$

by Littlewood-Paley theory. On the other hand, by Fourier transform, it is easy to show that

$$\|T_{\mu,\nu}(f)\|_2 \leq C_{n,m} 2^{-|\mu|-|\nu|} \|\Omega\|_2 \|f\|_2 . \tag{13}$$

So, by interpolation, we get

$$\|T(f)\|_p \leq C_{n,m,p} \|\Omega\|_2 \|f\|_p \qquad \text{for } p \in (\frac{2\sigma}{2\sigma-1}, 2). \tag{14}$$

Now, for $\Omega \in L^q(S^{n-1}\times S^{m-1})$ and $1 < q \leq 2$, write

$$T(f)(x,y) = \|\tau_{\Omega,r,s}(f)(x,y)\|_{L^2(R^+\times R^+, \frac{drds}{rs})}$$

where

$$\tau_{\Omega,r,s}(f)(x,y) = \int_{S^{n-1}\times S^{m-1}} \Omega(u',v') f(x-ru', y-sv') d\sigma(u') d\sigma(v'),$$

and, set

$$\tau^z_{\widetilde{\Omega},r,s}(f)(x,y) = \int_{S^{n-1}\times S^{m-1}} \widetilde{\Omega}_z(u',v') f(x-ru', y-sv') d\sigma(u') d\sigma(v')$$

where

$$\begin{aligned} \widetilde{\Omega}_z(x',y') = \; & \Omega_z(x',y') + \tfrac{1}{\sigma(S^{n-1})\sigma(S^{m-1})} \int_{S^{n-1}\times S^{m-1}} \Omega_z(u',v') d\sigma(u') d\sigma(v') \\ & - \tfrac{1}{\sigma(S^{n-1})} \int_{S^{n-1}} \Omega_z(u',y') d\sigma(u') - \tfrac{1}{\sigma(S^{m-1})} \int_{S^{m-1}} \Omega_z(x',v') d\sigma(v'). \\ \Omega_z(x',y') = \; & |\Omega(x',y')|^{\kappa z + \frac{q}{2}} \, sign(\Omega(x',y')) \end{aligned}$$

where $\kappa \in (1-\frac{q}{2}, \frac{q}{2})$ is to be determined and z is a complex number. When $Re(z) = 0$, $\widetilde{\Omega}_z \in L^2(S^{n-1}\times S^{m-1})$, by (14), we have

$$\left\| \left\| \tau^z_{\widetilde{\Omega},r,s}(f) \right\|_{L^2(R^+\times R^+, \frac{drds}{rs})} \right\|_{L^p(R^n\times R^m)} \leq C_{n,m,p} \|\Omega\|_q^{q/2} \|f\|_p \tag{15}$$

for $p \in (\frac{2\sigma}{2\sigma-1}, 2)$. When $Re(z) = 1$, $\widetilde{\Omega}_z \in L^{q/(\kappa+\frac{q}{2})}(S^{n-1} \times S^{m-1}) \subset L(\log^+ L)^2(S^{n-1} \times S^{m-1})$, by (11), we have

$$\left\| \left\| \tau^z_{\Omega,r,s}(f) \right\|_{L^2(R^+ \times R^+, \frac{drds}{rs})} \right\|_{L^2(R^n \times R^m)} \le C_{n,m} \|\Omega\|_{L \log^2(2+L)} \|f\|_2 . \tag{16}$$

Now, for $\gamma \in (\frac{2\sigma q'}{\sigma q'+2\sigma-2}, 2)$, taking $z_0 = (1-\frac{q}{2})/\kappa \in (\frac{2}{q}-1, 1)$ and $p \in (\frac{2\sigma}{2\sigma-1}, 2)$ such that $\frac{1}{\gamma} = \frac{1-z_0}{p} + \frac{z_0}{2}$, which is possible because for $\kappa \nearrow \frac{q}{2}$ and $p \searrow \frac{2\sigma}{2\sigma-1}$ (where "$\nearrow$" means "increasingly tends to", "$\searrow$" means "decreasingly tends to"),

$$\frac{1-z_0}{p} + \frac{z_0}{2} \nearrow \frac{\sigma q' + 2\sigma - 2}{2\sigma q'},$$

and, for $\kappa \searrow 1 - \frac{q}{2}$ and $p \nearrow 2$, $\frac{1-z_0}{p} + \frac{z_0}{2} \nearrow \frac{1}{2}$. Noting that $\Omega_{z_0} = \Omega$, so, by (15)-(16) and the interpolation of analytic family of operators, we get

$$\|T(f)\|_\gamma = \left\| \left\| \tau^{z_0}_{\Omega,r,s}(f) \right\|_{L^2(R^+ \times R^+, \frac{drds}{rs})} \right\|_{L^\gamma(R^n \times R^m)} \le C_{n,m,\gamma,\Omega} \|f\|_\gamma ,$$

which proves Theorem 2(ii) because the case $\gamma \in [2, \infty)$ is a corollary of Theorem 2(i).

2.3 Proof of Theorem 3

When $\alpha = 1$, by duality and Hölder's inequality, we have

$$T^1(f)(x,y) \le \|\Omega\|_q \sup_{r,s>0} \left(\int_{S^{n-1} \times S^{m-1}} |f(x - rx', y - sv')|^{q'} d\sigma(u') d\sigma(v') \right)^{1/q'} .$$

So, by the L^p-boundedness of spherical maximal function (see [11]),

$$\left\| T^1(f) \right\|_p \le C_{n,m,q,p} \|\Omega\|_q \|f\|_p$$

for $p > \frac{q'\sigma}{\sigma-1}$. On the other hand, for $\alpha = 2$, by Theorem 2,

$$\left\| T^1(f) \right\|_p \le C_{n,m,q,p} \|\Omega\|_q \|f\|_p$$

for $p > \frac{2\sigma q'}{\sigma q'+2\sigma-2}$.

Now, for $\alpha \in (1, 2)$ and $H(\cdot, \cdot; x, y) \in \Delta^\alpha$, consider

$$T_{H_z}(f)(x,y) = \int_{R^+ \times R^+} H_z(r,s;x,y) \int_{S^{n-1} \times S^{m-1}} \Omega(u',v') \cdot f(x - ru', y - sv') d\sigma(u') d\sigma(v') \frac{drds}{rs}$$

where

$$H_z(r,s;x,y) = |H(r,s;x,y)|^{(1-\frac{z}{2})\alpha} sign(H(r,s;x,y))$$

and z is a complex number. When $Re(z) = 0$, $\|H_z(\cdot,\cdot;x,y)\|_{\Delta^1} = \|H(\cdot,\cdot;x,y)\|_{\Delta^\alpha}^\alpha \le 1$, so

$$\|T_{H_z}(f)\|_p \le \left\|T^1(f)\right\|_p \le C_{n,m,q,p} \|\Omega\|_q \|f\|_p \tag{17}$$

for $p \in (\frac{q'\sigma}{\sigma-1}, \infty)$. When $Re(z) = 1$, $\|H_z(\cdot,\cdot;x,y)\|_{\Delta^2} = \|H(\cdot,\cdot;x,y)\|_{\Delta^\alpha}^{\alpha/2} \le 1$, so

$$\|T_{H_z}(f)\|_p \le \left\|T^2(f)\right\|_p \le C_{n,m,q,p} \|\Omega\|_q \|f\|_p \tag{18}$$

for $p \in (\frac{2\sigma q'}{\sigma q'+2\sigma-2}, \infty)$. Now, for $f \in C_c^\infty(R^n \times R^m)$, taking H such that $\|H(\cdot,\cdot;x,y)\|_{\Delta^\alpha} \le 1$ and

$$T^\alpha(f)(x,y) = T_H(f)(x,y).$$

Note that for $z_0 = 2(1-\frac{1}{\alpha})$, $T_{H_{z_0}}(f) = T_H(f)$, so, by (17) and (18) and interpolation of analytic family of operators, for $p \in (\frac{2\sigma q'}{\sigma q'+2\sigma-2}, \infty)$, taking $p_0 \in (\frac{q'\sigma}{\sigma-1}, \infty)$ and $p_1 \in (\frac{2\sigma q'}{\sigma q'+2\sigma-2}, \infty)$ such that $\frac{1}{p} = \frac{1-z_0}{p_0} + \frac{z_0}{p_1}$, we have

$$\|T^\alpha(f)\|_p = \left\|T_{H_{z_0}}(f)\right\|_p \le C_{n,m,q,p} \|\Omega\|_q \|f\|_p$$

which proves Theorem 3.

2.4 Proof of Theorem 4

We shall use the rotation method and only consider the product case. Let

$$\widetilde{L}^q(S^{n-1} \times S^{m-1}) = \left\{\Omega \in L^q(S^{n-1} \times S^{m-1}) : \Omega \text{ satisfies } (1)\right\}.$$

Similarly, we can define $L\widetilde{\log^+}L(S^{n-1}\times S^{m-1})$ and $L(\widetilde{\log^+}L)^2(S^{n-1}\times S^{m-1})$. Decompose $\widetilde{L}^q(S^{n-1} \times S^{m-1})$ into four parts

$$\widetilde{L}^q = \widetilde{L}^q_{o,o} + \widetilde{L}^q_{o,e} + \widetilde{L}^q_{e,o} + \widetilde{L}^q_{e,e}$$

where "o" means odd and "e" means even, $\widetilde{L}^q_{o,e}$ consists of all $\widetilde{L}^q(S^{n-1} \times S^{m-1})$ functions which is odd in the first variables and even in the second variables, etc. Say, Refers to [2]. So, we only need to consider the boundedness of

$$S_{\alpha,\beta}(f)(x,y) = \sup_{\|\Omega\|_q \le, \Omega \in \widetilde{L}^q_{\alpha,\beta}} \left| \int_{R^n\times R^m} \frac{\Omega(u',v')}{|u|^n |v|^m} f(x-u,y-v)dudv \right|$$

where $\alpha, \beta = o$ or e.

For $\alpha = \beta = o$, we have

$$\begin{aligned} &\int_{R^n\times R^m} \frac{\Omega(u',v')}{|u|^n|v|^m} f(x-u,y-v)dudv \\ &\qquad = \frac{1}{4}\int_{S^{n-1}\times S^{m-1}} \Omega(u',v') H_{u'}H_{v'}(f)(x,y)d\sigma(u')d\sigma(v') \end{aligned}$$

where

$$\begin{aligned} H_{u'}(f)(x,y) &= \int_R f(x-ru',y)\frac{dr}{r} \\ H_{v'}(f)(x,y) &= \int_R f(x,y-sv')\frac{ds}{s}. \end{aligned}$$

By duality,

$$S_{o,o}(f)(x,y) \le \frac{1}{4}\left(\int_{S^{n-1}\times S^{m-1}} |H_{u'}H_{v'}(f)(x,y)|^{q'} d\sigma(u')d\sigma(v')\right)^{1/q'}$$

which means that

$$\|S_{o,o}(f)\|_p \le C_p \|f\|_p \tag{19}$$

by L^p−boundedness of $H_{u'}$ and $H_{v'}$, where $q' \le p < \infty$.

To continue the proof, we need a Lemma from [2]. Let R'_i $(i = 1, \cdots, n)$ and R''_j $(j = 1, \cdots, m)$ denote the $i-th$ and $j-th$ Riesz transform on R^n and R^m respectively, and

$$\begin{aligned} K'_i(x,y) &= R'_i\left(p.v.\Omega(\cdot,y)\,|\cdot|^{-n}\,|y|^{-m}\right)(x) \\ K''_j(x,y) &= R''_j\left(p.v.\Omega(x,\cdot)\,|x|^{-n}\,|\cdot|^{-m}\right)(y). \end{aligned} \tag{20}$$

Note that the functions in (20) are all homogeneous of order $-n$ in x and homogeneous of order $-m$ in y, which means that there are ω'_i, ω''_j on $S^{n-1}\times S^{m-1}$ such that

$$\begin{aligned} K'_i(x,y) &= \omega'_i(x',y')\,|x|^{-n}\,|y|^{-m} \\ K''_j(x,y) &= \omega''_j(x',y')\,|x|^{-n}\,|y|^{-m}. \end{aligned}$$

Define $\widetilde{R}'_i$ and $\widetilde{R}''_j$ as follows

$$\begin{aligned} \widetilde{R}'_i(\Omega)(x',y') &= \omega'_i(x',y') \\ \widetilde{R}''_j(\Omega)(x',y') &= \omega''_j(x',y'). \end{aligned}$$

We have

Lemma 7 *$\widetilde{R}'_i$ and $\widetilde{R}''_j$ are bounded from $L\widetilde{\log^+ L}(S^{n-1}\times S^{m-1})$ to $\widetilde{L}^1(S^{n-1}\times S^{m-1})$, and $\widetilde{R}'_i\circ\widetilde{R}''_j$ is bounded from $L(\widetilde{\log^+ L})^2(S^{n-1}\times S^{m-1})$ to $\widetilde{L}^1(S^{n-1}\times S^{m-1})$, for $i = 1, \cdots, n$ and $j = 1, \cdots, m$. And, all of them are bounded from $\widetilde{L}^q(S^{n-1}\times S^{m-1})$ to itself for $q \in (1, \infty)$.*

Now, we continue the proof of Theorem 4.

For $\alpha = \beta = e$, note that $I = \sum_i \sum_j \widetilde{R}'_i \circ \widetilde{R}''_j \circ \widetilde{R}'_i \circ \widetilde{R}''_j$ and $\widetilde{R}'_i \circ \widetilde{R}''_j(\Omega) \in \widetilde{L}^q_{o,o}(S^{n-1} \times S^{m-1})$ for $\Omega \in \widetilde{L}^q_{e,e}(S^{n-1} \times S^{m-1})$ by Lemma 7, so

$$\begin{aligned}
S_{e,e}(f)(x,y) &= \sup_{\|\Omega\|_q \le 1, \Omega \in \widetilde{L}^q_{e,e}} \left| \int_{R^n \times R^m} \frac{\Omega(u',v')}{|u|^n |v|^m} f(x-u, y-v) du dv \right| \\
&= \sup_{\|\Omega\|_q \le 1, \Omega \in \widetilde{L}^q_{e,e}} \left| \sum_i \sum_j \int_{R^n \times R^m} \frac{\widetilde{R}'_i \circ \widetilde{R}''_j(\Omega)(u',v')}{|u|^n |v|^m} \widetilde{R}'_i \circ \widetilde{R}''_j(f)(x-u, y-v) du dv \right| \\
&\le \sum_i \sum_j \sup_{\|\Omega\|_q \le 1, \Omega \in \widetilde{L}^q_{e,e}} \left| \int_{R^n \times R^m} \frac{\widetilde{R}'_i \circ \widetilde{R}''_j(\Omega)(u',v')}{|u|^n |v|^m} \widetilde{R}'_i \circ \widetilde{R}''_j(f)(x-u, y-v) du dv \right| \\
&\le C_{n,m,q} \sum_i \sum_j S_{o,o}(R'_i \circ R''_j(f))(x,y)
\end{aligned}$$

which means that

$$\|S_{e,e}(f)\|_p \le C_{n,m,q,p} \|f\|_p \qquad (\text{for } q' \le p < \infty) \tag{21}$$

by (19). Similarly, we have

$$\begin{aligned}
\|S_{e,o}(f)\|_p &\le C_{n,m,q,p} \|f\|_p \qquad (\text{for } q' \le p < \infty) \\
\|S_{0,e}(f)\|_p &\le C_{n,m,q,p} \|f\|_p \qquad (\text{for } q' \le p < \infty).
\end{aligned} \tag{22}$$

(19), (21) and (22) give the desired results.

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